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Page 228, under "Stanton, Thomas E," note that Paper No. 2129 appears in Vol. 107, instead of Vol. 110.

Transactions, Vol. 115, 1950

Page 588, lines 36 and 43, change, "shearing stress" and "shearing stresses" to "deflection."

Page 588, lines 44, change "stress patterns" to "deflection patterns."

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Page 84, in Items 26 and 27, change "million gallons" to "thousand gallons."

Page 108, in Eq. 29*b* change "$- \cos A_{12}$" to "$- \cos^2 A_{12}$."

Page 110, in line six, following Eq. 35, change "$P_2 B_1$ ($= P_3 C_1$) as radius" to "$P_2 B_1$ and $P_3 C_1$ as radii."

Page 119, in the title of Table 1*A*, change "25%" to "20%."

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WATER POLLUTION CONTROL BY INTERSTATE COOPERATION

A SYMPOSIUM

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NOTE.—Published in November, 1949, *Proceedings*. Positions and titles given are those in effect when the Symposium was received for publication.

INTRODUCTION

Within the United States, three methods have been devised for controlling the pollution of interstate waters. They are: (1) Control by interstate agreements without enforcement authority; (2) control by reciprocal legislative agreements; and (3) control by interstate pollution compacts approved by Congress. Of these measures, the interstate agreements that do not provide for enforcement are the oldest, the Great Lakes Drainage Basin Sanitation Agreement dating back to 1928. By these agreements, of which there were three in 1948, the signatory states pledge themselves to cooperate with each other in regulating the pollution of streams whose waters they share. The present Symposium does not include a paper on this type of pollution abatement.

Control by reciprocal legislative agreement was initiated in 1937, that by interstate compact in 1936. The Interstate Commission on the Delaware River Basin whose objectives are described by Mr. Allen has remained the only agency that relies on reciprocal legislative agreement without concurrence of the federal government. By 1948 four other agencies had preferred to seek approval by Congress. Two of these commissions are represented in this Symposium. The New England Interstate Water Pollution Control Commission is described in its organization and activities by Messrs. Scott and Weston. The function of this commission is to deal primarily with inland waters that traverse one or more state boundaries in their course from the mountains to the sea. By contrast, jurisdiction of the Interstate Sanitation Commission, as shown by Mr. Hess, is limited, in general, to coastal, estuarial, and tidal waters.

Although the agencies that were created by interstate compact present much the same outer form, the four commissions that were in existence in 1948 differ widely in their functions. In general, two instruments appear to be evolving: (1) Control by commission order and specification of minimum treatment requirements, including the zoning of the waters; and (2) control by coordinated investigation, advice, and promotion of uniform legislative practices, leaving to the individual states the exercise of enforcement. The Interstate Sanitation Commission will be seen to represent the first instrument and the New England Interstate Water Pollution Control Commission has created the second instrument.

The stream sanitation activities of the Tennessee Valley Authority (TVA), as described by Mr. Clark, are concerned primarily with the evaluation of the sources, nature, and degree of pollution of the waters of the valley. The TVA leaves to the states through which the waters flow the control that is to be exerted.

The control of waters that the United States shares with Canada and Mexico is a matter of international treaty. The International Joint Commission on the boundary waters between the United States and Canada has been in existence since 1912. Although these international waters are also interstate waters, the present Symposium was not planned to include a discussion of the international aspects of stream pollution control.

Control of the pollution of interstate waters has been slow in evolving, but the mid-twentieth century, as shown by the papers included in this Symposium, has witnessed a strengthening of public demand for pollution abatement and an acceleration of pace in the creation of governmental agencies to secure its benefits.

NEW ENGLAND POLLUTION CONTROL COMPACT

By Warren J. Scott,[1] M. ASCE

History of the Compact

The New England Interstate Water Pollution Control Compact is the most recent legal instrument in that field to be ratified by Congress. It grew out of the desire of the New England state sanitary engineers to develop a working plan whereby the states might cooperate in the solution of their joint problems.

At the instigation of the state health officers and the heads of the water pollution control agencies, the state sanitary engineers in the New England States drew up a draft of proposed legislation in the latter part of 1946. Some changes were made before the draft reached its final form, in the early part of 1947. The next step was to secure approval of the state legislatures. Legislation was promptly introduced in several of the states, and following the usual hearings the compact was approved through appropriate legislative action by the Commonwealth of Massachusetts and the states of Connecticut and Rhode Island. The aid of the New England Representatives and Senators was enlisted to secure ratification by Congress in 1947, and the Compact Bill (United States Senate Bill No. 1418) "Passed Senate on July 23 and was unexpectedly passed by the House on the last day."[2] The prompt action is a testimonial to the good work of the New England Representatives and Senators in the national capital. Inasmuch as the compact carries a provision that it becomes effective when approved by any two contiguous states, it immediately became effective upon ratification by Congress.

Although Maine has fewer interstate pollution problems than the other New England States, Maine is an important part of the New England region, and it is believed that all the states have much to gain and nothing to lose by participation in the compact.

When the compact was drawn up, the question arose as to the possible participation of New York State, New England's important neighbor to the west and south. The states of New York and Vermont have a common interest in large Lake Champlain, which borders on both states, and in other waters flowing between the states. To a lesser degree, the states of Connecticut and Massachusetts have a common interest, with New York, in certain waters that flow across the western boundaries of these two New England states. Already Connecticut had joined with New York and New Jersey in the Interstate Sanitation Compact, which is concerned with the pollution of some of the waters of Long Island Sound and waters in and about the New York City and near-by New Jersey areas (see Symposium paper No. 3). Because of the possible interest of New York State in the New England compact, representatives of the New York State Department of Health, upon invitation, attended some of the preliminary New England conferences. As drafted, the compact provides for the participation of New York State if and when that state wishes. The compact specifically provides for the exclusion of the waters of the Interstate Sanitation District, which are covered by a separate compact.

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[1] Director, Bureau of San. Eng., Connecticut State Dept. of Health, Hartford, Conn.

[2] *Engineering News-Record*, August 7, 1947, p. 55.

Reasons for the Compact

The idea of the compact rests first on the fact that New England is composed of states with many similar characteristics, natural resources, and interests and with numerous streams flowing across state borders. Thus, the general approach to stream pollution abatement in the six states can proceed on parallel lines to quite an extent.

The New England states are faced with certain problems of pollution of interstate streams that must be solved. The drainage area of the largest river in New England—the Connecticut River—extends widely over Vermont, New Hampshire, Massachusetts, and Connecticut. There are other interstate streams of considerable magnitude. At some points tidal waters ebb and flow in harbors or river mouths touching on the borders of more than one state. It seems illogical that each state should work independently on its own program of pollution control unless each particular valley is taken into consideration as a whole. The sensible approach to pollution control of waters for domestic water supply, for bathing, for fishing, for boating, for industrial purposes, or for navigation is to classify the waters according to use and from that point to go forward with control programs. This phase of the subject is treated in Symposium paper No. 2.

The possible enactment of federal legislation in the future also serves to stimulate the formation of interstate pollution control compacts. Even those who favor, strongly, the passage of federal legislation in this field recognize that the most lasting progress in improving interstate streams will be made through joint agreement between the states. In fact, federal bills (such as United States Senate Bill No. 418) support the idea of interstate compacts and the formation of interstate agencies. There is no question in the minds of the adherents of the New England Interstate Water Pollution Control Compact that the states themselves are in a far better position to evaluate the needs in the region with due regard for the rights and interests of all concerned than is the federal government. The combination of the interested states to improve the condition of interstate streams, and, at the same time, to protect the interests of the states without federal dictation, is an important reason for the compact. In addition, it is recognized that the United States as a whole is concerned with the general problem of water conservation; and, if federal legislation is enacted, the compact group could well represent the area in any cooperative approach.

The compact applies to interstate streams, ponds, and lakes, and tributaries and tidal waters ebbing and flowing past the boundaries of any two states.

Membership

Each state is represented by five men, as in the case of the Interstate Sanitation Commission. Although the membership will be large when the maximum number of states eventually participate in the compact, it was felt that the viewpoints of all interests should be given ample recognition. The compact provides that a representative of the state health department and a representative of any existing state water pollution control board must be members of the commission. This requirement appeared necessary in order that there might be proper coordination with the state agencies actually engaged in pollution control work and concerned with the health aspects of pollution. The compact also provides that (unless a state decides otherwise through its enabling legislation): (*a*) One member shall represent an agency acting for fisheries or conservation; (*b*) one member shall represent industrial interests; and (*c*) one member shall represent municipal interests. This designation of representative interests has been adopted by all the participating states.

Voting Power

Although a majority of the members constitute a quorum, no action of the commission imposing any obligation on any signatory state or on any municipal agency or subdivision thereof, or on any person, firm, or corporation therein, is binding unless a majority of the members from such signatory state shall have voted in favor thereof. Thus, one state might nullify the votes of the other participating states. At first glance, this provision may seem undesirable, but it is considered to be satisfactory. The writer has had opportunity to observe, at first hand, the workings of the Interstate Sanitation Commission which has a similar arrangement. He has observed a group of men without selfish interests who have been studying the pollution abatement problem for many years. It is surprising how such a group working toward a common goal shakes off any sense of partisanship and acts in close accord. It is probable that the same attitude of cooperation would be found to exist among most such interstate groups.

Technical Work

The commission may appoint a secretary who must be a professional engineer versed in water pollution. Technical experts from each state are authorized to confer. Otherwise the compact does not provide for any technical staff. At its first meeting, the commission appointed a technical advisory board consisting of the chief engineers of the state departments of health, and also of existing state water pollution control agencies. In New Hampshire the same chief engineer serves both the state department of health and the New Hampshire Water Pollution Commission. In Massachusetts and Rhode Island, pollution control work is vested in the state departments of health. Vermont has set up a new water conservation board which cooperates with the state department of health in this field. In Connecticut the state department of health and the state water commission each administer certain laws relating to pollution abatement. Maine has established a sanitary water board on which the state department of health is represented.

Classification of Streams

The commission is directed to establish reasonable physical, chemical, and bacteriological standards of water quality satisfactory for various classifications of use of receiving waters. Each state agrees to submit its classification of its own waters to the commission for approval. The first major task of the commission after organization is to establish standards for various classifications of water use. After such standards are adopted, it then becomes the task of each state to work out the classification of its interstate waters according to their use, taking into account the prescribed standards. The classification then is to be submitted by each state to the commission for approval and the commission, of course, will be in a position to see that the classifications, when approved, will fit into a coordinated plan for each river valley. There is a provision of the compact to the effect that, from time to time, the commission may make such changes in definitions of classifications and in standards as may be required by changed conditions or as may be necessary for uniformity.

Agreement to Abate Pollution

In the compact, each state pledges to provide for the abatement of existing pollution and control of future pollution of interstate waters and to maintain the waters in a satisfactory condition consistent with the highest classified use of each body of water. The compact pledges each state, after approval of the classification of its interstate waters by the commission, to work toward the establishment of programs of treatment of sewage and industrial wastes that will meet the established standards. The compact commission is authorized to make recommendations for any legislative action deemed advisable, including amendments to the statutes of the signatory states which may be necessary to carry out the intent and purpose of the compact.

Commission Not Empowered to Issue Orders

Compacts drawn between states for the purpose of controlling pollution are varied in their character. The success of the New England compact rests largely on the cooperation and support of the participating states. The commission has no power to issue orders.

It would presumably be possible for one of the subscribing states to contend in a federal court that another state was not living up to the provisions of the compact; and the existence of the compact would place the complaining state in a commanding position in a court action, probably to a greater extent than if one state sought legal action against another state on the basis of damage claims that would have to be substantiated in the particular instance. However, if the compact accomplishments should deteriorate to such a point that court action became necessary between states, the compact would unquestionably be a failure.

Summary

The New England Interstate Water Pollution Control Compact provides that New York State may join the compact with relation to waterways common to New England and New York (exclusive of the waters of the Interstate Sanitation District), and accordingly the Empire State joined in 1949.

The compact is built around a classification of interstate waters according to their highest use, and (after the establishment of standards and classifications) depends on the capacity of the cooperating states to make it effective. Its success will rest on the cooperation and support of the states. There is no provision for the exercise of broad mandatory powers by the commission, nor for the establishment of an elaborate central technical organization. The technical work will be done by the existing state pollution control agencies, with coordination through a professional engineer secretary.

The compact does not set up a short-range program. Rather, it is a long-range plan that offers the New England States an excellent opportunity to develop and carry through well coordinated steps to improve the sanitary conditions of their interstate waters. The solution of interstate pollution problems by the states is preferable to the alternative of surrendering to the federal government the jobs of appraising the needs and of deciding on the remedies. If federal water pollution abatement legislation is enacted, however, the New England compact, with other interstate compacts in this field, affords the facilities for successful coordination between the states and the federal government.

CLASSIFICATION OF MASSACHUSETTS STREAMS

BY ARTHUR D. WESTON,[8] M. ASCE

INTRODUCTION

New England interstate streams differ somewhat from streams in many other sections of the United States in that these streams, particularly those in the State of Massachusetts, are seldom used both as sources of water supply for drinking and culinary purposes and as media for the disposal of sewage and industrial wastes. However, New England streams are used to a large extent both for industrial purposes and for recreation. Under present conditions the value of the streams for industry is greater than for recreation, although the income from recreation throughout New England as a whole probably is as great as, if not greater than, that from industry. Because of such stream usage the Conference of State Sanitary Engineers of New England met on December 18, 1935, and voted to advise the New England Regional Planning Commission of the National Resources Planning Board of the need for a New England program of stream classification, based on the present and proposed highest use of streams that could ultimately be agreed upon by the various states as a foundation on which to build agreements between the states on water pollution control. A committee was appointed by that commission soon after this recommendation, and its report, containing a proposed classification, was filed with the commission early in 1941. The committee expressed the opinion that it was not fair for the downstream state to demand a high standard of stream purity from the upstream state and do little toward water pollution control in its own territory. Accordingly, the committee recommended that:

1. The National Resources Planning Board request the governor of each New England state to appoint a state committee for the classification of the state's waters in accordance with the classifications proposed in this report, such committees to include official representatives of state agencies concerned with (*a*) health, (*b*) water pollution or water conservation, (*c*) fish and game, and (*d*) planning—together with a representative of industry.
2. Each New England state committee subsequently confer with neighboring state committees as to an agreement classifing interstate waters affecting these states.
3. The New England state committees (for purposes of cooperation in a program of classification of New England interstate waters) submit for review to the New England drainage basin committees of the National Resources Planning Board their findings as to such classifications. Such findings, however, were to involve no commitments between states as to the sanitary works needed.

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[8] Director and Chf. San. Engr., Div. of San. Eng., Massachusetts Dept. of Public Health, Boston, Mass.

4. Every effort be made by the New England States to establish sampling points in interstate waters for dissolved oxygen and bacteriological tests, such tests to be made with sufficient frequency to permit the determination of the quality index of the waters.

As a result of these recommendations, the chairman of the New England Regional Planning Commission communicated with the governors of the various states on February 5, 1941, suggesting the appointment of a committee from each state to work with similar committees of the other New England States for the purpose of arriving at some agreement on the classification of waters common to more than one state. The Massachusetts committee, appointed on February 12, 1941, by the Hon. Leverett Saltonstall, then Governor of Massachusetts, consisted of the writer as chairman and Raymond J. Kenney, commissioner of the Department of Conservation, Richard K. Hale, M. ASCE, of the Department of Public Works, and Otis D. Fellows of the State Planning Board. This committee soon met with the committee from Connecticut.

In a comparatively short time classifications were made of the streams common to Massachusetts and Connecticut which were agreed upon by the chief engineers of the Connecticut Water Commission and the Connecticut Department of Health and the Massachusetts Department of Public Health. These three agencies have mandatory powers over water pollution control in these states. Because of emergency activities during World War II, little was done by these agencies until 1946 when the engineering representatives of the state agencies in charge of water pollution control in the other New Engalnd States met with Massachusetts and Connecticut engineers; and finally on October 2, 1946, these engineers agreed on a system of classification. The engineers' report was submitted to their respective commissioners with the recommendation that the commissioners meet and agree on the proposed system and start negotiations through their respective state governments for the purpose of providing compact legislation in each of the New England States. The system of classifications thus agreed on, which is predicated on stream usage, is as follows:

Classification	Description
Class A......	Waters intended for drinking or for the cultivation of market shellfish.
Class B......	Waters used for bathing.
Class C......	Waters used for recreational boating, fishing, culture of seed oysters, or industrial supply after treatment.
Class D......	Waters used primarily for commercial navigation or for transportation of wastes without nuisance.

In connection with the first two of these classes, bacterial limits were placed on the water. For example, the bacterial quality of class A waters would be better than 50 coliform organisms per 100 ml of water examined, whereas the coliform density of class B waters would be better than 1,000 coliform organisms per 100 ml. There were certain other requirements in connection with these classifications; for example, class A waters and class B waters would be clear and would contain dissolved oxygen (D.O.) nearly to saturation; class C waters would be free of slick, odors, or visible floating solids and would contain D.O. in quantities adequate for fish life; class D waters also would be free of slick, odors, and visible floating solids and would have some D.O. present at all times.

It should be emphasized at this point that, although the recommendations of the engineers of the New England States may have represented the opinions of those state departments having the responsibility for water pollution control, they carried no legal significance except in so far as the laws of their states would permit water pollution control. The fact that the engineers agree on this system of classification, however, probably would have considerable weight in the determinations to be made by the states under compacts which have been made by the states of Massachusetts, Rhode Island, and Connecticut under legislative acts of these states passed in 1947 and which it is hoped will be entered into by the other New England States at no distant future. The State of New York, by an act passed in April, 1949, and the State of Vermont, by an act passed in May, 1949, are eligible to join the compact.

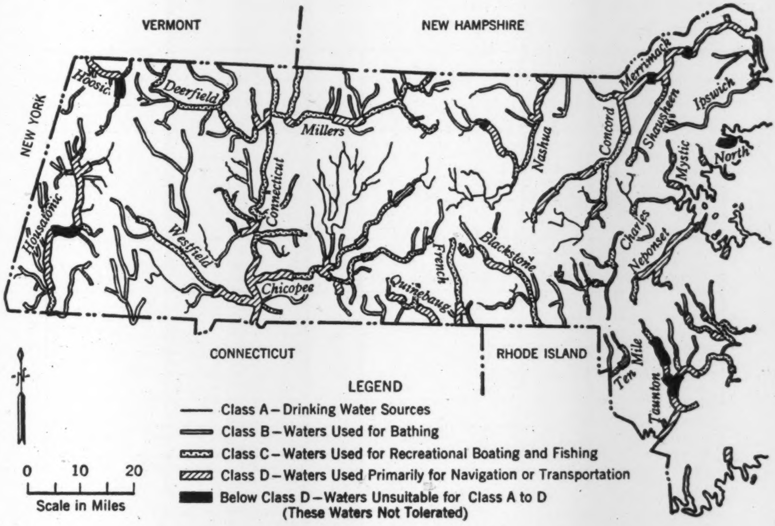
VERMONT
NEW HAMPSHIRE
NEW YORK
Hoosic
Deerfield
Millers
Connecticut
Westfield
Housatonic
Chicopee
Quinebaug
French
Blackstone
Nashua
Concord
Merrimack
Shawsheen
Ipswich
Mystic
North
Charles
Neponset
Ten Mile
Taunton
CONNECTICUT
RHODE ISLAND
LEGEND
Class A – Drinking Water Sources
Class B – Waters Used for Bathing
Class C – Waters Used for Recreational Boating and Fishing
Class D – Waters Used Primarily for Navigation or Transportation
Below Class D – Waters Unsuitable for Class A to D (These Waters Not Tolerated)
0 10 20
Scale in Miles

Fig. 1.—Classification of Massachusetts Streams According to Highest Use

The State of Massachusetts can be considered as the middle ground of the pollution of interstate rivers in New England. Accordingly, the map in Fig. 1 is presented to show the average condition of the waters of that state in 1941, according to the standard classification. In actual use, colors represent the agreed classifications as follows: Class A waters in green, class B waters in blue, class C waters in yellow, and class D waters in red. In Fig. 1, instead of colors, the various classes are identified by width of stream, the narrowest representation indicating the class A waters. Certain streams, shown solid in Fig. 1, would normally fall into class E waters if such a classification existed. The New England Conference of State Sanitary Engineers has recommended that such waters, devoid of oxygen and capable of causing a nuisance, should not exist in any New England state.

Of the various rivers in Massachusetts shared with some other state, those of most importance are the Housatonic River and the Hoosic River in the western part of the state, the Connecticut River which flows through the west central part, the French River which joins the Quinebaug River in Connecticut to form the Shetucket River and thence the Thames River; the Blackstone River which has an outlet through the State of Rhode Island; the Taunton River which has an outlet into the tidal waters of Massachusetts and of Rhode Island; and the Merrimack River which has an outlet into the sea through the State of Massachusetts.

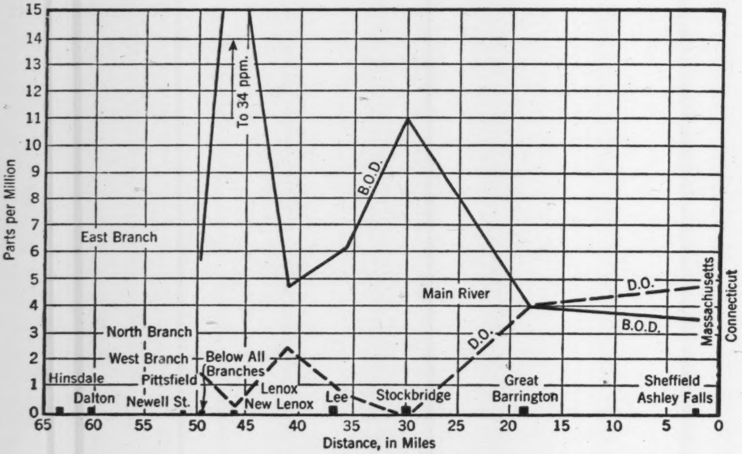

FIG. 2.—BIOCHEMICAL OXYGEN DEMAND (B.O.D.) AND DISSOLVED OXYGEN (D.O.), HOUSATONIC RIVER, JULY, 1941

HOUSATONIC RIVER

The Housatonic River, about 65 miles long in Massachusetts, has a total drainage area (at its mouth in Connecticut) of approximately 1,930 sq miles, of which 487 sq miles is in the State of Massachusetts. The drainage area above the Massachusetts-Connecticut line comprises 532 sq miles, of which 12 sq miles is in Connecticut and 33 sq miles is in New York. Under present conditions in Massachusetts this river is polluted by untreated and treated sewage and industrial wastes. During a period of low water conditions such as existed in July, 1941, the biochemical oxygen demand (B.O.D.) of this stream in certain parts of its course was as high as 34 ppm with practically zero D.O. (see Fig. 2). Nevertheless, the stream recovered before reaching the

State of Connecticut so as to rate a clsasification, at the state line, of at least class D. Practically equal or better bacterial improvement in this river was noticed during the dry period of October, 1947, when coliform densities of 100,000 per 100 ml of water examined were reduced to 100 per 100 ml near the state line. Because of its use as a conveyer of sewage effluent and industrial wastes throughout much of its course in the State of Massachusetts, this river must be classified as not better than class D. In view of the fact that Massachusetts laws permit action by the Department of Public Health to control stream pollution, it is to be expected that this stream may well reach the class C classification before entering the State of Connecticut.

Hoosic River

The Hoosic River drains part of the northwestern section of Massachusetts and the southwestern section of Vermont. It enters New York State a short distance below Williamstown, Mass., where it becomes an important tributary to the Hudson River at Mechanicville, N. Y. The course through Massachusetts has a drainage area of 204 sq miles, of which 39.5 sq miles is in Vermont.

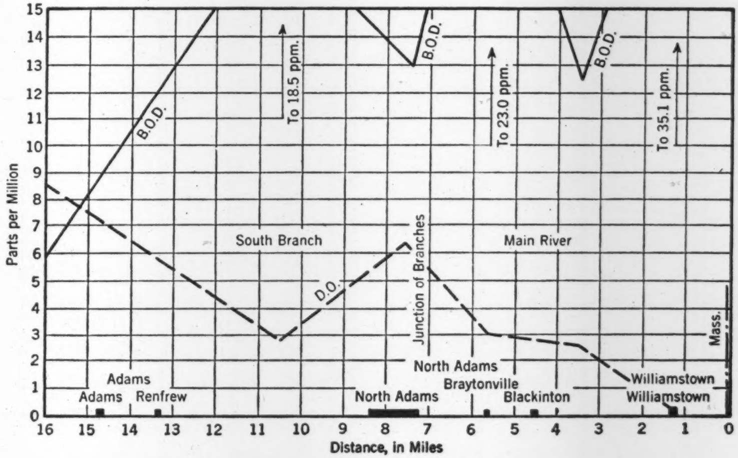

Fig. 3.—Biochemical Oxygen Demand (B.O.D.) and Dissolved Oxygen (D.O.), Hoosic River, August, 1939

This river receives considerable pollution from the untreated sewage of the towns of Adams and Williamstown, from the partly treated sewage of the City of North Adams, and from the industrial wastes principally from Adams and North Adams. The flow in this stream during the summer months of a dry year like 1941 has varied on a monthly basis from 0.372 cu ft per sec per sq mile to 0.529 cu ft per sec per sq mile; and it averaged 0.392 cu ft per sec per sq mile.

In flowing some 16 miles from a point above Adams to a point below Williamstown, during August, 1939, the Hoosic River varied in B.O.D. from about

6 ppm above Adams to 35 ppm in parts of its course in Williamstown. In D.O. content it varied from 0 ppm to about 8.7 ppm (see Fig. 3). Under the minimum conditions of flow in the Hoosic River in 1941 the B.O.D. load at Williamstown probably was in excess of the amount usual for a population of more than 86,000 persons. The census population of Adams, North Adams, and Williamstown totaled 39,115 in 1940. Obviously this stream does not fall within class D and probably will not until facilities for more complete treatment of domestic sewage and reasonable treatment of the industrial wastes have been provided.

CONNECTICUT RIVER

The Connecticut River has its source in the Connecticut Lakes in northern New Hampshire. It flows in a southerly direction between New Hampshire and Vermont, through the west central part of Massachusetts, and thence through the State of Connecticut to its mouth in Long Island Sound. Next to the Saint John River in Maine and New Brunswick, Canada, it is the largest river in New England. Of its total drainage area of some 11,345 sq miles, 105 sq miles is in Canada; 3,120 sq miles, in New Hampshire; 3,970 sq miles, in Vermont; 2,720 sq miles, in Massachusetts; and 1,430 sq miles, in Connecticut. From where the Connecticut River enters Massachusetts to where it leaves the state (a distance of about 67 miles) it falls about 135 ft. From the Enfield Dam near the Massachusetts-Connecticut line, still water extends nearly to the dam at Holyoke, Mass., a distance of about 17 miles.

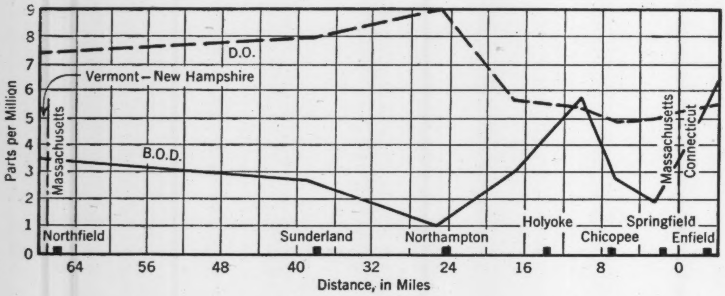

FIG. 4.—BIOCHEMICAL OXYGEN DEMAND (B.O.D.) AND DISSOLVED OXYGEN (D.O.), CONNECTICUT RIVER, AUGUST, 1941

According to the analyses available to the Massachusetts Department of Public Health, at the Massachusetts-Connecticut line, in August, 1941, the Connecticut River had a B.O.D. of only 3.5 ppm as compared with a B.O.D. of 3.4 ppm at Northfield near where it enters Massachusetts and a D.O. content of 5.3 ppm as compared to one of 7.4 ppm at Northfield (see Fig. 4).

During August, 1941, the flow in the vicinity of Montague (Turners Falls) near where the stream enters Massachusetts, and at Thompsonville, Conn., near where it leaves the state, was as given in Table 1.

The B.O.D. load in August, 1941, on this river, as it left Massachusetts was about 75,200 lb as compared to a B.O.D. load of 48,600 lb when the river entered the state, an increase of about 55%. It should be reasonably possible to increase the classification of this stream to class C.

TABLE 1.—FLOW OF THE CONNECTICUT RIVER AT THE POINT OF ENTRY INTO AND THE POINT OF EXIT FROM MASSACHUSETTS

Point of observation	Drainage area, in square miles	FLOW DURING AUGUST, 1941		
		In cubic feet per second per square mile	In cubic feet per second	In million gallons daily
Montague (Turners Falls), Mass.	7,163	0.370	2,652	1,714
Thompsonville, Conn.	9,661	0.413	3,985	2,576

FRENCH RIVER

Beginning at Spencer and Leicester, Mass., the French River flows in a southerly direction from 25 miles to 30 miles to Thompson, Conn., where it enters the Quinebaug River. It has a drainage area of only about 112 sq miles but receives considerable pollution from the towns of Webster and Dudley, Mass., and this pollution has been a source of complaint from the State of Connecticut.

Analyses and tests made by the Massachusetts Department of Public Health in October, 1941, showed that the B.O.D. of the stream at that time, near where it enters the State of Connecticut, was 14 ppm, with practically no D.O. present.

BLACKSTONE RIVER

The Blackstone River has its origin in the vicinity of Worcester, Mass., and flows in a southerly direction a distance of about 26 miles to Rhode Island. In that distance it drains a large section of the City of Worcester and much of an area of about eleven towns in Massachusetts. The population of these municipalities, according to the 1940 census, was about 249,000. About 311 sq miles of its total drainage area of 475 sq miles is in Massachusetts. Inasmuch as this river is the only outlet for much of the drainage of these municipalities, and inasmuch as its valley is highly industrialized, it receives much pollution in the form of raw or treated sewage and industrial wastes. Most of this pollution in recent years has had its source in the discharge from the sewage works of the City of Worcester and in the acid and iron wastes from the steel mills of that city. These latter wastes caused considerable damage to the sewage treatment plant during World War II.

In spite of the pollution received in its passage through the state, this stream, during the tests made by the Massachusetts Division of Sanitary Engineering in October, 1941 (during a period of reasonably low runoff) had a B.O.D., near where it entered Rhode Island, of only 3.7 ppm.

TAUNTON RIVER

The Taunton River is the largest stream in the southeastern section of Massachusetts. Its total drainage area above Slades Ferry Bridge, so called, in

Fall River, Mass., has been estimated to be about 529 sq miles. Below East Taunton, Mass., this stream is tidal and navigable for a distance of some 6 miles after which it enters a more or less broad channel to the head of Mount Hope Bay. This bay borders on both Massachusetts and Rhode Island.

The principal sources of pollution of this river, in Massachusetts, have been the domestic sewage and industrial wastes from Taunton and Fall River. This pollution (see Fig. 5) has been sufficiently pronounced in the City of Taunton to cause serious complaint in those sections where the water was devoid of oxygen and to cause great loss of fish life during run of alewives (see Fig. 6). Action through the courts by the Massachusetts Department of Public Health resulted in court stipulations from which have come the installation of industrial waste treatment works in the City of Taunton and engineering studies and plans for intercepting sewers and sewage treatment works, in that city, contracts for which have been let. The City of Fall River also has prepared plans for intercepting sewers and sewage treatment works, contracts have been awarded and the sewage treatment plant is nearly completed. At no time have the waters of Mount Hope Bay been found to be polluted to such an extent as to reduce the oxygen content materially, but the bacteriological condition of the water of the river near its mouth, at no great distance from the bay, has been serious enough as to warrant the closing of certain bathing beaches in the City of Fall River.

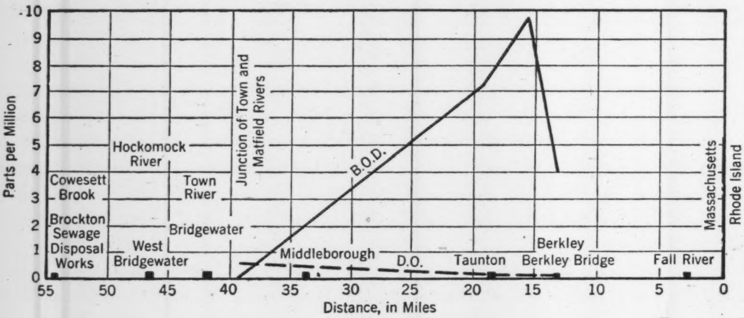

FIG. 5.—BIOCHEMICAL OXYGEN DEMAND (B.O.D.) AND DISSOLVED OXYGEN (D.O.), TAUNTON RIVER, AUGUST, 1944

MERRIMACK RIVER

The Merrimack River, which has its source in the junction of the Pemigewasset and Winnepesaukee rivers in the State of New Hampshire, has a total watershed of about 5,015 sq miles of which about 1,200 sq miles is in Massachusetts and about 3,800 sq miles is in New Hampshire. Of the 1,200 sq miles in Massachusetts about 669 sq miles is between the New Hampshire boundary and the sea. This section of the river, some 49 miles long, has given considerable concern because of pollution. The flow of this river, as recorded by the Essex Company at Lawrence, Mass., during September, 1942, averaged 0.435 cu ft per sec per sq mile. The water of the river (see Fig. 7) at the time of sampling, also in September, 1942, contained about 6 ppm of D.O. and had a B.O.D. of about 2 ppm at the Massachusetts–New Hampshire state line. By the time that the river reached a point below Lowell, Mass. (a distance of about 11 miles below the state line), the D.O. had been reduced to 0 ppm and the B.O.D. had increased from about 2.5 ppm to 30 ppm. At a point above Lawrence (a further distance of about 7 miles) near the intake to the City of Lawrence water treatment works, the B.O.D. had been reduced to nearly 5 ppm and the D.O. had increased to about 6.5 ppm. The pollution contributed by Lawrence and its vicinity increased the pollution load to 26 ppm and reduced the D.O. to somewhat less than 3 ppm. Below Haverhill, Mass., as a result of pollution from that city and of the pollution above the city, the D.O. was again reduced to 0 ppm and, even at Newburyport, Mass., where the waters are tidal, the D.O. was only a little more than 1 ppm. As it flows through Massachusetts, to about 49 miles below the New Hampshire boundary, the Merrimack River could not be classified better than class D, and in parts of its course it would be classified as class E.

Because of these conditions the General Court in session in 1946 appropriated funds for a joint board, consisting of the Department of Public Health and the Merrimack River Valley Sewerage Board, to investigate and study, and to prepare plans and maps for the disposal of sewage in the Merrimack River Valley. This appropriation was not sufficient to provide for more than a preliminary investigation of the cause and the effect of the pollution with estimates of cost of constructing remedial works; but, with the assistance of the laboratories of the Massachusetts Department of Public Health Sanitary Engineering Division, the consulting engineers conducted a very complete investi-

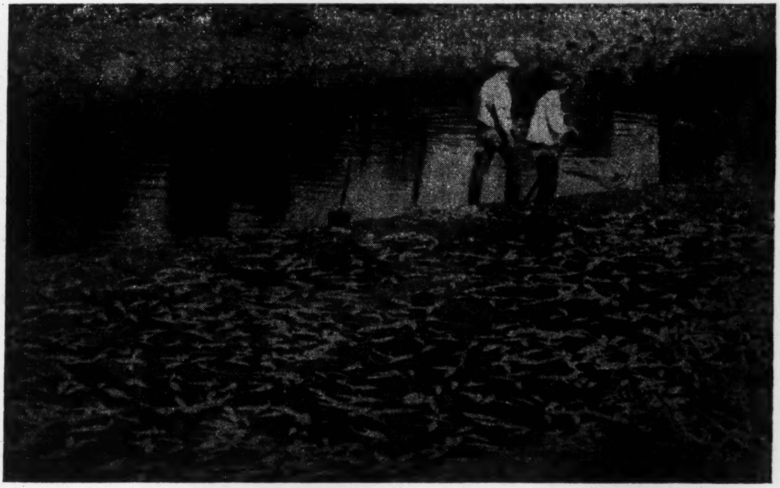

FIG. 6.—DEAD FISH IN THE TAUNTON RIVER

gation, the results of which have been printed in a legislative document (Massachusetts Senate Document No. 550, 1947). As a result of this investigation, the joint board recommended that the following specific construction projects be undertaken concurrently or separately as fast as funds can be made available:

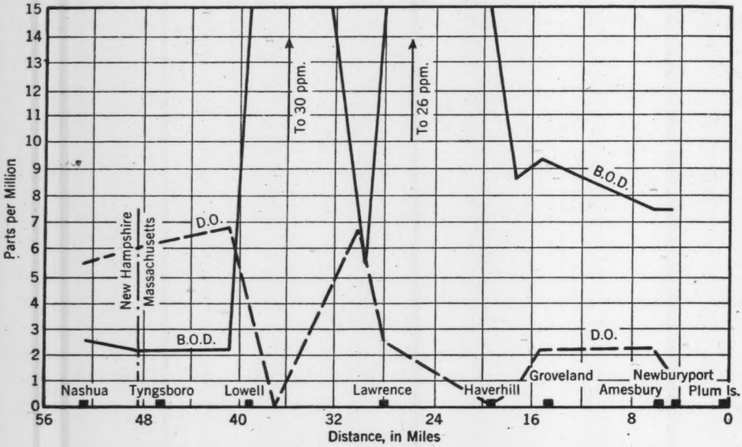

Fig. 7.—Biochemical Oxygen Demand (B.O.D.) and Dissolved Oxygen (D.O.), Merrimack River, September, 1942

Projects	Estimated construction cost
Lowell metropolitan region, intercepting sewers and activated sludge treatment plant with chlorination.........	$ 7,650,600
Lawrence metropolitan region, intercepting sewers and activated sludge treatment plant with chlorination.....	14,080,500
Haverhill metropolitan region, intercepting sewers and primary sedimentation treatment plant with chlorination..	4,164,500
Newburyport Harbor, intercepting sewers and primary treatment plants with chlorination at Amesbury, Newburyport, and Salisbury, Mass....................	1,685,500
Total....................................	$27,581,100

The enabling legislation to implement these projects was submitted to the voters in the district at the biennial state election in 1948 and was defeated.

It is interesting to note from these observations that, although the principal interstate streams common to Massachusetts are polluted (and in some cases grossly polluted) in their passage through the state, the self-purification of these streams, plus the present treatment (however inadequate) of sewage and industrial wastes discharged into them, has been sufficient in most cases to prevent a nuisance in any neighboring state. Similarly, it might be stated that, although some of these streams are grossly polluted in their courses before entering Massachusetts, none of them is sufficiently polluted to cause a nuisance. In other words, as they enter the state all these streams would be equal to, or better than, class D streams; and all (with the exception, at times, of the French River entering Connecticut and the Hoosic River entering Vermont) have been found to be equal at least to that classification.

Reasonable compliance with the rules and regulations enforceable by the Massachusetts Department of Public Health, as a result of the passage of Chapter 615 of the Acts of 1945, should convert all these streams, in most parts of their courses, to a standard at least equal to class C; but, in many cases, improvements equal even to class C will require facilities for complete treatment of sewage and chlorination of the effluent of treatment works. A grading beyond class C probably cannot be obtained unless the states can agree that bathing in river waters containing purified sewage and industrial waste is sufficiently necessary to warrant such classification.

The compact agreements now made possible by acts of legislatures in Massachusetts, Rhode Island, and Connecticut (and later in Vermont, New Hampshire, and Maine) offer an effective method of improving interstate streams which did not exist prior to 1947. The New England Conference of State Sanitary Engineers has expressed the opinion that it does

> "* * * not appear fair * * * for a downstream state to call for high standards of water in the upstream state and then do little in its own area to abate pollution * * *."

Just how the New England "Interstate Water Pollution Control Compact Commission" will respect this opinion is questionable. The standards of waters (classes A to D) agreed upon by the New England Water Pollution Control Commission on February 11, 1949, would indicate that there is little disagreement between the states.

SUMMARY

As early as 1935 the Conference of State Sanitary Engineers of New England met and agreed on a plan of stream classification for water pollution control consistent with the highest use, and agreed to undertake a systematic project of sanitary surveys of all New England interstate waters. This plan was accepted by the commissioners of the health departments and by the water commissioners having responsibilities in water pollution control. In carrying out its part of the plan, the Division of Sanitary Engineering of the Massachusetts Department of Public Health has made sanitary surveys. During a summer of minimum stream flow, like that of 1941, the principal interstate streams of the State of Massachusetts were found to be grossly polluted in certain sections of the state; but, before leaving the state, even in the absence of needed remedial sewage and industrial waste treatment works, some of these streams were in as satisfactory condition as they were upon entering the state. From the analysis of these surveys, however, there appears need for improvement to make these streams of greater value for recreational purposes and for fish life. Under acts of the state legislature the Massachusetts Department of Public Health is attempting to effectuate such improvement and it is hoped that by working with the compact commission this program for improvement will be expedited. As of September, 1949, the Massachusetts Department of Public Health has succeeded in bringing about the starting of twenty-eight municipal sewage treatment works or preparation of plans for such works.

SANITATION COMMISSION FOR NEW YORK, NEW JERSEY, AND CONNECTICUT

BY SETH G. HESS,[4] M. ASCE

The Interstate Sanitation Commission is an agency of the states of New York, New Jersey, and Connecticut, functioning through a tri-state compact and the legislation of the respective states. Its jurisdiction extends from Sandy Hook in New Jersey, through New York Harbor (including Arthur Kill, the Kill van Kull, and Newark Bay), up the Hudson River in New York State to the northern lines of Westchester and Rockland counties just above Bear Mountain Bridge. In Long Island Sound, it extends to New Haven, on the Connecticut shore, Port Jefferson, N. Y., on the north shore of Long Island, and on the Atlantic Ocean front to Fire Island Inlet. In general, this jurisdiction covers only the coastal, estuarial, and tidal waters.

Established in January, 1936, the commission is the first interstate agency for the control of pollution. The Interstate Sanitation Commission differs from most subsequent commissions, in that it functions under a tri-state compact and certain definite obligations, duties, and authority are vested in the commission by the three participating states. In addition to obligating the commission to the control and the abatement of pollution, the compact or treaty provides that existing state agencies cooperate with the commission in the accomplishment of its tasks. Accordingly, the commission has sought and received the fullest cooperation of the various state agencies; and, wherever possible, the commission has functioned through them.

The Tri-State Compact establishes definite chemical, bacteriological, and physical standards that must be met before any sewage or other polluting matter may be discharged into the waters of the Interstate Sanitation District. It is not necessary to prove the existence of a health menace. Accordingly, after consulting with the state agencies, it has been found, on occasions, that the commission can function more expeditiously under its laws than can the state agencies under their laws.

This compact authorizes the commission to issue orders to a municipality or other entity, requiring treatment of its sewage or other polluting matter, to meet established requirements within prescribed time limits. When necessary, the commission can enforce its orders by bringing action in its own name in the proper court or courts. In this respect, the Interstate Sanitation Commission differs markedly from most other interstate pollution control agencies.

Notwithstanding the authority vested in it, the commission has made every effort: (*a*) To avoid the use of preemptory power to obtain its objectives, by encouraging a cooperative attitude on the part of the municipalities and other entities; and (*b*) to cooperate with, and work through, state agencies wherever such methods give promise of success.

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[4] Director and Chf. Engr., Interstate Sanitation Comm., New York, N. Y.

The commission has functioned since January 24, 1936; and, despite the war interruption of almost 5 years during which no construction was possible, new plants have been placed in operation for the treatment of sewage from almost half of the population of the district.

The activities of the Interstate Sanitation Commission are numerous and varied and, for the purpose of this Symposium, it will suffice to enumerate and describe briefly only the more important functions. Public education is always of prime importance in undertaking any type of pollution abatement program. From its very inception, the commission recognized the need of carrying its story to the people. Speakers have appeared before various service organizations, such as the Rotary, Lions, Kiwanis, and various local clubs. Publicity has been further supplemented by newspaper articles on timely subjects which were affected by the commission's activities.

The principal duties of the Interstate Sanitation Commission can be divided into two major classifications—(1) abatement of pollution and (2) control of pollution.

(1) *Abatement of Pollution.*—The principal feature of the pollution abatement program concerns the construction of sewage treatment works by municipalities and other entities. In general, the members of the municipal governing bodies recognize the need for pollution abatement, but they are faced with two obstacles. The first is the difficulty of financing the proposed pollution abatement works, and the second concerns the reluctance of the taxpayer to assume the tax obligation and to recognize the advantages of pollution abatement. The commission has authority to order the construction of these necessary projects; but, since its policy has been to make every effort to accomplish these objectives through cooperation, it has worked closely with the municipal authorities to find ways and means of financing the necessary projects. Secondly, by education, the commission has sought to inform the public as to the advantages of sewage treatment works and the benefits derived.

In rounding out the over-all program, it has been the commission's objective to set up reasonable time schedules for each municipality, fixing the ultimate date of completion of the sewage treatment works and intermediate dates for various steps leading to the ultimate goal. During the unsettled times of World War II and the unstable construction market following the war, these time schedules had to be amended. As the construction industry has become more stabilized, there is less need for amending and postponing the dates established in the schedules in recent orders.

When municipalities show evidence of establishing their own schedules which conform to the over-all program of the district, the commission has refrained from issuing an order. Occasionally, it has been desirable to issue orders with the cooperation of the municipality, thereby fixing the established time schedule in a more formal manner. In a few cases the commission has found it necessary to make its own determination of a reasonable time schedule and to formalize it in the issuance of an order to a municipality. Should the municipality fail to give evidence of making an honest effort to conform to the time schedule, the commission is then forced to resort to legal action. There were only two such cases in the first 12 years of the commission's existence.

In general, the smaller municipalities have been given approximately 3 years for the preparation of plans and for the completion of construction. Obviously, some of the larger municipalities, of which New York, N. Y., is the outstanding example, will require more than 3 years to accomplish the huge amount of work involved in their programs. Notwithstanding the longer periods of time involved, almost without exception, the larger municipalities have prepared construction plans in record time. New York, faced with a difficult problem of financing, is working toward a satisfactory solution to its problem of implementing a comprehensive pollution abatement program which, in 1949, was about 50% complete, in 1953 will be 75% complete, and in 1959 will be treating all its sewage.

The second group of entities with which the commission must concern itself, are privately owned. Some of these are industries, and others are private real estate developments. By far the greater number of industries in the Interstate Sanitation District are located close to municipal sewers. In most cases, the industrial flow is of such quality and quantity that it can be discharged into the municipal sewers and treated at the municipal sewage treatment plant. There is strong evidence that, by providing adequate municipal sewage treatment, much of the industrial pollution in the Interstate Sanitation District will be eliminated. Concurrently industries not favorably located with respect to municipal sewage treatment plants, or whose effluent is of such a nature that it cannot be accepted by the municipality, have shown a very cooperative attitude. In one instance, an industry spent more than $1,000,000 in 4 years for the development of a process to abate pollution.

Private real estate development is no problem in the Interstate Sanitation District, since private treatment works are made a part of almost every new project.

Another phase of the commission's pollution abatement activities concerns the complaints registered and reported with the commission. If possible, a complaint is referred to the proper state or local agency; but, if the complaints are of an interstate nature, they are investigated and acted upon by the commission. Frequently, interstate conditions suggest joint conferences, as a result of which the complaints can be handled by local agencies.

(2) *Control of Pollution.*—The control of pollution is the second major phase in the activities of the Interstate Sanitation Commission. Existing sewage treatment works are required to provide treatment so that the effluent meets the standards established in the Tri-State Compact. The commission maintains a field staff and a laboratory for the purpose of control and to enable it to certify to each state that the plants in other states are functioning in the required manner.

The commission uses extreme care to avoid duplication of efforts. If a state agency is providing adequate inspection, sampling, and laboratory control, the commission merely verifies the work to a degree that will permit proper certification to the other states. If no adequate laboratory control is exercised by other agencies, the commission field staff spends 8 consecutive hours at the treatment plant taking composite samples and making periodic observations of chlorine application, residual chlorine, flow, and other details, as well as making determinations of the number of coliform organisms present in the effluent. Laboratory determinations include removal of suspended solids and B.O.D. Several such investigations are carried out during the year at each plant.

The result of these field investigations is of genuine value. The accumulated data form a background for the appraisal of the character of the sewage and the effectiveness of the treatment plants in the Interstate Sanitation District. Each year, requests are received for the publication of these data and they have been made an integral part of published annual reports.

Several additional phases of the control of pollution are part of the commission's activities, such as a careful daily review of the reported use of chlorine for a reduction of the coliform organisms. Any serious deviation from the compact requirements results in proper steps to assure compliance. The commission has likewise concerned itself with the proper use of by-passes and the establishment of controls, with the cooperation of other local and state agencies. In this connection, an important function is to prevent the misuse of by-passes, and their continued operation, following the emergency that called them into use.

Storm water overflows are important adjuncts to the plant required for pollution control. Many municipalities in the Interstate Sanitation District have combined sewers, whereas others have sewer systems of such a nature that they become severely overcharged during wet weather. The commission attempts to assure itself that overflows will be functioning only during the time when they are required. Likewise, it aims toward a more rational approach to the design of intercepting sewers in existing combined sewer systems. Many municipalities in the area, having combined sewer systems, plan to construct sewage treatment works. The intercepting sewer leading to the treatment works is designed to accommodate the dry weather flow plus the first flush of storm water. As a rule, the first flush of storm water is assumed to be a factor of the dry weather flow, or is assumed to be some arbitrary quantity unrelated to either the storm or the flow of sewage that will pass into the stream while storm water is permitted to overflow the regulator. As a result of the preliminary studies, it appeared possible to design interceptors to provide for the dry weather flow plus the intercepted runoff that occurs in a predetermined number of hours per year. To accomplish this, curves were developed to show the frequency of rainfalls at various intensities. The intensities involved are small and the frequency of storms are large as compared with the data presently available in published time-intensity curves. Accordingly, it was necessary for the commission to review rainfall data and develop time-intensity curves of frequencies occurring more than once a year and for intensities of from 0.6 in. per hr to 0.15 in. per hr. By the use of these data it would become possible to estimate the frequency of overflows as well as the actual volume of overflow. Although it is recognized that the sewage carried in the overflow is diluted by rain water, it nevertheless adds a pollution load to the receiving waters and is worthy of consideration in intercepting sewer design. A summary of the preliminary study on this subject is included in the 1947 Annual Report of the commission.

Summary.—Although the commission is clothed with authority to compel a municipality or other entity to take steps to abate pollution, the commission has operated through, and with, the assistance of state and local agencies concerned with pollution abatement as well as with federal agencies.

The cooperation between the commission and the various state and county agencies charged with pollution abatement is so close that it is sometimes difficult to discern where the efforts of the state or local agency have ended and where those of the commission have begun. The commission has used the power granted it only to supplement and to strengthen the authority of the state agency in bringing about a mutually desired result. The orders of the commission are based on reasonable dates of accomplishment, giving due regard to over-all conditions that may control, such as restricted materials and an unstable construction industry, as well as local conditions. The various state agencies concerned with pollution abatement are represented on the membership of the Interstate Sanitation Commission so that this commission provides a true and intimate integration of all the agencies having this common interest.

RIVERS AND HARBORS IN THE DELAWARE RIVER BASIN*

BY JAMES H. ALLEN,[5] M. ASCE

The degree to which the United States can conserve its natural resources, under its fundamental principles of constitutional government, will determine how long this nation can be a "beacon" for other nations. In its allotted sphere of activity, the Interstate Commission on the Delaware River Basin (Incodel) offers an example of a technique for such conservation that is consonant with well-proved democratic procedures. Incodel subscribes to, and functions in accordance with, the principle that the determination of policies and programs for the development of natural resources is primarily the responsibility of local and state governments, and of the people in the region most directly concerned. It freely admits that the central government—the federal government—has a tremendous stake in such programs; but it contends firmly that the entire problem must be approached on the basis of a cooperative partnership in which all interests participate alike. In this spirit, Incodel has been given authority to act, not only on sanitation problems, but on all aspects of river and harbor development.

When it was organized, in 1936, Incodel selected as its first objective the development of a sound and practical program for the sanitary improvement of the Delaware River and its tributaries. The plan was founded on the assumption that the state and local governments immediately interested are perfectly competent to perform the necessary tasks. The fact that little or no progress had been made in prior years is explained by the lack of a coordinating agency. It was formerly a common practice to induce industry, and people, into a region by "chamber of commerce" ballyhoo of all kinds, offering the privilege of dumping untreated water and filth into the nearest watercourse. When the ill effects of that haphazard practice became apparent, the separate agencies responsible for its correction began "passing the buck." In the circumstances, this reaction was natural; but, as long as it continued, progress was impossible. The answer was a program for the coordination and harmonization of all conflicting policies and efforts.

To this end, Incodel invited the chief engineers of the state health departments of New York, New Jersey, Pennsylvania, and Delaware to meet, present their individual problems and views, and formulate a joint plan of correction that would be mutually acceptable and advantageous to the entire region. The advisory committee thus formed met in monthly conferences for more than one and a half years, and produced a plan for correcting the sanitation problems of the Delaware Basin. The plan is simple and effective. It divides the Delaware River and its tributary watersheds into four zones, and prescribes a set of minimum standards for each zone as a guide for treating municipal sewage, industrial wastes, and other pollution material. These standards establish a practical balance between the legitimate use of the river as a part of a sanitation system and its use for other purposes—present and future, esthetic and economic. Submitted to Incodel, these standards were approved by the commission, and were incorporated into a reciprocal legislative agreement which has since been made law in each of the four states. A time schedule for building collection and treatment facilities was devised by the advisory committee of state sanitary engineers, and adopted by Incodel.

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[5] Exec. Secy., Interstate Comm. on the Delaware River Basin, Philadelphia, Pa.

The entire program is administered by the long established agencies of state and local government: The state departments of health issue and enforce the orders for compliance; the responsible municipalities and industries design, construct, maintain, and operate the necessary structures; Incodel merely supplements and coordinates the activities of all groups concerned. The Interstate Commission on the Delaware River Basin has not superseded or supplanted a single agency of government.

The result has been a remarkable record of progress and wholehearted cooperation. No longer do the states, their municipal subdivisions, and the industries eye one another with suspicion and jealousy. They are working together as a team for a common purpose and goal. As an example, consider the case of Philadelphia and Camden, on directly opposite shores of the Delaware River in Pennsylvania and New Jersey. In the years before Incodel was created, officials of each city excused their shortcomings on the plea that it was the responsibility of "the other fellow" to move first. Since that time, in contrast, both cities have a program of construction, the total cost of which will approximate $75,000,000 and each is eager to complete its contract before the other. Similar situations exist along other sections of the river and elsewhere in the region.

The commission has performed another outstanding service in harmonizing the conflicting interests of the federal government and the Commonwealth of Pennsylvania in so far as they concern the cleaning of the Schuylkill River, the largest tributary to the Delaware River. For more than a century the headwaters region of this stream has been a center of intensive anthracite coal mining operations. Its industrial wastes were carried downstream into the Delaware River, where they settled, to be a continual threat to navigation. This condition forced the Corps of Engineers, United States Army, to maintain channel dredges in that stretch of the river continuously, and neither the state nor the federal government could "see eye to eye" in discussing a corrective program.

As the result of a study, Incodel was able to show both parties how each would benefit by a joint project estimated to cost $35,000,000 to the state and $20,000,000 to the federal government.

That there are other ways of handling sanitation problems, or other problems of resource development, will be evident from the several papers of this Symposium. The procedures devised by Incodel may not be best for other areas, but, certainly, no method can be more in harmony with the principles of home rule and the American tradition.

STREAM SANITATION IN THE TENNESSEE VALLEY

By Robert N. Clark,[6] M. ASCE

The ultimate success of a regional program of resource development by a single governmental agency depends very largely on the expansion and the success of regional institutions and agencies. The greatest achievement will result from a policy of leadership and collaboration, coupled with a willingness and a desire by the governmental agency to withdraw from active participation in each element of the program as rapidly as local resources can assume responsibility. The TVA is committed to such a policy in many phases of its program of resource development, and this broad statement of general aims will help to develop an understanding of the TVA stream sanitation program, including its plans, its operation, and its objectives.

No explanation of the statement that the quality of stream water is one of the basic resources of a region is needed. Pollution of a stream that restricts the use of the water for reasonable purposes affects not only the downstream user, but, in a larger sense, the entire region. If a manufacturer finds that his choice of a manufacturing site is limited by pollution factors, his adverse decision sets in motion a series of widening consequences; and potential loss of a new industry represents not only the loss of pay rolls and taxes, but a potentially lower community standard involving both private enterprise and public service. Conversely, a reasonable protection of stream water quality promotes the welfare of both private and public interests, and benefits the region as a whole.

If the objective of a stream sanitation program is to improve water quality so that the most productive development of a river basin is possible, what are the steps to be taken toward that goal? Step 1, of course, is a definition of the problem—namely, a statement of the nature and the amount of polluting substances reaching the stream, and the capacity of the stream to receive pollution without damage. Step 2 involves the questions of what constitutes damage to a stream, and what are the limits of reasonable use. Step 3, after defining the problem, is to secure the needed degree of abatement of existing pollution and to set limits on new pollution. In each of these steps, the TVA participates, keeping in view at the same time the necessity of developing strong state organizations that can pick up the program and thus permit a progressive withdrawal of the TVA from active participation.

In step 1 (the evaluation of the sources, nature, and degree of pollution) the TVA has maintained a fairly comprehensive current record of river pollution by sewage and industrial wastes. The stream sanitation staff is in a position to make detailed studies of waste materials for the information of either regulatory agencies or any industry or municipality concerned. Before initiating any such studies, however, the ability of interested parties to participate is investigated, and the TVA adheres quite closely to the policy of providing only such services or equipment as are not otherwise readily available. For example, the City of Chattanooga, Tenn., retained a consulting engineer in January, 1947, to design a system of intercepting sewers and a sewage treatment plant. Both the city and the state health department needed basic information relative to the volume, strength, and character of the sewage—the city's need being for design data and the state's need being for a basis of review and approval of the plans to be submitted by the city. At the request of the state, the TVA began a program for which Chattanooga furnished sampling personnel, constructed the necessary weirs, and provided laboratory space, and to which the TVA contributed the services of chemists and bacteriologists, some laboratory equipment, and certain flow measuring devices, none of which were otherwise available. The TVA program was doubly beneficial—first because data were acquired at relatively low expense and, second, because a project leading directly toward pollution abatement was advanced.

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[6] Chf. Public Health Engr., TVA, Chattanooga, Tenn.

Step 2 (determination of the capacity of a stream to receive pollution without damage) leads to a more complicated relationship. The TVA does not attempt to establish criteria governing the degree of pollution with relation to stream flows or other stream characteristics, or with relation to the possible uses of the water. The development of criteria is a function of the several states, through their regulatory agencies. To the fullest possible extent, the TVA furnishes information on present or anticipated water uses, and also investigates the effect of existing pollution on the stream, to aid the state agencies if they desire it.

Some of these stream investigations are exceedingly interesting. A technique has been developed to trace a given "slug" of water downstream and to relate the sampling at successive points, not only to the position of the prism of water under study, but also to the time of water travel. This method permits the determination of rates of B.O.D. reduction and reoxygenation, and at the same time takes into account the effect of side pollution and many other factors. One such special study was the tracing of a given mass of water through Cherokee Reservoir in Tennessee. This study continued for about 90 days, which is the time required for water to pass through the reservoir. The results are not only interesting, but valuable, in that they permit a close estimation of the degree of treatment or recovery required with reference to any given source of pollution.

In making field studies, TVA activity is fully correlated with the work of the different states. In North Carolina, where a pollution study is under way, leading toward possible legislative action, the state studies are being concentrated on the area outside of the Tennessee watershed, and the TVA is providing service in its own area. In Alabama, during a study of bathing water quality, the field sampling was done mainly by the TVA, but samples were examined in the state health department laboratories.

Although the TVA does not establish the criteria for water quality in the Tennessee River system, it does have an opportunity to confer and advise with many different agencies. In such cases, a consistent policy has followed the principle that pollution control be based on the element of reasonable use.

Too many times pollution control programs have been shattered on the rock of unreasonable demands. For example, if one group favors a program that will improve fishing, it is reasonable to examine the subject and to determine the locations and conditions conducive to fish propagation, the areas favored by fishermen because of accessibility or for other reasons, and similar pertinent factors. Following such a study the investigator should determine the possible damage arising from excessive pollution in specified localities.

Step 3 in a pollution control program is abatement. There are two approaches to this problem, one built on a mandatory action, the other seeking voluntary cooperation. Experience has proved that voluntary action alone is not sufficiently dependable and that an effective program must have the backing of compulsory regulation if any betterment is to be expected. The TVA does not have regulatory powers so far as stream pollution is concerned, the legal authority for compulsory action in every case lying with the states. The TVA can put vital information in the hands of the states, however, and, with the background of state authority, can encourage voluntary action leading toward pollution abatement. For instance, in cases where new industries are contemplating locating in the Tennessee River Valley, the TVA advises the industry of existing state laws relating to stream pollution and lends its full influence to the support of such laws. In disposing of marginal lands, the TVA often writes a provision into the instrument of transfer holding the discharge of pollution into a stream or reservoir to the limits established by state laws.

Another type of activity is assistance to industrial groups in the development of methods for recovery or treatment of industrial wastes. Studies of specific wastes are generally undertaken only when the results will be of value, not to one plant, but to a related group of industries. Assistance may involve such activities as compilation and review of scientific literature, economic study of markets for recovered material, or field and laboratory studies of methods of waste treatment.

Still another type of activity deals with the quality of water supplies for both industrial and domestic purposes. Information is provided on request to any prospective water user relative to the chemical, and sometimes to the bacteriological, characteristics of most of the important streams of the Tennessee Valley. Reports on industrial water supplies are kept up to date and are amplified from time to time to insure maximum utility of the information. Rather extensive studies are sometimes made to trace back particularly objectionable or unusual types of pollution, particularly in the cases of public water supplies. An example is a study being conducted on Guntersville and Wheeler reservoirs in Alabama in an effort to trace the sources of objectionable tastes and odors. This investigation extends over hundreds of miles of river and includes investigations of both sources of odors and methods of treatment to reduce them.

Closely related to the quality of water supplies is the quality of water in recreational areas. The creation of a large chain of lakes in a region of striking scenic beauty quite naturally leads toward an extensive development of commercial and private recreational facilities. To permit a maximum use of the lakes without, at the same time, permitting an unwarranted exposure of the public health to hazardous conditions, a concerted effort is being made, particularly in Kentucky, Tennessee, and Alabama, to determine safe recreational areas. Studies of a cooperative nature, using the resources of the states and the TVA together, include sanitary surveys, sampling of individual areas, and river studies to trace the sources and persistence of bacteriological pollution.

Underlying all the varied detail of the TVA stream sanitation program, there is a dominant pattern that cannot be described fully in a few words, but that can be summed up as "cooperative effort." There are many obvious advantages in having a federal agency assist with problems of an interstate nature; what is not so obvious sometimes is the need to strengthen every component of state and local government, every local or regional industrial organization, and every consulting engineer, who is attempting to reduce pollution. No group is strengthened when a federal agency takes over its functions. It is more practical to facilitate a record of successful accomplishment, to build prestige, to interest correlative groups or activities, and finally, as the local service is established, for the federal agency to withdraw as completely as possible, leaving a strong local organization adapted to meet its own problems.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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Paper No. 2424

CORRECTION OF RESERVOIR LEAKAGE AT GREAT FALLS DAM

By A. H. Weber,[1] Assoc. M. ASCE

Synopsis

Leakage of more than 450 cu ft per sec from the Great Falls Reservoir (in White and Warren counties in Tennessee on the Caney Fork River, a tributary of the Cumberland River), acquired by the Tennessee Valley Authority (TVA) from a private utility company in 1939, was reduced 98% by injecting hot asphalt and cement into holes drilled to intercept the water-bearing rock fissures. A total of 608 holes was drilled and grouted along a cutoff line almost a mile long.

The leakage, which had been steadily increasing since the dam was raised in 1925, was a threat to the useful life of the Great Falls hydroelectric project. Before correction, the leakage represented about 14% of the water used by the two hydraulic turbines when in full operation.

This paper discusses the causes of leakage, and the limited usage of cement and asphalt in a grouting program designed to correct the leakage at a justified cost.

Introduction

Leakage from the Great Falls Reservoir increased progressively until a flow of more than 450 cu ft per sec at full reservoir was recorded just before corrective work was initiated in 1945. This leakage equaled approximately 14% of the over-all full-load water use of the two Francis hydraulic turbines installed in the power plant. The electrical energy loss amounted to about 20,000,000 kw-hr annually.

The average annual leakage increase since 1925 was about 23 cu ft per sec. Measurements made by the TVA after 1939 indicated that the average annual increase since 1939 was close to 30 cu ft per sec. Corrective work undertaken in 1945 revealed that the leakage channels were in an early stage of development and that continued acceleration of annual leakage increases could be expected.

Note.—Published in January, 1950, *Proceedings*. Position and title given are those in effect when the paper was received for publication.

[1] Constr. Engr., TVA, Chattanooga, Tenn.

The Great Falls hydroelectric project consists of: (1) A gravity-type concrete dam which impounds a "three-fingered" reservoir on the Caney Fork River and on two of its tributaries, the Collins and Rocky rivers; and (2) a powerhouse, 4,000 ft downstream from the dam, which is supplied with water from the reservoir through tunnels and penstocks. The powerhouse is equipped with two generating units that have a maximum discharge of 3,200 cu ft per sec. The project was originally placed in operation in 1916, with a low dam and one generating unit. In 1925 the dam was raised 35 ft to its present level, increasing the height of the reservoir to El. 805. The powerhouse was reconstructed, and the second generating unit was installed.

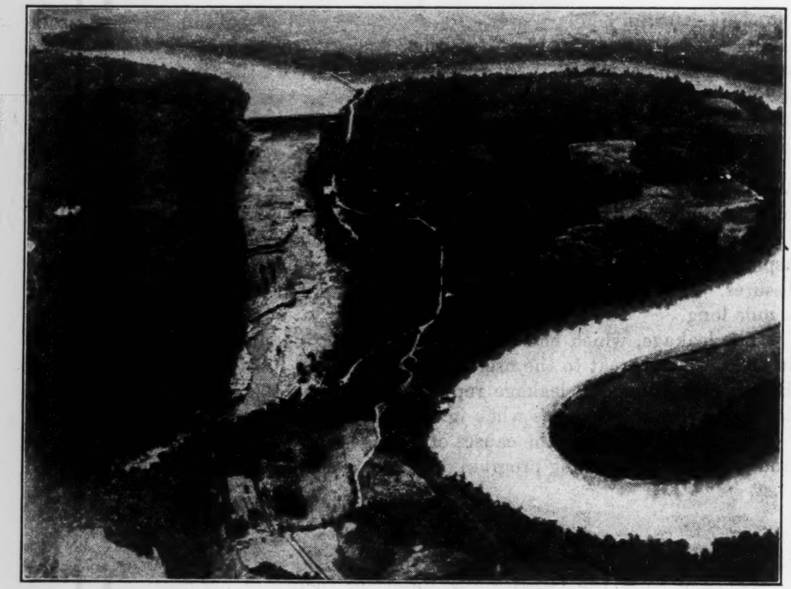


FIG. 1.—AIR VIEW OF THE GREAT FALLS PROJECT

The location of the powerhouse a considerable distance downstream from the dam is made possible by the topography of the area. The Caney Fork River between the dam and the powerhouse flows through a deep gorge. About midway between the dam and the powerhouse, the river has cut an inner gorge creating a series of falls and cascades, with the result that the river level at the powerhouse is about 75 ft lower than that at the dam. The Collins River enters the Caney Fork immediately above the dam and loops back more or less parallel to the Caney Fork as shown in Fig. 1.

The loop in the alinement of the Collins River creates a single ridge which varies in width from 600 ft to 3,500 ft and separates the lower 2 miles of the reservoir on the Collins River from the gorge of the Caney Fork River downstream from the dam. Through this ridge extensive leakage from the reservoir developed, and formed a series of waterfalls and cascades beginning near the river level just below the dam at the left abutment rising to a height of more than 60 ft near the powerhouse. A small amount of leakage also found outlets near the right abutment of the dam.

The formations present in the area are, in ascending order, the New Providence, the Fort Payne, and the Warsaw. The New Providence, a greenish siliceous shale, appears in the Caney Fork in the vicinity of the powerhouse. This formation lies well below the bottom of the reservoir, and its presence has no effect on the leakage problem.

Directly overlying the New Providence formation is the Fort Payne chert. This formation contains two members—the lower about 55 ft thick and the upper about 45 ft thick. The lower member of the Fort Payne is a medium-gray, fine-grained limestone, interbedded with an abundance of tough, dense, dark chert. Because of its highly siliceous nature, the member is resistant and relatively insoluble. The upper member is a medium-gray, fine-grained to medium-grained limestone. A considerable amount of both dense and porous chert is present, but the chert is less abundant than in the lower member. Cavities occur anywhere in the upper Fort Payne formation but they tend to concentrate toward the bottom. Practically all the leakage from the reservoir was through the upper Fort Payne formation.

Overlying the upper member of the Fort Payne formation is the Warsaw shale, a stratum of fissile, bluish-gray calcareous shale. Its depth varies but its usual thickness is about 20 ft. The stratum is relatively insoluble and impervious, and forms an effective barrier to the downward percolation of surface water. Surface water from the dividing ridge forms a system separate and distinct from that of the reservoir leakage. A geologic section plotted from drilling logs is shown in Fig. 2.

The transmission of leakage occurred through channels in the rock obviously developed long before the dam was constructed. Since the leakage strata are covered by the impervious bed of shale (which is entirely capable of protecting the underlying rock from the attacks of surface water), it appears that the channels were formed by river waters, probably in the following manner: As the Collins River cut downward, it passed through the shale and reached the soluble limestone beneath it. Joints and bedding planes permitted slow and minute entrance of the water into the rock; and, as the surface of the near-by Caney Fork River was considerably lower than that of the Collins River, a gradient existed, and the water eventually found its way through the divide. The minute openings were slowly enlarged by solvent action into appreciable channels. As the rivers continued to deepen their gorges, the subterranean channels that had been formed were left above river level and new ones developed at lower elevations. In this manner, a series of entrance passages came into existence, extending vertically from the top of the Fort Payne formation to the present river bed.

When the lake was first formed, the old channels were submerged again, but the residual clay in the channels largely prevented flow of appreciable

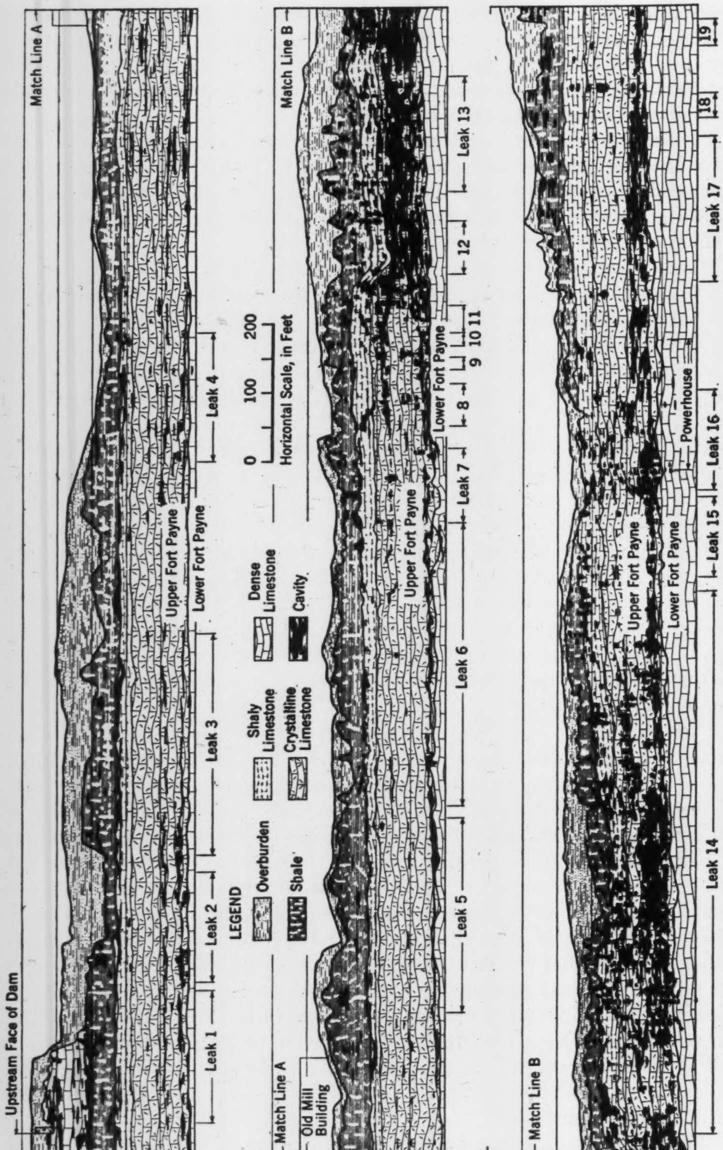

Upstream Face of Dam
Match Line A
Elevation, in Feet
830
810
790
770
750
730
Leak 1
Leak 2
Leak 3
Leak 4
Upper Fort Payne
Lower Fort Payne
LEGEND
Overburden
Shale
Shaly Limestone
Crystalline Limestone
Dense Limestone
Cavity
0
100
200
Horizontal Scale, in Feet
Match Line A
Old Mill Building
Match Line B
Leak 5
Leak 6
Leak 7
8
9
10
11
12
Leak 13
Upper Fort Payne
Lower Fort Payne
Match Line B
860
840
820
800
780
760
740
Leak 14
Leak 15
Leak 16
Leak 17
18
19
Powerhouse
Upper Fort Payne
Lower Fort Payne

FIG. 2.—GEOLOGIC SECTION ALONG LINE OF GROUT CURTAIN, BETWEEN THE DAM AND THE POWERHOUSE

volumes of water through the passages. In time, however, initial seeps enlarged into flows, as the clay eroded from the channels, with the result that progressively increased leakage found its way from the reservoir.

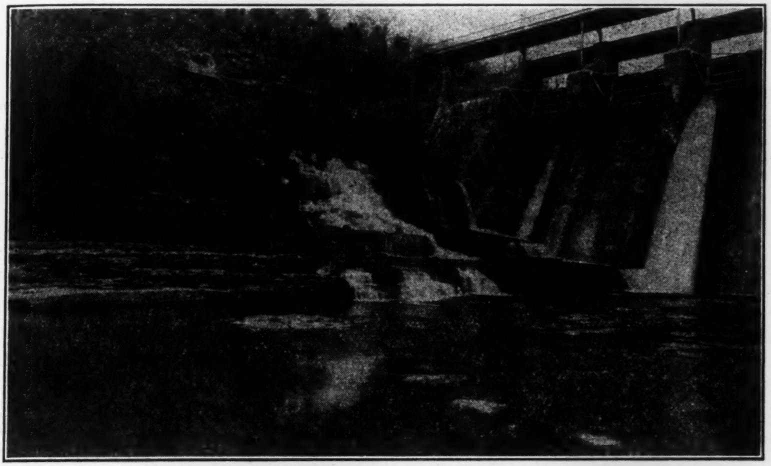


FIG. 3.—LEAKAGE NEAR RIGHT ABUTMENT OF DAM (LEAK NO. 20, LEAKAGE SYSTEM NO. 1) BEFORE CORRECTION

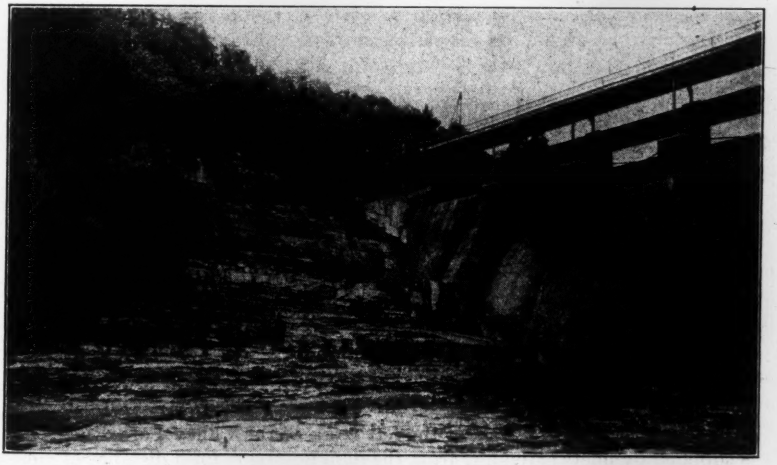


FIG. 4.—LEAKAGE NEAR RIGHT ABUTMENT OF DAM (LEAK NO. 20, LEAKAGE SYSTEM NO. 1) AFTER CORRECTION

There were three major areas, each separate and distinct from the others, from which leakage entered the Caney Fork River:

Leakage system No. 1 was on the right bank, and transmitted water around and under the right abutment of the dam. Water entered this system through inlets close to the dam and flowed under the right abutment. It escaped just below the dam near the contact of the Fort Payne and Warsaw shale as shown in Figs. 3 and 4. At high reservoir stages this leak had a volume of 20 cu ft per sec. It ceased to flow when the reservoir surface fell below El. 768.

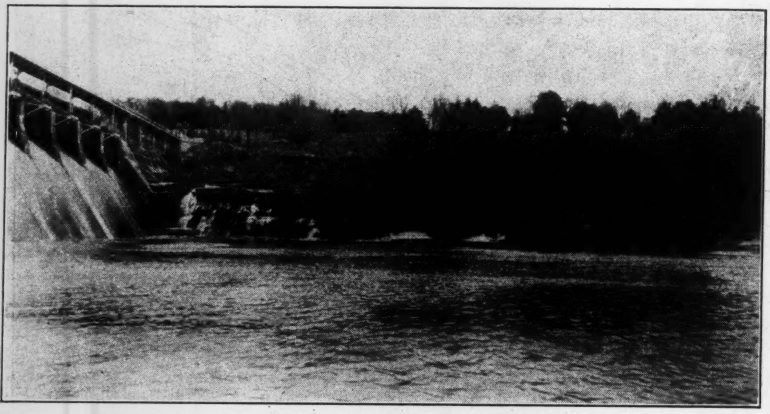


FIG. 5.—LEAKAGE NEAR LEFT ABUTMENT OF DAM (LEAKAGE SYSTEM NO. 2) BEFORE CORRECTION

FIG. 6.—LEAKAGE NEAR LEFT ABUTMENT OF DAM (LEAKAGE SYSTEM NO. 2) AFTER CORRECTION

Leakage system No. 2 was on the left bank of the Caney Fork River. It was supplied by inlets extending from the dam upstream for a distance of more than 5,000 ft. The major part of this leakage was discharged in three falls about 300 ft downstream from the dam. At minimum reservoir elevation this system of leakage discharged 62 cu ft per sec; and, at maximum pool, it discharged 150 cu ft per sec (see Figs. 5 and 6).

Leakage system No. 3 was in the narrow part of the divide in the vicinity of, and upstream from, the powerhouse. The leakage outlets of system No. 3, leaks Nos. 5 to 19, inclusive, between May, 1927, and January, 1944, and the leakage after correction, are shown in Fig. 7. The inlets for this system were in the vertical face of the rock bluff upstream from the intake structures. Leak No. 5 which was the largest single leak, and the one farthest upstream from system No. 3, flowed strongly at all operating reservoir levels. The other leaks dried up progressively as the reservoir fell, the highest one ceasing first. Leaks Nos. 14 to 19 dried up when the reservoir fell below El. 778. This system of leakage contributed a flow of about 68 cu ft per sec at minimum reservoir and a flow of 286 cu ft per sec at full reservoir. (The white matter that stands out in Fig. 7(*d*) is asphalt and should not be mistaken for water leakage.)

Plan for Leakage Correction

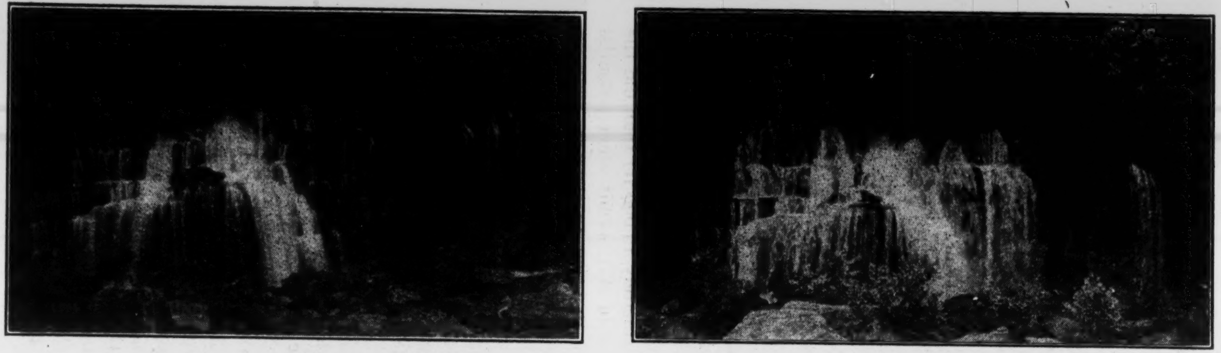
Experiments with asphalt and cement grouts, injected into holes drilled to intercept the leakage passages a short distance from the leakage outlets, were conducted by the TVA to test the feasibility of stopping this leakage without interfering with the normal operation of the reservoir. These experiments were successful in stopping most of leak No. 5, and the related data obtained indicated that the stopping of all the leakage was feasible economically.

The most natural concept of leakage correction on a problem of this magnitude would be to close up the inlets. However, at Great Falls, the inlets extended over a much longer traverse (approximately 10,000 ft) than did the outlets and the plugging of a known inlet would still permit other potential inlets to function. Since no physical harm can occur to the area between inlets and outlets, and since treatment near the outlets would in no way impair the hydroelectric structures, this apparently cheaper and more positive method was chosen.

The location of the grout holes had to satisfy two conditions: First, the holes had to be placed far enough from the leakage outlets so that effective grouting would not be interrupted by grout seepage from the leakage outlets too early in the grouting process; and, second, the holes had to be placed so that the vertical extent of the cutoff curtain would prevent leakage at all reservoir levels. The farther back from the outlets the holes were placed, the costlier would be the drilling, because the slope of the terrain rises sharply upward from the leakage outlets at the face of the bluff, whereas the level of the leakage channels as they approach the bluff remains about the same.

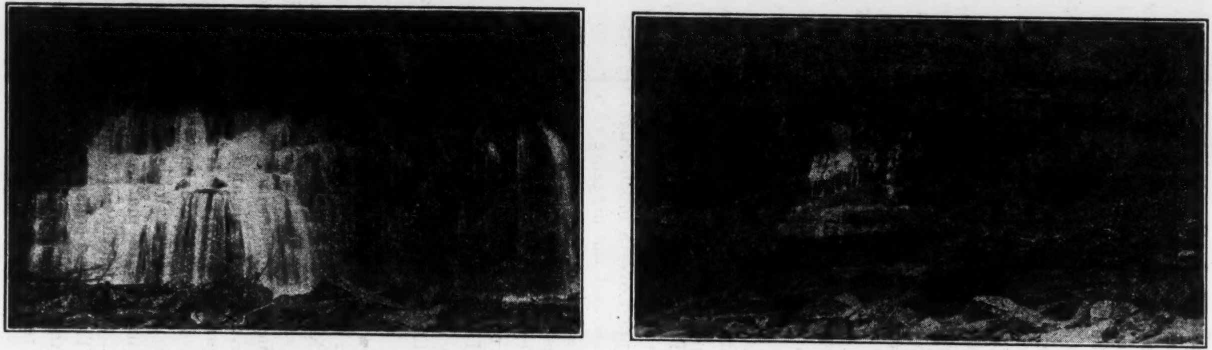
An average distance of 60 ft back from the bluff face was chosen for the experimental work. If the vertical extent of the cutoff were not involved, this distance would have been entirely adequate. Some few holes were as close as 35 ft to the leakage outlets, and they were asphalt grouted without serious loss of material before effective leakage correction. In general, a minimum distance of 50 ft was adequate for Great Falls.

The second consideration—the vertical extent of the cutoff—was controlled by the geologic formations at Great Falls. Where the shale dips below the full reservoir elevation, the vertical extent of the cutoff required was from the



(*a*) May 13, 1927; Headwater Elevation, 803.0

(*b*) August 23, 1936; Headwater Elevation, 791.2



(*c*) January 25, 1944; Headwater Elevation, 778.4

(*d*) March 26, 1946; Headwater Elevation, 799.4

Fig. 7.—Progressive Increase of Leak No. 5 Between 1927 and 1944 and After Correction in 1946

contact of the two Fort Payne members to the base of the shale. Where the base of the shale is above reservoir level, the drill holes were grouted to El. 806.

For the first stage holes a 40-ft spacing was selected with succeeding second stage, third stage, and fourth stage holes on 20-ft, 10-ft, and 5-ft centers, respectively. The cutoff line was completed as follows: 10% at a final spacing of 20 ft, 60% at a final spacing of 10 ft, and 30% at a final spacing of 5 ft.

The effectiveness of asphalt grout as a remedy for stopping flowing water in underground channels was demonstrated during the experimental work at Great Falls. Water velocities to 6 ft per sec, and one leak having a discharge of 60 cu ft per sec, did not interfere with blocking the water-bearing cavities with asphalt. Where leakage existed at all reservoir elevations, grouting with asphalt was scheduled. Where leakage did not exist at low reservoir stages, grouting with Portland cement was considered possible.

The largest leakage outlets in any system were at the lowest part of the dip, decreasing in size as they developed up the dip, generally in a downstream direction. The location of leakage outlets along the dip was also influenced by the structural sags in the rock, and the development of individual outlets was always greatest at the low point of the sag. Leakage measurements, made after a substantial part of leak No. 5 was stopped, disclosed that no appreciable reduction in the total leakage at low reservoir levels was effected; but, at high reservoir stages, the total leakage reduction was approximately equivalent to the quantity of water contributed by leak No. 5. This condition probably existed because some of the channels of leaks Nos. 6 to 13 (which were part of the same system) were not utilized at the lower reservoir levels and the water normally discharged by leak No. 5 was simply diverted into these inactive channels.

At full reservoir, the channels were completely filled; and, to pass the water usually discharged by leak No. 5, new channels had to be formed—a slow process—or the velocities in the remaining outlets had to increase. There was some increase in the velocity of the discharge noted, and the enlargement of the channels was also stimulated, as evidenced by the appearance of muddy water as the leakage channels were progressively blocked. Accordingly, the large leaks at the lowest point of any system were sealed first and the water was forced up the dip into the smaller channels at higher elevations. The highest water velocity (as measured by computing the rates of travel to an outlet of fluorescein dye injected into the drill holes) never exceeded 6 ft per sec; the average was less than 2 ft per sec.

Holes were first asphalt grouted on 40-ft centers, and then on 20-ft centers, with an occasional 10-ft hole grouted to check the water flow. The large leaks were corrected with fewer holes than the smaller leaks because connections to the channels of the larger leaks were made more easily, and a positive injection of asphalt into the more developed channels was possible.

Cement-grouting operations were begun opposite the upstream end of leak No. 14 of system No. 3 and progressed downstream from that point. All facilities were concentrated in this area in an attempt to seal these leakage passages before normal reservoir operation required filling above El. 778, the elevation at which leaks Nos. 14 to 19 began to function.

The effectiveness of the cement-grouting program was difficult to ascertain, and only by grout acceptance and grout recovery in the second stage and the third stage holes was it possible to obtain any measure of the results. However, on November 22, 1945, heavy rains on the Great Falls watershed filled the reservoir, and the work accomplished to that time received its initial test. All leaks that were asphalt grouted remained closed. Good results were observed in the areas grouted with cement. Leakage correction opposite holes on 40-ft centers was 80% effective; and opposite holes on 20-ft centers the correction was 95% effective. When holes were spaced on 10-ft centers, the correction was practically 100% effective.

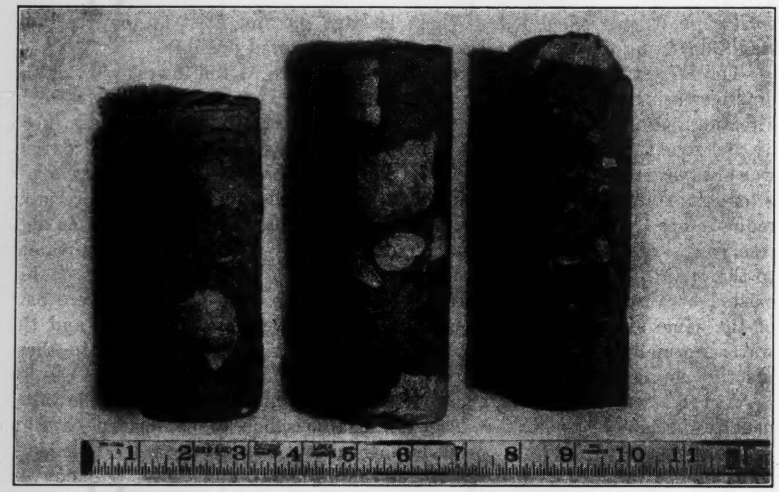


FIG. 8.—TYPICAL CORES DRILLED AFTER CONSOLIDATION GROUTING

Completion of the grouting program at low reservoir levels without interruption of the original work schedule was improbable during the flood season, and plans were changed to continue the work at existing reservoir levels using both asphalt and cement. Cement-grouting equipment and asphalt-grouting equipment were set up side by side so that a choice of material was available depending on the amount of leakage intercepted in the drill hole; but a completely cement-grouted hole on each side of an asphalt-grouted, or partly asphalt-grouted, hole was specified. Thus, cement grouts having water-cement ratios of 1.0 by volume or greater were pumped around the asphalt in an attempt to incase it, and to plug small openings into which asphalt probably would not penetrate. All remaining leakage was checked with this method.

DRILLING GROUT HOLES

Diamond drilling was employed for sinking all holes. Casings consisting of 3.5-in. standard black pipe as long as 30 ft, and averaging 10 ft in length, were required to reach the rock surface. In areas having numerous cavities and seams the drills were often blocked by sand, gravel, and blocky chert. These materials were usually most troublesome just above the contact of the upper and lower Fort Payne rock formation, and continued drilling caused the blocking of tools or the destruction of bits.

When this condition existed, drilling was stopped and the hole was pressure grouted with cement to consolidate the loose material. There was very little loss of cement through the leakage outlets, as most of the zone requiring this treatment had its inlets above reservoir level at the time the consolidation grouting was done. When drilling was resumed from 36 hours to 48 hours later, most holes were completed without further difficulty. However, if a second consolidation were necessary, it was done without hesitation. Typical cores showing the conglomerate texture obtained in reaming the cement grout from these holes are shown in Fig. 8.

The rate at which the diamond drill holes were completed determined the progress of the job. In all, six drills were available, but no more than five were in use at one time. The maximum number of drill shifts per day was fourteen. The drilling rates obtained are shown in Table 1. Diamond drilling costs were high because of the hardness of the rock existing at Great Falls. This hardness is explained by the large percentage of chert embedment in the Fort Payne formation. The average cost per foot of drilling for 32,536 lin ft was as follows:

TABLE 1.—Typical Drilling Rates

Item	Monthly Average		
	Minimum	Maximum	Average
Feet per shift hour	0.94	2.34	1.82
Feet per net drilling hour	1.33	3.01	2.27
Ratio, net hours to shift hours	0.71	0.83	0.80

Item	Dollars per linear foot
Drill Operation—	
Labor	0.94
Repairs (labor and material)	0.35
Drill tools	0.30
General plant and facilities	0.73
Casing 5,460 lin ft	0.20
Diamond drill bits	2.51
Gasoline (8,538 gal)	0.05
Miscellaneous	0.04
Moving and setting up (608 holes)	0.18
Total direct cost per linear foot	5.30

Asphalt Grouting

Asphalt was melted in five heaters of 500-gal capacity each, similar to those employed for highway work. The pumps were 4.5-in. by 3-in. by 4-in. air-driven, double-acting, reciprocating units equipped with iron ball valves, iron liners, and pistons. Oxidized petroleum asphalt was used for all grouting.

Deliveries were made in 30-ton to 40-ton carload lots scheduled for anticipated requirements so that dead storage was reduced to a minimum. Asphalt was packed in paper cartons, each containing about 1.5 cu ft of material.

Four asphalt-heating kettles were generally employed having a combined heating capacity of 80 cu ft per hr; three kettles were arranged in a bank, and from these the asphalt was pumped to the fourth kettle near the hole being grouted. Thus, by moving a single kettle approximately 80 ft each side of the three-kettle bank, holes 160 lin ft along the cutoff line could be grouted from one setup without danger of the asphalt freezing in the pipe lines. The asphalt-heating and asphalt-pumping arrangements are shown in Fig. 9. Holes were water tested and when no pressure was developed, or when connections to the leakage outlets were suspected, tests were made with fluorescein dye to determine the approximate velocity and the course of the water.

An important feature of the asphalt grouting as done at Great Falls was the position of the grout packer with respect to the water-bearing channels intercepted. The presence of other cavities in a hole above the main leakage channel, some active and some inactive, required the isolation of the lowest active cavity for successful grouting with the simplified equipment in use. The inclusion of several cavities below a packer resulted in the lowest cavities (usually the largest water-bearing channels) being unfilled with asphalt. This characteristic was observed many times at the beginning of the work by the emission of asphalt grout from the higher outlets, and in several places actual loss of asphalt to the surface occurred without sealing the water-bearing channels. Therefore, all packer settings for asphalt grouting were made as close to the water-bearing cavity as possible; and, where it was impossible to do so without crossing a higher cavity, pipe extensions were used below the packer to give the asphalt every opportunity of seeking the lowest water-bearing channel.

The rate of pumping varied considerably, and was largely a matter of judgment based on the behavior of the grout flow into the hole and its effect on the leakage through the connecting outlets. A pumping rate of 80 cu ft per hr was possible, but the average was usually between 40 cu ft per hr and 60 cu ft per hr. Rapid pumping was usually the rule where the water passages were small and the connecting outlets were scattered over a wide area. Where the water-bearing seams were large, the rate of pumping was usually slower. Pressures of 0.5 lb per ft of depth, measured from El. 810, were specified; but in the first stage and the second stage holes where most of the asphalt was placed pressure was of little consequence. When the connecting leakage was stopped, indicating that the "block" had been formed, pumping was discontinued regardless of registered pressure because cement grout in the same hole and in the next stage hole could be used to tighten the seal at the specified pressure.

The limited use of asphalt during any one continuous period of pumping was found to be advantageous in stopping the leaks (because it was soon evident that hot asphalt could travel distances beyond that necessary for effectiveness). By restricting the area covered by the asphalt and making repeated injections, the asphalt was permitted to build up the block within a

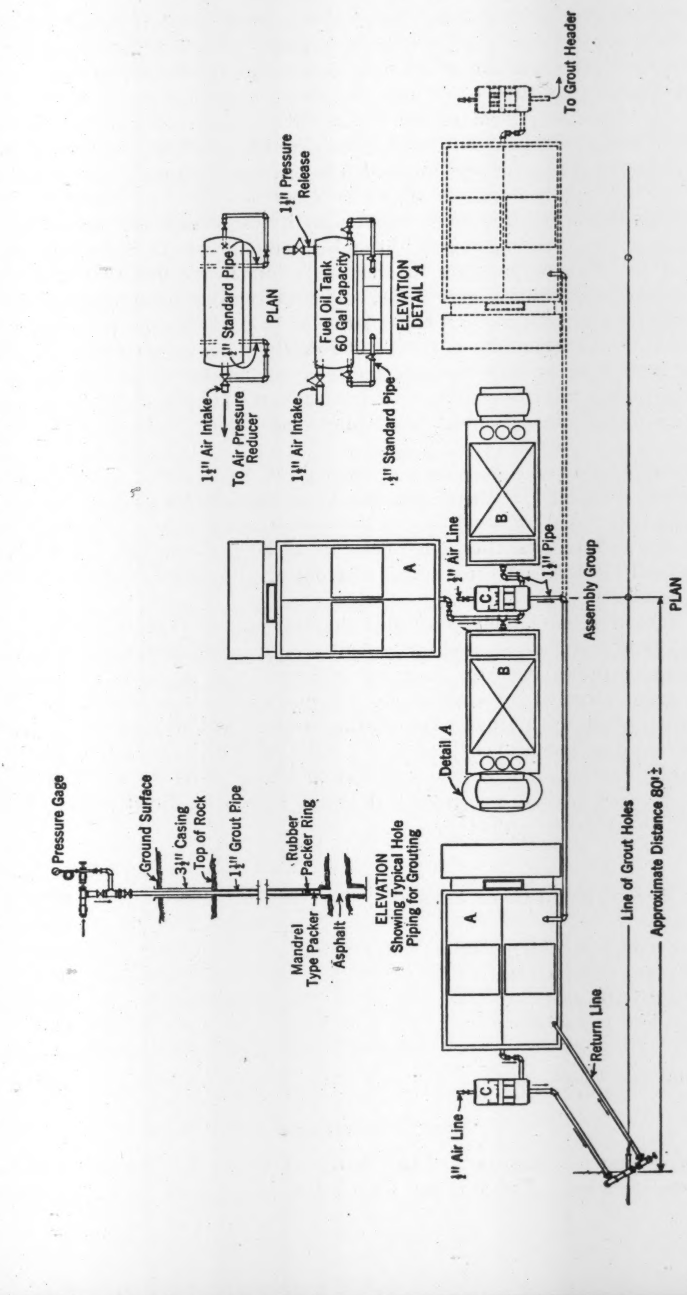

Pressure Gage
Ground Surface
3½" Casing
Top of Rock
1½" Grout Pipe
Rubber Packer Ring
Mandrel Type Packer
Asphalt
ELEVATION
Showing Typical Hole Piping for Grouting
1½" Air Intake
To Air Pressure Reducer
½" Standard Pipe
PLAN
1½" Pressure Release
Fuel Oil Tank 60 Gal Capacity
1½" Air Intake
½" Standard Pipe
ELEVATION
DETAIL A
To Grout Header
Detail A
A
B
C
1" Air Line
1½" Pipe
Assembly Group
B
A
C
1" Air Line
Return Line
Line of Grout Holes
Approximate Distance 80'±
PLAN

FIG. 9.—ASPHALT-GROUTING EQUIPMENT

reasonable distance of the hole. It also was found that injecting limited quantities of asphalt into a series of holes having the same outlet connections, before repeating the injection in any hole, gave rather spectacular results. For example, limited quantities of asphalt were pumped into a series of three holes connecting with a major part of the leakage at the right abutment (leak No. 20, Figs. 3 and 4), without noticeably changing the quantity of leakage; but, after a second application was injected into the same three holes, the entire leakage of 15 cu ft per sec was sealed in a period of about 12 hours. The results with this method were so rapid that it was continued, even though there was danger of losing a hole through interconnection. Some few holes were lost to pressure grouting in this way, but interconnection generally indicated that the cavities were sealed and that the desired results had been obtained. The quantity of asphalt placed at one continuous pumping was usually limited to from 600 cu ft to 1,000 cu ft. Greater quantities usually resulted in loss of asphalt through the leakage outlets—a condition which limited pumping tried to prevent. When asphalt did appear in the leakage outlets, pumping was stopped as nothing was gained from continued injection.

The temperature of asphalt varied between 300° F and 350° F. At temperatures lower than 300° F there was danger of asphalt freezing in the hole. Temperatures of 350° F were used where penetration of small cavities was desired, but in larger cavities a temperature of 300° F was ample. Asphalt was used only in the first three stages of grouting and only in cavities having flowing water.

The rate of usage largely controlled the unit cost of asphalt in place. A total of 548 crew shifts was used for asphalt grouting; and each crew contained a foreman, a pump operator, and six laborers. The average quantity of asphalt grout pumped by a single crew during a shift was 280 cu ft. The maximum placed by a single crew during a shift was 648 cu ft. Asphalt grouting was continuous over three 8-hour shifts daily whereas cement grouting was usually continuous over two consecutive 8-hour shifts daily. A total of 156,518 cu ft of asphalt was placed at Great Falls at a direct cost of $1.35 per cu ft, the distribution being:

Item	Dollars per cubic foot of asphalt
Asphalt, free on board Great Falls	0.77
Direct labor	0.25
Grouting equipment (depreciation and repairs)	0.06
General plant and facilities	0.19
Kerosene (87,400 gal)	0.05
Materials and miscellaneous	0.03
Total direct cost	1.35

Cement Grouting

Four cement-grouting units of the Boulder Dam (in Arizona and Nevada) type were installed. The Boulder Dam mixer is mounted on a steel frame having two levels. On the upper level is a charging hopper and horizontal cylindrical mixing tank. The mixing mechanism consists of a horizontal shaft with paddles rotated by an air motor. The cement grout is discharged from the mixing tank through a 3-in. drawoff cock into a mechanically-agitated sump located on the lower level of the steel frame. The sump is a cylindrical tank 30 in. high and 36 in. in diameter. The agitating mechanism consists of a vertical shaft with horizontal blades set near the bottom of the tank, and the shaft is rotated by an air motor similar to that used for the mixing tank. Each grouting unit was equipped with two 7-in. by 5-in. by 10-in. air-driven, double-acting grout pumps.

Cement-grouting methods and procedures were standardized as much as possible during the course of the work. Constant observation of grout behavior as the grout was pumped into the drill holes, and as it appeared at the leakage outlets, permitted the establishment of rather rigid rules which achieved effectiveness with a restricted use of cement. The first packer setting, whenever practical, was made not more than 10 ft above the contact of the two Fort Payne members. The remainder of the hole was grouted in additional steps isolating the more open cavities with the step limit set at 20 ft.

Each packer setting was water tested to determine the resistance of the hole as an aid in the selection of the water-cement ratio. The pressure specified was the same as that used for asphalt grouting and this rule of pressure was maintained for most of the grouting. The maximum pressure used was 40 lb per sq in. with the exception of some third stage and fourth stage holes having sand-filled cavities which were grouted at pressures of from 50 lb per sq in. to 70 lb per sq in.

Every effort was made to bring the hole to its refusal pressure as quickly as possible in the first stage and the second stage holes. Third stage and fourth stage holes for the most part became check holes, and refusal of these holes under a gradual increase of pressure with a decrease in grout acceptance was sought. Continuous pumping without developing pressure on a hole was limited to from 300 bags of cement to 500 bags of cement, depending on the size of the cavity being grouted. The pumping of larger quantities of cement usually resulted in loss of cement through the leakage outlets. General limitations and general rules for grouting were as contained in Table 2. These rules were set up primarily to check the limits of cement usage, and variation within this range was permitted as each hole often presented its own difficulties that necessitated manipulation of pressures and mixes to secure refusal.

Calcium chloride was often used to accelerate the set of cement in an attempt to limit its travel. From 2% to 6% (by weight) of cement was the amount of calcium chloride added with percentages of from 4% to 6% being the most helpful. The set was accelerated with calcium chloride in an effort to secure a "block" somewhere in the cavity against which neat cement mixes could be pumped without loss. Calcium chloride was prepared in solution and added to the mixing water at the grout mixer.

Once pressure was obtained, every effort was made to close the hole by continued pumping to obtain the most effective distribution of the cement under pressure. As soon as one half of the predetermined refusal pressure was indicated on the gage, the return line to the pump was connected and the flow of grout and pressure increase was controlled by stopcocks in the header pipe. Pressures were held for 20 min after refusal. After completion, holes requiring the use of heavy grouts (water-cement ratio by volume 0.8 or less) were rodded through the packer to make certain that the pipe and the hole remained open while grouting. Third stage and fourth stage holes and other tight holes were usually jetted to the previous completed packer setting and regrouted. Most of the holes did not accept cement on regrout and those that did usually accepted very small quantities. If rodding and jetting were not possible, because of obstructions, or the set of cement, and if there were reason to doubt the soundness of the grout (usually indicated by a rather abrupt refusal pressure), the hole was reamed and regrouted.

It was often necessary to inject the 300-bag to 500-bag limit of cement three or four times before filling the cavity in the first stage and the second stage holes sufficiently to develop pressure. One hole required seven injections before refusal was obtained. It was early discovered that without such restrictions vast quantities of cement could have been pumped without obtaining a closure.

As with asphalt, the unit cost of cement grouting was largely dependent on the rate of cement usage. A total of 866 crew shifts was required for the completion of the cement-grouting program; and each crew contained a foreman, a pump operator, and five or six cement handlers. The average quantity of cement grout pumped by a single crew during an 8-hour shift was 288 bags. The maximum placed by a single crew during a shift was 866 bags. A total of 249,468 bags of cement was used in cement grouting the cutoff at Great Falls at a cost of $1.19 per bag. The distribution of this cost is as follows:

TABLE 2.—GUIDE TO THE

(Note: At Any Pressure Drop at Constant Rate of Pumping, Begin
at Constant Rate of Pumping,

Line No.	WATER TEST DATA		STARTING ONLY			FIRST CHANGE		
	Quantity[a] (1)	Grout pressure[b] (2)	Ratio,[c] w/c (3)	Bag limit[d] (4)	Grout pressure[b] (5)	Ratio,[c] w/c (6)	Bag limit[d] (7)	Grout pressure[b] (8)
1	Any	0	0.8	30 to 50	0	0.6	30 to 50	0
2	Any	0 to 50	0.8	50 to 100	50	0.6	50 to 100	50
3	Any	50 to 100	1.0	50 to 100	50	0.8	50 to 100	50
4	>10	100	1.0	100 to 200	100	0.8	100 to 200	100
5	>10	100	1.6	100 to 200	100	1.0	100 to 200	100

[a] Pumping rate of flow, in gallons per minute. [b] Percentage of the grout pressure specified. [c] Ratio of chloride ($CaCl_2$) mixes. [f] 5% $CaCl_2$.

Item	Dollars rer bag of cement
Cement, free on board Great Falls	0.65
Calcium chloride	0.01
Direct labor	0.22
Grouting equipment	0.07
General plant and facilities	0.18
Materials and miscellaneous	0.03
Reaming holes (4,837 ft at $1.45 per ft)	0.03
Total direct expense	1.19

CONCLUSIONS

The grouting at Great Falls succeeded in stopping about 98% of the reservoir leakage. Most of the remaining water loss, approximately 9 cu ft per sec, is in areas outside the limits of the cutoff and is so located that its correction is not economically justified, unless leakage materially increases in the ensuing years.

The leakage remaining within the limits of the cutoff-grouting line was only a matter of seepage 3 years after completion of work, and there was no indication that the grout curtain had deteriorated. As can be readily seen from Table 3, showing the total material placed, further quantities of grout could be forced into the rock either with grouts of higher water-cement ratio under higher pressures or through closer spaced holes. Neither of these procedures would have contributed to the actual amount of leakage corrected. Rather than continue with grouting that would not accomplish any positive results, the decision was made to postpone any further work until additional treatment is warranted.

USE OF CEMENT GROUTS

with the Next Higher Water-Cement Ratio. At Any Pressure Rise
Continue the Mix in Use.)

SECOND CHANGE			THIRD CHANGE			Remarks	Line No.
Ratio,[c] w/c (9)	Bag limit[d] (10)	Grout pressure[b] (11)	Ratio,[c] w/c (12)	Bag limit[d] (13)	Grout pressure[b] (14)	(15)	
0.6[e]	100 to 200	0	0.6[f]	100 to 200	0	Discontinue grouting	1
0.6[e]	100 to 200	50	0.6[f]	100 to 200	50	Continue grouting to refusal	2
0.6	100 to 200	50	0.6[e]	100 to 200	50	Continue grouting to refusal	3
0.6	100 to 200	50	0.6[e]	100 to 200	50	Continue grouting to refusal	4
0.8	100 to 200	100	0.6	100 to 200	100	Continue grouting to refusal	5

water to cement, by volume. [d] Bags (1 cu ft) of cement. [e] 4% calcium chloride mix, $w/c = 0.6$ for all calcium

The leakage corrected added approximately 10% to the output of the Great Falls plant. The annual value of this power is equivalent to a return of more than 8% on the cost of the leakage correction. It is also estimated that, at the rate of annual leakage increase existing at the time of correction, an additional saving in power output of 1% per yr is being obtained.

TABLE 3.—ASPHALT AND CEMENT USAGE BY STAGES

Stage	Spacing[a] (ft)	Asphalt (cu ft)	BAGS OF CEMENT AT WATER-CEMENT RATIOS (w/c) OF					
			0.6[b]	0.6	0.8	1.0	1.6	2.0
First	40	115,291	25,628	31,504	26,204	18,274	2,526	116
Second	20	31,000	9,230	14,237	18,727	20,767	5,010	187
Third	10	10,227	3,163	6,963	10,909	23,423	10,275	3,332
Fourth	5	....	885	2,388	3,114	6,617	4,066	1,923
Total	....	156,518	38,906	55,092	58,954	69,081	21,877	5,558

[a] Distance between grout holes, in feet. [b] Calcium chloride ($CaCl_2$) mix.

The success of the leakage correction work at Great Falls was largely due to the topography of the Great Falls site. During the drilling of holes from 50 ft to 60 ft in rock, the effects obtained from grouting these holes were always visible from the bed of the Caney Fork River. Constant observation of the bluff containing the leakage outlets permitted visual inspection that made possible the establishment of rules and procedures for the limited usage of the grouting materials.

ACKNOWLEDGMENTS

The leakage correction work was under the general direction of C. E. Blee, M. ASCE, chief engineer, TVA. The writer was construction engineer for the project and H. D. Towers was construction superintendent. Geologic investigations were under the direction of B. C. Moneymaker, Assoc. M. ASCE, chief geologist. TVA.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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TRANSACTIONS

Paper No. 2425

INFLUENCE CHARTS FOR CONCRETE PAVEMENTS

BY GERALD PICKETT[1] AND G. K. RAY,[2] JUN. ASCE

SYNOPSIS

Influence charts are presented for the purpose of obtaining theoretical deflections and moments of pavement slabs under load. The charts are similar to, and are used in the same manner as, those presented by Nathan M. Newmark, M. ASCE, for the determination of theoretical stresses and displacements in elastic foundations.

Points at a slab edge, points near a slab edge, and points far from any slab edge are considered. Both the assumption of a subgrade modulus as originally proposed by H. M. Westergaard, M. ASCE, and the assumption of a deep elastic solid considered by Mr. Westergaard and others are treated. The charts are based on the theoretical equations derived by Mr. Westergaard and by A. H. A. Hogg.

INTRODUCTION

In deriving the basic equations for the stresses and deflections that occur in concrete pavements,[3] Mr. Westergaard used the assumption that the subgrade reaction per unit of area at any point is proportional to the slab deflection at that point. He has often stated that this assumption was "only an empirical makeshift justified by usable results." In his 1933 paper,[4] he introduced a correction in regard to the subgrade reaction to be applied to his formulas for interior loading for cases in which the load could be assumed to be uniformly distributed over a circular area. The object of this correction was to obtain formulas more nearly in accord with tests on concrete pavements for highways. Mr. Westergaard gave this matter further attention in his 1939 paper[5] which

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[1] Prof. of Applied Mechanics, Kansas State College, Manhattan, Kans.; formerly Research Physicist, Portland Cement Assn., Chicago, Ill.

[2] Engr., Highways and Municipal Bureau, Portland Cement Assn., Chicago, Ill.

[3] "Om Beregning af Plader paa elastisk Underlag med saerlight Henblik paa Spørgsmaalet om Spaendinger i Betonveje," by H. M. Westergaard, *Ingenioren*, Vol. 32, 1923, pp. 513–524.

[4] "Analytical Tools for Judging Results of Structural Tests of Concrete Pavements," by H. M. Westergaard, *Public Roads*, December, 1933, pp. 185–188.

[5] "Stresses in Concrete Runways of Airports," by H. M. Westergaard, *Proceedings*, Highway Research Board, National Research Council, Vol. 19, 1939, pp. 199–200.

dealt specifically with concrete runways for airports. He made a "plausible estimate" of what the correction should be to bring his formula for maximum stress for interior loading into agreement with what would be obtained if the assumed subgrade had been replaced by a "deep elastic solid." As far as is known, he has not prepared corrective terms applicable to other shapes of loaded areas or to stresses near the slab boundaries.

In 1938, Mr. Hogg[6] and D. L. Holl,[7] independently, solved the problem of a thin slab of infinite size supported by a semi-infinite elastic solid (corresponding to the deep elastic solid mentioned by Mr. Westergaard). Mr. Hogg expressed his formulas in series form, and made a few numerical computations.

From a study of experimental data, especially those obtained by the Swedish State Road Institute, S. G. Bergstrom[8] concluded that the subgrade acts more nearly like a semi-infinite solid (the assumption used by Mr. Hogg) than like a dense liquid (the assumption used by Mr. Westergaard when the corrective term is not used). Mr. Bergstrom, therefore, based all his calculations on the assumption of an elastic solid subgrade. However, being unable to obtain an exact mathematical solution for the cases of interest to him, Mr. Bergstrom used a method of approximation, and obtained numerical results for the case of a circular slab under a centrally applied load.

Most investigators believe that, of the two types of subgrade, the behavior of the semi-infinite elastic solid more nearly represents actual subgrade performance. Support for this belief is furnished by data from bearing tests with various sizes of circular bearing plates, such as those reported by L. A. Palmer and J. B. Thompson.[9] The constant C, characteristic of a solid subgrade, when computed from the data, is practically independent of the diameter of the bearing plate, as it should be, whereas the constant k, characteristic of a liquid subgrade, when computed from the data is found to be approximately inversely proportional to the diameter of the plate. In determining these constants from the load-deflection data, the following formulas are used:

$$k = \frac{q}{w} \qquad \text{(1a)}$$

and

$$C = \frac{q\,d}{w} \qquad \text{(1b)}$$

in which q is the unit load; d is the diameter of the plate; and w is the settlement of the plate.

On the other hand, because of variation of the subgrade material with depth, one cannot be certain that the behavior of a subgrade under a large slab can be predicted from its behavior under a much smaller bearing plate. In this connection, it can be shown that if the subgrade were an elastic solid layer of finite thickness, and were supported by a base of much more rigid material, then for bearing plates of diameters larger than the thickness of the elastic layer, the effect of the subgrade on the bearing plate would be more nearly like that of a dense liquid than that of a semi-infinite elastic solid. However, if smaller bearing plates were used on this elastic layer, the effect would be more like an elastic solid than a dense liquid.

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[6] "Equilibrium of a Thin Plate, Symmetrically Loaded, Resting on an Elastic Foundation of Infinite Depth," by A. H. A. Hogg, *Philosophical Magazine*, Series 7, Vol. 25, 1938, pp. 576–582.

[7] "Thin Plates on Elastic Foundation," by D. L. Holl, *Proceedings*, 5th International Cong. for Applied Mechanics, Cambridge, Mass., 1938, John Wiley & Sons, Inc., New York, N. Y., 1939, pp. 71–74.

[8] "Circular Plates with Concentrated Load on an Elastic Foundation," by S. G. Bergstrom, *Bulletin No. 6*, Swedish Cement and Concrete Research Inst., Stockholm, 1946.

[9] "Pavement Evaluation by Loading Tests at Naval and Marine Corps Air Stations," by L. A. Palmer and J. B. Thompson, *Proceedings*, Highway Research Board, National Research Council, Vol. 27, 1947, pp. 125–143.

It is beyond the scope of this paper to discuss the merits of either assumption further. It is the primary purpose of this paper to extend the work of Messrs. Westergaard and Hogg by presenting solutions of their equations in the form of influence charts. (Influence charts were prepared by Mr. Newmark in 1942 and 1947 to simplify calculations for similar problems.[10,11]) Although these charts may be used for any rigid type of pavement of uniform thickness, it is believed they will be used chiefly in connection with airport pavements. They greatly facilitate the determination of theoretical deflections and moments caused by loads on pavement slabs. In fact, with the charts one may readily obtain these values for any distribution of load that might be transmitted by airplane landing gears. It is only necessary to:

(1) Draw the imprint of tire or tires on transparent paper to a scale that depends on the properties of the slab and its supporting subgrade.

(2) Place the drawing on the appropriate chart in a position that depends on the location of the load with respect to the point for which values are desired; and

(3) Count the blocks of the chart covered by the diagram.

The value desired is then obtained as a product of the intensity of loading, a factor expressing properties of subgrade and slab, and the number of blocks covered by the diagram.

Eight influence charts were prepared from the basic equations of Messrs. Westergaard and Hogg. These charts were for both moments and deflections for four different cases. The cases are classified according to the point for which values may be determined and according to the subgrade assumption, as follows:

(1) Interior—liquid subgrade
(2) Interior—solid subgrade
(3) Edge—liquid subgrade
(4) One half of the radius of relative stiffness from an edge—liquid subgrade

Only the one case of solid subgrade was treated because it was the only one for which a basic formula was available at the time this project was under way. However, subsequently, the problem of an infinite slab on an elastic solid layer of finite thickness which in turn is supported by a rigid base and the problem of a finite rectangular slab on an elastic solid of infinite depth have been solved, and influence charts for these conditions have been prepared. No one should conclude that the writers favor a given subgrade assumption because they prepared influence charts for that assumption. Only after an adequate number

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[10] "Influence Charts for Computation of Stresses in Elastic Foundations," by N. M. Newmark, *Bulletin No. 338*, Univ. of Illinois Eng. Experiment Station, Urbana, Ill., 1942.

[11] "Influence Charts for Computation of Vertical Displacements in Elastic Foundations," by N. M. Newmark, *Bulletin No. 367*, Univ. of Illinois Eng. Experiment Station, Urbana, Ill., 1947.

of charts or similar computational aids have been prepared for various assumptions and theoretical values based on these assumptions have been compared with a sufficient amount of reliable experimental data can it be said definitely that one assumption more nearly represents subgrade performance under pavement slabs than does another assumption.

COMPUTATIONS PRELIMINARY TO PREPARATION OF CHARTS

The fundamental equations by Messrs. Westergaard and Hogg that formed the basis of the work will be given, followed by a general description of the computational procedure used. Certain tabulated summaries of the results will also be presented, but details of the work will be omitted. The following fundamental equations for deflection were used:

Interior—Liquid Subgrade.[12]—

$$w = \frac{P\,l^2}{4\,D}\,\mathrm{Re}\,H^1{}_0\left(\sqrt{i}\,\frac{r}{l}\right) \quad \text{(2)}$$

Interior—Solid Subgrade.[13]—

$$w = \frac{P\,l^2}{8\,D}\sum_{m=0,1}^{\infty}(-1)^m\left\{\frac{\frac{8}{3\sqrt{3}}\left(\frac{r}{2\,l}\right)^{6m}}{[(3\,m)!]^2} + \frac{\frac{4}{\pi}\left[\log\frac{\lambda\,r}{2\,l} - \left(1 + \frac{1}{2} + \cdots \frac{1}{3\,m+1}\right)\right]\left(\frac{r}{2\,l}\right)^{6m+2}}{[(3\,m+1)!]^2} - \frac{\frac{8}{3\sqrt{3}}\left(\frac{r}{2\,l}\right)^{6m+4}}{[(3\,m+2)!]^2} + \frac{2\left(\frac{r}{2\,l}\right)^{6m+5}}{[\Gamma\,(3\,m+3.5)]^2}\right\} \quad \text{(3)}$$

Edge—Liquid Subgrade.[14]—

$$w = \frac{2\,P}{\pi\,k\,l^2}\int_0^{\infty}\frac{\gamma\cos\frac{\alpha\,x}{l}\left[\cos\frac{\beta\,y}{l} + (1-\mu)\,\alpha^2\sin\frac{\beta\,y}{l}\right]e^{-\gamma y/l}\,d\alpha}{1 + 4\,(1-\mu)\,\alpha^2\,\gamma^2 - (1-\mu)^2\,\alpha^4} \quad \text{(4)}$$

Near an Edge—Liquid Subgrade.[15]—

$$w_{II} = \frac{2\,P}{\pi\,k\,l^2}\int_0^{\infty}\frac{\gamma\cos\frac{\alpha\,x}{l}\left(A\cos\frac{\beta\,y}{l} + B\sin\frac{\beta\,y}{l}\right)e^{-\gamma y/l}\,d\alpha}{1 + 4\,(1-\mu)\,\alpha^2\,\gamma^2 - (1-\mu)^2\,\alpha^4} \quad \text{(5)}$$

$$w_I = w'_{II} + \frac{P}{\pi\,k\,l^2}\int_0^{\infty}\frac{\cos\frac{\alpha\,x}{l}}{\beta^2+\gamma^2}\left(\beta\cos\frac{\beta\,y}{l}\sinh\frac{\gamma\,y}{l} - \gamma\sin\frac{\beta\,y}{l}\cosh\frac{\gamma\,y}{l}\right)d\alpha \quad \text{(6)}$$

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[12] "Stresses in Concrete Runways of Airports," by H. M. Westergaard, *Proceedings*, Highway Research Board, National Research Council, Vol. 19, 1939, pp. 200 and 201, Eqs. 33 and 34.

[13] "Equilibrium of a Thin Plate, Symmetrically Loaded, Resting on an Elastic Foundation of Infinite Depth," by A. H. A. Hogg, *Philosophical Magazine*, Series 7, Vol. 25, 1938, p. 581, Eq. 13.

[14] "New Formulas for Stresses in Concrete Pavements of Airfields," by H. M. Westergaard, *Transactions*, ASCE, Vol. 113, 1948, p. 434, Eq. 36.

[15] "Om Beregning af Plader paa elastisk Underlag med saerlight Henblik paa Spørgsmaalet om Spaendinger i Betonveje," by H. M. Westergaard, *Ingenioren*, Vol. 32, 1923, p. 523, Eqs. 62 and 63.

NOTATION

The symbols used in the preceding equations and in subsequent equations in the text, unless adequately defined at place of occurrence, are defined as follows:

w = deflection:

- w_I = deflection in the region where y is negative;
- w_{II} = deflection in the region where y is positive;
- w'_{II} = same expression as w_{II} but applied to region where y is negative;

P = the concentrated load;

$D = \dfrac{E\,h^3}{12(1-\mu^2)}$ is the flexural rigidity of the slab;

h = slab thickness;

E = Young's modulus of slab;

μ = Poisson's ratio for slab, assumed to be 0.15 in all computations;

$l = \left(\dfrac{D}{k}\right)^{\frac{1}{4}} = \left[\dfrac{E\,h^3}{12(1-\mu^2)k}\right]^{\frac{1}{4}}$ for liquid subgrade;

$l = \left(\dfrac{2\,D}{C}\right)^{\frac{1}{3}}$ for solid subgrade;

k = weight density of assumed liquid subgrade;

C = rigidity of assumed solid subgrade (equal to $E/(1-\mu^2)$ in which E and μ are elastic constants of an elastic solid);

Re = "the real part of";

Im = "the imaginary part of";

H^1_0 = the Hankel function of zero order that vanishes for an infinite positive imaginary argument;

r, θ = polar coordinates;

x, y = rectangular coordinates;

λ = 1.781072—is the antilogarithm of Euler's constant;

$\Gamma\,(\;)$ = a gamma function;

$$\beta = \sqrt{\frac{\sqrt{1+\alpha^4}-\alpha^2}{2}}\,;$$

$$\gamma = \sqrt{\frac{\sqrt{1+\alpha^4}+\alpha^2}{2}}\,;$$

$$A = \frac{1}{2}\left\{1 + 2\,\gamma^2\,g - (1-\mu)^2\,\alpha^4 + \left[1 + (1-\mu)^2\,\alpha^4 + g\sin\frac{2\,\beta\,c}{l} - 2\,\gamma^2\,g\cos\frac{2\,\beta\,c}{l}\right]e^{-2\gamma c/l}\right\};$$

$$B = \frac{1}{2}\left[2(1-\mu)\,\alpha^2 - g + \left(2\,\gamma^2\,g\sin\frac{2\,\beta\,c}{l} + g\cos\frac{2\,\beta\,c}{l}\right)e^{-2\gamma c/l}\right];$$

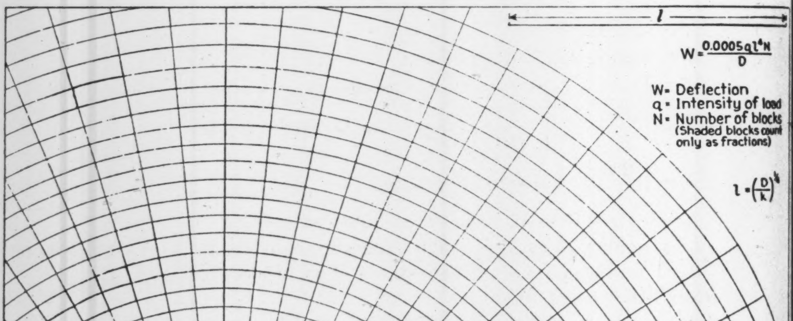
$$g = \frac{\mu\,\alpha^2}{2\,(\gamma^2+\beta^2)}\left[2\,\gamma^2 - (1-\mu)\,\alpha^2\right] + \frac{3\,(1-\mu)\,\alpha^2}{2} - \gamma^2;\ \text{and}$$

c = distance from x-axis to the edge of the slab.

In Eqs. 2 to 6, w is the deflection due to a concentrated load P at the origin of coordinates. Eqs. 5 and 6 are essentially like those presented by Mr. Westergaard in 1923.[15] Eq. 5 reduces to Eq. 4 for c equal to zero; that is, A becomes unity, and B becomes equal to $(1 - \mu)\,\alpha^2$.

Eq. 6 is particularly interesting because the integrand of the integral expression oscillates with increasing amplitude as the variable α increases. However, when this integrand is combined with that of w'_{II}, the sum of the two oscillates about the α-axis with decreasing amplitude. Therefore, the question of evaluating "divergent integrals" need not arise in any discussion of Eq. 6 or of any subsequent formulas based on it.

CHART I



(*a*) Subgrade Assumed to Be a Dense Liquid

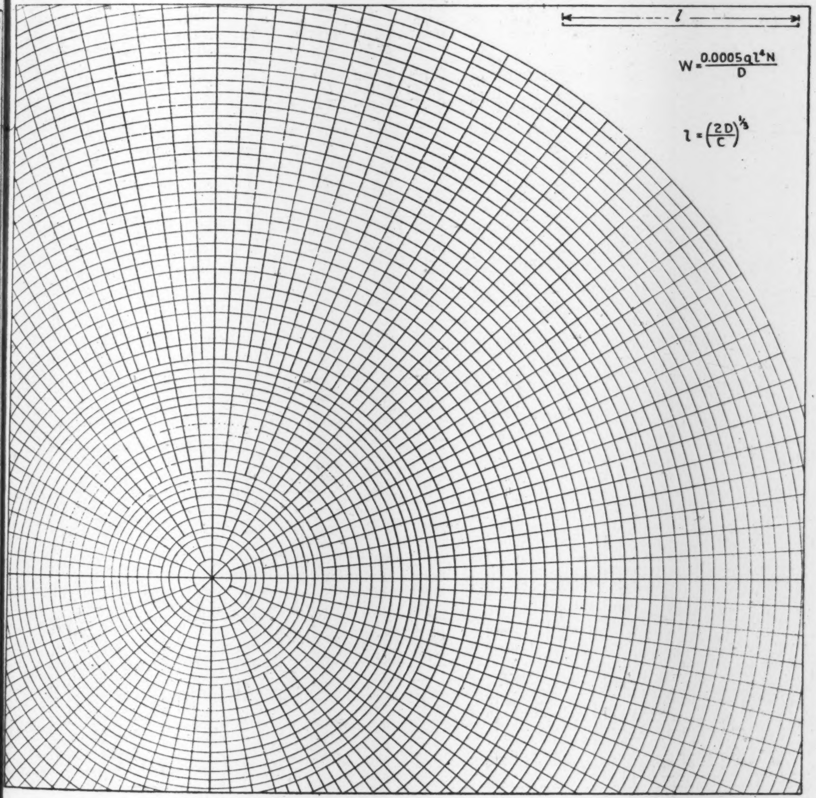
FIG. 1.—INFLUENCE CHART FOR DEFLECTION, $w = 0.0005 \frac{q l^4}{D} N$, AT AN I

DEFLECTIONS FROM DISTRIBUTED LOADS

From Maxwell's reciprocal relations, each of the equations for deflection may also be considered to be the deflection at the origin caused by a concentrated load at the variable coordinate points. Then, by use of the principle of superposition and by integration, one may find the deflection at the origin due to a distributed load. For example, if P in Eq. 2 is replaced by $q\, r\, dr\, d\theta$, and is integrated with respect to r from 0 to a, and with respect to θ from θ_1 to θ_2, the result is

$$w = \frac{q\,l^4}{D}\left(\frac{\theta_2 - \theta_1}{2\,\pi}\right)\left[\,1 + \frac{\pi}{2}\frac{a}{l}\,\mathrm{Im}\,\sqrt{i}\,H^1{}_1\left(\frac{\sqrt{i}\,a}{l}\right)\right] \qquad \text{(7)}$$

CHART 3



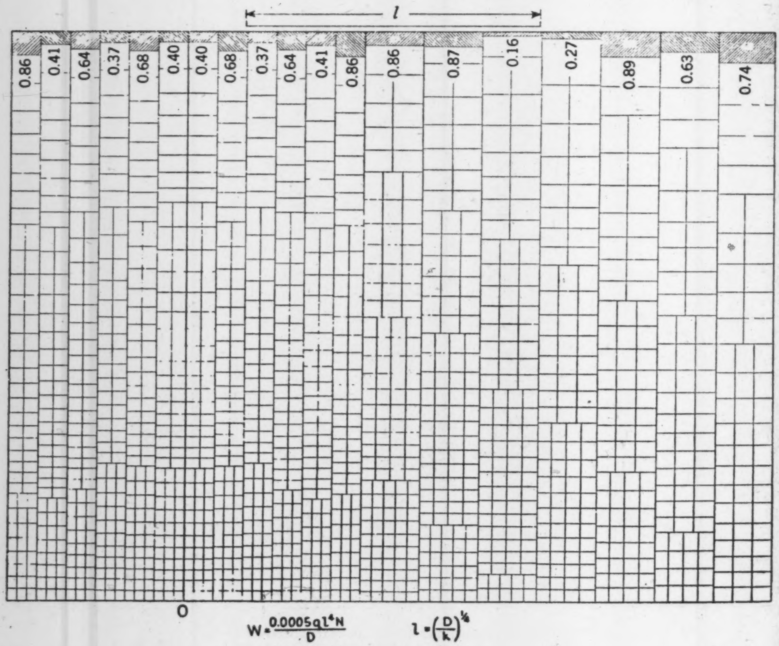
(*b*) Subgrade Assumed to Be an Elastic Solid

TERIOR POINT OF A CONCRETE SLAB (POISSON'S RATIO FOR CONCRETE, 0.15)

TABLE I.—DEFLECTION OF SLAB WHEN LOADED IN THE INTERIOR; VALUES OF THE RADII OF ARCS ENCLOSING GIVEN NUMBERS OF BLOCKS N IN THE FIRST QUADRANT OF THE APPROPRIATE DESIGN CHART

(a) LIQUID SUBGRADE; FIG. 1(a)						(b) SOLID SUBGRADE; FIG. 1(b)							
No. (N) (1)	$\frac{a}{l}$ (2)	No. (N) (1)	$\frac{a}{l}$ (2)	No. (N) (1)	$\frac{a}{l}$ (2)	No. (N) (1)	$\frac{a}{l}$ (2)	No. (N) (1)	$\frac{a}{l}$ (2)	No. (N) (1)	$\frac{a}{l}$ (2)	No. (N) (1)	$\frac{a}{l}$ (2)
0	0	94	0.750	270	1.463	0	0	110	0.6273	398	1.2837	782	1.9609
2	0.1012	110	0.821	286	1.524	2	0.0815	126	0.6744	430	1.3442	814	2.0146
6	0.1762	126	0.890	302	1.592	6	0.1414	142	0.7192	462	1.4037	846	2.0681
10	0.2285	142	0.957	318	1.659	10	0.1829	158	0.7620	494	1.4621	878	2.1215
14	0.2707	158	1.021	334	1.727	14	0.2167	174	0.8031	526	1.5197	910	2.1747
22	0.3424	174	1.085	350	1.797	22	0.2726	190	0.8428	558	1.5765	942	2.2277
30	0.4032	190	1.148	366	1.870	30	0.3193	206	0.8812	590	1.6328	974	2.2810
38	0.4566	206	1.211	382	1.945	38	0.3604	222	0.9186	622	1.6885	1,006	2.3343
46	0.5057	222	1.273	398	2.023	46	0.3976	238	0.9550	654	1.7436	1,038	2.3874
54	0.5516	238	1.336	414	2.105	54	0.4320	270	1.0253	686	1.7984	1,070	2.4406
62	0.5946	254	1.399	430	2.192	62	0.4641	302	1.0930	718	1.8529	1,102	2.4938
78	0.675	....	....	....	....	78	0.5232	334	1.1583	750	1.9070	1,134	2.5470
....	....	....	....	....	....	94	0.5771	366	1.2218	....	....	....	....

CHART 5



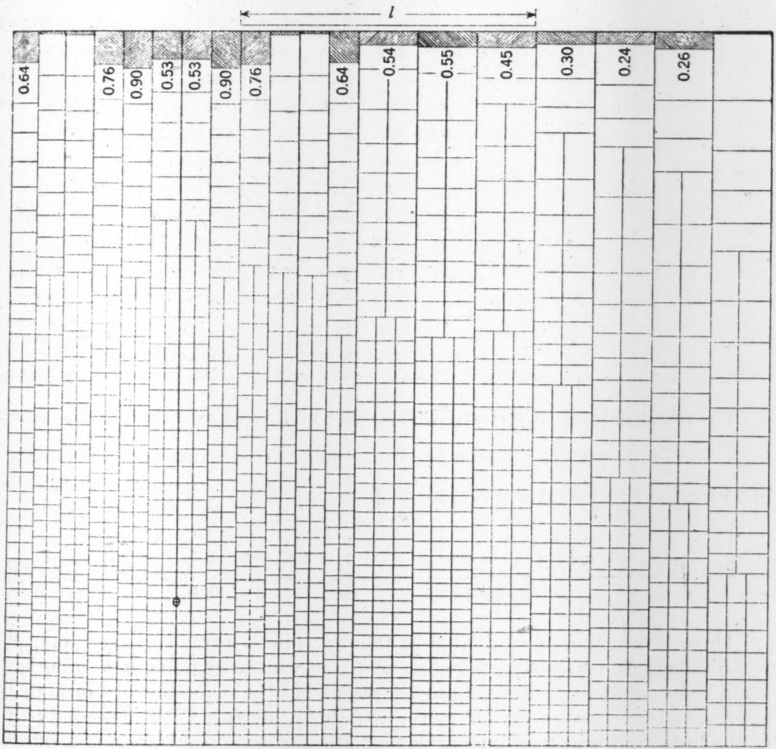
(*a*) Deflection at the Edge

FIG. 2.—INFLUENCE CHART FOR THE DEFLECTION, $w = 0.0005 \frac{q l^4}{D} N$, AT POINT O, DUE TO A POISSON'S RATIO FOR CONCRETE, 0.15; AND

In the preparation of Table 1(*a*) from which chart 1 (Fig. 1) was constructed, the quantity in brackets in Eq. 7 was set equal to 0.0005 times given integers and the resulting equations were solved by trial for a/l. The given integers determine the number of blocks to be enclosed by each circle of chart 1. Eq. 3 treated in like manner resulted in Table 1(*b*) from which chart 3 was constructed.

Tables 2(*a*) and 3(*a*), which form the basis for the two charts in Fig. 2 were prepared by use of a slightly different procedure. (Table 2(*a*) is similar to that introduced by Mr. Westergaard in 1926.[16]) In each of Eqs. 4, 5, and 6, the load P was replaced by the load $q\,dx\,dy$, then integrated with respect to x between a_1 and a_2, and with respect to y between b_1 and b_2—that is, the numbers of blocks for given regions of charts 5 and 7 (Fig. 2) were determined rather than

CHART 7



$W = \frac{0.0005 q l^4 N}{D}$ $\qquad l = \left(\frac{D}{k}\right)^{1/4}$

(*b*) Deflection at 0.5 *l* from the Edge

Load Near the Edge of a Concrete Slab (Subgrade Assumed to Be a Dense Liquid; Shaded Blocks Count Only as Fractions) the boundaries of a given group of blocks as was done for the two charts in Fig. 1. The previous method of solving by trial was not feasible here because of the nature of the equations involved.

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[16] "Stresses in Concrete Pavements Computed by Theoretical Analysis," by H. M. Westergaard. *Public Roads*, April, 1926, pp. 29 and 30, Figs. 8 and 9.

TABLE 2.—Deflections and Moments at the Edge of a Concret
of Area for the Regio

(a) Deflection at Point O, Fig. 2(a)

Range of Ratios,[b] a/l:

$\frac{b}{l}$[a]	0–0.1	0.1–0.2	0.2–0.3	0.3–0.4	0.4–0.5	0.5–0.6	0.6–0.8	0.8–1.0	1.0–1.2	1.2–1.4	1.4–1.6	1.6–1.8	1.8–2.0
2.0–1.8	1.086	1.082	1.05	1.02	1.013	0.97	0.92	0.85	0.757	0.63	0.505	0.41	0.30
1.8–1.6	1.281	1.271	1.259	1.234	1.20	1.17	1.16	1.02	0.88	0.76	0.64	0.505	0.40
1.6–1.4	1.728	1.718	1.696	1.663	1.61	1.53	1.47	1.33	1.18	1.03	0.86	0.724	0.583
1.4–1.2	2.267	2.254	2.230	2.183	2.128	2.05	1.951	1.78	1.60	1.37	1.177	1.00	0.805
1.2–1.0	2.969	2.92	2.859	2.807	2.737	2.64	2.503	2.281	2.032	1.77	1.529	1.30	1.06
1.0–0.8	3.669	3.644	3.595	3.528	3.432	3.33	3.14	2.848	2.543	2.218	1.923	1.65	1.363
0.8–0.6	4.547	4.514	4.449	4.353	4.207	4.07	3.85	3.490	3.111	2.733	2.356	2.00	1.68
0.6–0.5	5.285	5.244	5.161	5.046	4.89	4.69	4.433	4.009	3.563	3.13	2.700	2.316	1.948
0.5–0.4	5.807	5.759	5.67	5.532	5.36	5.17	4.857	4.381	3.895	3.406	2.94	2.525	2.131
0.4–0.3	6.374	6.314	6.20	6.05	5.861	5.63	5.267	4.754	4.222	3.701	3.196	2.739	2.320
0.3–0.2	6.973	6.902	6.75	6.56	6.35	6.12	5.713	5.139	4.554	3.999	3.454	2.956	2.505
0.2–0.1	7.602	7.506	7.356	7.15	6.89	6.612	6.14	5.529	4.900	4.287	3.710	3.181	2.689
0.1–0.0	8.263	8.150	7.960	7.73	7.406	7.12	6.65	5.924	5.241	4.589	3.967	3.42	2.895

[a] Symbol b denotes vertical distance from point O in Figs. 2 and 4

The result of integrations with respect to x and y was expressed in the form:

$$w = w\,(a_2, b_2) - w\,(a_1, b_2) - w\,(a_2, b_1) + w\,(a_1, b_1) \ldots\ldots\ldots (8)$$

in which the general term $w\,(a, b)$ is as follows:

$$w = \frac{2\,q\,l^4}{\pi\,D} \int_0^\infty \frac{\gamma^2 \left[(A + 2\,B\,\beta^2)\left(1 - \cos\frac{\beta\,b}{l}\,e^{-\gamma b/l}\right) + (2\,A\,\beta^2 - B)\sin\frac{\beta\,b}{l}\,e^{-\gamma b/l}\right]\sin\frac{\alpha\,a}{l}}{\alpha\,(\gamma^2 + \beta^2)\left[1 + 4\,(1 - \mu)\,\alpha^2\,\gamma^2 - (1 - \mu)^2\,\alpha^4\right]}\,d\alpha \;. (9)$$

These integrals were evaluated as follows: The integrand of each integral was divided into two parts. The first part was arbitrarily taken in a form that could be integrated by standard procedures, and at the same time in a form that approached the total integrand as the variable α increased. The second part of the integrand was then integrated numerically using the trapezoidal rule.[17] For this purpose, this second part of the integrand was evaluated for values of

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[17] "Higher Mathematics for Engineers and Physicists," by I. S. and E. S. Sokolnikoff, McGraw-Hill Book Co., Inc., New York, N. Y., 1941, p. 556.

α at 0.1 intervals in the range from 0 to 2.0 and at 0.5 intervals in the range from 2.0 to 5.0. In all cases, the second part was considered to be negligible for values of α in excess of 5.0.

SLAB RESTING ON A LIQUID SUBGRADE (NUMBER OF BLOCKS PER 0.01 l^2 INDICATED IN THE APPROPRIATE CHART)

(b) MOMENT AT POINT O, FIG. 4(a)													$\frac{b^{(a)}}{l}$
RANGE OF RATIOS,[b] a/l:													
0.1	0.2	0.3	0.4	0.5	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	
													2.0
3.13	3.0	3.0	2.9	2.7	2.4	1.92	1.33	0.74	0.19	−0.43	−0.6	−0.92	1.8
4.05	4.0	3.84	3.59	3.22	2.97	2.31	1.5	0.7	−0.07	−0.6	−0.9	−1.25	1.6
5.3	5.3	5.0	4.7	4.12	3.69	2.96	1.6	0.7	0.15	−0.8	−1.2	−1.7	1.4
6.99	6.75	6.51	5.99	5.4	4.75	3.4	2.0	0.85	−0.23	−1.0	−1.66	−2.26	1.2
9.15	8.90	8.39	7.67	6.8	5.9	4.3	2.48	0.89	−0.47	−1.3	−2.0	−2.5	1.0
12.14	11.71	10.87	9.84	8.26	7.1	5.0	2.62	0.5	−0.83	−2.0	−2.8	−3.5	0.8
16.27	15.51	14.19	12.36	10.42	8.3	5.5	2.53	0.13	−1.67	−2.80	−3.5	−4.03	0.6
20.64	19.45	17.29	14.5	11.94	9.2	5.81	2.08	−0.6	−2.3	−3.4	−4.2	−4.7	0.5
24.1	22.5	19.5	15.8	12.8	9.60	5.83	1.96	−0.8	−2.7	−3.8	−4.6	−5.0	0.4
28.81	26.08	21.74	17.20	13.19	9.72	5.6	1.54	−1.46	−3.2	−4.4	−5.0	−5.4	0.3
35.12	30.50	23.7	17.8	13.36	9.7	5.2	1.0	−1.8	−3.8	−4.9	−5.5	−5.9	0.2
43.86	35.5	25.0	18.1	13.13	9.4	4.72	0.3	−2.6	−4.4	−5.30	−6.1	−6.3	0.1
56.42	40.66	25.18	18.0	12.5	8.6	3.95	−0.6	−3.56	−5.2	−6.0	−6.5	−6.8	0.0

[b] Symbol *a* denotes horizontal distance from point O in Figs. 2 and 4.

MOMENTS FROM DISTRIBUTED LOADS

Since moments have direction within the plane of the slab, and since Maxwell's reciprocal relations do not, in general, apply between loads and moments, the derivations of the equations to be used in connection with the preparation of influence charts for moments require special consideration. As the basic formulas for interior loading, Eqs. 2 and 3, are based on the assumption that the slab is of infinite extent in all radial directions, it follows that the deflection at point $(r - dr,\theta)$ due to a load P at the origin is equal to the deflection at point (r,θ) due to the same load at point (dr,θ). Then, by Maxwell's reciprocal relations between loads and deflections, it follows that the slope dw/dr at point (r,θ) due to a load P at the origin is equal to the negative of the slope dw/dr at the origin due to the same load at point (r,θ). By similar reasoning, the reciprocal relation may be extended to higher derivatives for a slab of infinite extent in all radial directions.

TABLE 3.—DEFLECTIONS AND MOMENTS AT POINT O, 0.5 l FROM THE OF BLOCKS PER 0.01 l^2 OF AREA FOR THE

$\frac{b}{l}$[a]	(a) DEFLECTION AT POINT O, FIG. 2(b)												
	RANGE OF RATIOS,[b] a/l:												
	0–0.1	0.1–0.2	0.2–0.3	0.3–0.4	0.4–0.5	0.5–0.6	0.6–0.8	0.8–1.0	1.0–1.2	1.2–1.4	1.4–1.6	1.6–1.8	1.8–2.0
2.0–1.8	0.705	0.700	0.691	0.677	0.660	0.639	0.599	0.539	0.470	0.398	0.324	0.252	0.187
1.8–1.6	0.961	0.954	0.944	0.928	0.900	0.877	0.812	0.747	0.660	0.566	0.475	0.384	0.298
1.6–1.4	1.169	1.168	1.140	1.122	1.097	1.057	0.999	0.901	0.787	0.680	0.563	0.455	0.350
1.4–1.2	1.514	1.498	1.491	1.455	1.421	1.375	1.303	1.158	1.08	0.98	0.756	0.65	0.494
1.2–1.0	1.877	1.865	1.839	1.804	1.756	1.702	1.589	1.459	1.278	1.114	0.937	0.776	0.627
1.0–0.8	2.265	2.230	2.208	2.166	2.096	2.030	1.904	1.716	1.512	1.297	1.101	0.901	0.732
0.8–0.6	2.771	2.767	2.74	2.67	2.591	2.497	2.348	2.118	1.881	1.613	1.431	1.183	0.980
0.6–0.5	3.123	3.098	3.05	2.98	2.888	2.797	2.598	2.350	2.072	1.80	1.474	1.291	1.058
0.5–0.4	3.380	3.351	3.295	3.216	3.114	2.992	2.813	2.519	2.227	1.933	1.655	1.388	1.147
0.4–0.3	3.645	3.612	3.547	3.456	3.348	3.218	3.007	2.700	2.383	2.070	1.766	1.488	1.234
0.3–0.2	3.910	3.870	3.799	3.696	3.574	3.437	3.207	2.880	2.542	2.213	1.896	1.598	1.329
0.2–0.1	4.180	4.134	4.049	3.948	3.795	3.655	3.409	3.059	2.699	2.355	2.016	1.704	1.419
0.1–0	4.436	4.383	4.290	4.159	4.036	3.868	3.606	3.237	2.866	2.486	2.139	1.810	1.511
0–−0.1	4.667	4.612	4.515	4.389	4.240	4.074	3.800	3.412	3.017	2.630	2.261	1.917	1.599
−0.1–−0.2	4.868	4.818	4.723	4.596	4.444	4.272	3.988	3.575	3.185	2.769	2.386	2.030	1.710
−0.2–−0.3	5.065	5.017	4.927	4.799	4.645	4.470	4.176	3.767	3.315	2.906	2.504	2.130	1.791
−0.3–−0.4	5.268	5.221	5.133	5.001	4.847	4.662	4.357	3.918	3.468	3.031	2.613	2.234	1.890
−0.4–−0.5	5.464	5.418	5.326	5.202	5.037	4.844	4.541	4.082	3.617	3.162	2.737	2.338	1.980

[a] Symbol b denotes vertical distance from points O in Figs. 2 and 4.

The moment at point (r,θ) in a direction of the reference axis due to a load at the origin is given by

$$M = -\frac{D}{2}\left[(1+\mu)\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right)+(1-\mu)\left(\frac{d^2w}{dr^2}-\frac{1}{r}\frac{dw}{dr}\right)\cos 2\,\theta\right]\,..\,(10)$$

As a consequence of the foregoing reciprocal relations, Eq. 10 also represents the moment at the origin in the direction of the reference axis due to a load at point (r,θ) for a slab of infinite extent.

Substitute w from either Eqs. 2 or 3 into Eq. 10; let $P = q\,r\,dr\,d\theta$; and integrate r from 0 to a and θ from θ_1 to θ_2. The result is an expression for the moment M, in the direction of the reference axis, due to a load of intensity q uniformly distributed over a wedge bounded by $\theta = \theta_1$, $\theta = \theta_2$, and $r = a$. From the resulting expressions, Table 4 was prepared; and the two charts in Fig. 3 were then constructed from the information in the tables.

EDGE OF A CONCRETE SLAB RESTING ON A LIQUID SUBGRADE (NUMBER REGION INDICATED ON THE APPROPRIATE CHART)

(b) MOMENT AT POINT O, FIG. 4(b)													b[a]/l
RANGE OF RATIOS,[b] a/l:													
0.1	0.2	0.3	0.4	0.5	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	
													2.0
2.128	2.084	1.96	1.83	1.68	1.50	1.295	0.881	0.463	0.091	−0.264	−0.49	−0.711	1.8
2.796	2.737	2.628	2.473	2.239	2.083	1.658	1.098	0.559	0.062	−0.341	−0.642	−0.906	1.6
3.592	3.523	3.378	3.156	2.882	2.601	2.063	1.344	0.668	0.055	−0.446	−0.850	−1.090	1.4
4.782	4.661	4.431	4.120	3.732	3.251	2.544	1.566	0.658	−0.091	−0.679	−1.115	−1.380	1.2
6.132	5.967	5.638	5.25	4.65	4.0	3.035	1.785	0.683	−0.217	−0.903	−1.395	−1.718	1.0
7.964	7.697	7.191	6.500	5.7	4.9	3.516	1.919	0.589	−0.494	−1.25	−1.758	−2.2	0.8
10.415	9.85	9.0	8.1	7.0	5.8	3.95	1.9	0.4	−0.720	−1.65	−2.2	−2.65	0.6
12.894	12.1	11.0	9.5	7.7	6.25	4.1	1.9	0.25	−1.0	−1.9	−2.5	−3.0	0.5
14.8	13.8	12.2	10.2	8.075	6.364	4.173	1.8	0.0	−1.2	−2.1	−2.7	−3.25	0.4
17.1	15.9	13.281	10.7	8.3	6.4	4.05	1.6	−0.2	−1.4	−2.3	−2.95	−3.500	0.3
20.301	17.808	14.328	11.1	8.50	6.4	3.9	1.4	−0.4	−1.64	−2.5	−3.2	−3.70	0.2
24.350	20.132	15.042	11.234	8.55	6.5	3.8	1.2	−0.605	−1.86	−2.72	−3.4	−3.935	0.1
29.060	21.972	15.5	11.2	8.85	6.6	3.7	1.03	−0.74	−1.893	−2.874	−3.6	−4.274	0
29.656	22.536	16.3	11.8	9.3	6.9	3.8	1.0	−0.882	−2.066	−3.15	−3.9	−4.55	−0.1
26.140	21.831	16.7	12.513	9.674	7.346	4.179	1.0	−1.016	−2.299	−3.308	−4.153	−4.8	−0.2
23.279	20.80	16.9	13.233	10.214	7.606	4.464	1.0	−1.1	−2.5	−3.5	−4.4	−5.0	−0.3
21.702	19.85	17.032	13.844	10.854	8.065	4.599	1.0	−1.25	−2.7	−3.75	−4.5	−5.1	−0.4
20.359	19.039	16.720	13.907	11.121	8.355	4.774	1.065	−1.4	−2.846	−3.889	−4.686	−5.209	−0.5

[b] Symbol *a* denotes horizontal distance from points O, in Figs. 2 and 4.

In the basic formulas for loading at an edge or near an edge (Eqs. 4, 5, and 6), it is assumed that the slab extends indefinitely far in both the positive and the negative directions of x, indefinitely far in the positive direction of y, but only a finite distance c in the negative y-direction ($c = 0$ for Eq. 4). By a process of reasoning similar to that used previously, it may be shown that, for a slab with these boundaries, the following relations hold:

(1) The quantity $\dfrac{\partial^n w}{\partial x^n}$ at point (0,0) due to a load P at point (x,y) is equal to $\left(-\dfrac{\partial}{\partial x}\right)^n w$ at point (x,y) due to the same load at point (0,0).

(2) The quantity $\dfrac{\partial^n w}{\partial y^n}$ at point (0,0) due to a load P at point (x,y) is equal to $\left(\dfrac{\partial}{\partial c} - \dfrac{\partial}{\partial y}\right)^n w$ at point (x,y) due to the same load at point (0,0). There-

TABLE 4.—MOMENTS IN SLAB WHEN LOADED IN THE INTERIOR; LOCATION OF THE LINES BOUNDING THE BLOCKS OF THE FIRST QUADRANT OF THE APPROPRIATE DESIGN CHART (RADII, IN TERMS OF *l*, OF THE BOUNDING ARCS. EXCEPT WHERE NOTED, THE RADII OF CONSECUTIVE ARCS BETWEEN GIVEN RADIAL LINES ARE ARRANGED CONSECUTIVELY IN THE COLUMNS)

(*a*) SUBGRADE ASSUMED TO BE A DENSE LIQUID; FIG. 3(*a*)

Angles of Bounding Radii:

0°–5°	5°–10°	10°–15°	15°–20°	20°–25°	25°–30°	30°–35°	35°–40°	40°–45°	45°–50°	50°–55°	55°–60°	60°–65°	65°–70°	70°–75°	75°–80°	80°–85°	85°–90°
0.0173																	
0.0332									0.0311								
0.0503						0.0478						0.0456					
0.0768			0.0753			0.0729			0.0704			0.0686			0.0675		
0.0986			0.0964			0.0930			0.0895			0.0868			0.0856		
0.1535	0.1530	0.1520	0.1505	0.1488	0.1468	0.1445	0.1423	0.1401	0.1378	0.1360	0.1342	0.1326	0.1312	0.1300	0.1292	0.1287	0.1284
0.2011	0.2003	0.1987	0.1965	0.1938	0.1907	0.1876	0.1842	0.1809	0.1778	0.1748	0.1722	0.1698	0.1679	0.1663	0.1652	0.1644	0.1640
0.2458	0.2445	0.2423	0.2392	0.2354	0.2311	0.2267	0.2222	0.2177	0.2136	0.2097	0.2063	0.2032	0.2006	0.1986	0.1970	0.1959	0.1953
0.2888	0.2872	0.2842	0.2801	0.2752	0.2695	0.2637	0.2579	0.2523	0.2470	0.2422	0.2380	0.2342	0.2310	0.2284	0.2265	0.2253	0.2246
0.3317	0.3296	0.3257	0.3202	0.3138	0.3066	0.2995	0.2923	0.2853	0.2788	0.2730	0.2677	0.2634	0.2597	0.2566	0.2544	0.2529	0.2521
0.3748	0.3721	0.3671	0.3603	0.3521	0.3410	0.3345	0.3256	0.3172	0.3095	0.3026	0.2964	0.2910	0.2866	0.2831	0.2804	0.2787	0.2778
0.4190	0.4156	0.4093	0.4006	0.3905	0.3792	0.3688[a]	0.3583[a]	0.3484[a]	0.3393[a]	0.3312[a]	0.3241[a]	0.3181[a]	0.3131[a]	0.3091[a]	0.3060[a]	0.3040[a]	0.3030[a]
0.4651	0.4608	0.4527	0.4418	0.4294	0.4140	0.5078	0.4877	0.4701	0.4543	0.4407	0.4291	0.4192	0.4112	0.4048	0.3999	0.3969	0.3954
0.5140	0.5084	0.4980	0.4844	0.4690	0.4532	0.6584	0.6224	0.5926	0.5676	0.5467	0.5292	0.5160	0.5033	0.4944	0.4881	0.4838	0.4816
0.5670	0.5596	0.5461	0.5288	0.5090	0.4900	0.8423[a]	0.7732	0.7233	0.6843	0.6534	0.6286	0.6089	0.5931	0.5810	0.5722	0.5665	0.5637
0.6263	0.6179	0.5984	0.5760	0.5511	0.5294[a]	0.894	0.9616[a]	0.8705	0.8092	0.7643	0.7297	0.7031	0.6822	0.6664	0.6552	0.6477	0.6442
0.6964	0.6844	0.6567	0.6270	0.5985	0.7019	0.958	1.019	1.0523[a]	0.9489	0.8825	0.8348	0.7991	0.7720	0.7521	0.7378	0.7285	0.7240
0.7903	0.7642	0.7251	0.6835	0.6457	0.9577[a]	1.040	1.082	1.102	1.1160	1.0127	0.9461	0.8985	0.8641	0.8386	0.8210	0.8097	0.8042
0.9660[b]	0.9073	0.8167	0.7490	0.6969	1.1288	1.1699	1.169	1.161	1.3425[a]	1.1628	1.0668	1.0036	0.9591	0.9275	0.9056	0.8916	0.8849
1.171	0.9815[b]	1.0115[b]	0.8311	0.7563	1.210[b]	1.3198[b]	1.3136	1.235	1.425	1.3477	1.2018	1.1160	1.0591	1.0192	0.9925	0.9755	0.9674
1.254	1.1850	1.2201	0.9716	0.8280	1.4360	1.5560	1.4557[b]	1.3385	1.530	1.6212[a]	1.3588	1.2387	1.1648	1.1151	1.0821	1.0619	1.0522
1.320	1.2718	1.3057	1.0591[b]	0.9163	1.5322	1.6592	1.7065	1.518	1.709	1.736	1.5565	1.3770	1.2788	1.2167	1.1760	1.1511	1.1395
1.377	1.3376	1.3717	1.2704	1.1364[b]	1.6054	1.7382	1.8166	1.6220[b]	1.8220[b]	1.921	1.8500[a]	1.5379	1.4041	1.3247	1.2747	1.2445	1.2304
1.426	1.3945	1.4306	1.3586	1.3424	1.6710	1.8069	1.9027	1.8939	2.1239	2.0645[b]	1.970	1.7390	1.5450	1.4420	1.3799	1.3432	1.3263
1.472	1.4452	1.4836	1.4263	1.4345	1.7304	1.8690	1.9775	2.01345			2.174	2.0309[a]	1.7096	1.5713	1.4933	1.4484	1.4278
1.516	1.4921	1.5312	1.4866	1.5041	1.7856	1.9259	2.0449				2.3455[b]	2.144	1.9137	1.7179	1.6172	1.5611	1.5363
1.555	1.5347	1.5748	1.5410	1.5666	1.8334	1.9793						2.298	2.2014[a]	1.8899	1.7553	1.6850	1.6541
1.592	1.5750	1.6152[a]	1.5900	1.6228	1.8795	2.0298						2.6622[b]		2.1035	1.9138	1.8226	1.7836
1.630	1.6135	1.7627	1.6347[a]	1.6737[a]	1.9232									2.4050[a]	2.1030[a]	1.9800	1.9291
1.664[a]	1.6495[a]	1.8920	1.7930	1.8447	1.9661											2.1658[a]	2.0970[a]
1.793	1.7840	2.0108[a]	1.9308	1.9959[a]	2.0091												
1.909	1.9043		2.0260[a]														
2.019[a]	2.0167[a]																

TABLE 4.—(*Continued*)

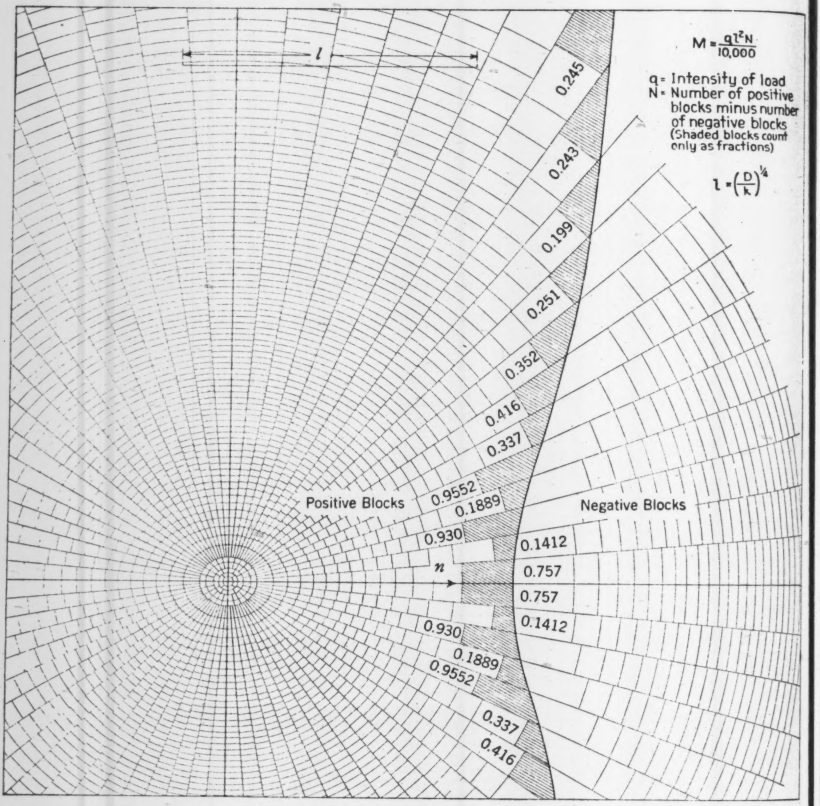
(*b*) Subgrade Assumed to Be an Elastic Solid; Fig. 3(*b*)

Angles of Bounding Radii:

0°–5°	5°–10°	10°–15°	15°–20°	20°–25°	25°–30°	30°–35°	35°–40°	40°–45°	45°–50°	50°–55°	55°–60°	60°–65°	65°–70°	70°–75°	75°–80°	80°–85°	85°–90°
0.0173																	
0.0332									0.0311								
0.0503						0.0478						0.0456					
0.0768			0.0753			0.0729			0.0704			0.0685			0.0675		
0.0985			0.0964			0.0929			0.0895			0.0868			0.0854		
0.1535	0.1529	0.1519	0.1504	0.1487	0.1467	0.1445	0.1423	0.1400	0.1379	0.1360	0.1341	0.1325	0.1312	0.1301	0.1292	0.1287	0.1284
0.2009	0.2001	0.1986	0.1964	0.1937	0.1906	0.1874	0.1840	0.1807	0.1776	0.1748	0.1721	0.1698	0.1679	0.1664	0.1652	0.1644	0.1640
0.2451	0.2441	0.2418	0.2387	0.2351	0.2310	0.2265	0.2220	0.2175	0.2134	0.2097	0.2062	0.2032	0.2007	0.1987	0.1971	0.1961	0.1956
0.2878	0.2866	0.2835	0.2794	0.2746	0.2692	0.2634	0.2576	0.2519	0.2466	0.2420	0.2377	0.2340	0.2309	0.2284	0.2266	0.2253	0.2246
0.3306	0.3289	0.3246	0.3192	0.3130	0.3061	0.2989	0.2917	0.2847	0.2783	0.2727	0.2675	0.2631	0.2594	0.2564	0.2542	0.2527	0.2519
0.3726	0.3704	0.3655	0.3588	0.3510	0.3426	0.3337[a]	0.3249[a]	0.3164[a]	0.3087	0.3022	0.2961	0.2908	0.2865	0.2830	0.2804	0.2787	0.2778
0.4158	0.4131	0.4069	0.3986	0.3890	0.3787	0.4701	0.4531	0.4383	0.3384[a]	0.3309[a]	0.3238[a]	0.3178[a]	0.3128[a]	0.3088[a]	0.3058[a]	0.3038[a]	0.3029[a]
0.4601	0.4568	0.4491	0.4389	0.4270	0.4147	0.6114	0.5821	0.5573	0.4527	0.4392	0.4278	0.4182	0.4104	0.4042	0.3996	0.3966	0.3951
0.5055	0.5015	0.4923	0.4799	0.4655	0.4509	0.7686	0.7198	0.6796	0.5635	0.5438	0.5269	0.5130	0.5019	0.4930	0.4866	0.4823	0.4801
0.5546	0.5497	0.5380	0.5266	0.5050	0.4876	0.9642[a]	0.8760	0.8127	0.6764	0.6480	0.6240	0.6051	0.5903	0.5785	0.5700	0.5645	0.5617
0.6089	0.6026	0.5870	0.5678	0.5458	0.5250	1.0307	1.0699[a]	0.9612	0.7951	0.7545	0.7213	0.6964	0.6777	0.6624	0.6516	0.6449	0.6413
0.6699	0.6617	0.6403	0.6152	0.5879	0.5634	1.1037	1.1297	1.1386[a]	0.9226	0.8657	0.8211	0.7885	0.7653	0.7459	0.7328	0.7242	0.7196
0.7392	0.7286	0.6990	0.6674	0.6324	0.6030	1.1941	1.1975	1.1907	1.0644	0.9843	0.9274	0.8855	0.8537	0.8294	0.8137	0.8031	0.7975
0.8183	0.8045	0.7640	0.7262	0.6802	0.6442	1.3309	1.2773	1.2482	1.2297	1.1133	1.0375	0.9820	0.9443	0.9141	0.8952	0.8825	0.8757
1.0150	0.9206	0.8460	0.7838	0.7312	0.6865	1.5067[b]	1.3795	1.3108	1.4355[a]	1.2577	1.1552	1.0862	1.0373	0.9997	0.9782	0.9627	0.9544
1.0545[b]	1.0733[b]	0.9691	0.8588	0.7873	0.7316	1.7833	1.5479	1.3821	1.4990	1.4260	1.2836	1.1947	1.1342	1.0907	1.0628	1.0448	1.0349
1.2767	1.2984	1.0988[b]	0.9582	0.8505	0.7804	1.8972	1.6936[b]	1.4659	1.5730	1.6350[a]	1.4261	1.3106	1.2355	1.1840	1.1498	1.1285	1.1166
1.3652	1.3892[a]	1.3407	1.1694[b]	0.9258	0.8332	1.9970	1.9971	1.5726	1.6555	1.6954	1.5894	1.4365	1.3426	1.2806	1.2397	1.2147	1.2003
1.4383[a]	1.6430	1.4331	1.4052	1.0256	0.8916	2.0850	2.1328	1.7393	1.7551	1.7638	1.7859[a]	1.5751	1.4569	1.3861	1.3331	1.3038	1.2886
1.6667	1.8257	1.5115	1.4997	1.2516[b]	0.9586	2.1629	2.2417	1.9364[b]	1.8876	1.8430	1.8426	1.7314	1.5804	1.4888	1.4308	1.3961	1.3794
1.8372	1.9815	1.5807[a]	1.5818	1.4941	1.0395	2.2330	2.3356	2.2774	2.1765	1.9316	1.9025	1.9124[a]	1.7150	1.6025	1.5334	1.4929	1.4732
1.9879	2.1230	1.7915	1.6534[a]	1.5941	1.1510	2.2976	2.4216	2.4357	2.2560[b]	2.0372	1.9679	1.9630	1.8647	1.7251	1.6421	1.5942	1.5715
2.1255	2.2559	1.9624	1.8726	1.6809	1.3623[b]	2.3587	2.5022			2.1749	2.0400	2.0163	2.0346	1.8582	1.7580	1.7014	1.6747
2.2545	2.3829	2.1144	2.0511	1.7566[a]	1.6210	2.4190				2.3900	2.1192	2.0724	2.2329	2.0050	1.8828	1.8155	1.7842
2.3783	2.5063[a]	2.2556	2.2106	1.9885	1.7258	2.4807					2.2080	2.1321	2.4738[a]	2.1694	2.0185	1.9379	1.9009
2.4986[a]		2.3902	2.3594	2.1785	1.8175	2.5380					2.3109	2.1961	2.5444	2.3574[a]	2.1675	2.0700	2.0259
		2.5201[a]	2.5016[a]	2.3496	1.8977[a]						2.4352	2.2649		2.4093	2.3342	2.2148	2.1617
				2.5099[a]	2.1472							2.3388		2.4633	2.5222[a]	2.3750[a]	2.3104
					2.3542							2.4203		2.5195	2.5733	2.4179	2.4753[a]
					2.5429[a]							2.5135				2.4620	2.5197
																2.5077	

[a] Values tabulated in the columns between superior letters *a* apply to every fourth arc only. Values for intermediate arcs may be found by interpolation. [b] Boundary between positive and negative blocks.

CHART 2



(*a*) Subgrade Assumed to Be a Dense Liquid

FIG. 3.—INFLUENCE CHARTS FOR THE MOMENT $M_n = \frac{q\,l^2\,N}{10,000}$ IN THE INTERIOR OF THE CON

BLOCKS; AND POISSON'S

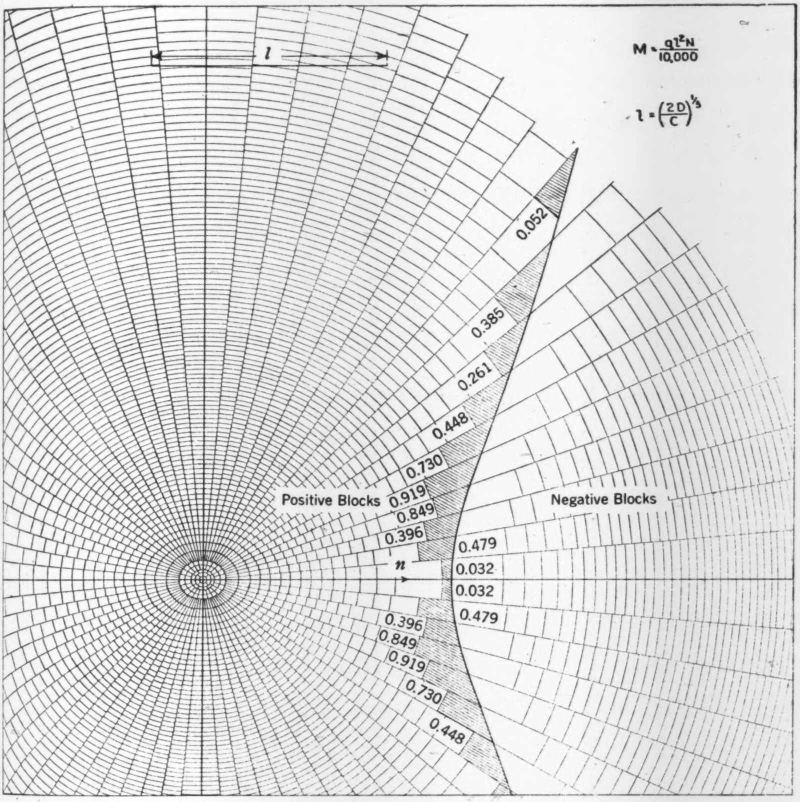
fore, since $M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)$, it follows that the moment M_x at point (0,0) due to a load at point (x,y) is given by

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \mu\left(\frac{\partial^2 w}{\partial y^2} - 2\,\frac{\partial^2 w}{\partial y\,\partial c} + \frac{\partial^2 w}{\partial c^2}\right)\right] \dots\dots\dots (11)$$

in which w is the deflection at point (x,y) due to the same load at point (0,0).

The expressions for deflection from Eqs. 4, 5, and 6 were each substituted into Eq. 11; P was placed equal to $q\,dx\,dy$ and the result was integrated with

CHART 4



(*b*) Subgrade Assumed to Be an Elastic Solid

CRETE PAVEMENT SLAB (N = NUMBER OF POSITIVE BLOCKS MINUS NUMBER OF NEGATIVE
RATIO FOR CONCRETE, 0.15)

respect to x from a_1 to a_2, and with respect to y from b_1 to b_2. From the resulting expressions, Tables 2(*b*) and 3(*b*) were computed. The charts of Fig. 4 were constructed from these tables. Expressions were put in the form:

$$M_x = M\,(a_2, b_2) - M\,(a_1, b_2) - M\,(a_2, b_1) + M\,(a_1, b_1) \quad \text{.....(12)}$$

The general term on the right side of Eq. 12—$M\,(a, b)$—depends on which equation was used for deflection. For Eqs. 4 and 5, the result can be written as follows:

$$M(a, b) = \frac{2\,q\,l^2}{\pi} \int_0^\infty \frac{\gamma^2 \sin \frac{\alpha\, a}{l} \left[S \left(1 - \cos \frac{\beta\, b}{l}\, e^{-\gamma b/l} \right) - T \sin \frac{\beta\, b}{l}\, e^{-\gamma b/l} \right]}{\alpha(\beta^2 + \gamma^2)\,[1 + 4\,(1 - \mu)\,\alpha^2\,\gamma^2 - (1 - \mu)^2\,\alpha^4]}\, d\alpha \,..(13)$$

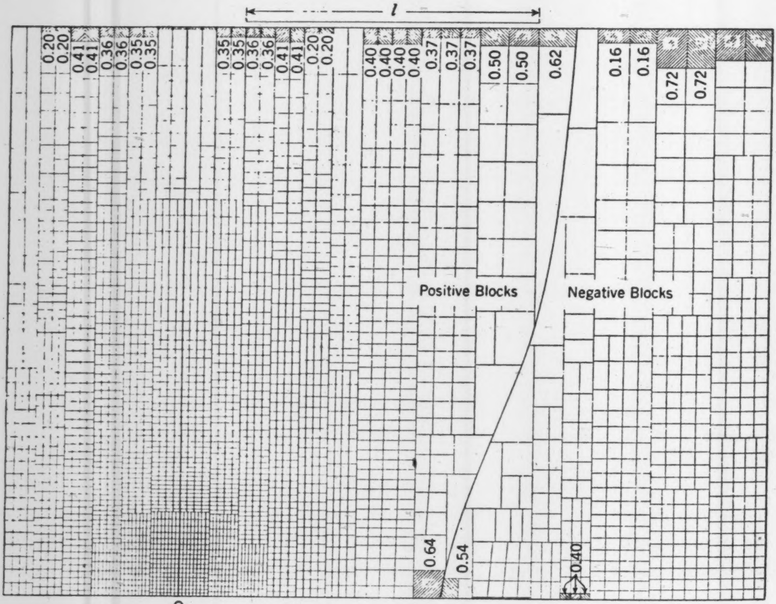
in which, for Eq. 5,

$$S = \Big\{ (1 - \mu)\,\alpha^2\,(A + 2\,B\,\beta^2) - \mu \left[2\,A\,\beta^2 - B + l^2 \frac{\partial^2}{\partial c^2} (A + 2\,B\,\beta^2) + \frac{2\,l}{\gamma} (\gamma^2 + \beta^2) \frac{\partial A}{\partial c} \right] \Big\} \,..(14a)$$

and

$$T = \Big\{ (1 - \mu)\,\alpha^2\,(-\,2\,A\,\beta^2 + B) - \mu \left[A + 2\,B\,\beta^2 + l^2 \frac{\partial^2}{\partial c^2} (-\,2\,A\,\beta^2 + B) + \frac{2\,l}{\gamma} (\gamma^2 + \beta^2) \frac{\partial B}{\partial c} \right] \Big\} \,..(14b)$$

CHART 6



(a) At the Edge

FIG. 4.—INFLUENCE CHARTS FOR THE MOMENT $M = \frac{q\,l^2\,N}{10{,}000}$, AT POINT O ($N$ = NUMBER OF SHADED BLOCKS COUNT

and, for Eq. 4 (that is, for $c = 0$),

$$S = (1 - \mu^2)\,\alpha^2\,[1 + 2\,(1 - \mu)\,\alpha^2\,\beta^2] \qquad (15a)$$

and

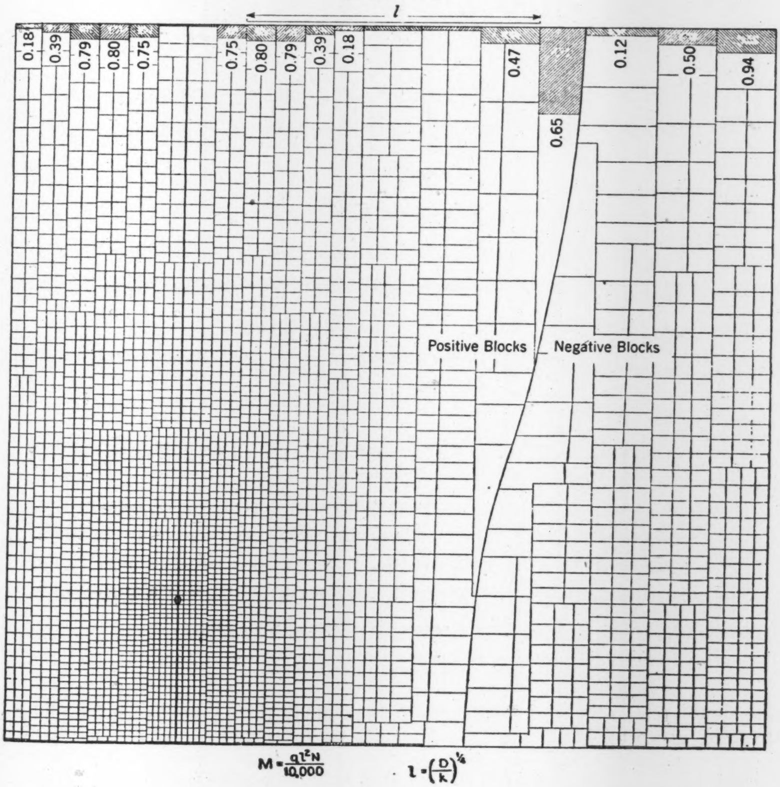
$$T = (1 - \mu^2)\,\alpha^2\,[(1 - \mu)\,\alpha^2 - 2\,\beta^2] \qquad (15b)$$

In the evaluation of the integrals (of which the one in Eq. 13 is representative), the integrands were divided into two parts as previously explained.

POISSON'S RATIO

Poisson's ratio was assumed to be 0.15 for all the computations. This choice was made because it is the value generally used by Mr. Westergaard.

CHART 8



(*b*) Parallel to, and 0.5 *l*, from the Edge

POSITIVE BLOCKS MINUS NUMBER OF NEGATIVE BLOCKS; POISSON'S RATIO FOR CONCRETE, 0.15; AND ONLY AS FRACTIONS)

INTERIOR - LIQUID SUBGRADE

DEFLECTION

$$w = \frac{ql^4}{D}\,\frac{\theta_2-\theta_1}{2\pi}\left[1+\frac{\pi}{2}\,\frac{a}{l}\;\mathrm{Im}\,\sqrt{i}\;H_1^1\!\left(\frac{\sqrt{i}\,a}{l}\right)\right]$$

Formula for Table 1a-

$$\frac{2000\,wD}{ql^4} = 2000\,\frac{(\theta_2-\theta_1)}{2\pi}\left[1+\frac{\pi a}{2l}\;\mathrm{Im}\,\sqrt{i}\;H_1^1\!\left(\frac{\sqrt{i}\,a}{l}\right)\right]$$

MOMENT

$$M = \frac{ql^2}{8}\,\mathrm{Re}\left[(1+\mu)(\theta_2-\theta_1)\,\frac{a}{l}\sqrt{i}\;H_1^1\!\left(\frac{\sqrt{i}}{l}\,a\right)+(1-\mu)(\sin 2\theta_2-\sin 2\theta_1)\left\{\frac{a\sqrt{i}}{2l}\,H_1^1\!\left(\frac{\sqrt{i}\,a}{l}\right)+H_0^1\!\left(\frac{\sqrt{i}\,a}{l}\right)-0.5\right\}\right]$$

Formula for Table 4a-

$$\frac{10000\,M}{ql^2} = \mathrm{Re}\left[(125.4455+\Theta_1)\frac{a\sqrt{i}}{l}\,H_1^1\!\left(\frac{\sqrt{i}\,a}{l}\right)+2\,\Theta_1\,H_0^1\!\left(\frac{\sqrt{i}\,a}{l}\right)-\Theta_1\right]$$

where $\Theta_1 = 531.25\,(\sin 2\theta_2-\sin 2\theta_1)$

INTERIOR - SOLID SUBGRADE

DEFLECTION

$$w=\frac{ql^4}{D}\,\pi\sum_{\substack{m\\0,1,2}}(-1)^m\left[\frac{\frac{8}{3\sqrt{3}}\left(\frac{a}{2l}\right)^{6m+2}}{[(3m)!]^2(6m+2)}+\frac{\frac{4}{\pi}\left\{\log\frac{\lambda a}{2l}-\frac{1}{6m+4}-\left(1+\frac{1}{2}+\cdots+\frac{1}{3m+1}\right)\right\}\left(\frac{a}{2l}\right)^{6m+4}}{[(3m+1)!]^2\,(6m+4)}-\frac{\frac{8}{3\sqrt{3}}\left(\frac{a}{2l}\right)^{6m+6}}{[(3m+2)!]^2(6m+6)}+\frac{2\left(\frac{a}{2l}\right)^{6m+7}}{[\Gamma(3m+3.5)]^2(6m+7)}\right]$$

Formula for Table 1b

$$\frac{wD}{ql^4}=\pi\left(\frac{a}{l}\right)^2\Big[.1924501+\left\{.0458085\left(\frac{a}{l}\right)^2-.497054\times10^{-6}\left(\frac{a}{l}\right)^8\right\}\log_{10}\left(\frac{a}{l}\right)-.0271743\left(\frac{a}{l}\right)^2-1.002344\times10^{-3}\left(\frac{a}{l}\right)^4$$

$$+.202100\times10^{-3}\left(\frac{a}{l}\right)^5-.2088217\times10^{-4}\left(\frac{a}{l}\right)^6+.496338\times10^{-6}\left(\frac{a}{l}\right)^8+.2175\times10^{-8}\left(\frac{a}{l}\right)^{10}-.2266\times10^{-9}\left(\frac{a}{l}\right)^{11}+.12948\times10^{-10}\left(\frac{a}{l}\right)^{12}\Big]$$

MOMENT

$$M=\frac{-ql^2}{16}\Bigg[\sum_{\substack{m\\1,2}}\frac{(-1)^m\frac{8}{3\sqrt{3}}\left(\frac{a}{2l}\right)^{6m}}{[(3m)!]^2}\left\{(1+\mu)(\theta_2-\theta_1)6m+(1-\mu)(\sin 2\theta_2-\sin 2\theta_1)(3m-1)\right\}$$

$$+\sum_{\substack{m\\0,1,2}}\frac{(-1)^m 4\left(\frac{a}{2l}\right)^{6m+2}}{\pi\,[(3m+1)!]^2}\cdot\left\{(1+\mu)(\theta_2-\theta_1)+\frac{1}{2}(1-\mu)(\sin 2\theta_2-\sin 2\theta_1)+\left[(6m+2)(1+\mu)(\theta_2-\theta_1)+3m(1-\mu)(\sin 2\theta_2-\sin 2\theta_1)\right]\left[\log\frac{\lambda a}{2l}-\left(1+\frac{1}{2}+\cdots+\frac{1}{3m+1}\right)\right]\right\}$$

$$-\sum_{\substack{m\\0,1,2}}\frac{(-1)^m\frac{8}{3\sqrt{3}}\left(\frac{a}{2l}\right)^{6m+4}}{[(3m+2)!]^2}\left\{(6m+4)(1+\mu)(\theta_2-\theta_1)+(3m+1)(1-\mu)(\sin 2\theta_2-\sin 2\theta_1)\right\}+\sum_{\substack{m\\0,1,2}}\frac{(-1)^m 2\left(\frac{a}{2l}\right)^{6m+5}}{[\Gamma(3m+3.5)]^2}\left\{(6m+5)(1+\mu)(\theta_2-\theta_1)+\frac{1}{2}(6m+3)(1-\mu)(\sin 2\theta_2-\sin 2\theta_1)\right\}\Bigg]$$

Formula for Table 4b

$$\frac{10\,000\,M}{ql^2}=\left(A\log_{10}\frac{a}{l}+B\right)\left(\frac{a}{l}\right)^2+C\left(\frac{a}{l}\right)^4+D\left(\frac{a}{l}\right)^5+E\left(\frac{a}{l}\right)^6+\left(F\log_{10}\frac{a}{l}+G\right)\left(\frac{a}{l}\right)^8+H\left(\frac{a}{l}\right)^{10}+I\left(\frac{a}{l}\right)^{11}+J\left(\frac{a}{l}\right)^{12}+\left(K\log_{10}\frac{a}{l}+L\right)\left(\frac{a}{l}\right)^{14}$$

where

$A=-18.3887\,(\theta_2-\theta_1)$

$B=4.91890\,(\theta_2-\theta_1)-84.5511\,\Theta_2$

$C=1.207100\,(\theta_2-\theta_1)+12.77989\,\Theta_2$

$D=-.354937\,(\theta_2-\theta_1)-4.50937\,\Theta_2$

$E=.0502958\,(\theta_2-\theta_1)+.709994\,\Theta_2$

$F=1.99530\times10^{-3}(\theta_2-\theta_1)+31.6873\times10^{-3}\,\Theta_2$

$G=-1.79745\times10^{-3}(\theta_2-\theta_1)-27.9717\times10^{-3}\,\Theta_2$

$H=-13.0979\times10^{-6}(\theta_2-\theta_1)-221.873\times10^{-6}\,\Theta_2$

$I=1.62595\times10^{-6}(\theta_2-\theta_1)+28.1689\times10^{-6}\,\Theta_2$

$J=-.109149\times10^{-6}(\theta_2-\theta_1)-1.92598\times10^{-6}\,\Theta_2$

$K=-1.23717\times10^{-9}(\theta_2-\theta_1)-22.4541\times10^{-9}\,\Theta_2$

$L=1.41704\times10^{-9}(\theta_2-\theta_1)+25.6026\times10^{-9}\,\Theta_2$

$\Theta_2 = \sin 2\theta_2-\sin 2\theta_1$ θ_1 and θ_2 are in degrees for A to L inclusive

FIG. 5.—WORKING FORMULAS FOR USE IN PREPARATION OF TABULAR DATA FOR CHART CONSTRUCTION

No doubt relatively simple corrections for the effect of Poisson's ratio similar to those given by Mr. Westergaard[18] could be developed.

SUMMARY OF THE COMPUTATIONS

Working formulas for computing values in Tables 1 to 4, in accordance with the procedure recommended in this paper, are presented in Figs. 5 and 6.

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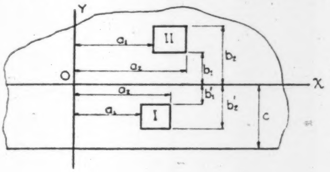
[18] "New Formulas for Stresses in Concrete Pavements of Airfields," by H. M. Westergaard, *Transactions*, ASCE, Vol. 113, 1948, p. 437, Table 1.

Influence Charts and Their Construction

An influence surface is similar to an influence line in that its ordinate at any point A gives the effect at a fixed point B of a unit load applied at point A. Influence surfaces are applicable to "two-dimensional" problems whereas influence lines are applicable to "one-dimensional" structures, such as beams or trusses.

Assume that the space between an influence surface and its reference plane is divided into cells with vertical walls, each cell being of unit volume and having



Boundaries of slab at $y=-c$, $y=+\infty$, $x=\pm\infty$

FOR LOAD ON REGION II

Deflection: $w_{II} = w_{II}(a_2,b_2) - w_{II}(a_1,b_2) - w_{II}(a_2,b_1) + w_{II}(a_1,b_1)$

Moment: $M_{II} = M_{II}(a_2,b_2) - M_{II}(a_1,b_2) - M_{II}(a_2,b_1) + M_{II}(a_1,b_1)$

FOR LOAD ON REGION I

Deflection: $w_I = w_I(a_2,b'_2) - w_I(a_1,b'_2) - w_I(a_2,b'_1) + w_I(a_1,b'_1)$

Moment: $M_I = M_I(a_2,b'_2) - M_I(a_1,b'_2) - M_I(a_2,b'_1) + M_I(a_1,b'_1)$

The general terms $w_{II}(a,b)$; $w_I(a,b)$; $M_{II}(a,b)$; $M_I(a,b)$ are as follows:

$w_{II}(a,b) = w'(a,b) - w''(a,b) + w'''(a,b)$; $\quad M_{II} = M'(a,b) - M''(a,b) + M'''(a,b)$

$w_I(a,b) = w'(a,b') - w''(a,b') - w'''(a,-b') + w^{IV}(a,b')$; $\quad M_I = M'(a,b') - M''(a,b') - M'''(a,-b') + M^{IV}(a,b')$

$$w'(a,b) = \frac{2q}{\pi k}\int_0^\infty\left[f_1(c)\left(1-e^{-\frac{\alpha b}{l}}\right) - f_2(c)\,\alpha\frac{b}{l}\,e^{-\frac{\alpha b}{l}}\right]\Theta\sin\frac{\alpha a}{l}\,\frac{d\alpha}{\alpha^2};\quad M'(a,b) = \frac{2ql^2}{\pi}\int_0^\infty\left[f_3(c)\left(1-e^{-\frac{\alpha b}{l}}\right) - f_4(c)\,\alpha\frac{b}{l}\,e^{-\frac{\alpha b}{l}}\right]\cdot\Theta\sin\frac{\alpha a}{l}\cdot d\alpha$$

$$w'(a,b) = \frac{2q}{\pi k}\left[f_1(c)\{0.4\,h_1(1.5,a,b) + 0.003\,h_1(0.44,a,b)\} - f_2(c)\{0.4\,h_2(1.5,a,b) + 0.003\,h_2(0.44,a,b)\}\right]$$

$$M'(a,b) = \frac{2ql^2}{\pi}\left[f_3(c)\{0.8\,h_3(1.5,a,b) + 0.006\,h_3(0.44,a,b)\} - f_4(c)\{0.8\,h_4(1.5,a,b) + 0.006\,h_4(0.44,a,b)\}\right]$$

$$w''(a,b) = \frac{q}{\pi k}\sum_n\left[f_1(c) - \left\{f_1(c) + f_2(c)\alpha_n\frac{b}{l}\right\}e^{-\alpha_n\frac{b}{l}}\right]\frac{\Theta_n}{\alpha_n^2}(\alpha_{n+1}-\alpha_{n-1})\sin\alpha_n\frac{a}{l};\quad M''(a,b) = \frac{ql^2}{\pi}\sum_n\left[f_3(c) - \left\{f_3(c) + f_4(c)\alpha_n\frac{b}{l}\right\}e^{-\alpha_n\frac{b}{l}}\right]\Theta_n(\alpha_{n+1}-\alpha_{n-1})\sin\alpha_n\frac{a}{l}$$

$$w'''(a,b) = \frac{q}{\pi k}\sum_n\left[g_1(n,c) - \left\{g_1(n,c)\cos\frac{\beta_n b}{l} + g_2(n,c)\sin\frac{\beta_n b}{l}\right\}e^{-\gamma_n\frac{b}{l}}\right]\psi_n\sin\alpha_n\frac{a}{l}$$

$$M'''(a,b) = \frac{ql^2}{\pi}\sum_n\left[g_3(n,c) - \left\{g_3(n,c)\cos\frac{\beta_n b}{l} + g_4(n,c)\sin\frac{\beta_n b}{l}\right\}e^{-\gamma_n\frac{b}{l}}\right]\psi_n\sin\alpha_n\frac{a}{l}$$

$$w^{IV}(a,b') = \frac{q}{2\pi k}\sum_n\left[1 - \cos\frac{\beta_n b'}{l}\cosh\frac{\gamma_n b'}{l} + \alpha_n^2\sin\frac{\beta_n b'}{l}\sinh\frac{\gamma_n b'}{l}\right]\left[\alpha_{n+1}-\alpha_{n-1}\right]\frac{\sin\alpha_n\frac{a}{l}}{\alpha_n(1+\alpha_n^4)}$$

$$M^{IV}(a,b') = \frac{ql^2}{2\pi}\sum_n\left[\alpha_n^2\left(1 - \cos\frac{\beta_n b'}{l}\cosh\frac{\gamma_n b'}{l}\right) + (0.85\,\alpha_n^4 - 0.15)\sin\frac{\beta_n b'}{l}\sinh\frac{\gamma_n b'}{l}\right]\left[\alpha_{n+1}-\alpha_{n-1}\right]\frac{\sin\alpha_n\frac{a}{l}}{\alpha_n(1+\alpha_n^4)}$$

$$\Theta = \alpha(0.4\,e^{-1.5\alpha} + 0.003\,e^{-0.44\alpha});\quad \Theta_n = \alpha_n(0.4\,e^{-1.5\alpha_n} + 0.003\,e^{-0.44\alpha_n});\quad \psi_n = \frac{\gamma_n^2(\alpha_{n+1}-\alpha_{n-1})}{\alpha_n(\beta_n^2+\gamma_n^2)(1+3.4\alpha_n^2\gamma_n^2 - 0.7225\alpha_n^4)}$$

$$h_1(\delta,a,b) = \tan^{-1}\left(\frac{\delta+\frac{b}{l}}{\frac{a}{l}}\right) - \tan^{-1}\left(\frac{\delta}{\frac{a}{l}}\right);\qquad h_2(\delta,a,b) = \frac{\frac{a}{l}\frac{b}{l}}{\left[\left(\delta+\frac{b}{l}\right)^2 + \left(\frac{a}{l}\right)^2\right]^2}$$

$$h_3(\delta,a,b) = \frac{a}{l}\left\{\frac{\delta}{\left[\delta^2+\left(\frac{a}{l}\right)^2\right]^2} - \frac{\left(\delta+\frac{b}{l}\right)}{\left[\left(\delta+\frac{b}{l}\right)^2+\left(\frac{a}{l}\right)^2\right]^2}\right\}\qquad h_4(\delta,a,b) = \frac{a}{l}\frac{b}{l}\,\frac{3\left(\delta+\frac{b}{l}\right)^2 - \left(\frac{a}{l}\right)^2}{\left[\left(\delta+\frac{b}{l}\right)^2+\left(\frac{a}{l}\right)^2\right]^3}$$

When $c = 0$ $\quad f_1(c) = 1.425$; $\quad f_2(c) = 0.425$; $\quad f_3(c) = 1.3929375$; $\quad f_4(c) = 0.4154375$

$g_1(n,c) = (1 + 1.7\alpha_n^2\beta_n^2)$; $\quad g_2(n,c) = (0.85\alpha_n^2 - 2\beta_n^2)$;

$g_3(n,c) = 0.9775(1 + 1.7\alpha_n^2\beta_n^2)\alpha_n^2$; $\quad g_4(n,c) = 0.9775(0.85\alpha_n^2 - 2\beta_n^2)\alpha_n^2$

When $c > 0$ $\quad f_1(c) = 0.669375$; $\quad f_2(c) = 0.3346875$, $\quad f_3(c) = 0.669375$; $\quad f_4(c) = 0.284484375$

$$g_1(n,c) = \frac{1}{2}\left[1 + 3.4\beta_n^2\alpha_n^2 - 0.7225\alpha_n^4 + 2\alpha_n^2 q + \left\{1 + 0.7225\alpha_n^4 + 2g\sin 2\frac{\beta_n c}{l} - 2\alpha_n^2 g\cos 2\frac{\beta_n c}{l}\right\}e^{-\frac{2\gamma_n c}{l}}\right]$$

$$g_2(n,c) = 0.85\alpha_n^2 + 0.7225\beta_n^2\alpha_n^4 - \beta_n^2 - g + \left\{-0.7225\beta_n^2\alpha_n^4 - \beta_n^2 + g\alpha_n^2\sin 2\frac{\beta_n c}{l} + g\cos 2\frac{\beta_n c}{l}\right\}e^{-\frac{2\gamma_n c}{l}}$$

$$g_3(n,c) = \phi_n\left[d_3 + a_3\tan\frac{\beta_n c}{l}\left(1 - \tanh\frac{\gamma_n c}{l}\right) + b_3\tanh\frac{\gamma_n c}{l} + c_3\tan^2\frac{\beta_n c}{l} + e_3\tan^2\frac{\beta_n c}{l}\tanh\frac{\gamma_n c}{l}\right]$$

$$g_4(n,c) = \phi_n\left[d_4 + a_4\tan\frac{\beta_n c}{l}\left(1 - \tanh\frac{\gamma_n c}{l}\right) + b_4\tanh\frac{\gamma_n c}{l} + c_4\tan^2\frac{\beta_n c}{l} + e_4\tan^2\frac{\beta_n c}{l}\tanh\frac{\gamma_n c}{l}\right]$$

$\phi_n = \cos^2\frac{\beta_n c}{l}\,e^{-\frac{\gamma_n c}{l}}\cosh\frac{\gamma_n c}{l}$; $\quad d_3 = 0.9775(\alpha_n^2 + 1.7\beta_n^2\alpha_n^4)$; $\quad a_3 = 2\alpha_n^2 q$; $\quad b_3 = a_4 + e_3$;

$c_3 = a_4 + d_3$; $\quad e_3 = (0.15 - 0.7225\alpha_n^4)(0.85\alpha_n^2 - 2\beta_n^2)$; $\quad d_4 = 0.9775(0.85\alpha_n^4 - 2\beta_n^2\alpha_n^2)$

$a_4 = (0.85\alpha_n^4 - 0.15)\,2g$; $\quad b_4 = e_4 - a_3$; $\quad c_4 = d_4 - a_3$; $\quad e_4 = (0.7225\alpha_n^4 - 0.15)(1 + 1.7\alpha_n^2\beta_n^2)$

$$q = 1.275\alpha_n^2 - \gamma_n^2 + \frac{0.15\alpha_n^2(2\gamma_n^2 - 0.85\alpha_n^2)}{2(\gamma_n^2+\beta_n^2)}\qquad \gamma_n = \sqrt{\frac{\sqrt{1+\alpha_n^4} + \alpha_n^2}{2}}\qquad \beta_n = \sqrt{\frac{\sqrt{1+\alpha_n^4} - \alpha_n^2}{2}}$$

Fig. 6.—Formulas for Deflections at Point O (Inset) and for Moments in the x-Direction at Point O, Due to a Uniform Unit Load q over Regions I and II

as its base an element of the area of the reference plane. The reference plane thus divided into elemental areas would be an influence chart. An influence chart, therefore, represents the integral of the volume between an influence surface and its reference plane. Influence surfaces and influence charts, therefore, give answers to the same problems but in different form. Influence surfaces are better for showing the effects of concentrated loads but influence charts are preferable for showing the effects of distributed loads.

As indicated previously, each chart in this paper was constructed from a table of computed values. For example, Table 1(a) shows that for chart 1 the arc with a radius of 0.1012 l encloses two blocks in the first quadrant or one block for each 45° of arc. In Eq. 7, if $\pi/2$ is substituted for $\theta_2 - \theta_1$, and 0.1012 is substituted for a/l, $\frac{2{,}000\, w\, D}{q\, l^4} = 2$, the number of blocks enclosed. (For values of a/l greater than unity, published tables of Hankel functions were used; but, for smaller values of a/l, more accurate values than could be obtained from the tables were found by using the first terms of the series expansion of these functions.)

Likewise, in the right side of the formula for Table 4(a) given in Fig. 5, if 0 is substituted for θ_1, 5° (expressed in radians) for θ_2, and 0.1535 for a/l, the value of 2 is obtained. Had 0.2011 been substituted for a/l, the value of 3 would have been obtained. Thus, for chart 5 between radial lines of 0° and of 5°, and between the two arcs with radii of 0.1535 and 0.2011 (two consecutive values in Col. 1 of Table 4(a)), there is exactly one $(3 - 2)$ full block.

Since practically every line in charts 1, 2, 3, and 4 is located by values given in Tables 1 and 4, the construction of these charts was relatively simple. The construction of charts 5, 6, 7, and 8 from Tables 2 and 3 is not so clearly indicated. These tables give the number of blocks per unit of area for specified areas of the influence charts. This information corresponds to that given by an influence surface since the value given when reduced to proper scale is the average ordinate of the corresponding influence surface over the specified area.

As an example of the information given by these tables, Table 2(a) denotes that, for the region between $b = 1.8\, l$, $b = 2.0\, l$, $a = 0$, and $a = 0.1\, l$ of chart 5, there must be 1.086 blocks per 0.01 l^2 of area, or a total of 2.172 blocks. In the construction of charts 5, 6, 7, and 8, care was exercised so that not only did each region have the correct number of blocks, but each block had the proper area for its position within the chart. This result was accomplished by interpolation.

USE OF THE INFLUENCE CHARTS

The use of the influence charts is demonstrated by the following examples:

It is required to find the stress at point O on a free edge of an 18-in. concrete pavement slab due to a load of 133,000 lb on a landing gear with twin wheels in tandem placed as shown in Fig. 7. Use the assumption of a dense liquid subgrade.

The essential data are: The lateral tire spacing is 31.5 in. and the fore and aft spacing is 62.0 in. Also, the tire pressure is 140 lb per sq in.; the modulus of elasticity of concrete is 4,000,000 lb per sq in.; Poisson's ratio for the concrete is 0.15; and k, the modulus of the subgrade reaction, is 200 lb per cu in.

First, l, the radius of relative stiffness, must be determined. From the equation—

$$l = \sqrt[4]{\frac{E h^3}{12\,(1 - \mu^2)\,k}} \qquad \text{.....................(16)}$$

—$l = 56.16$ in. Next the tire imprint dimensions must be determined. The total imprint is equal to the wheel load divided by the tire pressure, or 33,250/140 = 237.5 sq in. Using an imprint outline of a rectangle with semicircular ends such that the width is equal to six tenths of the length, the following dimensions are computed: Length = 21.32 in. and width = 12.79 in. A tire imprint tracing is then drawn using the graphical scale shown on the proper chart (in this case, chart 6) and the tracing is superimposed on chart 6 with the wheels in the position indicated in the foregoing requirement, point O of tracing over point O of chart. Fig. 8 shows a part of chart 6 with an outline of the twin tandem tire prints placed as required.

The number of blocks enclosed in each tire area is then determined by first counting all whole blocks and estimating all fractional blocks within the imprint area. In this case, the total number of blocks in all four tire prints is 439.2. Notice that the chart is symmetrical about the vertical axis so that, in this problem, it is necessary to count only the blocks in the two tire prints to the right of point O and to multiply the result by two.

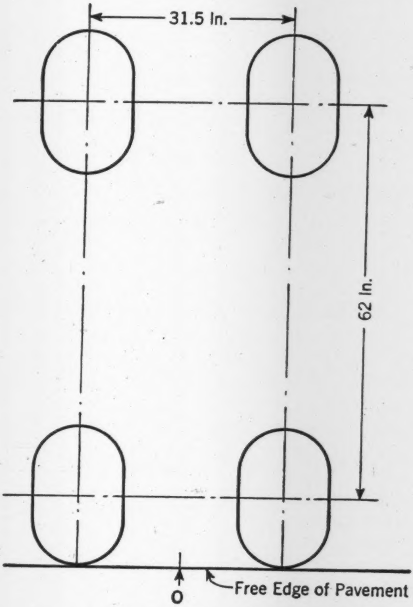


FIG. 7.—TWIN TANDEM LANDING GEAR AT THE EDGE OF A PAVEMENT

Moment M is computed by the formulas given in the caption of each figure. The formula for chart 6 is $M = \frac{q\,l^2\,N}{10{,}000}$. Since $q = 140$, $l = 56.16$, and $N = 439.2$, $M = \frac{140 \times (55.16)^2 \times 439.2}{10{,}000}$, or 19,393 in.-lb.

The flexural stress can now be determined by multiplying the moment by the section modulus of the slab, $\frac{6}{h^2}$. Therefore, the stress at point O on a free edge of an 18-in. slab $= \frac{6 \times 19{,}393}{(18)^2}$, or 359.1 lb per sq in.

The stress may be obtained at point O with the tire imprint placed anywhere on the chart. The location of a load that gives maximum possible stress at the edge of a slab may be found by placing the wheel imprint tracing in different positions on the chart and counting the enclosed blocks in each position.

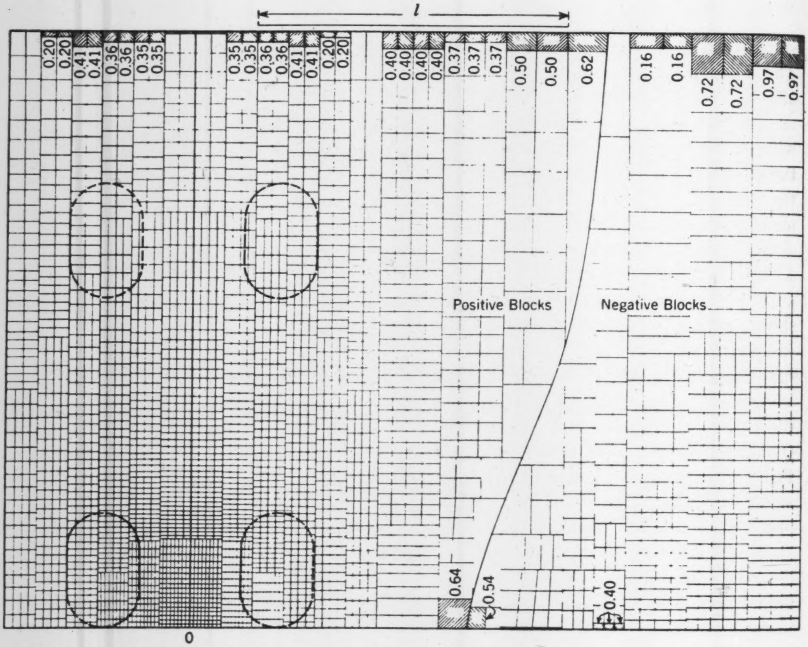

l
0.20
0.20
0.41
0.41
0.36
0.36
0.35
0.35
0.35
0.35
0.36
0.36
0.41
0.41
0.20
0.20
0.40
0.40
0.40
0.40
0.37
0.37
0.37
0.50
0.50
0.62
0.16
0.16
0.72
0.72
0.97
0.97
Positive Blocks
Negative Blocks
0.64
0.54
0.40
0

Fig. 8.—Tracing of Twin Wheels in Tandem Superimposed on Chart 6

All charts are used in the same manner with the graphical scale and the formula given in the captions.

Acknowledgments

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Paper No. 2426

IMPROVEMENTS AT THE BACK RIVER SEWAGE WORKS, BALTIMORE, MD.

By C. E. Keefer,[1] M. ASCE

Synopsis

The history of practically all large sewage works that have been in service for any considerable period has been that improvements and additions are required from time to time. In some instances, the need for these additions has been caused by an increase in sewage flow or organic load. In other cases, old treatment units have outlived their efficient usefulness and have been replaced by more modern equipment. The work described in this paper falls into both categories. During the 18-year period from 1930 to 1947, inclusive, $3,859,500 was spent on new improvements at the Back River sewage works in Baltimore, Md., which serves most of the city and a considerable section of the Baltimore County Metropolitan District adjacent to the city. This project provides for primary and secondary treatment of the sewage and includes: (1) A building with mechanically cleaned bar screens; (2) three grit chambers; (3) five preliminary settling tanks; (4) activated-sludge units; (5) equipment for chlorinating the final effluent; (6) a sludge control station containing pumps for handling raw and digested sludge and gas boilers for heating the digestion tanks, together with metering equipment used in conjunction with the operation of the sludge tanks; (7) five covered sludge digestion tanks; (8) a 200,000-cu-ft gas holder; (9) four vacuum filters; (10) an electric substation; and (11) the necessary sewer conduits and pipe lines connecting these improvements.

Introduction

Before 1906 Baltimore was one of the few large cities in the United States without a sewerage system. At that time the city was provided with several storm drains which discharged into the harbor. In most cases these were far from adequate. In part of the business section of the city and in a few of the

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[1] Deputy Sewerage Engr., Bureau of Sewers, Baltimore, Md.

better residential districts the sewage flowed through sewers, which emptied into the storm drains or streams tributary to the harbor.

By far the larger part of the city, however, had no sewers. There were more than 90,000 cesspools in the city in 1906, many of which were not regularly cleaned, with the result that they overflowed into alleys and streets. Furthermore, the wastes from kitchen sinks, bathtubs, and lavatories flowed down the streets to the nearest inlet. These polluting liquids eventually reached the harbor, a landlocked body of water, which frequently gave off considerable stench. This condition had existed for years until, in 1904, the first allotment of $10,000,000 was obtained for building a sewerage system, which included a sewage treatment works. A plant was constructed for separate sludge digestion, with sedimentation tanks and trickling filters, at a cost of $2,777,000 and was put in service in 1911.

During World War I, and immediately thereafter, practically no construction work was done at the sewage plant. However, from 1920 to 1930, a number of additions were made, consisting of a preliminary settling tank, sludge digestion tanks, sludge drying beds, and three pumping stations to handle sludge and the drainage from sludge beds. Several buildings for miscellaneous purposes were constructed, and other important work was completed; but much remained to be done.

The Sewage Treatment Works.—The sewerage system of Baltimore is of the separate type. The topography of the city is such that approximately 64% of the sewage flows by gravity to the sewage works. The remainder is pumped into the outfall sewer, which has a capacity of 291 mgd.

The flow increased from 55.09 mgd in 1925 to 60.65 mgd in 1930, the year when the improvements described in this paper were begun. The sewage is domestic in character with a relatively high biochemical oxygen demand (B.O.D.) and suspended solid content. When it reaches the treatment works it is in a stale condition.

The plant (Fig. 1) is situated on a tract of land containing 547 acres, about 2 miles east of the eastern boundary of the city on the west shore of Back River, a tributary of Chesapeake Bay. The sewage formerly entered the treatment works at a screen house, which was provided with two rows of bar screens with $\frac{7}{8}$-in. clear openings. From the screen house the sewage flowed to four manually cleaned preliminary settling tanks, each with a capacity of 3,800,000 gal. Each group of two tanks was provided with a vertical, centrifugal pump for removing supernatant and sludge from the tanks. During periods of normal flow two tanks were in operation while the other two were being cleaned. The effluent from the settling tanks entered a building containing drum-type revolving screens, wrapped with No. 20 mesh, Monel metal, wire cloth. The screened effluent then flowed to the trickling filters, which cover 30 acres and are 8.5 ft deep. The filtered effluent is settled in two humus tanks and then flows through two 60-in., wood stave, circular conduits into the Back River.

The sludge from the preliminary settling tanks and the humus tanks was pumped into fifty-one open digestion tanks with a working capacity of 2,406,700

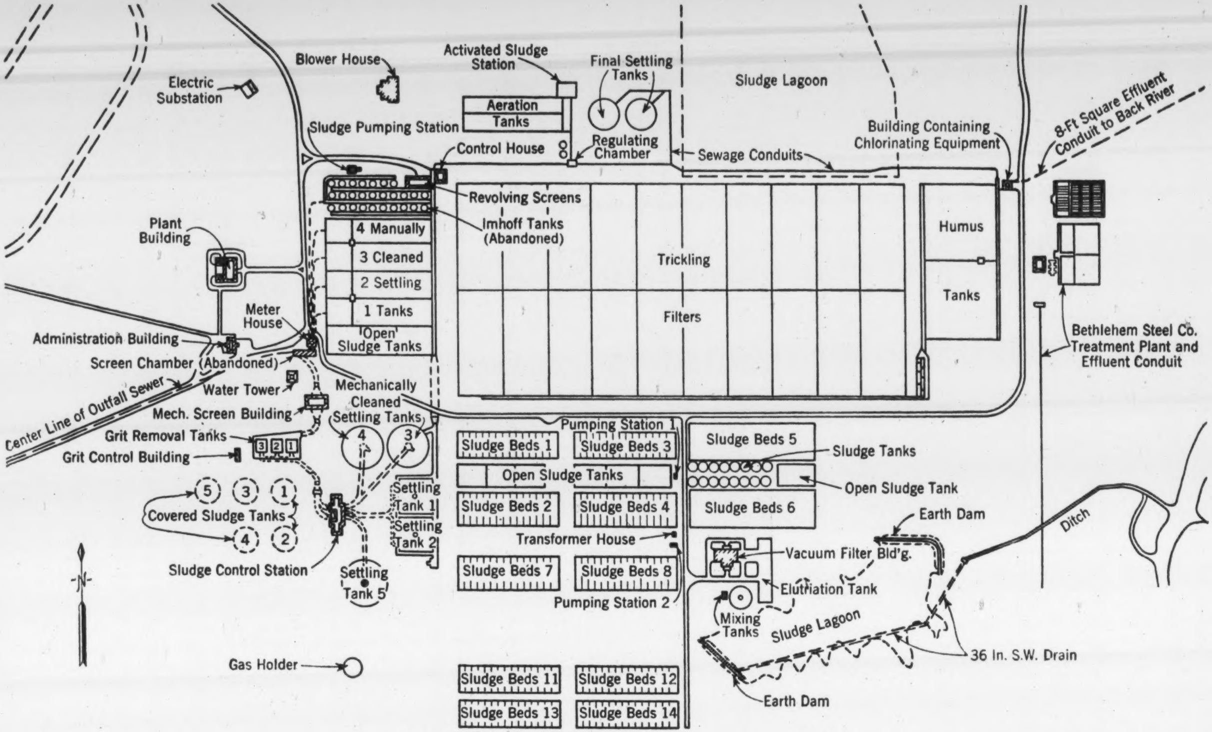
Electric Substation
Blower House
Activated Sludge Station
Final Settling Tanks
Sludge Lagoon
Aeration Tanks
Sludge Pumping Station
Control House
Regulating Chamber
Sewage Conduits
Building Containing Chlorinating Equipment
8-Ft Square Effluent Conduit to Back River
Plant Building
Revolving Screens
Imhoff Tanks (Abandoned)
4 Manually
3 Cleaned
2 Settling
1 Tanks
Open Sludge Tanks
Trickling
Filters
Humus
Tanks
Meter House
Administration Building
Screen Chamber (Abandoned)
Water Tower
Center Line of Outfall Sewer
Mech. Screen Building
Mechanically Cleaned Settling Tanks
4
3
Bethlehem Steel Co. Treatment Plant and Effluent Conduit
Pumping Station 1
Grit Removal Tanks
3
2
1
Grit Control Building
Sludge Beds 1
Sludge Beds 3
Sludge Beds 5
Sludge Tanks
Open Sludge Tanks
Open Sludge Tank
Settling Tank 1
Settling Tank 2
5
3
1
Covered Sludge Tanks
4
2
Sludge Beds 2
Sludge Beds 4
Sludge Beds 6
Earth Dam
Ditch
Transformer House
Vacuum Filter Bld'g.
Sludge Control Station
Settling Tank 5
Sludge Beds 7
Sludge Beds 8
Elutriation Tank
Pumping Station 2
Mixing Tanks
Sludge Lagoon
N
36 In. S.W. Drain
Gas Holder
Sludge Beds 11
Sludge Beds 12
Sludge Beds 13
Sludge Beds 14
Earth Dam

FIG. 1.—BACK RIVER SEWAGE WORKS, BALTIMORE, MD.

cu ft, or 3.65 cu ft per capita in 1930. The digestion tanks were supplemented by two lagoons, where digestion and considerable drying took place. Digested sludge was dried on 578,223 sq ft of sand beds and finally put into a storage pile where it was removed by farmers for fertilizing purposes.

New Improvements.—The need of adequate and more modern treatment facilities at the sewage works was apparent some years prior to 1930. From then until the end of 1947 the number of persons served increased from 660,000 to 947,600, with an accompanying increase in sewage flow from 60.65 mgd to 118.52 mgd. The following is a list of the major treatment units constructed during this period: Mechanical screening plant, grit chambers, preliminary settling tanks, activated-sludge units (aeration tanks, final settling tanks, sludge thickening tanks, and blower house and blower equipment), equipment for chlorinating final effluent, sludge control station, sludge digestion tanks, gas holder, vacuum filter building and equipment, and electric substation. Details of each of these units are described in sequence.

Mechanical Screening Plant

Although the original, manually cleaned, coarse screens served their purpose for many years, they became inadequate with augmented flows. One of the chief disadvantages of the old screen house was that the operating floor was often flooded with sewage during high flows, at which times the screens were difficult to clean. Moreover, there were no satisfactory facilities for the prompt disposal of the screenings. The material was stored in a small wooden shed and removed by a neighboring farmer for fertilizing purposes.

After an inspection of screening plants in the United States, Canada, and Europe, the straight-line type of screens was adopted. The best disposition of the screenings was studied. Burying the material or using it as a fertilizer was considered inadvisable. Practically the only other method of disposal was by incineration, to which there were several objections. There remained, however, one other method of disposal, about which little was known in the United States. At the pumping station serving Radial System No. 11 in Berlin, Germany, the screenings were pulverized by crushing rolls and then returned to the unscreened sewage. As this method of disposal offered a number of attractive features, a grinder was tested at the Baltimore sewage works. These tests[2] indicated that the screenings could easily be ground with a current consumption of about $\frac{1}{4}$ kw-hr per cu ft of ground material. The estimated cost of handling the screenings—including digestion in sludge tanks, drying on sand beds, and disposal to farmers—was $0.46 per cu ft. The cost of incinerating the screenings, including fixed charges, was estimated to be $0.41 per cu ft. As this difference in cost would amount to about only $300 a year in favor of incineration, it was decided to install grinders in the screen building.

The new screen building is an attractive one-story structure, 92 ft by 50 ft in plan. A rectangular underground conduit, 12 ft by 10 ft, was laid from the old screen house to the new building (Fig. 1). This conduit is designed to carry 291 mgd, the maximum capacity of the outfall sewer from the city. The conduit separates into three covered channels just before connecting with the building below the ground; and they unite with three screen chambers, each of which is 10 ft wide and has an average depth of 13.5 ft. A screen has been installed in each of the channels, two of which (Fig. 2) were provided in 1932, immediately upon the completion of the screen building, and the third in 1937. In the walls of each channel on the upstream side of the screens there is a weir, 14 ft long, the top of which is 2 ft below the screen room floor and 6 in. above the normal maximum elevation of the sewage in the channel. If the screens become clogged, the sewage can flow over this weir and around the screens instead of flooding the floor of the building.

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[2] "Pulverizing of Sewage Screenings at Baltimore, Maryland," by C. E. Keefer, *Proceedings*, ASCE, September, 1929, p. 1759.

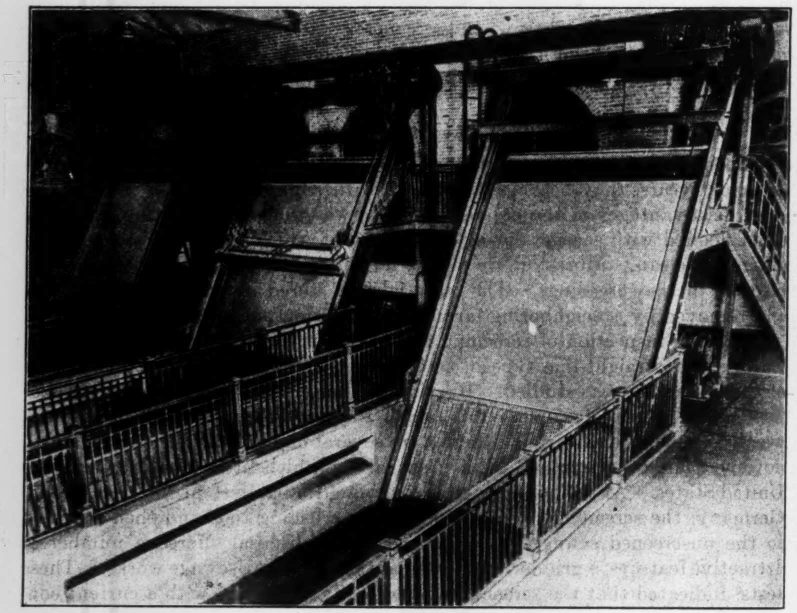

Fig. 2.—Mechanically Cleaned Screens

Each screen is cleaned mechanically and is designed to treat a flow of 150 mgd with a loss of head of not more than 6 in. The screening plant will treat a maximum flow of 300 mgd with one screen out of service. Each screen is set at a 60° angle with the horizontal and is 10 ft wide with a total vertical height of 29 ft. The total submerged surface area of each unit during maximum flows is 123 sq ft, and the area of the clear openings between the bars is 81.4 sq ft. Genuine wrought iron is used for the bars, which have tapered cross sections, 3 in. by $\frac{3}{8}$ in., with clear openings of $\frac{3}{4}$ in. The top of the bars terminates near the operating floor and connects with a smooth steel plate, which is in the same plane with the bars. The screens are pivoted just above the operating floor so that they can be lifted above the sewage in case repairs are necessary.

A mechanically operated rake is provided to clean each screen. The screenings are dragged up the bars and over the surface of the dead plate into a vertical hopper. Solenoid operated valves admit water under pressure into each hopper just prior to the discharge of the screenings into the hopper. The bottom of each hopper connects with a pulverizer with a capacity of 1.5 tons per hr. Each of the first two grinders that were installed was driven by a 25-hp motor at a speed of 1,140 rpm. Experience with these units indicated that better results would have been obtained if they were operated by larger motors at a higher speed. Therefore, the third grinder was provided with a 30-hp motor, running at a speed of 1,760 rpm. The grinders are watertight to prevent the escape of water, which flushes the pulverized material through the machine into the sewage on the downstream side of the screens. Each screen is driven by a 4-hp motor and was designed to be started automatically or manually. Each cleaning cycle of the screening rake takes from about 5.5 min to 6 min.

Stop planks of steel construction, in double grooves, are provided in the conduits upstream and downstream from the screens. Experience indicates that single grooves for stop planks would have been equally satisfactory with an appreciable saving in the size and the cost of the building. Office, toilet, storage, and transformer rooms have been built. With the exception of the office and the toilet rooms the building as originally designed was not heated. However, as considerable condensation collected on the walls and on the mechanical equipment, two unit gas heaters were later provided to correct this condition. A 3-ton traveling crane has been installed for handling stop planks and parts of the screens.

Grit Chambers

When the sewage plant was first constructed, no provision was made to remove grit from the sewage, primarily because Baltimore has a separate system of sewers. Operating experience gradually indicated the wisdom of providing grit removal equipment. Construction was finally started in October, 1937, on three grit removal tanks (Fig. 3). Each tank is designed to treat a maximum flow of 100 mgd with a theoretical detention period of 100 sec at the aforementioned flow. Each tank is 50 ft square and 9 ft deep, and is designed to remove particles approximately 0.25 mm in size.

The sewage flows from the mechanical screen building to the grit chambers in a reinforced-concrete box conduit, 12 ft wide by 10 ft high. This conduit unites with three smaller conduits, each 8.5 ft wide by 7.58 ft high, through which the sewage flows to the grit chambers. A motor-operated, steel sluice gate, 84 in. by 84 in., is provided in each conduit for controlling the flow to the grit chambers. The effluent from each tank discharges through a covered conduit, 8 ft by 8 ft, in which there is a motor-operated steel sluice gate. These three conduits discharge into a concrete conduit, 12 ft by 10 ft.

A rotating mechanism, supported at the center of each grit chamber and also on the tank walls, squeegees the deposited material to a classifier just outside the tank, which separates the organic solids from the inorganic solids.

The former material is moved up a concrete incline and discharges on a horizontal belt conveyor 14 in. wide. This conveyor terminates in a near-by, one-story brick building, which contains a bucket elevator for discharging the grit into trucks, a control room with the necessary electrical apparatus to operate the grit removal equipment, and a driveway for the trucks.

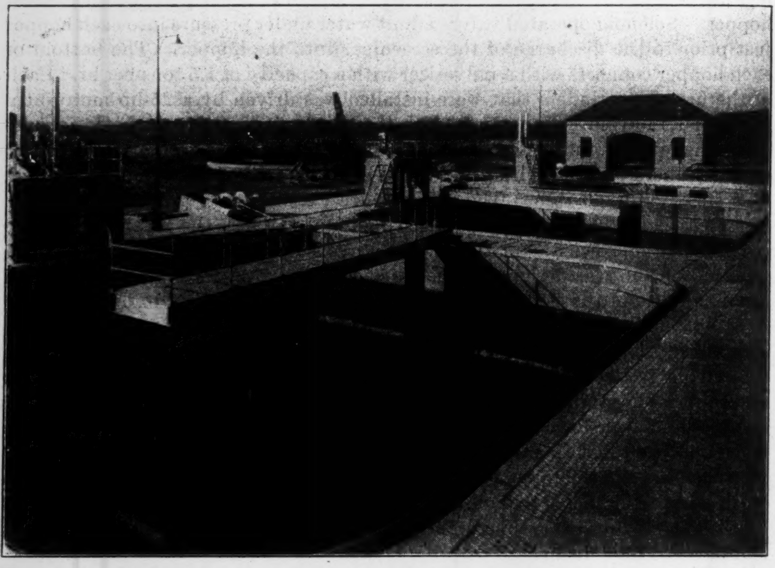

Fig. 3.—View Facing West Toward Grit Chambers and Grit Control Building

Preliminary Settling Tanks

The four manually cleaned preliminary settling tanks, three of which were put in service in 1911 and the fourth in 1921, were inadequate for a number of reasons. As they were cleaned manually, they had to be taken out of service to remove the sludge; consequently, twice the number of units was required. Furthermore, they were cleaned once every 4 or 5 days, whereas for the best results the sludge should have been removed as soon as it collected. The retention of this material in the tanks had a deleterious effect on the effluent. Moreover, the pH-value of the sludge decreased from 6.8 or 7.0 to 4.9 or 5.0 by the time it was removed. This acid condition made the material difficult to decompose, and required more digestion space. The preliminary settling tank layout includes six tanks, each capable of treating an average flow of 25 mgd with a theoretical detention period of 2 hours and a maximum flow of 50 mgd. The construction of five of these tanks was begun in 1930 and completed in 1938, providing for an average flow of 125 mgd and a maximum flow of 250 mgd.

The sewage flows to these tanks from the grit chambers first through a conduit 12 ft wide by 10 ft high and then through five concrete pipes, each 54 in. in diameter. These pipes pass through the basement of the sludge control station, to be described subsequently, and connect with the five settling tanks. A sixth pipe is designed to serve the future settling tank, which is being constructed.

Tanks 1 and 2 (Fig. 4), the first to be built, are each 140 ft square with a water depth of 13 ft along the sides and a freeboard of about 2 ft. The surface loading during average flow conditions (Table 1) is 1,280 gal daily per sq ft of tank surface. Where the influent pipes (which are 30 in. in diameter) connect with the basins, manually operated sluice gates have been installed for regulating or shutting off the flow. After passing through these gates, the sewage enters narrow channels along the wall adjacent to the inlet side of each tank. One side of each channel has an adjustable submerged weir, over which the sewage discharges into the tank. The opposite wall of the tank is also provided with adjustable weirs. The lengths of the inlet and the outlet weirs in each basin are 78 ft and 102.5 ft, respectively, or 55.7% and 73.2% of the

TABLE 1.—BASIS OF DESIGN, PRELIMINARY SETTLING TANKS (FLOW TO EACH TANK, 25 MGD, AND THEORETICAL DETENTION PERIOD, 2 HOURS)

Rates	Tank Nos.	
	1, 2	3, 4, 5
Settling rate, in gallons per square foot per day	1,280	1,085
Effluent overflow rate, in thousand gallons per day per foot of weir	244	46.5

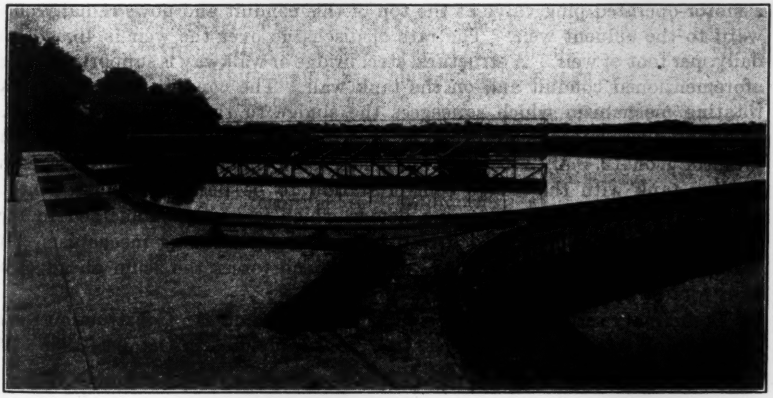

FIG. 4.—PRELIMINARY SETTLING TANKS 1 AND 2

width of each tank. The effluent overflow rate is 244,000 gal daily per foot of weir.

Each basin is provided with mechanically operated equipment for removing sludge continuously. This equipment consists of steel truss work, which carries the scraping blades for moving the settled sludge to the sludge discharge pipe at the center of the tank. Each clarifier is equipped with facilities for the continuous skimming of sewage. Attached to one side of the tank mechanism on a level with the surface of the sewage there is a vertical steel plate, 8 in. wide, which skims the surface as the mechanism revolves.

The bottoms of the tanks are constructed of plain concrete in two layers, the lower being 5 in. thick and the upper 3 in. thick. The floor slopes toward the center of the clarifier with a fall of 1 on 12. An 8-in. cast-iron discharge pipe connects with a depression at the center of the tank. The outer walls of the tanks, against which there is backfill, were designed to resist the horizontal thrust of the sewage with the backfill removed. In order to resist the deteriorating action of the elements, especially at the surface of the sewage, the upper 4.5 ft of all the concrete walls was made of a richer mix, consisting of 1 part of cement and 4¼ parts of aggregate.

The construction of preliminary settling tanks 3 and 4, Fig. 1 (which were built under the same contract), was begun in February, 1936, and completed in January, 1937. Each of these tanks is 170 ft in diameter, with a sewage depth of 12.5 ft at the walls and a freeboard of 2 ft. These tanks have a surface loading of 1,085 gal daily, per square foot of surface, during average flow conditions. The sewage enters each tank through a 66-in., pre-cast, reinforced-concrete conduit, which is under the tank floor and terminates at the center of the tank in a 54-in. vertical conduit. The sewage discharges into the tank through a motor-operated plug valve at the top of this conduit and flows radially outward to the effluent weir. The rate of discharge over the weir is 46,500 gal daily, per foot of weir. A structural steel bridge or walkway is supported on the aforementioned conduit and on the tank wall. The conduit also supports a rotating mechanism which squeegees the sludge to the center of the tank. The tank floor (which is 8 in. thick) slopes toward the center of the tank, with a fall of 1 on 12. A 12-in. cast-iron sludge pipe connects the center of each settling tank with the sludge control station to be described subsequently. Scum is removed by a skimming device which is partly submerged in the sewage and is supported and driven by the sludge cleaning mechanism. It extends radially inward from the tank wall and forces the scum ahead of it as it rotates about the tank center.

Preliminary settling tank 5 is similar to tanks 3 and 4. Construction work on this tank was begun in August, 1937, and completed in October, 1938. It was put in operation in December, 1938.

ACTIVATED-SLUDGE UNITS

After considerable preliminary study it was decided, in 1937, to add to the facilities for oxidizing the preliminary settling tank effluent. For some years prior to that time it was frequently necessary to by-pass a portion of the settled sewage around the trickling filters. This practice was followed whenever the rate of flow exceeded about 110 mgd, the maximum capacity of the filters.

The requirements in favor of additional trickling filters as against activated-sludge units were studied. The use of activated-sludge units was approved for two reasons: In the first place, as the area adjacent to the sewage plant was changing from a rural to a suburban community, it was considered advisable to construct treatment facilities that would be free from objectionable odors. Secondly, the effluent from activated-sludge units is better than that from trickling filters. It has always been the purpose of the City of Baltimore to produce an effluent with a high degree of purity in order to protect the watercourse into which it flows. Therefore, it was considered more advisable to build the activated-sludge units.

The basic data used in the design of the activated-sludge units are as follows:

Item	Description	Quantity
0	Sewage Flow, in Million Gallons per Day—	
1	Average	20
2	Maximum	40
	Aeration Tanks (Spiral Flow)—	
3	Number of tanks	2
4	Theoretical detention period (based on an average sewage flow plus 25% of return sludge), in hours	5
5	Average solids content, in percentage of mixed liquor	0.3
	Volume of Return Sludge (Based on Average Sewage Flow), in Million Gallons Daily—	
6	Average (good sludge with 1.5% solids)	5
7	Maximum (bulked sludge with 0.75% solids)	10
8	Installed output of air (three blowers) with sewage flow of 40 mgd, in cubic feet per gallon	1.78
	Final Settling Tanks—	
9	Number of tanks	2
10	Theoretical detention period (based on an average sewage flow plus 25% of return sludge), in hours	2.4
11	Surface loading (based on an average sewage flow plus 25% of return sludge), in thousand gallons per square foot per day	1
12	Effluent overflow rate, in thousand gallons per foot of weir, per day	26.2
13	Assumed removal of suspended solids, in parts per million	60
	Activated Sludge Withdrawal, in Million Gallons Daily—	
14	Average (good sludge with 1.5% solids)	5.08
15	Maximum (bulked sludge with 0.75% solids)	10.16
	Excess Activated Sludge (Based on Average Sewage Flow), in Thousand Gallons Daily—	
16	Average (good sludge with 1.5% solids)	80

Item	Description	Quantity
17	Maximum (bulked sludge with 0.75% solids)	160
	Sludge Concentration Tanks—	
18	Number of tanks	2
	Average Volume of Excess Sludge (One Tank in Service)—	
19	Theoretical detention period, in hours	13.75
20	Surface loading, in gallons per square foot per day	300
21	Solids, in percentage of the concentrate	4
	Maximum Volume of Excess Sludge (Two Tanks in Service)—	
22	Theoretical detention period, in hours	24
23	Surface loading, in gallons per square foot per day	300
24	Solids, in percentage of concentrate	4
25	Depth of tank allowed for supernatant, in feet	3
26	Average volume of thickened excess sludge (4% solids) to be pumped to raw sewage or to digestion tanks (based on treating an average flow of 20 mgd of sewage), in million gallons per day	30
27	Average volume of supernatant from thickened excess sludge (based on treating an average flow of 20 mgd of sewage), in million gallons per day	50

The plant was designed to treat an average flow of 20 mgd and a peak flow of 40 mgd (items 1 and 2). The trickling filters along with these units could then treat an average flow of 130 mgd and a maximum flow of 150 mgd. The activated-sludge units were designed so that an average flow of 40 mgd could be treated by constructing additional units at a later date.

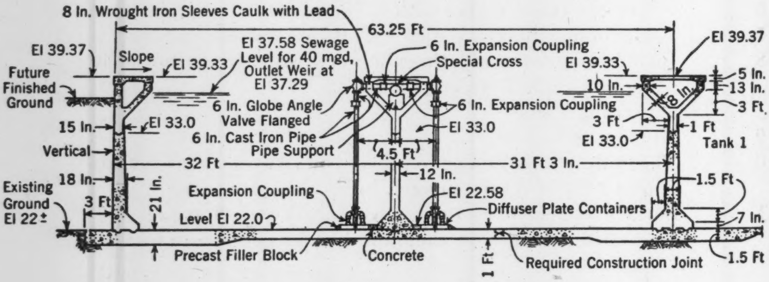
8 In. Wrought Iron Sleeves Caulk with Lead
63.25 Ft
El 39.37
Slope
El 39.33
El 37.58 Sewage Level for 40 mgd, Outlet Weir at El 37.29
6 In. Expansion Coupling
Special Cross
El 39.33
El 39.37
Future Finished Ground
10 In.
8 In.
5 In.
13 In.
3 Ft
6 In. Globe Angle Valve Flanged
6 In. Expansion Coupling
15 In.
El 33.0
6 In. Cast Iron Pipe
Pipe Support
El 33.0
3 Ft
1 Ft
El 33.0
Tank 1
Vertical
4.5 Ft
32 Ft
31 Ft 3 In.
18 In.
12 In.
1.5 Ft
Existing Ground El 22±
3 Ft
21 In.
Expansion Coupling
El 22.58
Diffuser Plate Containers
Level El 22.0
7 In.
Precast Filler Block
Concrete
1 Ft
Required Construction Joint
1.5 Ft

FIG. 5.—CROSS SECTION OF AERATION TANK

Aeration Tanks.—The activated-sludge plant consists of two aeration tanks, two final settling tanks, an activated-sludge station, two sludge thickening tanks, an operating gallery, and a blower house. Each aeration tank (Fig. 5) (which contains two "flow-through" channels, each 30.25 ft wide) is 383 ft long with a working depth of 15.29 ft. The settled sewage and the return activated sludge enter the inlet end of each tank through two cast-iron elbows, and the mixed liquor discharges over a weir, 37.28 ft long. The plant is designed on the basis that normally the return sludge will contain 1.5% solids and will amount to 25% of the sewage flow, and that, during periods of bulking, the sludge will contain 0.75% solids and will amount to 50% of the sewage flow. With a sewage flow of 20 mgd the volumes of sludge under the aforementioned conditions are 5 mgd and 10 mgd, respectively (items 6 and 7).

The aeration tanks have an average theoretical detention period of 5 hours (item 4) and are of the spiral-flow type. Each tank contains four vertical concrete cross baffles, each 12 ft wide, extending the full depth of the tank. Their function is to prevent unaerated mixed liquor from forming along the horizontal centers of the tanks.

Two rows of diffuser plates are provided on the bottom of each tank adjacent to one wall. Each tank contains 1,232 diffusers (12 in. by 12 in.) made of fused crystalline alumina grains. The plates have a permeability rating varying from 36 to 44 with an average of 40. The ratio of tank sewage surface area to plate area is 15.6. The diffuser plates are retained in a bed of cement mortar in pre-cast concrete plate holders by three-pronged malleable iron clamps. The air pipe supplying the plates in each tank is cast iron and varies in diameter from 12 in. to 20 in. This pipe is supported in a V-shaped valley in the top of the longitudinal wall extending through the center of each tank. From this pipe, 6-in., vertical, cast-iron pipes carry the air through 3-in. cast-iron pipes to the diffuser plate holders. The diffuser plates were installed so that the permeability rating of all the plates served by the same vertical air pipe varied as little as possible to insure that each plate in adjacent plate holders would pass essentially the same amount of air.

Final Settling Tanks.—East of the aeration tanks are two final settling tanks (see Fig. 1). Each is 126 ft in diameter with a water depth of 13.33 ft along the sides and a freeboard of 2.52 ft. The effluent from the aeration tanks flows first to a common aerated open channel and then through two 42-in. cast-iron pipes to the centers of the final settling tanks. When the sewage flow averages 20 mgd (to which 5 mgd of activated sludge is added), the average theoretical detention period in the tanks is 2.4 hours and the surface loading is 1,000 gal per sq ft per day (items 10 and 11). The effluent overflow rate amounts to 26,200 gal daily per foot of weir during average flow conditions (item 12). Each tank was provided with two pairs of submerged photoelectric cells which can be set at any suitable elevation. The purpose of these cells was to determine the depth of sludge in the tanks. The effluent from each tank discharges through a 3-ft-square, reinforced-concrete conduit and then discharges into a larger conduit.

The volume of activated sludge that collects in the settling tanks was assumed to be 5.08 mgd with the sludge in good condition (containing 1.5% solids) and with a sewage flow of 20 mgd (item 14). This volume consists of 5.0 mgd of return sludge and 0.08 mgd of sludge resulting from the removal of 60 ppm of suspended solids from the sewage (item 13), leaving 10 ppm of suspended solids in the final effluent. When the sludge is bulked with an assumed solid content of 0.75%, the volume of activated sludge would be double the aforementioned amount, or 10.16 mgd (item 15).

The activated sludge, which settles in the final tanks, is squeegeed to the center of each tank by a mechanism that makes 1 revolution in 40 min. The sludge flows through an 18-in. cast-iron pipe, one from each tank, into a suction well in the activated-sludge station adjacent to the aeration tanks. Each pipe line contains a rate controller for regulating the flow of sludge. Two centrifugal pumps, with a rated capacity of 5.2 mgd each, lift the activated sludge from the suction well into an upper chamber, whence it flows through two 18-in. cast-iron pipe lines (each of which contains a rate controller) to the two aeration tanks.

Sludge Thickening Tanks.—From the aforementioned upper chamber the excess activated sludge flows through a 6-in. cast-iron pipe line to two thickening tanks. These tanks were designed to thicken 160,000 gal daily of excess sludge containing 1.5% solids (items 16 and 17). This quantity was assumed to result from the treatment of an average flow of 40 mgd of settled sewage. Under the aforementioned sludge flow conditions, with one tank in service, the theoretical detention period is 13.75 hours and the surface loading is 300 gal per sq ft per day. The volume of concentrated sludge pumped from the tank was assumed to be 60,000 gal daily containing 4% solids.

Each sludge concentration tank is 26 ft in diameter with a water depth of 16 ft along the sides. Sludge enters each tank at the center, and the effluent discharges over a V-notched weir at the periphery. The effluent flows to two centrifugal pumps, one of which is a spare, and is lifted into the influent channel to the activated-sludge plant. The sludge is thickened in the aforementioned tanks by a picket-fence mechanism. The thickened sludge flows to two reciprocating pumps (variable speed), each of which has a maximum capacity of 140 gal per min and is pumped through a 6-in. cast-iron force main either into the raw sewage ahead of the preliminary settling tanks or into the raw-sludge suction well of the sludge control station.

To prevent the activated sludge from becoming septic in the thickening tanks the supernatant in these tanks is chlorinated by two vacuum-type chlorinators, each capable of delivering from 15 lb per hr to 300 lb per hr of chlorine. The chlorine, dissolved in water, is introduced into each tank at eight points approximately 17 in. below the surface of the supernatant. To control bulking, chlorine can be added to the return activated sludge as it flows into the receiving well from the final settling tanks.

In the basement of the activated-sludge station a motor-driven centrifugal pump with a capacity of 5,400 gal per min is available for dewatering the aeration tanks and the final settling tanks. As there is a possibility of flooding the pumping unit if the piping in the aforementioned basement should break, the drive motor is designed to operate submerged under 15 ft of water.

The following venturi meter equipment was installed in the activated-sludge station:

1. Two meters, 36 in. by 18 in., for measuring the sewage flow to the aeration tanks;
2. Two rate controllers, 18 in. by 18 in., for measuring the return sludge flowing to the aeration tanks; and

3. Two rate controllers, 18 in. by 18 in., for measuring the sludge flowing from the final settling tanks.

Control tables, made of ebony-asbestos slabs, were provided on the main floor of the activated-sludge station. These tables (three in number) contain gages that indicate the flow of sewage, sludge, and air to each tank; the flow of return sludge from each of the final settling tanks; and the ratios of air to sewage and return sludge to sewage.

Immediately east of, and adjacent to, the aeration tanks there is a covered pipe gallery (Fig. 6) which joins the basement of the activated-sludge station. This gallery contains the sewage, sludge, air, and mixed liquor piping that connects with the various tanks. The sludge and other pumps are either in this gallery or in the basement of the activated-sludge station.

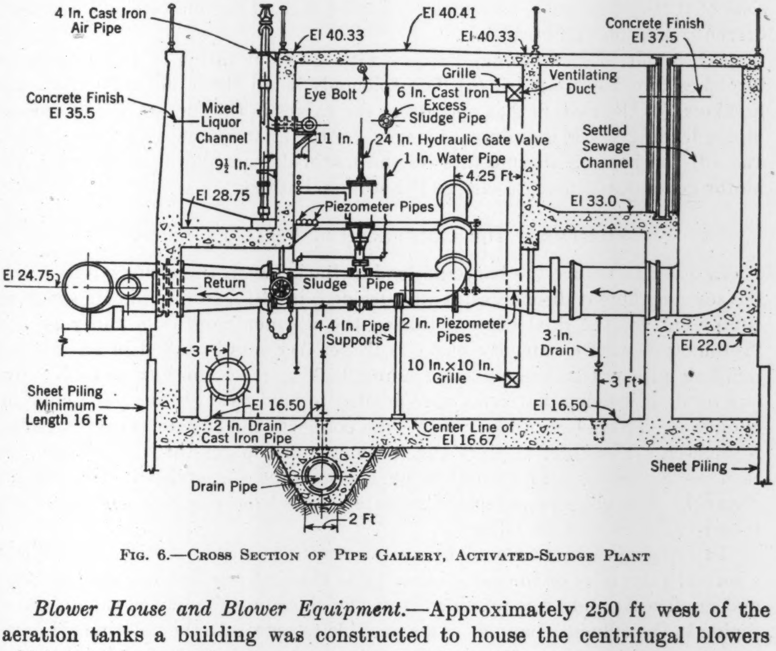
4 In. Cast Iron Air Pipe
El 40.33
El 40.41
El 40.33
Concrete Finish El 37.5
Concrete Finish El 35.5
Grille
Ventilating Duct
Eye Bolt
6 In. Cast Iron Excess Sludge Pipe
Mixed Liquor Channel
11 In.
24 In. Hydraulic Gate Valve
Settled Sewage Channel
9½ In.
1 In. Water Pipe
4.25 Ft
El 28.75
Piezometer Pipes
El 33.0
El 24.75
Return
Sludge
Pipe
4-4 In. Pipe Supports
2 In. Piezometer Pipes
3 In. Drain
El 22.0
3 Ft
10 In.×10 In. Grille
3 Ft
Sheet Piling Minimum Length 16 Ft
El 16.50
2 In. Drain Cast Iron Pipe
Center Line of El 16.67
El 16.50
Sheet Piling
Drain Pipe
2 Ft

FIG. 6.—CROSS SECTION OF PIPE GALLERY, ACTIVATED-SLUDGE PLANT

Blower House and Blower Equipment.—Approximately 250 ft west of the aeration tanks a building was constructed to house the centrifugal blowers which furnish air to the aeration tanks. This building, a one-story structure, 94 ft by 80 ft in plan, contains three motor-driven centrifugal blowers with space for one future unit. Each blower delivers 16,500 cu ft per min of air against a pressure of 7.5 lb per sq in. With all three blowers in service 1.78 cu ft of air per gal of sewage is available at a peak sewage flow of 40 mgd. The blowers are directly connected to 2,300-volt, air cooled, squirrel-cage motors, operating at 3,600 rpm.

The air handled by the blowers enters the basement of the building and passes first through three primary filters and then through secondary filters. The primary filters, each with a capacity of 19,500 cu ft per min of free air, are of the automatic, viscous, impingement, self-cleaning type. The secondary filters are of the dry cellulose type with a capacity of 58,500 cu ft per min of free air. From the secondary filters the air flows through three 30-in. cast-iron pipes to the blowers. In each of these pipes there is a venturi tube with a maximum capacity of 20,000 cu ft per min of free air and a motor-operated intake throttling valve. The discharge pipe from each blower connects with a 42-in. cast-iron air main, which is laid underground to the aeration tanks. A 10-in. motor-operated gate valve is installed in the discharge pipe from each blower. This valve opens automatically when a blower is started and closes when full operating pressure is built up in the discharge pipe.

Auxiliary equipment used in conjunction with the lubricating system consists of three water cooled oil coolers, one for each of the blowers and a motor-driven centrifugal oil purifier.

A 125-volt, 60-cell storage battery with a 1-hour rating of 144 ampere-hr furnishes direct current to: (*a*) The solenoids in all the 2,300-volt oil circuit breakers, (*b*) the motors that operate to the valves in the suction and discharge piping leading to the centrifugal blowers, (*c*) the limit switches on these valves, and (*d*) the bearing thermostats on the motor-driven blower units. A 5-kw motor generator is used to charge the storage battery.

EQUIPMENT FOR CHLORINATING FINAL EFFLUENT

In order to reduce the bacterial load in Back River, the estuary into which the sewage effluent discharges, it was decided, in 1945, to provide chlorinating equipment for the final effluent during the warmer months of the year. A "rotometer" with a capacity of 6,500 lb per day was installed in an existing building adjacent to the effluent channel leading from the humus tanks, and two outdoor wooden platforms were erected to hold twenty-four 1-ton chlorine cylinders. Extra-heavy black steel pipe connects the chlorine tanks with the rotometer. The chlorine from the rotometer discharges through 1-in. piping to a water ejector and then through a 3-in. pipe suspended in the effluent channel. The chlorine, dissolved in water, flows from this pipe through sixty-four ¼-in. holes into the effluent.

In order that there would be no interruption in chlorinating the effluent a second rotometer of the same capacity as the first one was installed in 1946, and duplicate piping was provided from the chlorine cylinders to the discharge piping in the effluent channel. It is possible to segregate the chlorine cylinders into two groups and operate the rotometers in parallel, or chlorine from all the cylinders can be fed to one rotometer.

SLUDGE CONTROL STATION

About halfway between the preliminary settling tanks and the heated sludge digestion tanks, a sludge control station was constructed. The equipment in this station was designed to serve the aforementioned settling tanks and five sludge digestion tanks. The piping and the apparatus in the building, however, are arranged so that ultimately six settling tanks and eight digestion tanks can be served.

The basement of the building is sufficiently large to contain six venturi tubes, 54 in. by 30 in.; five of these have been installed to measure the sewage flowing to the preliminary settling tanks. Each meter is of a special design with its invert in the same horizontal line. This design was selected because it was feared that solids, which would roll along the bottom of the conduit leading to the meter, might be retained on the upstream side of the throat of the meter. A subsequent examination of the venturi tubes, 42 in. by 21 in., in service at the sewage works for twenty-five years indicated that this apprehension was unfounded. In the future, meters of a standard design will be used since they are considerably cheaper.

Brass pipes, 1¼ in. in diameter, connect the annular rings on the meters with 10-in., vertical, cast-iron float tubes. Crosses with removable plugs are provided at all changes in the line and grade of these pipes to facilitate cleaning. At the top of each float tube there is a pipe provided with a needle valve for continuously introducing a small stream of clean water into the tube.

At one end of the sludge control station there is a well, divided into two compartments. One compartment receives raw sludge from the settling tanks, and the other ripe sludge from the digestion tanks. From each of the settling tanks an 8-in. cast-iron sludge pipe connects with this well. Furthermore, a similar pipe, carrying scum from the settling tanks, empties into the well. Each of these pipes connects with a wrought-iron swivel pipe, the outer end of which can be raised to a level above the hydraulic grade in the settling tanks. The digested-sludge well is equipped with 10-in. swivel pipes of the same design as those in the raw-sludge well. Each pipe connects with a 10-in. cast-iron discharge pipe, terminating at the bottom of a sludge digestion tank.

Adjacent to the well there is a room, containing the pumping equipment. It is 55 ft by 22 ft in plan with a main gallery floor and a lower floor for the pumping equipment. There are seven horizontal centrifugal pumping units, all operated by 440-volt alternating-current motors. Two of the pumps (which have a capacity of 350 gal per min) are used to handle raw sludge. By reducing the speed the capacity can be decreased to 75 gal per min or 100 gal per min. Each pump has sufficient capacity to handle the raw sludge from a sewage flow of 140 mgd. The pumps discharge into a 10-in. cast-iron force main, containing a venturi tube with a maximum capacity of 720,000 gal daily. The venturi tube connects with a cast-iron manifold, and 10-in. pipes (each containing an hydraulically operated gate valve) extend from this manifold to the digestion tanks.

There are also two centrifugal pumps for handling digested sludge. Each of these has a capacity of 1,000 gal per min. The sludge discharges through a 10-in. cast-iron force main that contains a venturi tube with a maximum capacity of 2.30 mgd.

All four sludge pumps are especially designed to permit the passage of 4-in. spheres. The discharge piping from each of the digested-sludge pumps is designed so that sludge from the heated digestion tanks can be pumped to a vacuum filter building or into other tanks or lagoons for further digestion or storage. The sludge piping in the building is covered with wool-felt insulation, ¾-in. thick, to prevent condensation from forming on the pipes.

There are two centrifugal pumps, each with a capacity of 400 gal per min, which circulate hot water through pipe coils in the sludge digestion tanks for heating purposes. These two units can handle sufficient water to heat six sludge tanks, each with a capacity of 200,600 cu ft. It is planned, at some future date, to install eight of these tanks, and at that time the capacity of each hot water pump can be increased to heat the additional tanks by providing a larger impeller. Each unit is driven by a 3-hp motor, operating at a speed of 1,740 rpm. As the motor contains four independent sets of windings, the pump can also be run at speeds of 1,160 rpm, 870 rpm, and 580 rpm. These reduced speeds are used during warm weather when less heat is required.

On the gallery floor of the pump room there is a large switchboard on which electrical equipment and recording instruments are mounted. All pumping equipment on the lower floor is controlled from this switchboard. The recording instruments consist of liquid-level and methane recorders. The former record the level of the sludge in the digestion tanks, and the latter record the amount of methane in the gases evolved from each of these tanks.

The sludge control station contains a boiler room with equipment for heating the sludge digestion tanks; six gas-fired boilers have been provided, each with a capacity of 2,304,000 Btu per hr. The flues from the boilers are made of 12-gage copper.

The existing boilers generate low pressure steam, which passes to two heat exchangers. The return water from the digestion tanks is pumped through the heat exchangers by the hot water pumps in the pump room. Each heat exchanger is designed to heat 400 gal of water per min from 100° F to 136° F.

Meters are available in the boiler room for recording and registering the amount of water used to heat the sludge digestion tanks. Furthermore, there are electrically operated thermometers for recording the temperatures of the sludge in the digestion tanks, the heating water entering and leaving these tanks, and the outdoor atmosphere.

Adjacent to the boiler room is a room containing five 6-in., rotary-type, displacement gas meters, each with a normal capacity of 16,980 cu ft of gas per hr. The meters are designed to operate with the gas flowing in either direction. The advantage of this design is that, when sludge is being withdrawn from the digestion tanks, gas from the gas holder can flow back into the digestion tank through the gas meter. Thus, the formation of a vacuum or the sucking of air into the space below the roof of the digestion tank is obviated. Each digestion tank is provided with its own gas meter.

To guard against the possibility of a gas explosion in the sludge control station an electrically operated, combustion gas alarm was installed. Radiating from this alarm there are six ⅛-in. copper pipes which terminate at various points in the building. A small pump forces the air adjacent to the open ends of the pipes to the combustion alarm, which rings a bell if any inflammable gas is present.

SLUDGE DIGESTION TANKS

Just west of the sludge control station five covered sludge digestion tanks have been built. The first two were constructed from 1933 to 1935; the third, from 1937 to 1939; and the last two, in 1946–1947. The basis of the design of these tanks is as follows:

Item	Description	Quantity
1	Capacity of each tank, in thousand cubic feet	200.6
2	Percentage of tank space occupied by sludge	80
3	Percentage of tank space occupied by supernatant liquor	20
4	Average moisture of sludge, in percentages	93
5	Quantity of dry solids in each tank, in thousand cubic feet	11.2
6	Moisture, in percentage of raw sludge added	95
7	Volatile matter, in percentage of raw sludge (dry basis)	75
	Quantity of Raw Solids Added Daily to Each Tank—	
8	In percentage of dry solids in tanks	4
9	In cubic feet (dry basis)	448
10	In thousand gallons (wet basis, containing 95% water)	67
11	Volume of gas produced, in cubic feet per pound of fresh volatile matter	7
12	Gas produced daily by each tank, in thousand cubic feet	150
13	Maximum gas production, in percentage of the average	250
14	Methane content of gas, in percentage of gas volume	70
15	Estimated number of persons served per tank, in thousands	242

As indicated, each tank has a capacity of 200,600 cu ft and was assumed to serve 242,000 persons (items 1 and 15). The raw sludge added to the tanks was estimated to contain 5% dry solids, 75% of which was considered to be volatile matter. The quantity of sludge added to each tank was estimated to be 67,000 gal daily, which was based on the dry solids added daily, amounting to 4% of the dry solids in the tank. It was assumed that 7 cu ft of gas would be produced per pound of fresh volatile solids added, or 150,000 cu ft daily per tank.

The tanks (Fig. 7) are 100 ft in diameter and 25 ft deep at the walls. Their roofs are made of reinforced concrete and rest on the tank walls and on concrete columns at the center. The roofs of tanks 1 and 2 differ somewhat from those of tanks 3, 4, and 5. Each roof is supported by thirty-two horizontal steel beams placed in a radial position with their outer ends resting on the tank wall. In tanks 1 and 2 these are 125-lb, 24-in. beams, terminating approximately 14.5 ft from the center of the tank and hung from sixteen structural steel vertical members. These latter members are supported by steel trusses, which in turn are secured to a steel column, resting on the central concrete column.

In tanks 3, 4, and 5, however, the horizontal steel beams are 36 in. deep and weigh 250 lb per ft. The inner end of each beam is supported by two 2½-in. vertical steel bolts, which are suspended from a structural steel spider at a distance of 4.75 ft from the center of the tank. With this type of construction it is possible to suspend the stirring mechanism on a steel drum, which encloses the central column and passes through an annular slot in the tank roof.

This slot is provided with a cylindrical steel trough filled with water to prevent the escape of gas. The concrete roofs of the digesters were made quite thin to keep the dead load as light as possible. A vent in the roof made of 16-gage, galvanized sheet steel, with an opening 7 ft by 1.33 ft in plan, permits the escape of gas into a gas dome. There is a 6-in. cast-iron gas pipe from each digester to the sludge control station.

Each tank roof is covered with three-ply asphalt felt roofing to prevent the escape of gas. This roofing is covered with a layer of concrete 3 in. thick so that gas cannot get under the membrane and raise it; four ports, 12 in. in diameter, have been furnished in the roof for sampling the sludge.

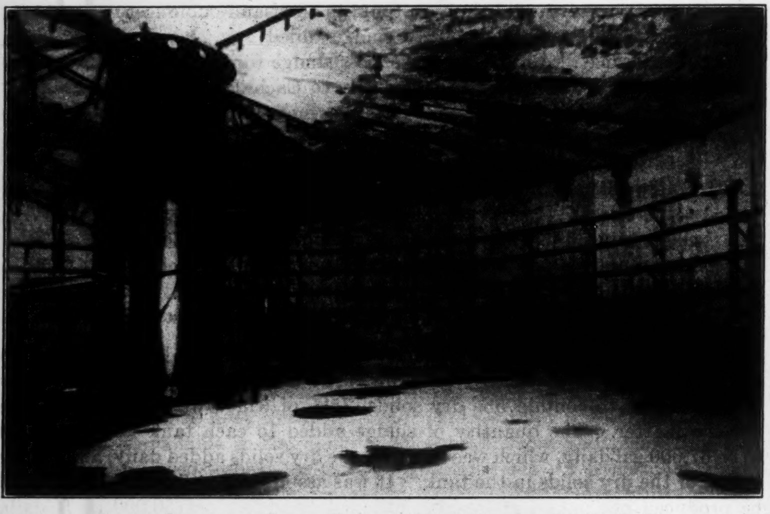

FIG. 7.—INTERIOR OF SLUDGE DIGESTION TANK; CAPACITY, 200,600 CU FT

Since the foundation under the tanks consisted of soft muck, each central column rests on fifty-seven pre-cast concrete piles, 18 in. by 18 in., which support a maximum load of 4,088,000 lb. The tank walls are supported by two concentric rows of wooden piles, spaced 3 ft apart along the periphery of the tank footing. Tongue-and-groove sheet piling, 12 in. by 12 in., was driven around the outer edge of the footing to prevent the pressure of dirt around the digester from being transmitted to the soil beneath the tank floor and heaving the bottom. A layer of coarse gravel, approximately 15 in. thick, was provided under the entire tank floor. Cast-iron pipes, 6 in. in diameter, were laid in the gravel for drainage purposes. The floor of each digester consists of concrete 8 in. thick. A mechanism, which rotates about the vertical center line of the tank, is provided to stir the sludge gently and to force the digested material to the discharge pipe.

Adjoining each tank there is a manhole containing three 10-in. sluice gates, placed 12 ft, 16 ft, and 20 ft above the bottom of the tank for withdrawing supernatant sludge liquor from different depths. There is also another manhole adjoining each tank, which contains a manometer for recording the elevation of the sludge in the digester. Digested sludge flows from the tank through a 10-in. cast-iron pipe to the sludge well in the sludge control station. Raw sludge is pumped to each of the tanks through two 8-in. cast-iron pipes laid on top of each digester.

Heat to Be Added to Digesters.—The following data relate to the heating of the sludge digesters. It was assumed that a temperature of 80° F would be maintained in the digesters in January, the coldest month in the year. The heat added is used to raise the temperature of the raw sludge pumped into the tanks and to compensate for losses by conduction. By far the larger amount of heat is required to raise the temperature of the incoming cold sludge. The heat needed for this purpose was estimated to be 831,000 Btu per hr and was based on the assumption that 23,750 lb of raw sewage at 45° F would be added hourly to each digester (based on daily additions equal to 4% of the dry solids in the digester). The temperature maintained in the digester was assumed to be 80° F.

The following is the estimated heat loss from each tank, in British thermal units per hour:

Through roof	99,000
Through walls	5,800
Through bottom	188,000
Total	292,800

The heat loss through the roof was based on an average air temperature of 34° F. Cost estimates were prepared to determine the desirability of insulating the roof with building tile or other insulating materials. The additional cost, however, was not compensated for by the value of the gas saved to heat the tanks.

In computing the heat loss through the tank walls the following formula[3] was used:

$$U = \frac{t_1 - t_2}{\dfrac{r_1 \log r_2/r_c}{k} + \dfrac{r_1 \log r_c/r_1}{k_c}} \qquad (1)$$

in which U is the heat transmission per square foot of tank surface per hour; t_1 is the temperature of the sludge in the tank (80° F); t_2 is the temperature of the ground at a distance, D, beyond the wall of the tank (45° F); r_1 is the radius of the inside of the tank (50 ft); r_c is the radius of the outside of the tank (51.75 ft); $r_2 = r_1 + D$ is the horizontal distance from the center line of the tank where the normal temperature of the ground is not affected by t_1 (91.75 ft); k is the conductivity of the ground (0.9 Btu per sq ft per hr per foot of thickness per 1° F temperature difference); and k_c is the conductivity of concrete (0.69 Btu per sq ft per foot of thickness per 1° F difference in temperature of the two surfaces).

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[3] "Theory of Heat Losses from Pipes Buried in Ground," by John R. Allen, *Transactions*, Am. Soc. of Heating and Ventilating Engrs., Vol. 26, 1920, p. 335.

In computing the loss of heat through the bottom of the digesters it was assumed that the ground beneath them would not act as an insulator because there would be a flow of ground water through the drains under the tanks.

The area of heating coils in the digesters was based on the assumption that the heat transmitted through the pipe would be approximately 20 Btu per sq ft per hr per 1° F temperature difference between the inside and the outside of the pipe. The heating coil (which is 1,240 ft long and 6 in. in diameter) has an area of 1,975 sq ft. This pipe, unpainted, is hung in the form of a spiral on the inside of the digester about 2 ft from the walls. All the hot water piping (which is underground) from the sludge control station to the five digesters is insulated with asbestos sponge felt, 1 in. thick. The pipes are enclosed in reinforced-concrete conduits. Gravel and drain tile have been placed under these conduits to prevent ground water from backing up into them.

GAS HOLDER

A waterless gas holder was built with a capacity of 200,000 cu ft. The gas piping between the digestion tanks and the holder is designed so that gas will flow from the holder into the empty space formed below the digester roofs when sludge is drawn from the digesters. If a gas holder were not provided, a partial vacuum would be produced in the top of the digester and air would be sucked into this space with the danger of forming an explosive mixture of gas and air.

A waterless holder was constructed, because the cost of maintenance and operation is less than with a water-type holder. With the latter type provision must be made to heat the water in the seal cups and the tank. Furthermore, the painting of a water-type holder is much more of a problem.

The gas holder, which is built on a concrete foundation, is 63 ft 9 in. in diameter and 75 ft high at the eaves. The gas is stored under 8 in. of pressure below a circular piston, which moves up and down on top of the gas. The piston is made of $\frac{3}{16}$-in. steel plates, secured to steel trusses. Since it is very essential to know the quantity of stored gas, a gage has been provided on the outside of the holder and also in the sludge control station. An alarm rings if the quantity is less than 5,000 cu ft. On the 12-in. inlet pipe to the holder there is a gas burner. When the piston in the holder reaches its highest position, the burner ignites automatically and remains lighted until the piston has dropped 3 ft. In case the burner fails to light, the piston is forced upward by the gas, which escapes into the atmosphere through a series of holes in the shell of the holder.

VACUUM FILTER BUILDING AND EQUIPMENT

In 1931 experiments were started at the Back River plant to determine if primary digested sludge could be dewatered on vacuum filters. After conducting extensive experiments for more than 2 years,[4,5,6] it was decided to use vacuum filters at the Back River plant.

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[4] "The Dewatering of Sludge by Vacuum Filtration," by C. E. Keefer and E. C. Cromwell, *Sewage Works Journal*, November, 1932, p. 929.

[5] "Absorption and Flocculation As Applied to Sewage Sludges," by A. L. Genter, *ibid.*, July, 1934, p. 689.

[6] "The Vacuum Filtration of Elutriated Sludge," by C. E. Keefer and Herman Kratz, Jr., *ibid.*, September, 1934, p. 845.

The filter layout was designed to serve an estimated population of 1,200,000 in 1970 with a production of 60.5 tons daily (300-day operating year) of digested dry solids. This quantity consists of 55 tons of digested primary sludge and 5.5 tons of digested activated sludge. The digested sludge was assumed to contain 4.0% dry solids.

The filtering equipment was installed in a one-story brick building, 97 ft by 56 ft in plan. Two filters (Fig. 8), each with a surface area of 500 sq ft, were initially provided in 1939, to be followed by two more filters of the same size in 1946. Sludge from any of a number of open digestion tanks flows by gravity into a covered well in the basement of the filter building. Adjacent to this well there are three motor-driven centrifugal sludge pumps, each with a capacity of 500 gal per min. These pumps deliver the sludge from the well through a 10-in. cast-iron force main (in which there is a venturi tube) into a tank equipped with a horizontal paddle mixer. A 20-in. cast-iron gravity pipe line connects the effluent channel of the trickling filters with this mixing tank. Sludge and filter effluent, which is used for elutriation purposes, can thus be mixed in this tank in almost any ratio. The mixture of sludge and filter effluent then flows to a circular elutriation tank, which is 80 ft in diameter and has a working depth of 15 ft at the side wall. This tank is designed to have a theoretical detention period of 4 hours with the upper 11.5 ft allotted for sedimentation and the lower 3.5 ft for sludge storage. Although only one tank was provided at first, it is planned to construct another of the same size in the future when larger volumes of sludge are dewatered.

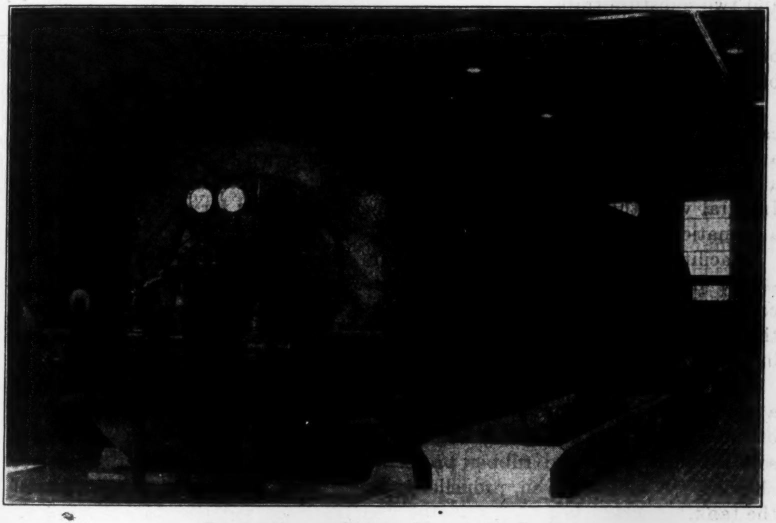

FIG. 8.—VIEW OF VACUUM FILTERS, 500 SQ FT

The mixture of sludge and water enters the center of the elutriation tank, and the effluent or elutriate overflows a V-notched weir at the periphery of the tank. The elutriated sludge flows through 12-in. cast-iron pipes into two wells in the basement of the filter building. Each well contains a gravity-discharge, vertical elevator with a capacity that can be varied from 960 cu ft per hr to 3,840 cu ft per hr. The sludge from each elevator discharges into a rubber-lined steel coagulating tank, 12.5 ft by 1.67 ft in plan and 3.75 ft deep. The coagulated sludge from these tanks flows by gravity to any of the four vacuum filters.

Vacuum is applied to the filters by three pumps, one of which is for stand-by service. Each pump has a displacement of 2,000 cu ft per min. In the piping connecting the vacuum pumps with the filters there are two vacuum receivers and two moisture traps.

Compressed air is furnished the filters by three rotary-type blowers. Each blower delivers 750 cu ft per min of free air at a pressure of 2 lb per sq in. One blower is for stand-by service.

A horizontal belt conveyor with an ultimate capacity of 48 tons of filter cake per hour has been provided just below the operating floor of the building and between the four filters. This belt discharges just outside the building on a portable belt conveyor, which deposits the cake in a pile east of the building. Integral with the horizontal conveyor there are scales for weighing the cake automatically.

Facilities are provided in the vacuum filter building for coagulating the sludge by chlorinated copperas. It was decided to use this coagulant as copperas is quite cheap in Baltimore. In designing the equipment for making the chlorinated copperas it was assumed that 2.0 lb (anhydrous basis) per 100 lb of dry sludge solids would be required for coagulating purposes.

A rubber-lined steel mixing tank with an internal diameter of 9.42 ft and a depth of 6.83 ft was provided in the basement of the building for the reception of the copperas. The lining consists of a layer of hard rubber and a layer of soft rubber. The hard rubber protects the tank against the action of free chlorine. A motor-driven, propeller-type chemical unit mixes the contents of the tank.

Two centrifugal chemical pumps, each with a capacity of 100 gal per min, take their suction from the mixing tank and, after circulating the copperas through a system of rubber-lined piping, discharge it again into the tank. One of these pumps is used as a stand-by. Each pump is capable of circulating the contents of the mixing tank in 32 min. Dry chlorine gas is used to chlorinate the copperas. The gas is measured by a rotometer with a capacity ranging from 115 lb per day to 2,340 lb per day. Theoretically 100 lb of chlorine is necessary to oxidize, completely, 782 lb of $FeSO_4 \cdot 7H_2O$. During the early period of operation of the filter plant, dry chlorine gas was fed into the piping between the storage tank and the chemical pumps. The pumps mixed the chlorine gas and the copperas, and by the time the latter discharged into the storage tank the reaction between the two chemicals was completed. After trying this piping arrangement for several months better results were obtained by feeding the chlorine into the discharge piping from the chemical pumps.

The chlorine is stored in 1-ton cylinders in a room in the basement of the filter building. At one end of the room there is a platform scale capable of holding four 1-ton cylinders. A motor-driven centrifugal fan was installed to ventilate the two rooms containing the chlorine cylinders and the chlorinating equipment. This fan (which produces an air change every 2.6 min) takes its suction near the floor and discharges the air or chlorine fumes outside the building. Several push-button stations were installed at different points in the building for starting or stopping the fan quickly.

The chlorinated copperas, which is made in 3,260-gal batches, is stored in the basement in four rubber-lined steel tanks with a working capacity of 2,165 gal each. Two of these tanks were provided in 1939 when the original installation was made, and the other two were installed in 1946. On the main floor of the building there are six variable speed, diaphragm, acid resisting pumps, each with a capacity of 1 gal per min, which take their suction from the chlorinated copperas storage tanks and discharge into the coagulating tanks. These pumps are arranged in two groups with three to a group, and are operated by one of the sludge elevators through a chain drive. In the original installation the coagulant was mixed with the sludge in the coagulating tanks by diffused air, blown through porous diffuser tubes. As this method of mixing proved unsatisfactory, the diffuser tubes were removed and replaced by one motor-driven propeller-type mixer in one tank and two mixers in the other. The mixers, which rotate at speeds of 34 rpm, are placed near the outlet end of each tank. The discharge end of the rubber hose from the coagulant pumps terminates just ahead of the propeller mixers. A glass indicator has been provided in each hose so that the operator can readily see if the coagulant is being pumped.

Electric Substation

Prior to the construction of the activated-sludge units 2,300-volt electric current was purchased from the local power company. At the time these units were being planned, it was realized that the demand for power would increase to 800 kw or more when they would go into operation. A demand of this magnitude warranted the installation of 33,000-volt incoming service, which is considerably cheaper than current purchased at the lower voltage. An electric substation, therefore, containing the necessary transformers and auxiliary equipment, was built.

This substation consists of a one-story brick building 53 ft by 21 ft in plan with a yard enclosed by a picket fence in the rear. Two 33,000-volt, three-wire, overhead power lines serve the station, one of which is for stand-by purposes. Each power line connects with a 34.5-kv, single-throw three-pole, air-break switch. In the yard at the rear of the building a 600-ampere oil circuit breaker and three 1,000-kva, single-phase, oil immersed (33,000/2,400) transformers were provided.

Inside the substation metal clad switchgear equipment has been installed for serving five outgoing, 2,300-volt feeders. Ultimately there will be six. Each feeder connects with two adjacent sections of the switchgear, each section housing an oil circuit breaker and its accessory equipment. There are two 2,300-volt buses, to either of which the load side of the circuit breakers can be connected. Suitable interlocks are provided so that the buses cannot be paralleled.

MISCELLANEOUS ITEMS

Heating Facilities.—With the exception of the electric substation the six buildings described in this paper are heated by gas-fired unit heaters or gas-fired steam boilers. As a rule convector-type radiation enclosed in semirecessed cabinets is installed in the smaller rooms and unit heaters are provided in the larger rooms.

Plant Telephone System.—The sewage plant has been provided with a private telephone system for a number of years. This system was extended to the six new buildings. A 24-pair, underground telephone cable was laid to each building, in which a selective-ringing, selective-talking telephone set was installed.

CONSTRUCTION WORK

Practically all the work was done under twenty-five contracts, which varied in amount from $6,273 to $873,064. A small part—the installing of the mechanically cleaned screens, the laying of some of the electric conduits, and a few minor items—was done by day labor.

CONSTRUCTION COST

The total cost of constructing the treatment units exclusive of engineering expense amounted to $3,859,500, divided as follows:

Mechanical Screen Building:	
Substructure	$ 16,029
Superstructure	33,620
Three screens	24,246
Total	$ 73,895
Three grit chambers	99,036
Five preliminary settling tanks	407,412
Activated-sludge units	1,001,819
Chlorinating equipment	6,490
Sludge control station	215,267
Five sludge digestion tanks	643,255
Gas holder	44,951
Vacuum filter installation	394,410
Electric substation	40,213
Grading	214,602
Sewer conduits	373,421
Sludge pipes	106,231
Gas pipes	22,862
Hot water pipes	49,079
Storm drains	34,982
Electric conduits and cables	42,922
Roads and sidewalks	14,015
Miscellaneous piping	62,706
Miscellaneous	11,932
Total	$3,859,500

The unit cost of the buildings and the tanks is given in Table 2. The cost of the buildings varied from $0.26 per cu ft for the mechanical screen building to $0.50 per cu ft for the blower house. The blower house was more expensive because it contained heavy expensive foundations, had a complete plumbing and heating system, and was provided with considerable miscellaneous equipment. The primary settling tanks, which varied in cost from $0.22 per cu ft to $0.28 per cu ft, were considerably cheaper than the final settling tanks, which cost $0.38 per cu ft. Sludge digestion tanks 4 and 5, built after World War II, cost $0.91 per cu ft, which was 112% more than the cost of tanks 1 and 2 built from 1933 to 1935. The costs of the grit chambers, settling tanks, elutriation tank, sludge thickening tanks, and digesters also include the cost of the mechanisms in these tanks.

TABLE 2.—UNIT COSTS OF BUILDINGS AND OTHER STRUCTURES

Item	Year built	Total cost	Volume (cu ft)	Cost (cu ft)
Mechanical screen building	1931–1932	$49,569[a]	192,794	$0.26
Grit chamber building	1937–1938	11,548	26,431	0.44
Grit chambers	1937–1938	72,888[b]	67,500	1.08
Preliminary Settling Tanks:				
Nos. 1 and 2	1930–1931	163,638[b]	638,908	0.26
Nos. 3 and 4	1936–1937	149,493[b]	670,092	0.22
No. 5	1937–1938	94,281[b]	335,046	0.28
Activated-Sludge Units:				
Blower house	1938–1939	123,926[e]	248,604	0.50
Aeration tanks	1938–1939	488,836	667,543	0.73
Final settling tanks	1938–1939	127,812[b]	340,010	0.38
Sludge thickening tanks	1938–1939	18,201[b]	19,614	0.93
Sludge control station	1932–1933	74,202[c]	203,552	0.32
Sludge Digestion Tanks:				
Nos. 1 and 2	1933–1935	172,758[b]	401,200	0.43
No. 3	1937–1939	106,400[b]	200,600	0.53
Nos. 4 and 5	1946–1947	364,097[b]	401,200	0.91
Vacuum Filter Equipment:				
Filter building	1937–1939	128,707[d]	316,170	0.41
Elutriation tank	1937–1939	34,011[b]	98,458	0.35
Gas holder	1935	44,951	200,000	0.23
Electric substation	1939–1940	15,221[e]	32,380	0.47

[a] Does not include cost of screens. [b] Includes cost of mechanical equipment. [c] Does not include cost of mechanical and electrical equipment and pile foundation. [d] Does not include cost of mechanical and electrical equipment. [e] Does not include cost of electrical equipment.

RESULTS ACCOMPLISHED

The additions and enlargements have materially improved the sewage plant. The odors produced by the plant have been greatly reduced as a result of the elimination of scum on the preliminary settling tanks. The disposal of screenings is no longer a problem, and the removal of grit from the sewage has eliminated, to a considerable extent, the deposition of this material in tanks and pipe lines.

The preliminary settling tanks produce a less acid sludge which is more amenable to digestion. Although the operation of the activated-sludge units has been handicapped by frequent sludge bulking, they have consistently produced an effluent containing less than 15 ppm of 5-day B.O.D. and suspended solids. The heated sludge tanks under loadings of 0.17 lb of dry solids per cubic foot per day have reduced the volatile content of the sludge an average of 50% in 15 days with the production of 10 cu ft of gas to 14 cu ft of gas per pound of volatile matter in the sludge.

During the 8-year period from 1940 to 1948, inclusive, the vacuum filters have dewatered sludge at an average rate of 4.64 lb of dry solids per square foot of filter area per hour with the use of 5.4 lb of chlorinated copperas per 100 lb of dry solids. The adoption of vacuum filters for the dewatering of sludge, in lieu of sludge drying beds, has resulted in a material reduction in the operating personnel, has permitted the drying of sludge throughout the year, and has decreased the production of odors. Chlorinating the final effluent during the warmer months of the year has afforded protection to the river against pathogenic organisms.

ENGINEERING ORGANIZATION

The engineering work was done under the general supervision of the following chief engineers of the Baltimore Department of Public Works: The late C. F. Goob (from 1927 to 1931), the late Bernard L. Crozier, M. ASCE (from 1931 to 1938), the late Frank K. Duncan, M. ASCE (from 1938 to 1939), the late George Cobb (from 1939 to 1943), and Nathan L. Smith, M. ASCE (from 1943 to 1947).

The director of the Department of Public Works since January 19, 1948, is Paul L. Holland, M. ASCE. The design and the construction work was under the direction of the following sewerage engineers of the Bureau of Sewers: Milton J. Ruark (from 1926 to 1931), George E. Finck, M. ASCE (from 1931 to 1947), and John J. Hunt (from 1947 to the present). The entire work was under the direct supervision of the writer. The firm of Whitman, Requardt and Associates designed the grit chambers and the activated-sludge units. The construction work was supervised by Edward R. Anderson (from 1930 to 1946) and Harold E. Burton (from 1946 to the present). Robert J. Trautman, Assoc. M. ASCE, has been connected with the work since 1930 and has served as associate engineer since January 1, 1945.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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TRANSACTIONS

Paper No. 2427

MATHEMATICAL ANALYSIS OF AN AERIAL SURVEY

By Lo-Ho[1]

Synopsis

Aerial photographic methods are now widely used in topographic surveying, and in recent years efforts have been made to place them under mathematical control by analytical procedures of computation. In this field Earl Church,[2] Assoc. M. ASCE, and P. H. Underwood,[3] M. ASCE, are the leaders. The purpose of this paper is to present an analytical solution based on the direction-cosine principles of space geometry. The underlying mathematical principles are first developed and then applied to aerial survey problems. The methods of analysis and adjustment and the determination of the effect of the atmospheric refraction are new.

Data Used

The data immediately available in an aerial survey are the projections of the ground points on the negative plate, as transmitted through the objective lens; but the angles at the front nodal point, subtended by the separations of the objects at the instant of exposure, will be used in the following analysis. Furthermore, the positions of three or more ground points are also required, and these will be provided by the ground survey. The space angles at the front nodal point are assumed equal to their corresponding angles at the rear nodal point. The latter may be determined if the perpendicular distance from the rear nodal point to the plane of the negative plate, and the coordinates of the base of this perpendicular with respect to the fiducial axes on the negative plate, can be determined. An analysis of the geometrical relation between the rear nodal point and the fiducial axes is a part of this paper. The effect of atmospheric refraction is also analyzed; but the measurement of the coordinates of points in the negative by a comparator or other special instrument, and the

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[1] Prof. of Surveying and Geodesy, Tangshan Eng. College, Tangshan, North China.

[2] "Manual of Photogrammetry," Pitman Pub. Corp., New York, N. Y., and Chicago, Ill., 1944, p. 536.

[3] "Space Resection Problems in Photogrammetry," by P. H. Underwood, *Transactions*, ASCE, Vol. 112, 1947, p. 921.

calibration of the camera to determine values of distortion, are not considered in the present paper.

MATHEMATICAL PRINCIPLES

For convenience of reference, a summary of the mathematical principles directly involved in the following analysis will be presented first. Some well-known formulas are given without proof; but, for those not commonly known, the method of derivation will be outlined.

The method of adjustment described consists largely of the determination of corrections to be applied to computed or observed values. When the true values are connected by one or several conditional equations, the approximate relations between the corrections may be obtained by differentiation of such conditional equations, since the corrections to be applied are quite small in comparison with the values to be corrected. The resulting formulas will all be linear equations containing the corrections as unknowns and the differentials as exponents whose values may be found from the known values of the quantities. This general method will be used to derive many of the required equations.

Reference Axes.—The normal line through point O on the sphere OSWNE, Fig. 1, is adopted as the *Z*-axis; the north–south line through O, in the tangential plane of the sphere at point O, is the *Y*-axis; and the east–west line is the *X*-axis. The positive directions are all indicated by arrowheads as shown. The center of the sphere is designated point C, and its radius by *R*.

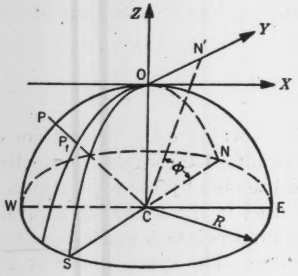

FIG. 1

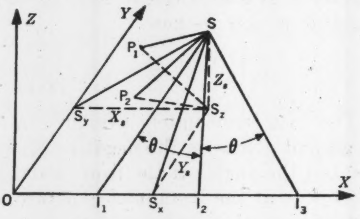

FIG. 2

Normal Distance of a Point to the Sphere.—Let the coordinates of point P be X_p, Y_p, and Z_p, and let a normal line from point P on the sphere be drawn to point P_f as shown in Fig. 1. Then

$$\overline{P_fP} = [X^2_p + Y^2_p + (Z_p + R)^2]^{\frac{1}{2}} - R$$

$$= Z_p + \frac{X^2_p + Y^2_p}{2\,(Z_p + R)} - \frac{(X^2_p + Y^2_p)^2}{8\,(Z_p + R)^3} + \cdots \ldots \ldots \quad (1)$$

which may be used to determine the elevation of a point defined by coordinate distances.

Length of the Arc OP_f.—The curved distance, OP_f, on the sphere is given by

$$\overline{OP_f} = R \sin^{-1} \frac{(X^2_p + Y^2_p)^{\frac{1}{2}}}{R + \overline{P_fP}} \tag{2}$$

Eq. 2 may be used to determine the length of line OP at sea level.

Equation of a Normal Line.—The equation of the radial line PC is

$$\frac{X}{X_p} = \frac{Y}{Y_p} = \frac{Z + R}{Z_p + R} \tag{3}$$

which may be used to determine the equation of the plumb line from a point having known coordinates.

Coordinates of a Point from Its Elevation and Distance.—Let the azimuth of line OP, measured from the north point in clockwise direction be α; its curved distance OP_f be s; and its elevation P_fP be h. Then

$$X_p = (R + h) \sin \frac{s}{R} \sin \alpha \tag{4a}$$

$$Y_p = (R + h) \sin \frac{s}{R} \cos \alpha \tag{4b}$$

and

$$Z_p = (R + h) \cos \frac{s}{R} - R \tag{4c}$$

Eqs. 4 may be used to determine the coordinates of points located by a ground survey.

Azimuth Angle of a Line.—Let N′ be a point on the Y-axis and let the vertical angle of the line CN′ be ϕ as shown in Fig. 1. Then the coordinates of N′ are 0, $R \cot \phi$, and 0; and the equation of the plane containing points N′, C, and P is

$$\begin{vmatrix} X & Y & Z & 1 \\ 0 & R \cot \phi & 0 & 1 \\ 0 & 0 & -R & 1 \\ X_p & Y_p & Z_p & 1 \end{vmatrix} = 0 \tag{5}$$

Let P_1 be another point with coordinates X_1, Y_1, and Z_1. The equation of the plane containing points C, P, and P_1 is found to be

$$\begin{vmatrix} X & Y & X & 1 \\ 0 & 0 & -R & 1 \\ X_p & Y_p & Z_p & 1 \\ X_1 & Y_1 & Z_1 & 1 \end{vmatrix} = 0 \tag{6}$$

Eq. 5 can be used to determine the meridian plane through a given point; Eq. 6 gives the vertical plane through two given points; and the angle between the two planes is the azimuth of line PP_1, which may be determined from the equations of the two planes by the usual method.

Space Angles at a Point.—The coordinates OS_x, OS_y, and S_xS of point S, Fig. 2, with respect to the rectangular axes are X_s, Y_s, and Z_s, respectively. Similarly the coordinates of points P_1 and P_2 in the (XY)-plane are denoted by X_1, Y_1, X_2, and Y_2, respectively. Angle P_1SP_2 may be determined from these coordinates thus:

$$\overline{S_sP_1} = \sqrt{(X_1 - X_s)^2 + (Y_1 - Y_s)^2} \tag{7a}$$

$$\overline{S_sP_2} = \sqrt{(X_2 - X_s)^2 + (Y_2 - Y_s)^2} \tag{7b}$$

$$\overline{P_1P_2} = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2} \tag{7c}$$

$$\overline{SP_1} = \sqrt{Z_s^2 + (\overline{S_sP_1})^2} \tag{7d}$$

$$\overline{SP_2} = \sqrt{Z_s^2 + (\overline{S_sP_2})^2} \tag{7e}$$

and

$$\cos \angle \overline{P_1SP_2} = \frac{(\overline{SP_1})^2 + (\overline{SP_2})^2 - (\overline{P_1P_2})^2}{2\,\overline{SP_1}\,\overline{SP_2}} \tag{7f}$$

Eqs. 7 can be used to determine the space angle at an exposure station. Lines SS_x and SS_y are perpendicular to axes OX and OY, respectively.

Coordinates in Terms of Axial Distances.—Let points I_1, I_2, and I_3, Fig. 2, be points on the X-axis forming equal space angles I_1SI_2 and I_2SI_3 to be denoted by θ; and let the angle S_xSI_2 be denoted by E_x. Then, from the relation, $\frac{\overline{SI_3}}{\overline{SI_1}} = \frac{\overline{I_2I_3}}{\overline{I_1I_2}} = \frac{\cos(\theta - E_x)}{\cos(\theta + E_x)}$, it may be found that

$$\tan E_x = \frac{\overline{I_2I_3} - \overline{I_1I_2}}{\overline{I_1I_3}} \cot \theta \tag{8}$$

Since $\overline{S_xI_2} = \overline{SS_x} \tan E_x$ and $\overline{S_xI_2} + \overline{I_2I_3} = \overline{SS_x} \tan(\theta + E_x)$, it may be found that

$$\overline{SS_x} = \frac{\overline{I_2I_3}}{\tan(\theta + E_x) - \tan E_x} \tag{9}$$

and

$$\overline{S_xI_2} = \frac{\overline{I_2I_3} \tan E_x}{\tan(\theta + E_x) - \tan E_x} \tag{10}$$

From Fig. 2, it is evident that

$$X_s = \overline{OI_2} - \overline{S_xI_2} \tag{11}$$

and

$$Z_s = \sqrt{(\overline{SS_x})^2 - Y_s^2} = \sqrt{(\overline{SS_y})^2 - X_s^2} \tag{12}$$

Formulas similar to Eqs. 8 to 11 may be derived, if three points on the Y-axis subtend equal space angles at point S. Eqs. 8 through 12 can be used to determine the perpendicular distance from the rear node of a lens to the plane of film emulsion, and the coordinates of the foot can be determined with respect to the fiducial axes.

Equation of Straight Line.—The coordinates of any point P_n will be denoted by X_n, Y_n, and Z_n and their small corrections by x_n, y_n, and z_n. Direction cosines of a line in the directions of X, Y, and Z will be denoted by L, M, and N and their small corrections by l, m, and n, respectively. The equation of line P_1P_2 is

$$\frac{X - X_1}{L} = \frac{Y - Y_1}{M} = \frac{Z - Z_1}{N} \qquad (13)$$

in which

$$L = \frac{X_2 - X_1}{r} \qquad (14a)$$

$$M = \frac{Y_2 - Y_1}{r} \qquad (14b)$$

and

$$N = \frac{Z_2 - Z_1}{r} \qquad (14c)$$

In Eqs. 14,

$$r = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2} \qquad (15)$$

If small corrections x_1, y_1, and z_1 are added to the coordinates of point P_1, it can be found that

$$l = \frac{1}{r}[-(1 - L^2)\,x_1 + L\,M\,y_1 + L\,N\,z_1] \qquad (16a)$$

$$m = \frac{1}{r}[L\,M\,x_1 - (1 - M^2)\,y_1 + M\,N\,z_1] \qquad (16b)$$

and

$$n = \frac{1}{r}[L\,N\,x_1 + M\,N\,y_1 - (1 - N^2)\,z_1] \qquad (16c)$$

Eq. 16c can be written

$$n = -\frac{L\,l + M\,m}{N} \qquad (17)$$

Distance from a Point to a Line.—The perpendicular distance D_n from any point P_n to the line defined by Eq. 13 is given by

$$\begin{aligned} D^2_n = {} & (X_1 - X_n)^2 + (Y_1 - Y_n)^2 \\ & + (Z_1 - Z_n)^2 - [L\,(X_1 - X_n) \\ & + M\,(Y_1 - Y_n) + N\,(Z_1 - Z_n)]^2 \qquad (18) \end{aligned}$$

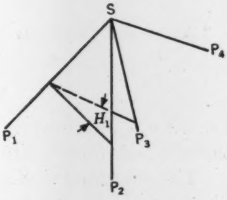

FIG. 3

Angle Between Two Intersecting Lines.—Let the direction cosines of two lines SP_1 and SP_2, Fig. 3, be denoted by L_1, M_1, N_1, L_2, M_2, and N_2, respectively. The angle A_{12} included by the two lines (P_1S and SP_2, Fig. 3) is given by

$$\cos A_{12} = L_1\,L_2 + M_1\,M_2 + N_1\,N_2 \qquad (19a)$$

The correction a_{12} (in radians) of angle A_{12} due to small corrections of the direction cosines is given by

$$a_{12} = -\operatorname{cosec} A_{12}\,(L_2\,l_1 + L_1\,l_2 + M_2\,m_1 + M_1\,m_2 + N_2\,n_1 + N_1\,n_2) \qquad (19b)$$

If the corrections of the direction cosines are due to the corrections x, y, and z of the coordinates of point S alone, it may be found that

$$a_{12} = \frac{\operatorname{cosec} A_{12}}{r_1 r_2} \{[(r_1 L_1 + r_2 L_2) - (r_1 L_2 + r_2 L_1) \cos A_{12}]\, x$$
$$+ [(r_1 M_1 + r_2 M_2) - (r_1 M_2 + r_2 M_1) \cos A_{12}]\, y$$
$$+ [(r_1 N_1 + r_2 N_2) - (r_1 N_2 + r_2 N_1) \cos A_{12}]\, z\} \quad ..(20)$$

in which r_1 and r_2 are the original lengths of the lines SP$_1$ and SP$_2$, respectively.

Distance Between Two Lines.—The perpendicular distance between two nonintersecting lines, $\frac{X - X_1}{L_1} = \frac{Y - Y_1}{M_1} = \frac{Z - Z_1}{N_1}$ and $\frac{X - X_2}{L_2} = \frac{Y - Y_2}{M_2} = \frac{Z - Z_2}{N_2}$, is given by

$$d = \frac{\begin{vmatrix} X_1 - X_2 & Y_1 - Y_2 & Z_1 - Z_2 \\ L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \end{vmatrix}}{[(M_1 N_2 - M_2 N_1)^2 + (L_1 N_2 - L_2 N_1)^2 + (L_1 M_2 - L_2 M_1)^2]^{\frac{1}{2}}} \quad ..(21a)$$

The condition of intersection of the two lines is controlled by

$$\begin{vmatrix} X_1 - X_2 & Y_1 - Y_2 & Z_1 - Z_2 \\ L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \end{vmatrix} = 0 \quad ....(21b)$$

Small corrections x_1, x_2, y_1, y_2, z_1, z_2, l_1, l_2, m_1, m_2, n_1, and n_2 to be added to the coordinates and the direction cosines of two nearly intersecting lines to make them intersect are connected by

$$\begin{vmatrix} X_1 - X_2 & Y_1 - Y_2 & Z_1 - Z_2 \\ L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \end{vmatrix} + \begin{vmatrix} X_1 - X_2 & Y_1 - Y_2 & z_1 - z_2 \\ L_1 & M_1 & n_1 \\ L_2 & M_2 & n_2 \end{vmatrix}$$
$$+ \begin{vmatrix} X_1 - X_2 & y_1 - y_2 & Z_1 - Z_2 \\ L_1 & m_1 & N_1 \\ L_2 & m_2 & N_2 \end{vmatrix} + \begin{vmatrix} x_1 - x_2 & Y_1 - Y_2 & Z_1 - Z_2 \\ l_1 & M_1 & N_1 \\ l_2 & M_2 & N_2 \end{vmatrix} = 0 \quad ..(22)$$

Eq. 22 was obtained by neglecting small terms involving powers of the corrections, which powers are higher than the first.

The coordinates X_m, Y_m, and Z_m of the middle point P$_m$ of the line perpendicular to the foregoing two lines may be found to be

$$X_m = \tfrac{1}{2}(X_1 + X_2)$$
$$- \frac{\begin{vmatrix} X_1 - X_2 & Y_1 - Y_2 & Z_1 - Z_2 \\ 2 L_1 L_2 & L_1 M_2 + L_2 M_1 & L_1 N_2 + L_2 N_1 \\ M_1 N_2 - M_2 N_1 & L_2 N_1 - L_1 N_2 & L_1 M_2 - L_2 M_1 \end{vmatrix}}{2[(M_1 N_2 - M_2 N_1)^2 + (L_2 N_1 - L_1 N_2)^2 + (L_1 M_2 - L_2 M_1)^2]} \quad ..(23a)$$

$$Y_m = \tfrac{1}{2}(Y_1 + Y_2) - \frac{\begin{vmatrix} X_1 - X_2 & Y_1 - Y_2 & Z_1 - Z_2 \\ L_1 M_2 + L_2 M_1 & 2 M_1 M_2 & M_1 N_2 + M_2 N_1 \\ M_1 N_2 - M_2 N_1 & L_2 N_1 - L_1 N_2 & L_1 M_2 - L_2 M_1 \end{vmatrix}}{2\left[(M_1 N_2 - M_2 N_1)^2 + (L_2 N_1 - L_1 N_2)^2 + (L_1 M_2 - L_2 M_1)^2\right]} \quad \text{(23b)}$$

and

$$Z_m = \tfrac{1}{2}(Z_1 + Z_2) - \frac{\begin{vmatrix} X_1 - X_2 & Y_1 - Y_2 & Z_1 - Z_2 \\ L_1 N_2 + L_2 N_1 & M_1 N_2 + M_2 N_1 & 2 N_1 N_2 \\ M_1 N_2 - M_2 N_1 & L_2 N_1 - L_1 N_2 & L_1 M_2 - L_2 M_1 \end{vmatrix}}{2\left[(M_1 N_2 - M_2 N_1)^2 + (L_2 N_1 - L_1 N_2)^2 + (L_1 M_2 - L_2 M_1)^2\right]} \quad \text{(23c)}$$

which may be used to determine the most probable point of intersection of two observed lines that should meet at a point if there were no errors of observation.

Three Concurrent Lines.—Let the space angles between the three concurrent lines SP_1, SP_2, and SP_3, as shown in Fig. 3, be denoted by A_{12}, A_{13}, and A_{23}, respectively; and let H_1 denote the dihedral angle between the two planes SP_1P_2 and SP_1P_3. By the law of cosines in spherical trigonometry,

$$\cos H_1 = \frac{\cos A_{23} - \cos A_{12} \cos A_{13}}{\sin A_{12} \sin A_{13}} \quad \text{(24)}$$

If H_1 remains unchanged, but A_{12} and A_{13} are corrected by $a_{12} = -\rho \tan A_{12}$ and $a_{13} = -\rho \tan A_{13}$, respectively, the correction to be applied to A_{23} is given by

$$a_{23} = \rho \operatorname{cosec} A_{23} \left(2 \cos A_{23} - \frac{\cos A_{13}}{\cos A_{12}} - \frac{\cos A_{12}}{\cos A_{13}}\right) \quad \text{(25)}$$

When the direction cosines L_1, M_1, N_1, L_2, M_2, and N_2 and the space angles A_{13} and A_{23} are known, the direction cosines of line SP_3 may be found by solving the equations:

$$\cos A_{13} = L_1 L_3 + M_1 M_3 + N_1 N_3 \quad \text{(26a)}$$

$$\cos A_{23} = L_2 L_3 + M_2 M_3 + N_2 N_3 \quad \text{(26b)}$$

and

$$L^2{}_3 + M^2{}_3 + N^2{}_3 = 1 \quad \text{(26c)}$$

From Eqs. 26*a* and 26*b*, the magnitudes L_3 and M_3 may be found in terms of N_3; and, by substitution in Eq. 26*c*, a quadratic is obtained which may be solved for the root N_3.

If points P_1 and P_2 are fixed in position, if point S is moved a little to a new position having $(X + x)$, $(Y + y)$, and $(Z + z)$ as its new coordinates, and if the space angles A_{13} and A_{23} are given small corrections a_{13} and a_{23}, respectively, the corrections l_1, m_1, n_1, l_2, m_2, and n_2 may be found in terms of x, y, and z by Eqs. 16 and l_3, m_3, and n_3 may be found in terms of x, y, and z through the solution of the equations:

$$L_1 l_3 + M_1 m_3 + N_1 n_3 = -\sin A_{13}\, a_{13} - L_3 l_1 - M_3 m_1 - N_3 n_1 \quad\text{..(27a)}$$

$$L_2 l_3 + M_2 m_3 + N_2 n_3 = -\sin A_{23}\, a_{23} - L_3 l_2 - M_3 m_2 - N_3 n_2 \quad\text{..(27b)}$$

and

$$L_3 l_3 + M_3 m_3 + N_3 n_3 = 0 \quad\text{..................(27c)}$$

which are obtained by differentiating Eqs. 26.

Four Concurrent Lines.—Let the space angles between the four lines SP_1, SP_2, SP_3, and SP_4, in Fig. 3, be denoted by A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, and A_{34}. Since the dihedral angles between the planes SP_1P_2, SP_1P_3, and SP_1P_4 can be expressed in terms of the space angles by Eq. 24 and the sum or difference of two is equal to the third, by equating the cosine of the one to the cosine of the sum of the other two, it is found that

$$\cos A_{34} = \cos A_{13} \cos A_{14} + \frac{K_{123} K_{124}}{\sin^2 A_{12}} + \frac{(\cos A_{23} - \cos A_{12} \cos A_{13})(\cos A_{24} - \cos A_{12} \cos A_{14})}{\sin^2 A_{12}} \quad\text{..(28)}$$

in which

$$K_{123} = (1 + 2 \cos A_{12} \cos A_{13} \cos A_{23} - \cos^2 A_{12} - \cos^2 A_{13} - \cos^2 A_{23})^{\frac{1}{2}} \quad\text{..(29a)}$$

and

$$K_{124} = (1 + 2 \cos A_{12} \cos A_{14} \cos A_{24} - \cos A_{12} - \cos^2 A_{14} - \cos^2 A_{24})^{\frac{1}{2}} \quad\text{..(29b)}$$

The second term on the right side of Eq. 28 will have a negative sign when points P_3 and P_4 are on the opposite sides of plane SP_1P_2. The small corrections of the six space angles are connected by

$$a_{34} = c_{12}\, a_{12} + c_{13}\, a_{13} + c_{14}\, a_{14} + c_{23}\, a_{23} + c_{24}\, a_{24} \quad\text{.........(30)}$$

in which the differential coefficients are

$$c_{12} = 2 \cot A_{12} \cot A_{34} - \frac{K_{124} (\cos A_{12} - \cos A_{13} \cos A_{23})}{K_{123} \sin A_{12} \sin A_{34}} - \frac{K_{123} (\cos A_{12} - \cos A_{14} \cos A_{24})}{K_{124} \sin A_{12} \sin A_{34}} - \frac{\cos A_{14} \cos A_{23} + \cos A_{13} \cos A_{24}}{\sin A_{12} \sin A_{34}} \quad\text{..(31a)}$$

$$c_{13} = \frac{\sin A_{13} (\cos A_{14} - \cos A_{12} \cos A_{24})}{\sin^2 A_{12} \sin A_{34}} - \frac{K_{124} \sin A_{13} (\cos A_{13} - \cos A_{12} \cos A_{23})}{K_{123} \sin^2 A_{12} \sin A_{34}} \quad\text{..(31b)}$$

$$c_{14} = \frac{\sin A_{14} (\cos A_{13} - \cos A_{12} \cos A_{23})}{\sin^2 A_{12} \sin A_{34}} - \frac{K_{123} \sin A_{14} (\cos A_{14} - \cos A_{12} \cos A_{24})}{K_{124} \sin^2 A_{12} \sin A_{34}} \quad ..(31c)$$

$$c_{23} = \frac{\sin A_{23} (\cos A_{24} - \cos A_{12} \cos A_{14})}{\sin^2 A_{12} \sin A_{34}} - \frac{K_{124} \sin A_{23} (\cos A_{23} - \cos A_{12} \cos A_{13})}{K_{123} \sin^2 A_{12} \sin A_{34}} \quad ..(31d)$$

and

$$c_{24} = \frac{\sin A_{24} (\cos A_{23} - \cos A_{12} \cos A_{13})}{\sin^2 A_{12} \sin A_{34}} - \frac{K_{123} \sin A_{24} (\cos A_{24} - \cos A_{12} \cos A_{14})}{K_{124} \sin^2 A_{12} \sin A_{34}} \quad ..(31e)$$

If the angles A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, and A_{34} are determined by the coordinates of points S, P_1, P_2, P_3, and P_4, the value of A_{34}, obtained from the triangle SP_3P_4, will be identical with that determined from Eq. 28.

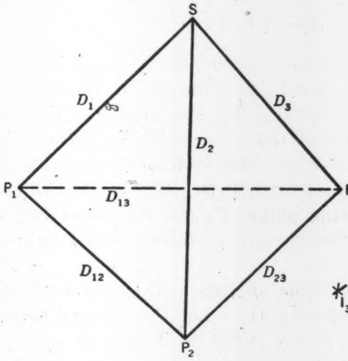

Fig. 4

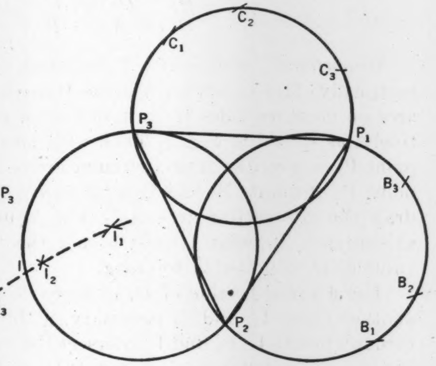

Fig. 5

If the conditional formula (Eq. 28) is satisfied, the direction cosines L_4, M_4, and N_4 of line SP_4 are determined from the values of the direction cosines in the other three lines by the simultaneous linear equations:

$$\cos A_{14} = L_1 L_4 + M_1 M_4 + N_1 N_4 \quad \ldots\ldots\ldots\ldots (32a)$$

$$\cos A_{24} = L_2 L_4 + M_2 M_4 + N_2 N_4 \quad \ldots\ldots\ldots\ldots (32b)$$

and

$$\cos A_{34} = L_3 L_4 + M_3 M_4 + N_3 N_4 \quad \ldots\ldots\ldots\ldots (32c)$$

Angles and Sides of a Tetrahedron.—The space angles of the tetrahedron at the vertex S, Fig. 4, are denoted by A_{12}, A_{13}, and A_{23}; and the lengths of the sides, SP_1, SP_2, SP_3, P_1P_2, P_1P_3, and P_2P_3, are denoted by D_1, D_2, D_3, D_{13}, D_{13}, and D_{23}, respectively. The relations between these angles and sides are

$$D^2_{12} = D^2_1 + D^2_2 - 2\,D_1\,D_2 \cos A_{12} \qquad (33a)$$

$$D^2_{13} = D^2_1 + D^2_3 - 2\,D_1\,D_3 \cos A_{13} \qquad (33b)$$

and

$$D^2_{23} = D^2_2 + D^2_3 - 2\,D_2\,D_3 \cos A_{23} \qquad (33c)$$

which are usually used to determine D_1, D_2, and D_3 from the known values of A_{12}, A_{13}, A_{23}, D_{12}, D_{13}, and D_{23}. The small corrections d_1, d_2, and d_3 of D_1, D_2, and D_3, respectively, are connected by

$$d_2 = \frac{D_1 - D_2 \cos A_{12}}{D_1 \cos A_{12} - D_2}\,d_1 \qquad (34a)$$

$$d_3 = \frac{D_1 - D_3 \cos A_{13}}{D_1 \cos A_{13} - D_3}\,d_1 \qquad (34b)$$

and

$$d_2 = \frac{D_3 - D_2 \cos A_{23}}{D_3 \cos A_{23} - D_2}\,d_3 \qquad (34c)$$

Combining Eqs. 34*b* and 34*c*,

$$d_2 = \frac{D_1 - D_3 \cos A_{13}}{D_1 \cos A_{13} - D_3} \times \frac{D_3 - D_2 \cos A_{23}}{D_3 \cos A_{23} - D_2}\,d_1 \qquad (35)$$

Geometrical Solution of a Tetrahedron.—To determine D_1, D_2, and D_3 geometrically, first construct a plane triangle $P_1P_2P_3$ to scale and draw circular arcs on its three sides to contain the vertex angles A_{12}, A_{13}, and A_{23}, respectively, as shown in Fig. 5. Then with an estimated value of D_1 as radius and point P_1 as a center draw a circular arc to intersect the circles on either side of point P_1 at points B_1 and C_1 as shown. Next, with P_2B_1 ($= P_3C_1$) as radius, draw the intersecting arcs at I_1, Fig. 5, using points P_2 and P_3, respectively, as centers. If point I_1 falls within the third circle as shown, the assumed value of D_1 is probably too long.

Use a revised value of D_1 to repeat the same operation, thus determining another point I_2, and, if necessary, a third point I_3. Let the smooth curve through points I_1, I_2, and I_3 intersect the outer arc of the third circle at point I, Fig. 5. The distances P_2I and P_3I are the required values of D_2 and D_3, respectively. With line P_2I representing the value of D_2, it is an easy matter to determine D_1. The foregoing method can be used to determine D_1, D_2, and D_3 to three decimal places.

Analytical Solution of a Tetrahedron.—The approximate value D'_1 determined by the geometrical method is used for D_1 in determining the value D'_2 of D_2 by Eq. 33*a* and the value D'_3 of D_3 by Eq. 33*b*. Then D'_3 is introduced as the value of D_3 to find the value D''_2 of D_2 by Eq. 33*c*, and finally the correction d_1 is found by

$$D'_2 + \frac{D'_1 - D'_2 \cos A_{12}}{D'_1 \cos A_{12} - D'_2}\,d_1 = D''_2 + \frac{D'_1 - D'_3 \cos A_{13}}{D'_1 \cos A_{13} - D'_3} \times \frac{D'_3 - D''_2 \cos A_{23}}{D'_3 \cos A_{23} - D''_2}\,d_1 \qquad (36)$$

Eq. 36, solved for d_1, becomes

$$d_1 = \frac{D'_2 - D''_2}{\dfrac{D'_1 - D'_3 \cos A_{13}}{D'_1 \cos A_{13} - D'_3} \times \dfrac{D'_3 - D''_2 \cos A_{23}}{D'_3 \cos A_{23} - D''_2} - \dfrac{D'_1 - D'_2 \cos A_{12}}{D'_1 \cos A_{12} - D'_2}} \tag{37}$$

The values obtainable from the foregoing process of calculation are

$$D_1 = D'_1 + d_1 \tag{38a}$$

$$D_2 = D'_2 + \frac{D'_1 - D'_2 \cos A_{12}}{D'_1 \cos A_{12} - D'_2} d_1 \tag{38b}$$

and

$$D_3 = D'_3 + \frac{D'_1 - D'_3 \cos A_{13}}{D'_1 \cos A_{13} - D'_3} d_1 \tag{38c}$$

More accurate values can be found by using the improved value of D_1 to repeat the process.

If the coordinates X_1, Y_1, Z_1, X_2, Y_2, Z_2, X_3, Y_3, and Z_3 of points P_1, P_2, and P_3, and the lengths of the sides D_1, D_2, and D_3 are known, coordinates X, Y, and Z of point S are found by solving the equations:

$$D^2_1 = (X - X_1)^2 + (Y - Y_1)^2 + (Z - Z_1)^2 \tag{39a}$$

$$D^2_2 = (X - X_2)^2 + (Y - Y_2)^2 + (Z - Z_2)^2 \tag{39b}$$

and

$$D^2_3 = (X - X_3)^2 + (Y - Y_3)^2 + (Z - Z_3)^2 \tag{39c}$$

from which it follows that

$$\begin{aligned} &2\,(X_1 - X_2)\,X + 2\,(Y_1 - Y_2)\,Y + 2\,(Z_1 - Z_2)\,Z \\ &\qquad = X^2_1 + Y^2_1 + Z^2_1 - X^2_2 - Y^2_2 - Z^2_2 - D^2_1 + D^2_2 \end{aligned} \tag{40a}$$

and

$$\begin{aligned} &2\,(X_1 - X_3)\,X + 2\,(Y_1 - Y_3)\,Y + 2\,(Z_1 - Z_3)\,Z \\ &\qquad = X^2_1 + Y^2_1 + Z^2_1 - X^2_3 - Y^2_3 - Z^2_3 - D^2_1 + D^2_3 \end{aligned} \tag{40b}$$

Thus, X and Y are found in terms of Z in the following form:

$$X = c_1 Z + k_1 \tag{41a}$$

and

$$Y = c_2 Z + k_2 \tag{41b}$$

in which c_1, c_2, k_1, and k_2 are numerical constants. Substituting the equivalents of X and Y into one of the original equations and solving it, two values of Z are obtained which correspond to two positions of point S, one above and one below the base $P_1P_2P_3$.

Refraction of Air.—According to the theory of atmospheric refraction, the total change of direction of the light ray from its position at infinity to point S_1 in the atmosphere, as shown in Fig. 6, is given approximately by

$$C_1 = (g_1 - 1) \tan \zeta_1 \tag{42a}$$

in which C_1 is the angle between the tangent line of the path of light at point S_1 and its original direction; ζ_1 is the angle between the tangent and the plumb line; and g_1 is the refractive index of air at point S_1. Similarly, at another point S_2 on the same path of light,

$$C_2 = (g_2 - 1) \tan \zeta_2 \tag{42b}$$

The change of direction of the curve from points S_1 to S_2 is equal to

$$C_{12} = (g_2 - 1) \tan \zeta_2 - (g_1 - 1) \tan \zeta_1 \tag{42c}$$

If the path of the light ray is very flat (as is actually the case), ζ_1 and ζ_2 may be considered as equal, and, therefore,

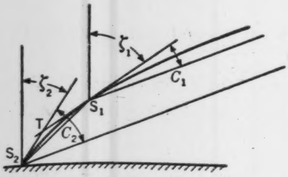

FIG. 6

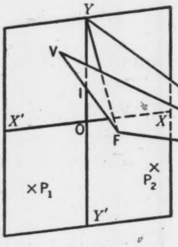

FIG. 7

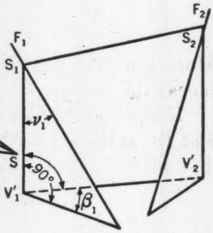

FIG. 8

$$C_{12} = (g_2 - g_1) \tan \zeta_1 \tag{42d}$$

In that case it may be reasonably assumed that the angle between the chord S_1S_2 and the tangent at point S_1 is proportional to C_{12}. Consequently,

$$\angle\, \overline{TS_1S_2} = -\rho \tan \zeta_1 \tag{43}$$

in which ρ is an unknown constant whose value mainly depends on the height of point S_1 above point S_2. The negative sign is used to account for the fact that the zenith distance of the direction of the chord S_2S_1 is smaller than ζ_1, the direction of the tangent TS_1. Since angle TS_1S_2 is always less than the astronomical refraction in numerical value, the value of ρ in minutes of arc is less than 1, if the zenith distance does not exceed 75°.

PROBLEMS IN AERIAL SURVEY

Camera Constants.—The relation between the rear nodal point and the fiducial axes on the negative plate is essentially as shown in Fig. 2. If the images I_1, I_2, and I_3 of three objects which are at equal angles from each other appear on both fiducial axes, the actual angles θ at the exposure station between the real objects are measured with a theodolite, the lengths OI_1, OI_2, and OI_3 along the fiducial axes on the negative plate can be determined accurately (corrected for distortion if necessary), and the three coordinates of the rear nodal point may be calculated by Eqs. 8 through 12. On account of the errors of measurement and distortion, the values determined from different sets of data may be slightly different.

When the coordinates of the rear nodal point are exactly known, the space angle at the node subtended by any two image points on the negative plate can be calculated if the coordinate distances of the points can be measured and corrected for distortion.

Position of Exposure Station.—When the coordinate system is so assumed that its (XY)-plane is tangent to the spheroid at a point of given latitude and longitude and the point of tangency is used as origin, the coordinates of a point defined by elevation, direction, and distance from the origin may be calculated by Eqs. 4. The straight distance between two points of known coordinates can be computed easily. If the space angles subtended at the exposure station by three points of known positions can be determined, the position of the exposure station which forms a tetrahedron with the three ground points is determined by the geometrical method or the analytical method (see subsequently in Example 1).

Refraction Correction.—Since the light ray emanating from station S_2 to the camera at station S_1, Fig. 6, follows the same path as the light ray traveling in the opposite direction, the observed direction of point S_2 from point S_1 is deviated an amount given by Eq. 43. To determine this error of refraction, the refractive constant ρ in Eq. 43 must be determined for the exposure station. As an essential requirement, at least four points, P_1, P_2, P_3, and P_4—all with known positions—must be included in one photograph. Let X, Y, and Z be the coordinates of the exposure station S calculated from the space angles A_{12}, A_{13}, and A_{23} and the known positions of points P_1, P_2, and P_3. From the coordinates of S, P_1, P_2, P_3, and P_4 and the center of curvature C of the spheroidal surface, the direction cosines of the lines SP_1, SP_2, SP_3, SP_4, and SC may be found. The angles A_1, A_2, A_3, and A_4 between the plumb line SC and the other four lines can be calculated and may be considered as their observed zenith distances, to which corrections equal to $-\rho \tan A_1$, $-\rho \tan A_2$, $-\rho \tan A_3$, and $-\rho \tan A_4$, respectively, must be added. The correction a_{12} for the space angle A_{12} due to the corrections $-\rho \tan A_1$ and $-\rho \tan A_2$ may be found in terms of ρ by Eq. 25 on the assumption that the dihedral angle between the vertical planes SCP_1 and SCP_2 is not appreciably changed by the effect of refraction. Similarly, the other corrections a_{13}, a_{14}, a_{23}, a_{24}, and a_{34} can all be found in terms of ρ (see subsequently in Example 6).

To account for the corrections a_{12}, a_{13}, and a_{23}, corrections x, y, and z must be added to the calculated coordinates X, Y, and Z of the exposure station S so that Eq. 20 is satisfied. From the three equations thus obtained, the corrections x, y, and z can be found in terms of ρ (see subsequently in Example 7).

Since the corrections of the direction cosines of the lines SP_1, SP_2, and SP_3 can be found in terms of x, y, and z and therefore in terms of ρ by Eqs. 16, and a_{14}, a_{24}, and a_{34} can all be expressed in terms of ρ, the corrections l_4, m_4, and n_4 to be applied to the direction cosines of the line SP_4 may be found in terms of ρ by the three linear equations derived from Eq. 19*b* (see subsequently in Example 8).

The perpendicular distance D_4 from point P_4, which is defined by the known coordinates X_4, Y_4, and Z_4 to line SP_4 (which line, in turn, is defined by a point with coordinates $(X + x)$, $(Y + y)$, and $(Z + z)$ and direction cosines $(L_4 + l_4)$, $(M_4 + m_4)$, and $(N_4 + n_4)$) is found in terms of ρ by Eq. 18. Because of the presence of other errors of measurement, it is not likely that $D^2{}_4 = 0$ can be satisfied by a real value of ρ; but there is always a value of ρ for which $D^2{}_4$ is minimum. Consequently,

$$\frac{d(D^2{}_4)}{d\rho} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (44a)$$

may be used to determine ρ. Similarly if several other points P_5, P_6, and P_7 (also of known positions) appear in the same photograph, the value of ρ, determined from

$$\frac{d(D^2{}_4 + D^2{}_5 + D^2{}_6 + D^2{}_7)}{d\rho} = 0 \ldots\ldots\ldots\ldots\ldots (44b)$$

should be more reliable.

Equation to Determine ρ.—Let L_4, M_4, and N_4 be the direction cosines of the line which makes angles A_{14} and A_{24} with the lines SP_1 and SP_2. When the coordinates of point S are changed to $(X + x)$, $(Y + y)$, and $(Z + z)$ and the angles A_{14} and A_{24} are changed to $(A_{14} + a_{14})$ and $(A_{24} + a_{24})$ to account for the effect of refraction, the corrections x, y, z, l_4, m_4, and n_4 may all be found in terms of ρ as illustrated subsequently by Examples 7 and 8. Since —$D^2{}_4 = (X + x - X_4)^2 + (Y + y - Y_4)^2 + (Z + z - Z_4)^2 - [(L_4 + l_4) \times (X + x - X_4) + (M_4 + m_4)(Y + y - Y_4) + (N_4 + n_4)(Z + z - Z_4)]^2$— is an expression involving the fourth power of ρ, the expression:

$$\frac{dD^2{}_4}{d\rho} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (45)$$

gives an equation of the form:

$$a + a_1\rho + a_2\rho^2 + a_3\rho^3 = 0 \ldots\ldots\ldots\ldots\ldots (46)$$

the solution of which gives the value of ρ.

Orientation of Negative Plates.—As shown in Fig. 7, point S is the rear nodal point of the camera lens; X, X′, Y, and Y′ are the fiducial marks on the negative plate, point F is the plate intercept of a perpendicular from point S, and point V is the piercing point, on the negative plate, of the plumb line SV. Since the space angles at point S subtended by points Y, P_1, and P_2 and the equations of the lines SP_1 and SP_2 are determinable, the equation of line SV is computed by Eqs. 26. Similarly, the equation of SF may be found also. The equation of the plumb line SV is determined from the calculated coordinates of point S and the coordinates of the center of curvature of the spheroidal surface.

The angles between the three lines SY, SF, and SV are found from their equations; and the angle YFV is found from these angles by Eq. 24. Angle FYY′ can be found from the coordinates of points Y and F; and, as in the case shown, the angle between the lines FV and YY′, is equal to

$$\angle\,\overline{FIY'} = \angle\,\overline{FYY'} + \angle\,\overline{YFV} \ldots\ldots\ldots\ldots\ldots (47)$$

Since the equation of the plane of SV and SF can be determined from the equations of the two lines, and the equation of the plane of the meridian through point S is found from Eq. 5, the azimuth of line SF is found from the dihedral angle between these two planes.

The coordinates of point F, the azimuth of line SF, and angles VSF and FIY′ completely define the orientation of the negative plate at the instant of exposure.

As shown in Fig. 8, points S_1 and S_2 are two exposure stations and V'_1 and V'_2 represent the feet of the plumb lines on the spheroidal surface. The elevations V'_1S_1 and V'_2S_2 can be found from the calculated coordinates of points S_1 and S_2 by Eq. 1. The azimuth of the line S_1S_2 is found from the dihedral angle between the meridian plane of point S_1 and the vertical plane containing points S_1 and S_2.

Since the equations of lines S_1S_2 and $S_1V'_1$ and the plate perpendicular S_1F_1 can be found, it is possible to compute the angle ν_1, Fig. 8, which may be considered as the vertical angle of the line of sight of the camera, and the angle $V'_1S_1S_2$. The horizontal angle β_1 is determined from the three space angles at point S_1 by Eq. 24. Similarly, the vertical and horizontal angles of the plate perpendicular S_2F_2 at point S_2 are found. These angles, together with the angles of orientation of the two negative plates, completely define their relative position in the air.

Position of Photographed Point.—The space angles at the exposure station S_1 made by a point P_n of unknown position with known points P_1, P_2, and P_3 can be determined from their images on the negative plate by Eqs. 7 and they can be corrected for atmospheric refraction by Eq. 25, when the position of station S_1 and the refractive constant ρ are known. From these space angles the direction cosines of the line S_1P_n may be determined from Eqs. 26 or 32 according to the number of known points available.

Similarly, if point P_n is also photographed at another exposure station S_2 and the equation of the line S_2P_n can be determined, the accuracy of all the data may be tested by computing the perpendicular distance between the two lines S_1P_n and S_2P_n with Eq. 21*a*. If this distance is reasonably small, its midpoint may be considered as the probable position of point P_n and its coordinates may be determined by Eqs. 23.

Four or More Points of Known Positions in One Photograph.—In this case more data are available than necessary and the method of least squares is used to determine the coordinates of the exposure station S.

The coordinates of point S are first calculated from the space angles of points P_1, P_2, and P_3. Then the lengths and direction cosines of lines SP_1, SP_2, SP_3, and SP_4 are calculated; and the small corrections a_{12}, a_{13}, and a_{23} of the space angles A_{12}, A_{13}, and A_{23} due to corrections x, y, and z which must be added to the calculated coordinates of point S are found from Eq. 20.

Let the values of angles A_{14}, A_{24}, and A_{34}, calculated from the direction cosines of the four lines, be denoted by I_{14}, I_{24}, and I_{34} and their increases in terms of x, y, and z, found by Eq. 20, be denoted by i_{14}, i_{24}, and i_{34}, respectively. The corrections a_{14}, a_{24}, and a_{34} when added to the observed values A_{14}, A_{24}, and A_{34}, respectively, must yield

$$I_{14} + i_{14} = A_{14} + a_{14} \tag{48a}$$

$$I_{24} + i_{24} = A_{24} + a_{24} \tag{48b}$$

and

$$I_{34} + i_{34} = A_{34} + a_{34} \tag{48c}$$

Therefore,

$$a_{14} = I_{14} - A_{14} + i_{14} \tag{49a}$$

$$a_{24} = I_{24} - A_{24} + i_{24} \tag{49b}$$

and

$$a_{34} = I_{34} - A_{34} + i_{34} \tag{49c}$$

which give the angular corrections a_{14}, a_{24}, and a_{34}, also, as linear functions of x, y, and z.

As it is possible to restrain the angular corrections if the sum Q, defined by the condition that—

$$Q = W_{12}\,(a_{12})^2 + W_{13}\,(a_{13})^2 + W_{23}\,(a_{23})^2 + W_{14}\,(a_{14})^2 + W_{24}\,(a_{24})^2 + W_{34}\,(a_{34})^2 \tag{50}$$

—shall be a minimum, the values of x, y, and z can be determined from the three normal equations derived from $\frac{dQ}{dx} = 0$, $\frac{dQ}{dy} = 0$, and $\frac{dQ}{dz} = 0$, respectively (see subsequently in Example 5). In Eq. 50, W is the weight of a corresponding observed value. The foregoing method can be extended to other cases involving five or six ground points of known positions.

Four or More Points of Known Positions and One or Several Points of Unknown Positions Common to Two Aerial Photographs.—Let the positions of four ground points P_1, P_2, P_3, and P_4 be given, with that of P_5 undetermined, and let the available data be composed of the space angles A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}, A_{15}, A_{25}, and A_{35} and A'_{12}, A'_{13}, A'_{14}, A'_{23}, A'_{24}, A'_{34}, A'_{15}, A'_{25}, and A'_{35} at the exposure stations S and S′, respectively. In the solution, the coordinates of points S and S′ are first computed from the given coordinates of points P_1, P_2, and P_3 by the use of the space angles at S and S′ upon these points. Then, corrections to these coordinates x, y, z, x', y', and z', respectively, and other corrections are computed as explained in the following text.

For the points of known positions, corrections a_{12}, a_{13}, a_{23}, a_{14}, a_{24}, a_{34}, a'_{12}, a'_{13}, a'_{23}, a'_{14}, a'_{24}, and a'_{34} of their space angles may be expressed in terms of x, y, and z and x', y', and z', respectively, by Eq. 20. For point P_5 (position unknown), the direction cosines of lines SP_5 and $S'P_5$ are found from their space angles fixed by three points of known positions such as P_1, P_2, and P_3 by Eqs. 32; and their corrections l_5, m_5, n_5, l'_5, m'_5, and n'_5 are also to be determined. The correction a_{15} may be expressed in terms of l_1, m_1, n_1, l_5, m_5, and n_5 by Eq. 19b; l_1, m_1, and n_1 can be obtained in terms of x, y, and z; and n_5 can be found in terms of l_5 and m_5 by Eq. 17. Therefore, a_{15} can be expressed in terms of x, y, z, l_5, and m_5. Similarly, a_{25} may be expressed in terms of the same five corrections. The correction a_{35} should be determined from a_{12}, a_{13}, a_{23}, a_{15}, and a_{25} by Eq. 30 and therefore may also be expressed in terms of the foregoing five corrections. Likewise a'_{15}, a'_{25}, and a'_{35} may be expressed in terms of x', y', z', l'_5, and m'_5.

To make the two lines SP_5 and $S'P_5$ intersect, however, corrections x, y, z, x', y', z', l_5, m_5, l'_5, and m'_5 must satisfy a conditional equation based on Eq. 22 in which n_5 and n'_5 are replaced by their equivalents derived from Eqs. 16. By this condition of intersection, one of the corrections such as m'_5 can be found in terms of the other nine.

The foregoing analysis shows that the basic corrections in this case are x, y, z, x', y', z', l_5, m_5, and l'_5 which may be determined from a set of normal equations if the sum of the weighted squares of the corrections of the space angles is a minimum. The method is general and applicable to the case with $(3 + q)$-points of known positions and u points of unknown positions; but the number of basic corrections, being equal to $3\ (2 + u)$ is dependent on u only. The method can also be modified to include such data as A_{45} and A'_{45}. If the refractive constant ρ is known, the space angles should first be corrected for refraction by Eq. 25, and then they should be used as the data to start the aforementioned method of analysis.

If (because of some special arrangement to be explained subsequently) the position of the exposure station S has been determined previously, the corrections x, y, and z may be considered equal to zero. The number of the basic corrections is then decreased by 3, thus decreasing the work of numerical calculation greatly.

Analysis of Partly Overlapping Aerial Survey.—Fig. 9 represents an aerial survey with a 60% overlap between photographs G_1, G_2, and G_3. The method of successive approximation can be applied to cases of this kind. It is assumed at the beginning that: (1) At least three points of known positions, such as P_1, P_3, and P_4, are available for the determination of the exposure station S_1 at which photograph G_1 is taken; (2) in photograph G_2 only two points of known positions such as P_3 and P_4 can be identified; and (3) each of the succeeding photographs contains at least two well-defined points in common with two previous photographs. It is also assumed that: (4) The angles of tilt are small and (5) the altimeter records are reasonably reliable. Under these assumed conditions, the exposure station may be represented approximately, in plan, by the principal point, and in elevation by the altimeter reading. If the principal point of a vertical photograph can be identified in a previous photograph by its surrounding topographic details, its position in plan is estimated or determined by the radial-line method of plotting.

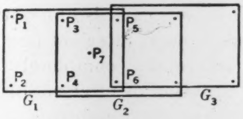

FIG. 9

From the known coordinates of points P_3 and P_4, and from the estimated coordinates of the exposure station S_2 at which photograph G_2 is taken, the direction cosines of the lines S_2P_3 and S_2P_4 and their including angle I''_{34} may be calculated. The change i''_{34} of I''_{34} due to corrections x_2, y_2, and z_2 to be added to the estimated coordinates X_2, Y_2, and Z_2 of point S_2 may be found by Eq. 20. Since the sum $(I''_{34} + i''_{34})$ must be equal to the observed value A''_{34} of the space angle at point S_2,

$$i''_{34} = A''_{34} - I''_{34} \qquad (51)$$

which is a linear equation in x_2, y_2, and z_2. If it is believed that x_2 and y_2 are relatively small because X_2 and Y_2 are more reliably determined than Z_2, Eq. 51 can be used to find z_2 by omitting the terms involving x_2 and y_2. In that case, the calculation should be begun again with X_2, Y_2, and $(Z_2 + z_2)$ as the initial values. It is essential that good initial values be obtained by a simple method so that the entire process of calculation can be shortened.

The direction cosines of line S_2P_5 may be calculated from the computed values of the direction cosines of lines S_2P_3 and S_2P_4 and the space angles A''_{35} and A''_{45} by Eqs. 26. The changes in the direction cosines of lines S_2P_3 and S_2P_4 due to the corrections x_2, y_2, and z_2, and the corrections l''_5, m''_5, and n''_5 to be applied to the calculated values of the direction cosines of line S_2P_5 must not alter the values of the space angles A''_{35} and A''_{45}. Therefore,

$$L''_5 l''_3 + L''_3 l''_5 + M''_5 m''_3 + M''_3 m''_5 + N''_5 n''_3 + N''_3 n''_5 = 0 \quad (52a)$$

$$L''_5 l''_4 + L''_4 l''_5 + M''_5 m''_4 + M''_4 m''_5 + N''_5 n''_4 + N''_4 n''_5 = 0 \quad (52b)$$

and

$$L''_5 l''_5 + M''_5 m''_5 + N''_5 n''_5 = 0 \quad (52c)$$

in which all the capital letters denote numerical constants determined from known or assumed values. Corrections l''_3, m''_3, n''_3, l''_4, m''_4, and n''_4 can be found in terms of x_2, y_2, and z_2 by Eqs. 16; therefore corrections l''_5, m''_5, and n''_5 can also be obtained in terms of x_2, y_2, and z_2 by Eqs. 16. The line S_2P_5 must intersect line S_1P_5, however; and the equation of the latter has been determined completely in the analysis of the previous survey. Hence, the equation of intersection based on Eq. 22 involves only x_2, y_2, z_2, l''_5, m''_5, and n''_5 as the unknowns; and, as the latter three corrections may be found in terms of the corrections of coordinates, the final result is again a linear equation in x_2, y_2, and z_2. Similarly, the intersection of the lines S_1P_6 and S_2P_6 gives another equation in x_2, y_2, and z_2. These three equations are used to determine the values of the three corrections.

As the errors of the initial values X_2, Y_2, and Z_2 may be very large, the corrected values $(X_2 + x_2)$, $(Y_2 + y_2)$, and $(Z_2 + z_2)$ are usually only better approximations. With these revised values of the coordinates of point S_2 as the new base for starting, the foregoing procedure is repeated (see subsequently in Example 9).

Since there may be other well-defined points such as P_7 common to photographs G_1 and G_2, and since from each common point a linear equation in x_2, y_2, and z_2 can be derived, more equations may be obtained than are required for the determination of the three unknowns. In such a case, in the second course of approximation, the method of least squares is used.

After the coordinates of point S_2 are thus calculated, the equations of the lines from point S_2 to all points in photograph G_2 may be computed, and the positions of the points common to photographs G_1 and G_2 may be determined by solving Eqs. 23.

The same method of calculation may be used for obtaining the positions of the succeeding exposure stations and points, until the survey ends or closes on another point of known position.

Aerial Triangulation.—Present practice in aerial surveying is mainly based on the use of partly overlapping (generally 60%) exposures as shown in Fig. 9. The method of computing such a survey is not straightforward although results may be obtained which are as reliable as the angular data.

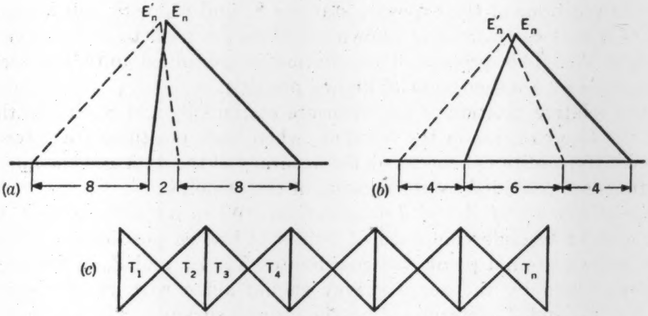

FIG. 10.—AERIAL TRIANGULATION

A fully duplicated, and also partly overlapping, aerial survey may be made with two cameras so arranged that their fields of view have a longitudinal overlap of 20% as shown in Fig. 10(*a*) or of 60% as shown in Fig. 10(*b*). When cameras arranged in such a combination are operated simultaneously by electric control so that the exposures of each camera have a 60% overlap of their own, each photograph of one camera will be nearly duplicated by some photograph of another.

Let a series of triangles $T_1, T_2, \cdots, T_n$ on a strip of ground surface (see Fig. 10(*c*)) be so spaced that adjacent pairs will appear together in the same photograph; and let the simultaneous exposures of the two cameras at the *j*th exposure station be denoted by E_j and E'_j. If triangle T_1 is composed of points of known positions and may be used to determine the position of the exposure

TABLE 1*A*.—SCHEDULE OF SEQUENCE IN THE USE OF TRIANGLES IN AERIAL TRIANGULATION COMPUTATIONS; 25% OVERLAP

Exposures	Backsights	Foresights
E_1	T_1	T_2
E'_3	T_1	T_2
E_2	T_2	T_3
E'_4	T_2	T_3
…………	…………	…………
…………	…………	…………
E_n	T_n	T_{n+1}
E'_{n+2}	T_n	T_{n+1}

TABLE 1*B*.—SCHEDULE OF SEQUENCE IN THE USE OF TRIANGLES IN AERIAL TRIANGULATION COMPUTATIONS; 60% OVERLAP

Exposures	Backsights	Foresights
E_1	T_1	T_2
E'_2	T_1	T_2
E_2	T_2	T_3
E'_3	T_2	T_3
…………	…………	…………
…………	…………	…………
E_n	T_n	T_{n+1}
E'_{n+1}	T_n	T_{n+1}

station, the two sets of exposures can be compiled as in Tables 1*A* and 1*B*, showing the sequence in which calculations are made. As in the case of a ground survey, the sights used to determine the positions of the instrument are called the "backsights," and those used to determine the points sighted are called the "foresights." In either case the backsights of triangle T_1 are used to find the positions of the exposure stations S_1 and S'_3 or S_1 and S'_2 and the foresights from two stations of known positions are used to find the vertexes of triangle T_2. This process of calculation is continued until the series of triangles ends on another point of known position.

As the relative position of the exposure stations S_n and S'_n is identical to that of the two cameras in the airplane, when their positions are determined independently, additional checks on the accuracy of the data and the numerical calculations are provided by the spacing of the cameras.

Error of Closure of Aerial Triangulation.—When an aerial survey begins from a ground triangle composed of points of known positions and ends on another known ground point, the coordinates X_a, Y_a, and Z_a of the terminal point determined by the aerial survey should agree with the more reliable values X_g, Y_g, and Z_g determined by the ground survey. The errors in three directions are: $e_x = X_a - X_g$, $e_y = Y_a - Y_g$, and $e_z = Z_a - Z_g$, respectively, and the error of closure is equal to $e_c = \sqrt{e^2_x + e^2_y + e^2_z}$. Let the distance between one of the starting points and the terminal point be D. The ratio $\frac{e}{D}$, reduced to the form $\frac{1}{N}$, will serve as a measure of the accuracy of the survey.

If it is assumed that the errors e_x, e_y, and e_z are accumulated equally from all parts of the survey, the corrections to be added to the calculated coordinates of an intermediate point should be: $\frac{-e_x D_i}{D}$, $\frac{-e_y D_i}{D}$, and $\frac{-e_z D_i}{D}$, respectively, in which D_i is the distance of the point from the starting point.

TABLE 2.—COORDINATES OF GROUND POINTS, IN METERS
(1 Meter = 3.28083 Ft)

Point	*X*	*Y*	*Z*
P_1......	2,578.45	3,876.38	68.49
P_2......	5,769.38	1,643.21	135.27
P_3......	6,043.75	5,872.15	146.73
P_4......	2,385.00	5,934.00	76.30

ILLUSTRATIVE EXAMPLES

Data.—The data in Table 2 are assumed for the purpose of making sample calculations that will illustrate the general features of the methods and offer some indication as to the accuracy of the results to be expected. The positions of two exposure stations are also assumed so that the field angles to the various ground points are, for station S_1,

Angle	Value
$P_1S_1P_2$	67° 31′
$P_1S_1P_3$	69° 47′
$P_2S_1P_3$	75° 38′
$P_1S_1P_4$	29° 08.011′
$P_2S_1P_4$	89° 53.239′
$P_3S_1P_4$	55° 47.648′

and, for station S_2,

Angle	Value
$P_1S_2P_2$	44° 04.508′
$P_1S_2P_3$	62° 05.500′
$P_1S_2P_4$	29° 24.033′
$P_2S_2P_3$	49° 10.383′
$P_2S_2P_4$	68° 52.333′

Example 1. To Find the Coordinates X_1, Y_1, *and* Z_1 *of Exposure Station* S_1.—The lengths D_1, D_2, and D_3 of lines S_1P_1, S_1P_2, and S_1P_3 must first be determined from the equations:

$$D^2_1 - 2(0.3824147)\, D_1\, D_2 + D^2_2 - 15{,}173{,}542 = 0 \qquad (53a)$$

$$D^2_1 - 2(0.3455712)\, D_1\, D_3 + D^2_3 - 15{,}997{,}524 = 0 \qquad (53b)$$

and

$$D^2_2 - 2(0.2481263)\, D_2\, D_3 + D^2_3 - 17{,}959{,}339 = 0 \qquad (53c)$$

The results are summarized in Table 3. The final values are accurate to the third decimal place. The coordinates of point S_1 (in meters) are then found to be: $X_1 = 4{,}953.549$, $Y_1 = 3{,}827.388$, and $Z_1 = 2{,}698.353$.

TABLE 3.—COMPUTATION OF LENGTHS D BY EQS. 53 (IN METERS)

Trial No.	D'_1		D'_2		D'_3		D''_2	d_1
	Assumed (1)	Corrected (2)	Computed (3)	Corrected (4)	Computed (5)	Corrected (6)	Computed (7)	Computed (8)
1...	+3,505	+3,544.60	+3,504.87	+3,465.27	+3,487.09	+3,447.09	+3,424.18	+39.6
2...	+3,545	+3,543.957	+3,463.818	+3,464.918	+3,445.691	+3,446.791	+3,466.005	− 1.043
3...	+3,543.957	+3,543.963	+3,464.916	+3,464.910	+3,446.796	+3,446.791	+3,464.904	+ 0.0057

Example 2. To Find the Elevation of Station S_1.—Let the radius of curvature of the spheroidal surface be $R = 6{,}363{,}200$ meters (1 meter = 3.28083 ft). It is obvious that the third term in Eq. 1 can be neglected in this case and that the elevation P_fS_1, therefore, is $2{,}698.353 + \dfrac{19{,}593{,}000}{6{,}365{,}900} = 2{,}701.431$ m.

TABLE 4.—COMPUTATION OF DIRECTION COSINES

Line	L	M	N
S_1P_f....	−0.0007781	−0.0006012	−0.9999995
S_1P_1.....	−0.6701815	+0.0138241	−0.7420682
S_1P_2.....	+0.2354551	−0.6303707	−0.7397255
S_1P_3.....	+0.3162945	+0.5932365	−0.7402895
S_1P_4.....	−0.6069250	+0.4977735	−0.6195673

Example 3. To Find the Direction Cosines of Lines S_1P_f, S_1P_1, S_1P_2, S_1P_3, S_1P_4, *and the Included Angles.*—The direction cosines of the plumb line S_1P_f are determined from the coordinates of station S_1, and the center of curvature ($X = 0$, $Y = 0$, and $Z = -6{,}363{,}200$). Similarly, the direction cosines of the other four lines may be found from the known coordinates (see Table 4). From the direction cosines the following space angles may be com-

puted by Eq. 19a:

Angle	Value
$P_fS_1P_1$	42° 02.893′
$P_fS_1P_2$	42° 16.518′
$P_fS_1P_3$	42° 17.720′
$P_fS_1P_4$	51° 42.172′

and, as measured to P_4,

Angle	Value
$P_1S_1P_4$	29° 08.681′
$P_2S_1P_4$	89° 54.417′
$P_3S_1P_4$	55° 48.389′

Example 4. To Find the Direction Cosines of Line S_1P_4 *from the Direction Cosines of Lines* S_1P_1, S_1P_2, *and* S_1P_3.—The direction cosines of line S_1P_4 found from the calculated values of angles $P_1S_1P_4$, $P_2S_1P_4$, and $P_3S_1P_4$ by Eqs. 32 are: $L_4 = -0.6069253$, $M_4 = +0.4977740$, and $N_4 = -0.6195673$.

TABLE 5.—DIFFERENCES BETWEEN THE ASSUMED AND COMPUTED VALUES OF ANGLES AT STATION S_1

Angle	Assumed	Computed	Error
$P_1S_1P_4$....	29° 08.011′	29° 08.681′	0° 00.670′
$P_2S_1P_4$....	89° 53.239′	89° 54.417′	0° 01.178′
$P_3S_1P_4$....	55° 47.648′	55° 48.389′	0° 00.741′

Example 5. To Find the Most Probable Position of S_1.—Table 5 demonstrates the differences between the assumed and the calculated values of the angles $P_1S_1P_4$, $P_2S_1P_4$, and $P_3S_1P_4$. If these errors are to be distributed to the six space angles at station S_1 by introducing corrections x_1, y_1, and z_1 to the calculated coordinates of station S_1, it may be found from Eq. 20 that

$$a_{12} = -0.30007\,x_1 - 0.39364\,y_1 - 0.97174\,z_1 \tag{54a}$$

$$a_{13} = -0.26216\,x_1 + 0.40511\,y_1 - 1.0170\,z_1 \tag{54b}$$

and

$$a_{23} = 0.42573\,x_1 - 0.03279\,y_1 - 1.1427\,z_1 \tag{54c}$$

and Eqs. 20 and 49 will yield

$$a_{14} = 0.670 - 0.27669\,x_1 + 0.26535\,y_1 - 0.27829\,z_1 \tag{55a}$$

$$a_{24} = 1.178 - 0.41046\,x_1 - 0.01784\,y_1 - 1.2136\,z_1 \tag{55b}$$

and

$$a_{34} = 0.741 - 0.30059\,x_1 + 0.50608\,y_1 - 0.63048\,z_1 \tag{55c}$$

The angular corrections are all reduced to minutes of arc by multiplying the results by 3,437.7, which is the number of minutes in one radian. The normal equations derived from these corrections, with equal weights, are

$$0.6786\,x_1 - 0.2203\,y_1 + 0.8364\,z_1 = 0.8916 \tag{56a}$$

$$-0.2203\,x_1 + 0.6470\,y_1 - 0.3633\,z_1 = -0.5317 \tag{56b}$$

and

$$0.8364\,x_1 - 0.3633\,y_1 + 5.2323\,z_1 = 2.0830 \tag{56c}$$

from which $x_1 = 0.9105$, $y_1 = -0.3851$, and $z_1 = 0.2258$. These are the corrections to be added algebraically to the coordinates of S_1, respectively.

Example 6. To Find the Corrections of the Space Angles for Atmospheric Refraction.—By using the calculated values of the space angles between the lines S_1P_f, S_1P_1, S_1P_2, S_1P_3, and S_1P_4, Eq. 25 yields: $a_{12} = -1.3368\ \rho$, $a_{13} = -1.3948\ \rho$, $a_{23} = -1.5523\ \rho$, $a_{14} = -0.5873\ \rho$, $a_{24} = -2.0283\ \rho$, and $a_{34} = -1.0970\ \rho$—in which ρ is in minutes.

Example 7. To Find the Corrections of the Coordinates of S_1 *in Terms of* ρ.—By equating the errors of refraction in angles P_1SP_2, $P_1S_1P_3$, and $P_2S_1P_3$ to those due to the corrections x_1, y_1, and z_1 to be added to the calculated coordinates of station S_1, three equations (in meters)—

$$0.30007\ x_1 + 0.39364\ y_1 + 0.97174\ z_1 = 1.3368\ \rho \quad \text{(57a)}$$

$$0.26216\ x_1 - 0.40511\ y_1 + 1.0170\ z_1 = 1.3945\ \rho \quad \text{(57b)}$$

and

$$-0.42573\ x_1 + 0.03279\ y_1 + 1.1427\ z_1 = 1.5523\ \rho \quad \text{(57c)}$$

—may be obtained which give: $x_1 = 0.0231\ \rho$, $y_1 = 0.0042\ \rho$, and $z_1 = 1.3669\ \rho$.

Example 8. To Find the Corrections of the Direction Cosines in Terms of ρ.—From the values of x_1, y_1, and z_1 found in Example 7 and from other known values, by Eqs. 16, the corrections of the direction cosines of S_1P_1, S_1P_2, and S_1P_3 may be determined as shown in Table 6 in which l_4, m_4, and n_4 are found from the three linear equations derived from Eq. 19*b*.

TABLE 6.—CORRECTIONS FOR DIRECTION COSINES, EXAMPLE 8

Line	l	m	n
S_1P_1.....	$+0.0001882\ \rho$	$-0.0000057\ \rho$	$-0.0001701\ \rho$
S_1P_2.....	$-0.0000752\ \rho$	$+0.0001822\ \rho$	$-0.0001792\ \rho$
S_1P_3.....	$-0.0000985\ \rho$	$-0.0001737\ \rho$	$+0.0001813\ \rho$
S_1P_4.....	$+0.0001706\ \rho$	$-0.0001457\ \rho$	$-0.0002847\ \rho$

Example 9. To Find the Coordinates of Station S_2 *from Space Angles, and the Known Positions of Points* P_1 *and* P_2.—The initial values assumed for the coordinates of station S_2 are: $X_2 = 4{,}400$, $Y_2 = 6{,}210$, and $Z_2 = 3{,}160$. With these and other known values the solution of Eq. 51 yields: $536.9\ x_2 - 2{,}160.7\ y_2 - 1{,}499.2\ z_2 = 150{,}214$, as a correction for angle $P_1S_2P_2$. Similarly, Eq. 22 yields

TABLE 7.—COORDINATES OF STATION S_2 COMPUTED FROM SPACE ANGLES

Trial No.	X_2			Y_2			Z_2		
	As-sumed (1)	Correc-tions (2)	Corrected (3)	As-sumed (4)	Correc-tions (5)	Corrected (6)	As-sumed (7)	Correc-tions (8)	Corrected (9)
1.....	+4,400	−58.5	+4,341.5	+6,210	−69.7	+6,140.3	+3,160	−19.8	+3,140.2
2.....	+4,341	+17.6	+4,358.6	+6,140	+10.4	+6,150.4	+3,140	− 8.2	+3,131.8
True value..	....	....	+4,358.00	....	....	+6,150.00	....	....	+3,132.00

$$0.1691\ x_2 + 0.1015\ y_2 + 0.5552\ z_2 = -28.02$$

and

$$0.2796\ x_2 - 0.5559\ y_2 - 0.4415\ z_2 = 31.12.$$

for making lines S_2P_3 and S_2P_4 intersect the lines S_1P_3 and S_1P_4, respectively. The solution of these three equations gives: $x_2 = -58.5$, $y_2 = -69.7$, and $z_2 = -19.8$. The foregoing results and those of a further process of approximation are shown in Table 7 which indicates a steady convergence to the true values.

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SLOPE DEFLECTION EQUATIONS FOR CURVED MEMBERS

BY KEITH T. FOWLER,[1] JUN. ASCE

SYNOPSIS

The purpose of this paper is to illustrate a rapid method of determining the slope deflection equations for curved or haunched members through the use of the neutral point concept or the column analogy. A complete derivation and three examples are included. The equations obtained are valid for any deflection or rotation confined to a single plane. No restriction is placed on the applied loads as long as they lie in the plane.

INTRODUCTION

The determination of the slope deflection equations for curved or haunched members is complicated by the fact that each structure requires a particular equation—that is, no simple expression such as the equation for a straight member with a constant moment of inertia is obtainable. The nature of the applied forces has no effect on the form of the equations. Through the use of the neutral point concept or the column analogy it is possible to derive expressions which are perfectly general—the variables being computed in separate calculations.

The method of analysis is intended primarily for structures similar to those illustrated by Fig. 1. The limiting feature of this theory is that each elastic joint requires three simultaneous equations if it is free to deflect in two directions and to rotate.

Notation.—The following letter symbols are adopted for use in this paper and for the guidance of discussers:

d = deflection terms in Eqs. 41 (special case);
E = Young's modulus of elasticity;
H = horizontal reaction; H_{FE} = fixed-end reaction;
I = rectangular moment of inertia of structural member;

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[1] Structural Engr., International Harvester Co., Chicago, Ill.

M = bending moment defined by appropriate subscript; M_{FE} = fixed-end moment; M_{AB} represents the moment reaction at joint A of member AB;

s = distances measured along the neutral axis of a member; ds = small increments of distance s;

U = bending strain energy produced by generally applied forces

$$= \int \frac{M^2\,ds}{2EI};$$

V = vertical reaction; V_{FE} = fixed-end reaction;

X = the x-translation of an abutment;

x = axis and abscissa distances about the neutral point or centroid of the analogous column;

Y = the y-translation of an abutment;

y = axis and ordinate distances about the neutral point or centroid of the analogous column; and

θ = rotation of an abutment.

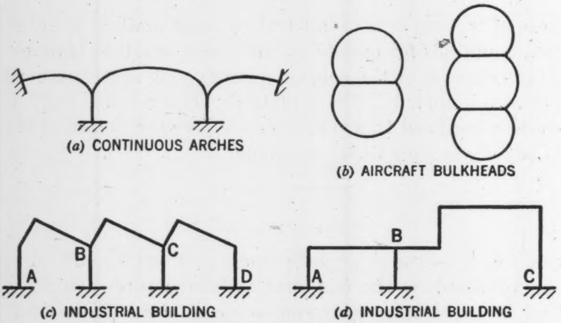


FIG. 1.—TYPICAL STRUCTURES

The sign convention used is:

(1) Deflections, horizontally to the right, vertically upward, and rotationally clockwise with regard to the reference axes, are positive; and

(2) Force reactions, acting horizontally to the right and vertically upward, and clockwise moment reactions, are positive.

GENERAL DESCRIPTION

The basic principle involved in this method is the use of distortions of the neutral point or centroid of the analogous column in expressing the moments, thrusts, and shears in terms of the joint displacements. Fig. 2 illustrates a structure incorporating the possible joint deflections and rotations.

The procedure of analysis is as follows:

(1) Divide the structure into the least possible number of individual units and compute the fixed-end reactions due to the applied forces by any convenient method. In the structure illustrated by Fig. 1(c), members AB and CD would be treated as single units; the same procedure would be applied to members AB

and BC in Fig. 1(*d*). All reactions must be expressed in agreement with the sign convention.

(2) Assume a convenient direction for the reference axes. The fixed-end reactions must be parallel to the reference axes.

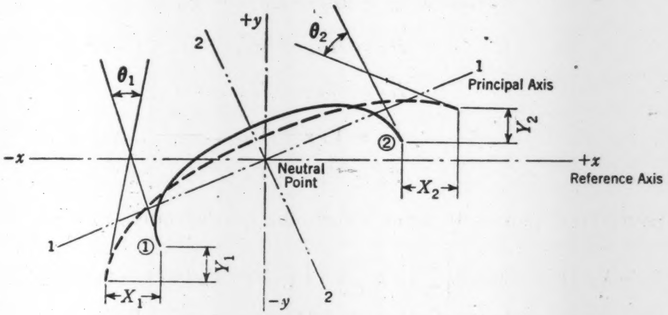

+y
θ_2
2
θ_1
1
Principal Axis
Y_2
②
-x
+x
Reference Axis
Neutral Point
X_2
1
①
Y_1
X_1
-y
2

FIG. 2.—POSSIBLE JOINT DEFLECTIONS AND ROTATIONS

(3) Compute the following terms for the centroidal reference axes of each unit (all reference axes being parallel): (*a*) $\sum \frac{ds}{E\,I} = A$; (*b*) $\sum\left(y^2\,\frac{ds}{E\,I}\right) = I_x$; (*c*) $\sum\left(x^2\,\frac{ds}{E\,I}\right) = I_y$; and (*d*) $\sum\left(x\,y\,\frac{ds}{E\,I}\right) = I_{xy}$.

(4) Write the equations for the horizontal and vertical reactions, and the moment reactions at the ends of each member in terms of the fixed-end reactions and the unknown deflections and rotations.

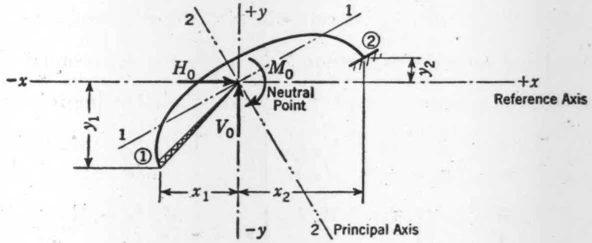

2
+y
1
②
H_0
M_0
y_2
-x
+x
Reference Axis
Neutral Point
y_1
1
V_0
①
x_1
x_2
-y
2
Principal Axis

FIG. 3.—REACTIONS ON A DEFORMED STRUCTURE

(5) Let, at each joint, $\sum M = 0, \sum H = 0, \sum V = 0$.

(6) Solve the required number of simultaneous equations for the unknown deflections and rotations at the various joints and substitute the values into the equations for the reactions acting on each member as obtained in step (4).

The theoretical derivation of the equations involved in the foregoing steps, is arranged, for simplicity, in the Appendix. For example, equations for the reactions on a deformed structure, incorporating Eqs. 26 to 34 in the Appendix, are evolved as follows: Referring to Fig. 3—

$$M_1 = M_{FE} + M_0 - H_0 y_1 + V_0 x_1 \tag{1a}$$

$$M_2 = M_{FE} - M_0 + H_0 y_2 - V_0 x_2 \tag{1b}$$

$$H_1 = H_{FE} + H_0 \tag{1c}$$

$$H_2 = H_{FE} - H_0 \tag{1d}$$

$$V_1 = V_{FE} + V_0 \tag{1e}$$

and

$$V_2 = V_{FE} - V_0 \tag{1f}$$

The neutral point forces, in terms of the unknown deflections and rotations, are

$$V_0 = \left\{(X_1 - X_2 - \theta_1 y_1 + \theta_2 y_2)\left[\sum\left(x\,y\,\frac{ds}{E\,I}\right)\right] + (Y_1 - Y_2 + \theta_1 x_1 - \theta_2 x_2)\left[\sum\left(y^2\,\frac{ds}{E\,I}\right)\right]\right\} \div \left\{\left[\sum\left(y^2\,\frac{ds}{E\,I}\right)\right]\left[\sum\left(x^2\,\frac{ds}{E\,I}\right)\right] - \left[\sum\left(x\,y\,\frac{ds}{E\,I}\right)\right]^2\right\} \tag{2a}$$

$$H_0 = \left\{(X_1 - X_2 - \theta_1 y_1 + \theta_2 y_2)\left[\sum\left(x^2\,\frac{ds}{E\,I}\right)\right] + (Y_1 - Y_2 + \theta_1 x_1 - \theta_2 x_2)\left[\sum\left(x\,y\,\frac{ds}{E\,I}\right)\right]\right\} \div \left\{\left[\sum\left(y^2\,\frac{ds}{E\,I}\right)\right]\left[\sum\left(x^2\,\frac{ds}{E\,I}\right)\right] - \left[\sum\left(x\,y\,\frac{ds}{E\,I}\right)\right]^2\right\} \tag{2b}$$

and

$$M_0 = \frac{\theta_1 - \theta_2}{\sum \frac{ds}{E\,I}} \tag{2c}$$

The inelastic bracket always extends from point 1, Fig. 3, to the neutral point. In cases where the principal axes coincide with the reference axes, Eqs. 2 are greatly simplified since the expression $\sum\left(x\,y\,\frac{ds}{E\,I}\right) = I_{xy}$ is equal to zero.

In the column analogy concept, Eqs. 2 take the following forms:

$$M_1 = M_{FE} - \left\{\frac{W}{A} + \left[\frac{M_x I_y - M_y I_{xy}}{I_x I_y - (I_{xy})^2}\right] y_1 + \left[\frac{M_y I_x - M_x I_{xy}}{I_x I_y - (I_{xy})^2}\right] x_1\right\} \tag{3a}$$

$$M_2 = M_{FE} + \left\{\frac{W}{A} + \left[\frac{M_x I_y - M_y I_{xy}}{I_x I_y - (I_{xy})^2}\right] y_2 + \left[\frac{M_y I_x - M_x I_{xy}}{I_x I_y - (I_{xy})^2}\right] x_2\right\} \tag{3b}$$

$$H_1 = H_{FE} + \left[\frac{M_x I_y - M_y I_{xy}}{I_x I_y - (I_{xy})^2}\right] \tag{3c}$$

$$H_2 = H_{FE} - \left[\frac{M_x I_y - M_y I_{xy}}{I_x I_y - (I_{xy})^2}\right] \tag{3d}$$

$$V_1 = V_{FE} - \left[\frac{M_y I_x - M_x I_{xy}}{I_x I_y - (I_{xy})^2}\right] \tag{3e}$$

and

$$V_2 = V_{FE} + \left[\frac{M_y I_x - M_x I_{xy}}{I_x I_y - (I_{xy})^2} \right] \dots\dots\dots\dots\dots (3f)$$

in which

$$W = \theta_2 - \theta_1 \dots\dots\dots\dots\dots\dots\dots\dots (4a)$$

$$M_x = X_1 - X_2 - \theta_1 y_1 + \theta_2 y_2 \dots\dots\dots\dots\dots (4b)$$

and

$$M_y = Y_2 - Y_1 - \theta_1 x_1 + \theta_2 x_2 \dots\dots\dots\dots\dots (4c)$$

The results from either the neutral point method or the column analogy method are identical. The neutral point method is used in the examples. No attempt is made to present examples which represent practicable structural designs or loadings. Three examples are included to illustrate the flexibility of the method and to show the treatment for special cases. The structures are assumed to be homogeneous, in which case E may be taken as equal to unity. The amounts of inertia of the various members are only relative.

EXAMPLE A

Example A is illustrated by Fig. 4, in which the three arches are identical. The calculation of the fixed-end reactions and of the constants for the structure is not a necessary part of this paper. Fig. 5 illustrates the dimensions of the arches, the centroid ordinates of the segment points being as follows:

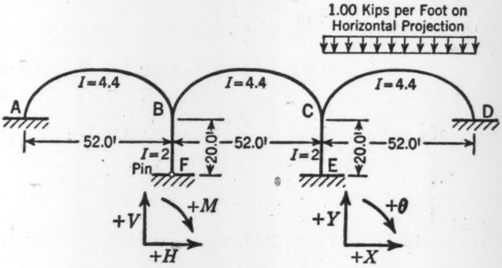


FIG. 4.—CONTINUOUS IDENTICAL ARCHES

Point	x	y
1	2.2	5.7
2	6.5	5.1
3	10.7	4.2
4	14.8	2.6
5	18.6	0.5
6	21.9	−2.3
7	24.4	−5.8
8	25.7	−10.0

In Example A, $E = 1$; $ds/I = 1$; and, by the formulas in step (3), $A = 16$; $I_x = 444.4$; $I_y = 4{,}922$; and $I_{xy} = 0$. Fig. 6 illustrates the fixed-end reactions for the arch CD (step (1)). Example A has four unknowns—X_B, X_C, θ_B, and θ_C. Vertical deflections do not enter the problem since member shortening has been neglected.

Arch AB.—Point A, Fig. 4, is designated as point 1. The values θ_A, Y_A, and Y_B are equal to zero. Substituting appropriate values in Eqs. 2,

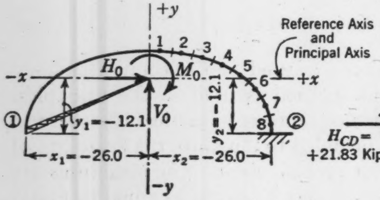

+y
-x
+x
-y
1
2
3
4
5
6
7
8
①
②
H_0
M_0
V_0
$y_1 = -12.1$
$y_2 = -12.1$
$x_1 = -26.0$
$x_2 = -26.0$
Reference Axis and Principal Axis

FIG. 5.—ARCH DIMENSIONS AND PROPERTIES

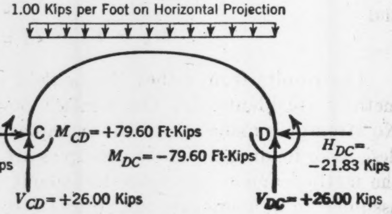

1.00 Kips per Foot on Horizontal Projection
C
D
H_{CD}= +21.83 Kips
M_{CD}= +79.60 Ft-Kips
M_{DC}= −79.60 Ft-Kips
H_{DC}= −21.83 Kips
V_{CD}= +26.00 Kips
V_{DC}= +26.00 Kips

FIG. 6.—FIXED-END REACTIONS ON ARCH CD

$$V_0 = -\frac{\theta_B x_B}{\sum\left(x^2 \frac{ds}{I}\right)} = -\frac{\theta_B (26.0)}{4{,}922.6} = -0.00529\,\theta_B \quad (5a)$$

$$H_0 = \frac{-X_B + \theta_B y_B}{\sum\left(y^2 \frac{ds}{I}\right)} = \frac{-X_B + \theta_B(-12.1)}{444.4}$$

$$= -0.00225\,X_B - 0.0272\,\theta_B \quad (5b)$$

and

$$M_0 = -\frac{\theta_B}{\sum \frac{ds}{I}} = -\frac{\theta_B}{16} = -0.0625\,\theta_B \quad (5c)$$

Then, from Eqs. 1,

$$M_{AB} = M_0 - H_0(-12.1) + V_0(-26.0) = -0.2535\,\theta_B - 0.0272\,X_B \quad (6a)$$

$$M_{BA} = -M_0 + H_0(-12.1) - V_0(26.0) = 0.5295\,\theta_B + 0.0272\,X_B \quad (6b)$$

$$H_{AB} = H_0 = -0.00225\,X_B - 0.0272\,\theta_B \quad (6c)$$

$$H_{BA} = -H_0 = 0.00225\,X_B + 0.0272\,\theta_B \quad (6d)$$

$$V_{AB} = V_0 = -0.00529\,\theta_B \quad (6e)$$

and

$$V_{BA} = -V_0 = 0.00529\,\theta_B \quad (6f)$$

Arch BC.—Point B, Fig. 4, is designated as point 1. Only the vertical deflections Y_B and Y_C are equal to zero. From the same general equations used for arch AB,

$$V_0 = \frac{\theta_B x_B - \theta_C x_C}{\sum\left(x^2 \frac{ds}{I}\right)} = \frac{\theta_B(-26.0) - \theta_C(26.0)}{4{,}922.6}$$

$$= -0.00529\,\theta_B - 0.00529\,\theta_C \quad (7a)$$

$$H_0 = \frac{X_B - X_C - \theta_B y_B + \theta_C y_C}{\sum\left(y^2 \frac{ds}{I}\right)} = \frac{X_B - X_C - \theta_B(-12.1) + \theta_C(-12.1)}{444.4}$$

$$= 0.00225\, X_B - 0.00225\, X_C + 0.0272\, \theta_B - 0.0272\, \theta_C \ldots (7b)$$

and

$$M_0 = \frac{\theta_B - \theta_C}{\sum\frac{ds}{I}} = \frac{\theta_B - \theta_C}{16} = 0.0625\, \theta_B - 0.0625\, \theta_C \ldots\ldots\ldots (7c)$$

From Eqs. 1,

$$M_{BC} = M_0 - H_0(-12.1) + V_0(-26.0)$$
$$= 0.5295\, \theta_B - 0.2535\, \theta_C + 0.0272\, X_B - 0.0272\, X_C \ldots (8a)$$

$$M_{CB} = -M_0 + H_0(-12.1) - V_0(26.0)$$
$$= -0.2535\, \theta_B + 0.5295\, \theta_C - 0.0272\, X_B + 0.0272\, X_C \ldots (8b)$$

$$H_{BC} = H_0 = 0.00225\, X_B - 0.00225\, X_C + 0.0272\, \theta_B - 0.0272\, \theta_C \ldots\ldots\ldots (8c)$$

$$H_{CB} = -H_0 = -0.00225\, X_B + 0.00225\, X_C - 0.0272\, \theta_B + 0.0272\, \theta_C \ldots (8d)$$

$$V_{BC} = V_0 = -0.00529\, \theta_B - 0.00529\, \theta_C \ldots\ldots\ldots\ldots\ldots (8e)$$

and

$$V_{CB} = -V_0 = 0.00529\, \theta_B + 0.00529\, \theta_C \ldots\ldots\ldots\ldots\ldots (8f)$$

Arch CD.—Arch CD, Fig. 4, contains fixed-end reactions which must be included in the equations. Referring to Fig. 6, Eqs. 2 (considering point C as point 1) yield

$$V_0 = \frac{\theta_C x_C}{\sum\left(x^2 \frac{ds}{I}\right)} = \frac{\theta_C(-26.0)}{4{,}922.6} = -0.00529\, \theta_C \ldots\ldots\ldots (9a)$$

$$H_0 = \frac{X_C - \theta_C y_C}{\sum\left(y^2 \frac{ds}{I}\right)} = \frac{X_C - \theta_C(-12.1)}{444.4} = 0.00225\, X_C + 0.0272\, \theta_C \ldots (9b)$$

and

$$M_0 = \frac{\theta_C}{\sum\frac{ds}{I}} = \frac{\theta_C}{16} = 0.0625\, \theta_C \ldots\ldots\ldots\ldots\ldots\ldots (9c)$$

Hence, by Eqs. 1,

$$M_{CD} = M_{FE} + M_0 - H_0(-12.1) + V_0(-26.0)$$
$$= 79.60 + 0.5295\, \theta_C + 0.0272\, X_C \ldots (10a)$$

$$M_{DC} = M_{FE} - M_0 + H_0(-12.1) - V_0(26.0)$$
$$= -79.60 - 0.2535\, \theta_C - 0.0272\, X_C \ldots (10b)$$

$$H_{CD} = H_{FE} + H_0 = 21.83 + 0.00225\, X_C + 0.0272\, \theta_C \ldots\ldots\ldots (10c)$$

$$H_{DC} = H_{FE} - H_0 = -21.83 - 0.00225\, X_C - 0.0272\, \theta_C \ldots\ldots (10d)$$

$$V_{CD} = V_{FE} + V_0 = 26.00 - 0.00529\,\theta_C \tag{10e}$$

and

$$V_{DC} = V_{FE} - V_0 = 26.00 + 0.00529\,\theta_C \tag{10f}$$

Column BF.—The equations for a vertical column with one end pinned are illustrated by Fig. 16 in the Appendix. Therefore, from Eqs. 38, also in the Appendix,

$$M_{BF} = 3\left(\frac{2}{20}\right)\left(\theta_B - \frac{X_B}{20}\right) = 0.3000\,\theta_B - 0.0150\,X_B \tag{11a}$$

$$H_{BF} = -\frac{M_{BF}}{20} = -0.0150\,\theta_B + 0.00075\,X_B \tag{11b}$$

and

$$H_{FB} = -H_{BF} = 0.0150\,\theta_B - 0.00075\,X_B \tag{11c}$$

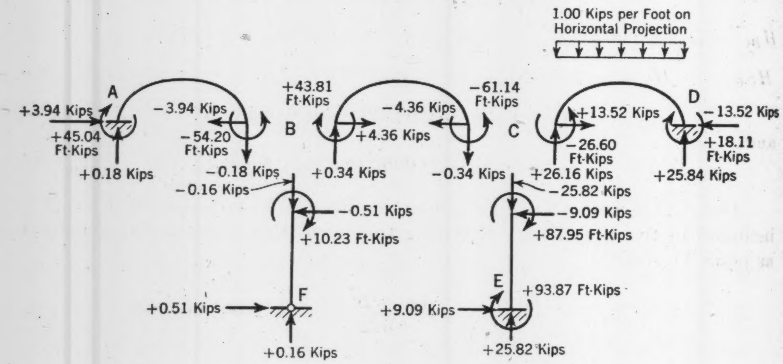

1.00 Kips per Foot on Horizontal Projection
A
+3.94 Kips
+45.04 Ft-Kips
+0.18 Kips
-3.94 Kips
-54.20 Ft-Kips
-0.18 Kips
B
+43.81 Ft-Kips
+4.36 Kips
+0.34 Kips
-4.36 Kips
-61.14 Ft-Kips
-0.34 Kips
C
+13.52 Kips
-26.60 Ft-Kips
+26.16 Kips
D
-13.52 Kips
+18.11 Ft-Kips
+25.84 Kips
-0.16 Kips
-0.51 Kips
+10.23 Ft-Kips
F
+0.51 Kips
+0.16 Kips
-25.82 Kips
-9.09 Kips
+87.95 Ft-Kips
E
+93.87 Ft-Kips
+9.09 Kips
+25.82 Kips

FIG. 7.—CONCLUDING STEPS, SOLUTION OF EXAMPLE A

Column CE.—The reactions in terms of the joint deflections and rotations for a vertical fixed-end column are expressed by Eqs. 36 in the Appendix (refer to Fig. 15):

$$M_{CE} = 2\left(\frac{2}{20}\right)\left(2\,\theta_C - \frac{3\,X_C}{20}\right) = 0.4000\,\theta_C - 0.0300\,X_C \tag{12a}$$

$$M_{EC} = 2\left(\frac{2}{20}\right)\left(\theta_C - \frac{3\,X_C}{20}\right) = 0.2000\,\theta_C - 0.0300\,X_C \tag{12b}$$

$$H_{CE} = -\frac{M_{CE} + M_{EC}}{20} = -0.0300\,\theta_C + 0.0030\,X_C \tag{12c}$$

and

$$H_{EC} = -H_{CE} = 0.0300\,\theta_C - 0.0030\,X_C \tag{12d}$$

Four unknowns are involved in the solution of Example A. The required simultaneous equations are as follows:

$$\sum M_B = 0 = 1.3590\,\theta_B - 0.2535\,\theta_C + 0.0394\,X_B - 0.0272\,X_C \quad (13a)$$

$$\sum M_C = 0 = -0.2535\,\theta_B + 1.4590\,\theta_C - 0.0272\,X_B + 0.0244\,X_C + 79.60 \quad (13b)$$

$$\sum H_B = 0 = 0.0394\,\theta_B - 0.0272\,\theta_C + 0.00525\,X_B - 0.00225\,X_C \quad (13c)$$

and

$$\sum H_C = 0 = -0.0272\,\theta_B + 0.0244\,\theta_C - 0.00225\,X_B + 0.0075\,X_C + 21.83 \quad (13d)$$

Solving Eqs. 13 simultaneously, $\theta_B = -33.25$ counterclockwise; $\theta_C = -29.62$ counterclockwise; $X_B = -1{,}345$ to the left; and $X_C = -3{,}328$ to the left. The results obtained by substituting the computed deflections and rotations into the equations for the various joint reactions are illustrated by Fig. 7.

EXAMPLE B

The structure illustrated by Fig. 8 is sometimes encountered in modern aircraft design. Before analyzing the problem it is necessary to fix the structure in space by drawing axes through one of the joints. At the fixed joint the deflections and rotation are equal to zero; and, in this example, $X_B = X_A = \theta_B = Y_B$. The linear shortening of beam AB can be neglected; so the solution to the example requires the determination of only θ_A and Y_A. Two simultaneous equations will suffice for any type of loading on the structure. The three-celled fuselage bulkhead illustrated by Fig. 1(b) would involve seven unknown quantities for any manner of loading.

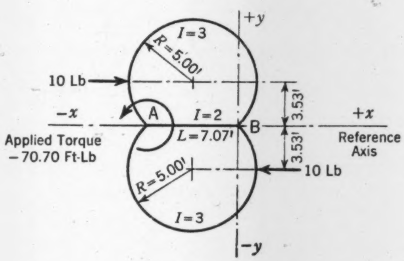


FIG. 8.—EXAMPLE B

Fig. 9 illustrates the fixed-end reactions (step (1)) and the properties of the individual units about their centroids. Assuming $E = 1$, the formulas in step (3) yield: $A = 7.87$; $I_x = 59.70$; and $I_y = 119.20$. Because the principal axes of each member are parallel to the reference axes, the term $\sum\left(x\,y\,\frac{ds}{I}\right)$ is zero in all cases.

Upper Shell AB.—From Eqs. 2, the neutral point forces, in terms of the joint rotations and deflections, are

$$V_0 = \frac{Y_A + \theta_A\,x_A}{\sum\left(x^2\,\frac{ds}{I}\right)} = \frac{Y_A + \theta_A\,(-3.53)}{119.20} = 0.00839\,Y_A - 0.0296\,\theta_A \quad (14a)$$

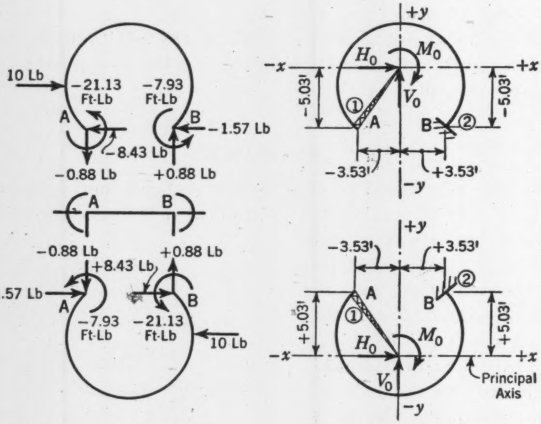
$$H_0 = \frac{-\theta_A\, y_A}{\sum\left(y^2 \dfrac{ds}{I}\right)} = \frac{-\theta_A\,(-5.03)}{59.70} = 0.08425\,\theta_A \ldots\ldots\ldots \text{(14}b\text{)}$$

and

$$M_0 = \frac{\theta_A}{\sum \dfrac{ds}{I}} = \frac{\theta_A}{7.87} = 0.1270\,\theta_A \ldots\ldots\ldots\ldots\ldots\ldots \text{(14}c\text{)}$$

Hence, from Eqs. 1,

$$\begin{aligned} M_{AB} &= -\,21.13 + M_0 - H_0\,(-5.03) + V_0\,(-3.53) \\ &= -\,21.13 - 0.0296\,Y_A + 0.6551\,\theta_A \ldots \text{(15}a\text{)} \end{aligned}$$


10 Lb
-21.13 Ft-Lb
-7.93 Ft-Lb
A
B
-1.57 Lb
-8.43 Lb
-0.88 Lb
+0.88 Lb
A
B
-0.88 Lb
+0.88 Lb
+8.43 Lb
+1.57 Lb
A
B
-7.93 Ft-Lb
-21.13 Ft-Lb
10 Lb
+y
-x
+x
H_0
M_0
V_0
-5.03'
-5.03'
①
A
B
②
-3.53'
+3.53'
-y
+y
-3.53'
+3.53'
②
A
B
①
+5.03'
+5.03'
H_0
M_0
-x
+x
V_0
Principal Axis
-y

(a) FIXED-END REACTIONS (b) ELASTIC PROPERTIES

FIG. 9.—STEP 1, EXAMPLE B

$$\begin{aligned} M_{BA} &= -\,7.93 - M_0 + H_0\,(-5.03) - V_0\,(3.53) \\ &= -\,7.93 - 0.0296\,Y_A - 0.4465\,\theta_A \ldots \text{(15}b\text{)} \end{aligned}$$

$$H_{AB} = -\,8.43 + H_0 = -\,8.43 + 0.08425\,\theta_A \ldots\ldots\ldots \text{(15}c\text{)}$$

$$H_{BA} = -\,1.57 - H_0 = -\,1.57 - 0.08425\,\theta_A \ldots\ldots\ldots \text{(15}d\text{)}$$

$$V_{AB} = -\,0.88 + V_0 = -\,0.88 + 0.00839\,Y_A - 0.0296\,\theta_A \ldots \text{(15}e\text{)}$$

and

$$V_{BA} = +\,0.88 - V_0 = 0.88 - 0.00839\,Y_A + 0.0296\,\theta_A \ldots\ldots \text{(15}f\text{)}$$

Lower Shell AB.—The neutral point forces and moment, from Eqs. 4, are

$$V_0 = \frac{Y_A + \theta_A\, x_A}{\sum\left(x^2 \dfrac{ds}{I}\right)} = \frac{Y_A + \theta_A\,(-3.53)}{119.20} = 0.00839\,Y_A - 0.0296\,\theta_A \ldots \text{(16}a\text{)}$$

$$H_0 = \frac{-\theta_A\, y_A}{\sum\left(y^2\,\dfrac{ds}{I}\right)} = \frac{-\theta_A\,(5.03)}{59.70} = -\,0.08425\,\theta_A \ldots\ldots\ldots(16b)$$

and

$$M_0 = \frac{\theta_A}{\sum \dfrac{ds}{I}} = \frac{\theta_A}{7.87} = 0.1270\,\theta_A \ldots\ldots\ldots\ldots\ldots\ldots(16c)$$

Then (by Eqs. 1),

$$M_{AB} = -\,7.93 + M_0 - H_0\,(5.03) + V_0\,(-3.53)$$
$$= -\,7.93 - 0.0296\,Y_A + 0.6551\,\theta_A \ldots(17a)$$

$$M_{BA} = -\,21.13 - M_0 + H_0\,(5.03) - V_0\,(3.53)$$
$$= -\,21.13 - 0.0296\,Y_A - 0.4465\,\theta_A \ldots(17b)$$

$$H_{AB} = 1.57 + H_0 = 1.57 - 0.08425\,\theta_A \ldots\ldots\ldots\ldots(17c)$$

$$H_{BA} = 8.43 - H_0 = 8.43 + 0.08425\,\theta_A \ldots\ldots\ldots\ldots(17d)$$

$$V_{AB} = -\,0.88 + V_0 = -\,0.88 + 0.00839\,Y_A - 0.0296\,\theta_A \ldots(17e)$$

and

$$V_{BA} = 0.88 - V_0 = 0.88 - 0.00839\,Y_A + 0.0296\,\theta_A \ldots\ldots\ldots(17f)$$

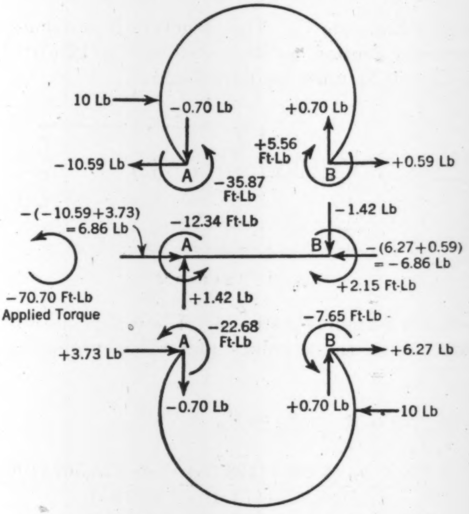

10 Lb
−0.70 Lb
+0.70 Lb
+5.56 Ft-Lb
−10.59 Lb
A
B
+0.59 Lb
−35.87 Ft-Lb
−(−10.59+3.73) =6.86 Lb
−12.34 Ft-Lb
−1.42 Lb
A
B
−(6.27+0.59) = −6.86 Lb
+2.15 Ft-Lb
−70.70 Ft-Lb
Applied Torque
+1.42 Lb
−7.65 Ft-Lb
−22.68 Ft-Lb
A
B
+3.73 Lb
+6.27 Lb
−0.70 Lb
+0.70 Lb
10 Lb

FIG. 10.—CONCLUDING STEPS, SOLUTION OF EXAMPLE B

Beam AB.—Eqs. 35 (as illustrated by Fig. 15) supply the necessary relationships for the computation of

$$M_{AB} = 2\left(\frac{2}{7.07}\right)\left(2\,\theta_A - \frac{3\,Y_A}{7.07}\right) = 1.131\,\theta_A - 0.240\,Y_A \ldots(18a)$$

$$M_{BA} = 2\left(\frac{2}{7.07}\right)\left(\theta_A - \frac{3\,Y_A}{7.07}\right) = 0.566\,\theta_A - 0.240\,Y_A \ldots\ldots (18b)$$

$$V_{AB} = -\left(\frac{M_{AB} + M_{BA}}{7.07}\right) = -\,0.240\,\theta_A + 0.068\,Y_A \ldots\ldots (18c)$$

and

$$V_{BA} = -\,V_{AB} = 0.240\,\theta_A - 0.068\,Y_A \ldots\ldots\ldots\ldots\ldots\ldots\ldots (18d)$$

The conditions for equilibrium which supply the necessary simultaneous equations are: $\sum M_A = -\,70.70$ and $\sum V_A = 0$. Also, $\sum M_B = 0$; $\sum V_B = 0$; and $\sum H_A + \sum H_B = 0$. Hence,

$$\sum M_A = 0 = 2.4412\,\theta_A - 0.2992\,Y_A + 41.64 \ldots\ldots\ldots\ldots (19a)$$

and

$$\sum V_A = 0 = -\,0.2992\,\theta_A + 0.08478\,Y_A - 1.76 \ldots\ldots\ldots (19b)$$

Solving Eqs. 19 simultaneously $\theta_A = -\,25.59$ and $Y_A = -\,69.45$ ($X_A = X_B = \theta_B = Y_B = 0$).

Fig. 10 illustrates the reactions at the ends of the members produced by the applied forces.

EXAMPLE C

Fig. 11 illustrates Example C. The structure is conveniently subdivided into two frames and one column, as illustrated by Fig. 12(*a*). Hence, only the unknown values X_B and θ_B must be determined.

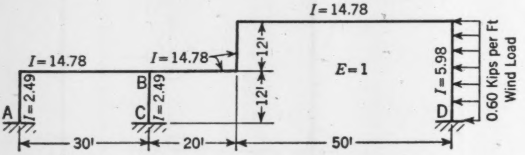


FIG. 11.—EXAMPLE C

Frame AB.—By the formulas in step (3): $A = 6.85$; $I_x = 109.22$; $I_y = 473.6$; and $I_{xy} = 128.55$. With these values and other properties in Fig. 12(*b*), Eqs. 2 yield:

$$V_0 = \frac{(-\,X_B + \theta_B\,y_B)I_{xy} + (-\,\theta_B\,x_B)I_x}{I_x\,I_y - (I_{xy})^2}$$

$$= \frac{[-\,X_B + \theta_B\,(4.23)]\,(128.55) - \theta_B\,(25.56)\,(109.22)}{(109.22)\,(473.6) - (128.55)^2}$$

$$= -\,0.0637\,\theta_B - 0.00364\,X_B \ldots (20a)$$

$$H_0 = \frac{(-\,X_B + \theta_B\,y_B)\,I_y + (-\,\theta_B\,x_B)\,I_{xy}}{I_x\,I_y - (I_{xy})^2}$$

$$= \frac{[-\,X_B + \theta_B\,(4.23)]\,(473.6) - \theta_B\,(25.56)\,(128.55)}{(109.22)\,(473.6) - (128.55)^2}$$

$$= -\,0.0363\,\theta_B - 0.01342\,X_B \ldots (20b)$$

and

$$M_0 = \frac{-\theta_B}{A} = \frac{-\theta_B}{6.85} = -0.146\,\theta_B \ldots\ldots\ldots\ldots\ldots (20c)$$

Therefore,

$$M_{AB} = M_0 - H_0\,(-7.77) + V_0\,(-4.44) = -0.145\,\theta_B - 0.0881\,X_B \ldots (21a)$$

$$M_{BA} = -M_0 + H_0\,(4.23) - V_0\,(25.56) = 1.6204\,\theta_B + 0.0362\,X_B \ldots (21b)$$

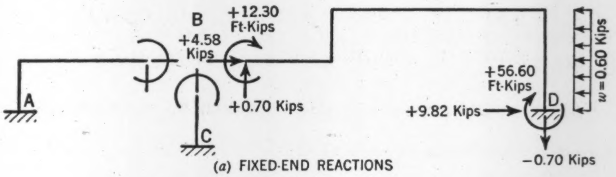
$$H_{AB} = H_0 = -0.0363\,\theta_B - 0.01342\,X_B \ldots\ldots\ldots\ldots (21c)$$

$$H_{BA} = -H_0 = 0.0363\,\theta_B + 0.01342\,X_B \ldots\ldots\ldots\ldots (21d)$$

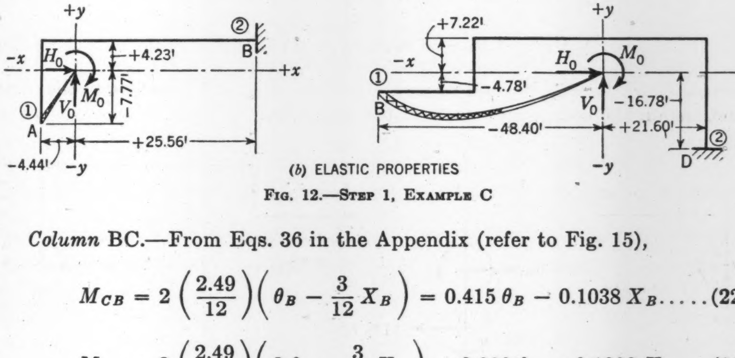
$$V_{AB} = V_0 = -0.0637\,\theta_B - 0.00364\,X_B \ldots\ldots\ldots\ldots (21e)$$

and

$$V_{BA} = -V_0 = 0.0637\,\theta_B + 0.00364\,X_B \ldots\ldots\ldots\ldots (21f)$$


B
+12.30
Ft-Kips
+4.58
Kips
+0.70 Kips
A
C
+56.60
Ft-Kips
D
+9.82 Kips
−0.70 Kips
w = 0.60 Kips

(a) FIXED-END REACTIONS


+y
②
B
−x
+x
H_0
M_0
+4.23'
−7.77'
①
V_0
A
+25.56'
−4.44'
−y
+7.22'
−x
①
B
−4.78'
H_0
M_0
V_0
+y
+x
−16.78'
−48.40'
+21.60'
②
D
−y

(b) ELASTIC PROPERTIES

Fig. 12.—Step 1, Example C

Column BC.—From Eqs. 36 in the Appendix (refer to Fig. 15),

$$M_{CB} = 2\left(\frac{2.49}{12}\right)\left(\theta_B - \frac{3}{12}X_B\right) = 0.415\,\theta_B - 0.1038\,X_B \ldots\ldots (22a)$$

$$M_{BC} = 2\left(\frac{2.49}{12}\right)\left(2\,\theta_B - \frac{3}{12}X_B\right) = 0.830\,\theta_B - 0.1038\,X_B \ldots (22b)$$

and

$$H_{BC} = -H_{CB} = -\left(\frac{M_{CB} + M_{BC}}{12}\right) = -0.1038\,\theta_B + 0.0173\,X_B \ldots (22c)$$

Frame BD.—By Eqs. 2, the neutral point reactions in terms of the unknown joint rotation and deflection are

$$V_0 = \frac{[X_B - \theta_B\,(-4.78)]\,(-277.6) + \theta_B\,(-48.40)\,(501.77)}{(501.77)\,(5{,}320.3) - (-277.6)^2}$$

$$= -0.0099\,\theta_B - 0.0001075\,X_B \ldots (23a)$$

$$H_0 = \frac{[X_B - \theta_B\,(-4.78)]\,(5{,}320.3) + \theta_B\,(-48.40)\,(-277.6)}{(501.77)\,(5{,}320.3) - (-277.6)^2}$$
$$= 0.01505\,\theta_B + 0.00206\,X_B \quad (23b)$$

and

$$M_0 = \frac{\theta_B}{9.56} = 0.1045\,\theta_B \quad (23c)$$

Hence, by Eqs. 1,

$$M_{BD} = 12.3 + M_0 - H_0\,(-4.78) + V_0\,(-48.40)$$
$$= 12.3 + 0.6555\,\theta_B + 0.01506\,X_B \quad (24a)$$

$$M_{DB} = 56.6 - M_0 + H_0\,(-16.78) - V_0\,(21.60)$$
$$= 56.6 - 0.1427\,\theta_B - 0.0322\,X_B \quad (24b)$$

$$H_{BD} = 4.58 + H_0 = 4.58 + 0.01505\,\theta_B + 0.00206\,X_B \quad (24c)$$

$$H_{DB} = 9.82 - H_0 = 9.82 - 0.01505\,\theta_B - 0.00206\,X_B \quad (24d)$$

$$V_{BD} = 0.70 + V_0 = 0.70 - 0.0099\,\theta_B - 0.0001075\,X_B \quad (24e)$$

and

$$V_{DB} = -\,0.70 - V_0 = -\,0.70 + 0.0099\,\theta_B + 0.0001075\,X_B \quad (24f)$$

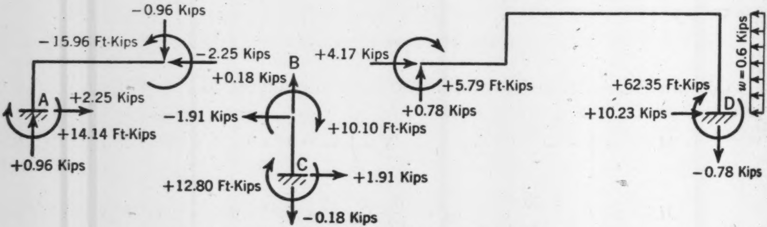
The required simultaneous equations are

$$\sum M_B = 12.3 + 3.1059\,\theta_B - 0.0525\,X_B = 0 \quad (25a)$$

and

$$\sum H_B = 4.58 - 0.0524\,\theta_B + 0.03278\,X_B = 0 \quad (25b)$$

Therefore, $\theta_B = -\,6.50$ and $X_B = -\,149.5$. Fig. 13 illustrates the final reactions on the component parts of the structure.



−0.96 Kips
−15.96 Ft-Kips
−2.25 Kips
B
+4.17 Kips
+0.18 Kips
+5.79 Ft-Kips
+62.35 Ft-Kips
w=0.6 Kips
A
+2.25 Kips
−1.91 Kips
+0.78 Kips
+10.23 Kips
D
+14.14 Ft-Kips
+10.10 Ft-Kips
+0.96 Kips
C
+12.80 Ft-Kips
+1.91 Kips
−0.78 Kips
−0.18 Kips

FIG. 13.—CONCLUDING STEPS, SOLUTION OF EXAMPLE C

CONCLUSION

A method has been presented which is relatively convenient for analyzing continuous frames incorporating curved members. The extent of its application is dependent on the ability of the engineer to solve multiple simultaneous equations. Considerable accuracy is required if large numbers of equations are involved. (The method of iteration, which is in reality a successive approximation procedure, is particularly convenient for solving multiple simultaneous equations rapidly.)

Some of the features of the procedure may be summarized as follows:

1. No restriction is placed on the applied loading as long as it lies in the plane of the structure.
2. The equations automatically produce results which indicate the correct directions for the reactions. This feature is particularly convenient when using the equations to determine moment distribution constants.
3. Several loading conditions may be analyzed without altering the form of the equations—the fixed-end reactions being treated as separate steps.
4. When the structure is homogeneous, relative values of the moments of inertia and a modulus of elasticity equal to unity will suffice for all calculations.
5. As with the usual slope deflection solutions, the results are self-checking.

ACKNOWLEDGMENT

In the preparation of this paper L. E. Grinter, M. ASCE, research professor of civil engineering and mechanics at the Illinois Institute of Technology in Chicago, made many valuable suggestions as to content and manner of presentation and R. L. Stevens, Assoc. M. ASCE, professor of bridge and structural engineering at the Illinois Institute of Technology, checked the validity of the basic equations.

APPENDIX

The neutral point concept and the column analogy have been presented in several forms in engineering literature. Few of the available references on the subject discuss the effects of distorting the position of the neutral point, which is of prime importance in this paper. Deflections and rotations at the joints of a structure produce a considerable change in the reactions. This Appendix is concerned with the values added to or subtracted from the fixed-end reactions of a structure by changes in the relative position of the supports.

Let point 1, Fig. 14(*a*) move parallel to the x–x reference axis a distance X_0 relative to point 2. Deflections to the right are positive. No joint rotation or vertical deflection is permitted. Then, from Fig. 14(*a*),

$$X_0 = X_1 - X_2 \quad \text{(26)}$$

Fig. 14(*b*) illustrates the case in which point 1 deflects a vertical distance Y_0 relative to point 2, joint rotations and horizontal deflections being prevented. Considering deflections in the vertically upward direction as positive,

$$Y_0 = Y_1 - Y_2 \quad \text{(27)}$$

The final group of separate deflection studies is illustrated by Fig. 14(*c*), in which point 1 is rotated through the angle θ_0 relative to point 2. Clockwise rotations are positive. From Fig. 14(*c*),

$$\theta_0 = \alpha - \beta = \theta_1 - \theta_2 \quad \text{(28a)}$$

$$X_0 = -\theta_1 y_1 + \theta_2 y_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28b)$$

and

$$Y_0 = \theta_1 x_1 - \theta_2 x_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28c)$$

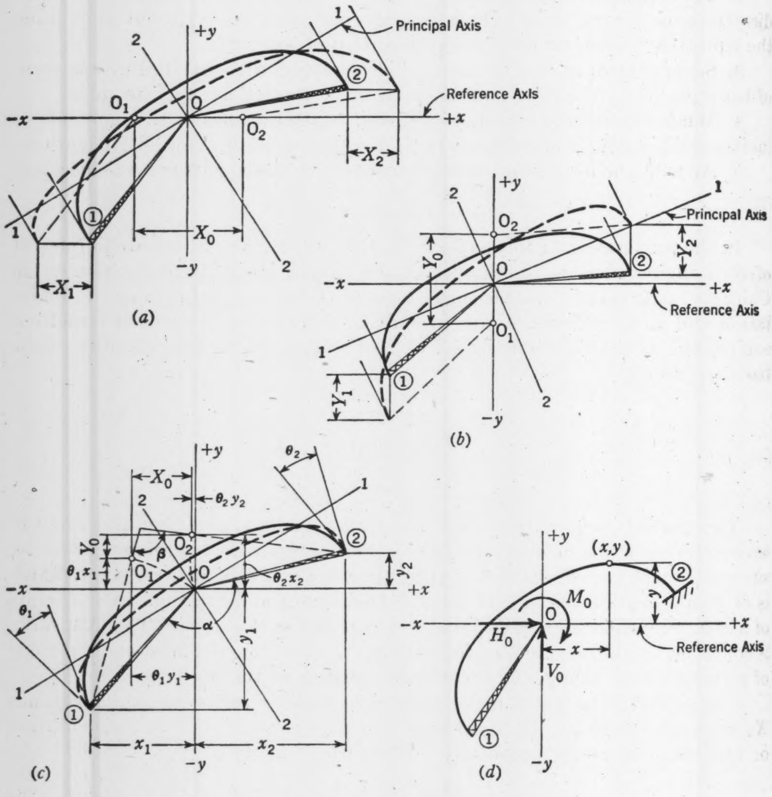

Principal Axis
Reference Axis
+y
-y
-x
+x
1
2
O
O_1
O_2
①
②
X_0
X_1
X_2
(a)
Principal Axis
Reference Axis
Y_0
Y_1
Y_2
(b)
θ_2
$\theta_2 y_2$
$\theta_1 x_1$
$\theta_2 x_2$
β
α
θ_1
$\theta_1 y_1$
y_1
y_2
x_1
x_2
(c)
(x,y)
M_0
H_0
V_0
x
y
Reference Axis
(d)

FIG. 14.—DERIVATION OF SLOPE DEFLECTION EQUATIONS

By combining Eqs. 26 through 28, the following general expressions are obtained:

$$\theta_0 = \theta_1 - \theta_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29a)$$

$$X_0 = X_1 - X_2 - \theta_1 y_1 + \theta_2 y_2 \ldots\ldots\ldots\ldots\ldots (29b)$$

and

$$Y_0 = Y_1 - Y_2 + \theta_1 x_1 - \theta_2 x_2 \ldots\ldots\ldots\ldots\ldots (29c)$$

If the joints of a structure are not rigidly held in position while the structure is subjected to loading, the axial and shear strain energies can be neglected without incorporating a significant error. The bending strain energy produced by gradually applied forces is expressed as U and equals

$$U = \int \frac{M^2\,ds}{2\,E\,I} \quad \text{(30)}$$

in which M is the bending moment at any point in the structure. Fig. 14(d) shows that the moment at any point (x,y) is equal to

$$M = M_0 + V_0\,x - H_0\,y \quad \text{(31)}$$

The neutral point forces are positive as shown. Hence,

$$U = \int_1^2 \frac{(M_0 + V_0\,x - H_0\,y)^2\,ds}{2\,E\,I} \quad \text{(32)}$$

It then follows that

$$X_0 = \frac{\partial U}{\partial H_0} = \int_1^2 \frac{(M_0 + V_0\,x - H_0\,y)\,(-\,y)\,ds}{E\,I}$$

$$= -\,V_0 \sum \left(x\,y\,\frac{ds}{E\,I}\right) + H_0 \sum \left(y^2\,\frac{ds}{E\,I}\right)$$

$$= -\,V_0\,I_{xy} + H_0\,I_x \quad \text{(33}a\text{)}$$

$$Y_0 = \frac{\partial U}{\partial V_0} = \int_1^2 \frac{(M_0 + V_0\,x - H_0\,y)\,(x)\,ds}{E\,I}$$

$$= V_0 \sum \left(x^2\,\frac{ds}{E\,I}\right) - H_0 \sum \left(x\,y\,\frac{ds}{E\,I}\right) = V_0\,I_y - H_0\,I_{xy} \quad \text{(33}b\text{)}$$

and

$$\theta_0 = \frac{\partial U}{\partial M_0} = \int_1^2 \frac{(M_0 + V_0\,x - H_0\,y)\,(1)\,ds}{E\,I} = M_0 \sum \frac{ds}{E\,I} = M_0\,A \quad \text{(33}c\text{)}$$

About any pair of centroidal axes the terms $\sum\left(y\,\frac{ds}{E\,I}\right)$ and $\sum\left(x\,\frac{ds}{E\,I}\right)$ are equal to zero. In cases where the reference axes coincide with the principal axes, the term $\sum\left(x\,y\,\frac{ds}{E\,I}\right)$ is also equal to zero.

Solving Eq. 33a and Eq. 33b simultaneously,

$$V_0 = \frac{X_0\,I_{xy} + Y_0\,I_x}{I_x\,I_y - (I_{xy})^2} \quad \text{(34}a\text{)}$$

and

$$H_0 = \frac{X_0\,I_y + Y_0\,I_{xy}}{I_x\,I_y - (I_{xy})^2} \quad \text{(34}b\text{)}$$

Also, from Eq. 33c,

$$M_0 = \frac{\theta_0}{A} \quad \text{(34}c\text{)}$$

Eqs. 1 through 4 in the paper incorporate Eqs. 34 into expressions for the reactions on a deformed structure. The general equations reduce to simple formulas when applied to a straight member with a constant moment of inertia if the member is parallel to one of the reference axes. In Fig. 15(*a*) a horizontal straight member that was initially fixed at the ends,

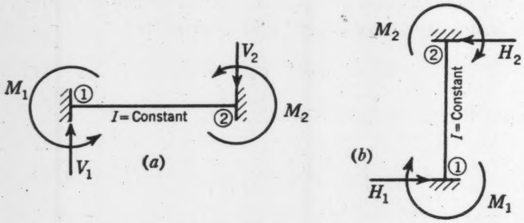


FIG. 15.—STRAIGHT MEMBER, INITIALLY FIXED AT THE ENDS

$$M_1 = M_{FE} + 2\,E\left(\frac{I}{L}\right)\left(2\,\theta_1 + \theta_2 - \frac{3\,Y_1}{L} + \frac{3\,Y_2}{L}\right) \ldots\ldots (35a)$$

$$M_2 = M_{FE} + 2\,E\left(\frac{I}{L}\right)\left(2\,\theta_2 + \theta_1 - \frac{3\,Y_1}{L} + \frac{3\,Y_2}{L}\right) \ldots\ldots (35b)$$

$$V_1 = V_{FE} - \left(\frac{M_1 + M_2 - \sum M_{FE}}{L}\right) \ldots\ldots\ldots\ldots (35c)$$

and

$$V_2 = V_{FE} + \left(\frac{M_1 + M_2 - \sum M_{FE}}{L}\right) \ldots\ldots\ldots\ldots (35d)$$

In Fig. 15(*b*), a vertical straight member that was initially fixed at the ends,

$$M_1 = M_{FE} + 2\,E\left(\frac{I}{L}\right)\left(2\,\theta_1 + \theta_2 + \frac{3\,X_1}{L} - \frac{3\,X_2}{L}\right) \ldots\ldots (36a)$$

$$M_2 = M_{FE} + 2\,E\left(\frac{I}{L}\right)\left(2\,\theta_2 + \theta_1 + \frac{3\,X_1}{L} - \frac{3\,X_2}{L}\right) \ldots\ldots (36b)$$

$$H_1 = H_{FE} + \left(\frac{M_1 + M_2 - \sum M_{FE}}{L}\right) \ldots\ldots\ldots\ldots (36c)$$

and

$$H_2 = H_{FE} - \left(\frac{M_1 + M_2 - \sum M_{FE}}{L}\right) \ldots\ldots\ldots\ldots (36d)$$

In Fig. 16(*a*), a vertical straight member with its upper end pinned,

$$M_1 = M_{FE} + 3\,E\left(\frac{I}{L}\right)\left(\theta_1 + \frac{X_1}{L}\right) \ldots\ldots\ldots\ldots (37a)$$

$$H_1 = H_{FE} + \left(\frac{M_1 - M_{FE}}{L}\right) \qquad \text{(37b)}$$

and

$$H_2 = H_{FE} - \left(\frac{M_1 - M_{FE}}{L}\right) \qquad \text{(37c)}$$

In Fig. 16(*b*), a vertical straight member with its lower end pinned,

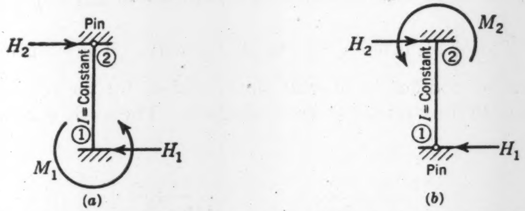

Pin
H_2
②
I = Constant
①
H_1
M_1
(a)
M_2
H_2
②
I = Constant
①
H_1
Pin
(b)

FIG. 16.—EQUATIONS FOR VERTICAL STRAIGHT MEMBERS WITH ONE END PINNED

$$M_2 = M_{FE} + 3\,E\left(\frac{I}{L}\right)\left(\theta_2 - \frac{X_2}{L}\right) \qquad \text{(38a)}$$

$$H_2 = H_{FE} - \left(\frac{M_2 - M_{FE}}{L}\right) \qquad \text{(38b)}$$

and

$$H_1 = H_{FE} + \left(\frac{M_2 - M_{FE}}{L}\right) \qquad \text{(38c)}$$

In Fig. 17(*a*), a horizontal straight member with its left end pinned,

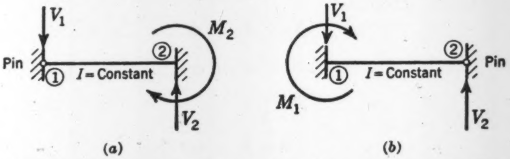

V_1
M_2
Pin
②
①
I = Constant
V_2
(a)
V_1
②
Pin
①
I = Constant
M_1
V_2
(b)

FIG. 17.—EQUATIONS FOR HORIZONTAL STRAIGHT MEMBERS WITH ONE END PINNED

$$M_2 = M_{FE} + 3\,E\left(\frac{I}{L}\right)\left(\theta_2 + \frac{Y_2}{L}\right) \qquad \text{(39a)}$$

$$V_2 = V_{FE} + \left(\frac{M_2 - M_{FE}}{L}\right) \qquad \text{(39b)}$$

and

$$V_1 = V_{FE} - \left(\frac{M_2 - M_{FE}}{L}\right) \qquad \text{(39c)}$$

and, in Fig. 17(*b*), a horizontal straight member with its right end pinned,

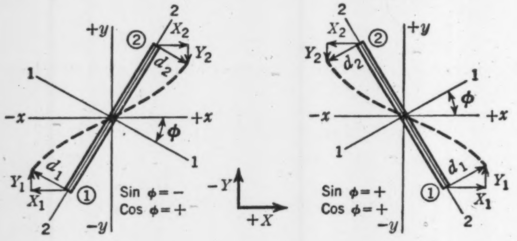
$$M_1 = M_{FE} + 3\,E\left(\frac{I}{L}\right)\left(\theta_1 - \frac{Y_1}{L}\right) \qquad \text{(40a)}$$

$$V_1 = V_{FE} - \left(\frac{M_1 - M_{FE}}{L}\right) \quad \text{(40b)}$$

and

$$V_2 = V_{FE} + \left(\frac{M_1 - M_{FE}}{L}\right) \quad \text{(40c)}$$

For an inclined straight member, the expression, $\sum\left(y^2 \frac{ds}{EI}\right)\sum\left(x^2 \frac{ds}{EI}\right) - \left[\sum\left(x\,y \frac{ds}{EI}\right)\right]^2$, is often a very small quantity. The equations illustrated in Fig. 15 can be modified to provide the equations for the reactions on such a member. Fig. 18 illustrates two possible cases. The angle ϕ is measured from



(a) SIN φ=− AND COS φ=+ (b) SIN φ=+ AND COS φ=+

FIG. 18.—INCLINED STRAIGHT MEMBERS

the horizontal reference axis to the major principal axis. The clockwise direction is positive. Hence,

$$X = d \cos \phi \quad \text{(41a)}$$

$$Y = d \sin \phi \quad \text{(41b)}$$

and

$$d = \frac{X}{\cos \phi} = \frac{Y}{\sin \phi} \quad \text{(41c)}$$

The deflection terms d_1 and d_2 replace the terms Y_1, Y_2, X_1, and X_2 in Eqs. 35 and 36. For example, Eq. 36a can be modified as follows:

$$\begin{aligned} M_1 &= M_{FE} + 2\,E\left(\frac{I}{L}\right)\left(2\,\theta_1 + \theta_2 + \frac{3\,d_1}{L} - \frac{3\,d_2}{L}\right) \\ &= M_{FE} + 2\,E\left(\frac{I}{L}\right)\left(2\,\theta_1 + \theta_2 + \frac{3\,X_1}{L\cos\phi} - \frac{3\,X_2}{L\cos\phi}\right) \quad \text{(42)} \end{aligned}$$

AMERICAN SOCIETY OF CIVIL ENGINEERS
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Paper No. 2429

COMPUTATION OF EQUITABLE CHARGES FOR TREATMENT OF MUNICIPAL SEWAGE

By Ellis E. Bankson,[1] M. ASCE

Synopsis

Early in 1949, with financial assistance from the City of Pittsburgh, Pa., the State of Pennsylvania, and the federal government, the Allegheny County Sanitary Authority completed preliminary plans for an $82,000,000 collection treatment system for municipal sewage and industrial wastes. The scope and procedure leading to the creation of the Allegheny County Sanitary Authority, and a concise statement of the recommendations of the Authority, have been stated by Jay du Von.[2] A fairly comprehensive statement of the basic financing problems, and their relation to rate making as applied to Allegheny County, has been presented elsewhere by the writer.[3] The present paper, supplementing the former, offers numerical examples of actual computations for the guidance of engineers with similar problems. Items of cost data are numbered consecutively for the convenience of discussers.

Introduction

The question of equity in a schedule of charges for the collection and treatment of municipal sewage and industrial waste is timely because of the pollution abatement activity throughout the United States, the magnitude of the Pennsylvania program, and the local resort to "revenue bonds" as the adopted basis for all construction funds. By "local" is meant the sphere of influence allotted to the Allegheny County Sanitary Authority, hereinafter called "the Authority." Many water rate schedules just grew, like Topsy, and some sewage rates may follow a similar course, but the following analysis is presented with a view to determining a true equity, in sewage charges, between the various classes of customer types or quantity of service.

Note.—Published in April, 1950, as *Proceedings-Separate No. 15.* Position and title given are those in effect when the paper was received for publication.

[1] Managing Engr., Allegheny County San. Authority, Pittsburgh, Pa.

[2] "Allegheny County Completes Preliminary Plans for Collection and Treatment System," by Jay du Von, *Water and Sewage Works*, April, 1949, p. 155.

[3] "Allegheny County, Pa., Sets Sewer-Service Charges," by Ellis E. Bankson, *The American City*, January, 1949, p. 88.

It would appear that the outstanding factor of this analysis, aside from the obvious elements of cost allocation, is that of ground-water infiltration into the sewer system of the community, or the relationship between such infiltration and the sewage waste from a particular customer property. It may be noted, however, that the construction and operating costs of the Pittsburgh Metropolitan program include only the intercepter sewers and the sewage works for primary treatment. This is in contrast to a possible complete sewer system and complete sewage treatment, in some other case of particular interest.

In the present case, the construction cost is estimated at somewhat more than $81,000,000 and the annual operating cost is estimated at $1,350,000. Debt service is computed on the basis of a serial, 40-year, revenue bond issue at a 3% interest rate, although the interest rate must, naturally, depend upon market conditions. The resulting interest cost and bond maturity schedule would be covered by a 4.5% rate on the total construction cost, or about $3,650,000 per yr. Inclusion of the aforesaid operating cost would create an annual burden of about $5,000,000, as adopted herein for the basis of cost allocation.

SOURCE OF DATA FOR SEWAGE FLOW

The basic data, and the reasoning required for estimating the total revenue bearing sewage to be treated by the Authority in 1954, are as follows:

Item	Description	Gallons per year
1	All sewage from metered water sales in 1945........	26,814,597,000
2	All sewage from private water supply in 1945.......	1,862,696,000
3	By change from flat rate to metered service by 1954.	1,447,280,000
4	Increase from domestic customers by 1954..........	1,534,000,000
5	Increase of commercial and industrial service by 1954	714,225,000
6	A total of metered revenue sewage by 1954.........	32,372,798,000
7	Estimate of service for 20,000 flat rates in 1954.....	2,000,000,000
8	A reasonable excess above these conservative figures (for the purpose of a fair analysis on rate structure)	2,127,202,000
9	A fair estimate of total revenue sewage in 1954......	36,500,000,000

Item 9 (in gallons per year) can be further classified, in comparison with potential customers in 1954, as follows:

Item	Class	Revenue sewage	Customers
10	Domestic....................	16,500,000,000	251,251
11	Commercial.................	11,000,000,000	16,331
12	Industrial..................	9,000,000,000	1,435
13	Total (item 9)............	36,500,000,000	269,017

For the purpose of analyzing an equitable rate structure, item 10 includes 2,000,000,000 gal for service to 20,000 domestic customers who may still be on a flat rate basis in 1954, and 2,000,000,000 gal have been added to item 12 as a reasonable anticipated extra service for industry.

The total estimated revenue sewage for 1954 is shown in items 9 and 13. From this beginning, the gross annual output capacity for 1954 is estimated as:

Item	Description	Million gallons
14	The total of revenue sewage for 1954	36,500
15	The average of ground-water infiltration (at 20,000 gal per day per mile of sewer)	16,000
16	Storm water flow and the daily peak of sewage (at about 82% of the ultimate storm water capacity)	20,500
17	Annual output capacity for 1954	73,000

The final computations for estimated unit costs are as follows:

Item	Description		Dollars	
18	Estimate of annual burden			5,000,000
	Deduction for Special Revenue and Special Costs:			
19	Annual gain by the "minimum" charge		40,000	
	Estimate of Surcharge for Extra Concentration of sewage—			
20	(*a*) Conservative estimate	60,000		
21	(*b*) Possible extra load	50,000		
22	Total surcharge (items 20 and 21)		110,000	
23	Allocation of annual cost for normal quality of sewage: 800,000 (item 60) minus 110,000 (item 22)		690,000	
24	Total deductions (items 19 to 23)			840,000
25	Allocation of annual cost for the treatment of sewage on the quantity basis (item 18 minus item 24)			4,160,000

Dividing $4,160,000 (item 25) by 73,000,000,000 (item 17), the unit cost is found to be 5.7¢ per thousand gal.

Unit Cost of Sewerage Service for an Average Household

The average quantity of domestic sewage in Allegheny County is indicated at about $\frac{16{,}500{,}000{,}000 \text{ gal (item 10)}}{1{,}500{,}000 \text{ population}}$, or 11,000 gal per yr per person. This would mean an average of 30 gal per day for each person, or 120 gal per day for a family of four. Admittedly, there is a wide range above and below this average value. To this figure should be added the ground-water flow and extra provision for storm water.

The total length of all sewers within the service area is about 2,200 miles, or 11,616,000 ft, for a sewerage service to some 269,017 customers (item 13), or about 43 ft of sewer for each average customer. These totals include all the commercial and industrial customers that may require a much greater sewer frontage than the average. If all costs are to be covered by current bills to active customers (as is probably required for a revenue bond issue), the total length should include vacant lots. On this basis, the average of sewer length for an average residence would be something less than 43 ft each.

An estimated population of 1,500,000 in 1954 would require $\frac{11{,}616{,}000}{1{,}500{,}000}$, or about 7.7 ft per person, which is 30.8 ft for a house of four persons, as an average. The average length per person in single residences, however, would doubtless exceed 8 ft; and the average per person living in large apartment buildings would be much less. Since the average household of the county contains a scant four persons each, the average length would be 4 × 8 = 32 ft of sewer for each average residence, by this method.

In this case, an infiltration of about 20,000 gal per day per mile of sewer (item 15), for rate making purposes, would mean about 3.8 gal per day per ft of sewer, or nearly 115 gal per day of infiltration for a household of four at 30 ft of sewer per house. The cost of storm water capacity should be allocated in an amount that is at least equal to that of ground-water infiltration, or more, because the 73,000,000,000 gal in item 17 is based on a storm water flow that is higher than the average of ground-water infiltration. The cost for storm water, therefore, would equal at least that for infiltration, or the equivalent cost of 115 gal per day per average house.

The average capacity in gallons required to service the normal household of four persons will be as follows:

Item	Description	Quantity
26	Infiltration	115
27	Domestic sewage	120
28	Storm water capacity	115
29	Total per sample household of four	350

—and the resulting quantity cost of the sewerage service is 350 gal (item 29) at 5.7¢ per thousand gal, or 2¢ per day for 0.12 thousand gal (see item 27). The cost per gallon of domestic sewage capacity is 2/0.12 = 16.6¢ per thousand gal.

The cost allocation for normal sewage quality is estimated at $690,000 per yr (item 23) for the 36,500,000,000 gal of customer sewage (item 9), or less than 1.9¢ per thousand gal of sewage from the customer property. The sum of the two units (16.6 plus 1.9) indicates a total of 18.5¢ per thousand of domestic sewage from customer property.

Division Between Quantity and Quality Costs

The entire cost of the sewage collection system is properly allocated to the "quantity" of sewage. In addition, certain features of the plant cost are attributable to "quantity" and would be substantially the same regardless of the sewage strength. Other features are a direct result of the quality of sewage —that is, the biochemical oxygen demand (B.O.D.) and the suspended solids content. An analysis of the capital expenditure for the various components of the central treatment plant is presented in order that the annual debt service for the plant may be distributed between quantity and quality of the sewage, as follows:

Distribution As to Quantity—

Item	Description	Cost, in dollars
30	Pump station	2,800,000
31	Sewage meter	50,000
32	Grit chambers and comminutors	670,000
33	Sedimentation tanks	2,610,000
34	Power and boilers	1,800,000
35	Plant conduits and outlets	545,000
36	Administration building, garage, exterior piping, and miscellaneous	906,000
37	Total (55% of item 47)	9,381,000

Distribution As to Quality—

Item	Description	Cost, in dollars
38	Pre-aeration tanks	520,000
39	Blower plant	436,000
40	Sludge digestion plant	3,619,000
41	Elutriation tanks	145,000
42	Mechanical dewatering plant	946,000
43	Sludge drying and incineration	1,553,000
44	Gas utilities	150,000
45	Chlorination plant	250,000
46	Total (45% of item 47)	7,619,000
47	Grand total, based on a plant designed for 180,000,000 gal per day (items 37 and 46)	17,000,000

Items 37 and 46 indicate that 55% of debt service is allocated to quantity cost and 45% to quality cost. However, the annual operating expenses of the treatment plant, as distinguished from the initial capital expenditures, are estimated as 45% chargeable to quantity cost and 55% chargeable to quality.

SUMMARY OF QUANTITY AND QUALITY COSTS

The cost allocation between collecting system and treatment plant is computed as follows: The initial construction cost of the collecting system is $64,477,000, or about 80% of the direct construction cost of $81,052,000, for the entire project. Therefore, the annual debt service cost arising from the collecting system is 80% of $3,650,000, or $2,920,000. In a similar manner, about $500,000 of the $1,350,000 for annual operating and maintenance cost (about 37%) is properly allocated to the collecting system. The remainder in each case would be properly allocated to the sewage treatment plant at $730,000 for debt service and $850,000 for operation cost as follows:

Annual Cost Attributable to Collecting System—

Item	Description	Dollars
48	Debt service (80% × $3,650,000)	2,920,000
49	Operating cost (37% × $1,350,000)	500,000
50	Total	3,420,000

Annual Cost Attributable to Treatment Plant—

Item	Description	Dollars
51	Debt service (20% × $3,650,000)	730,000
52	Operating cost (63% × $1,350,000)	850,000
53	Total	1,580,000

The former charge of $3,420,000 (item 50) is properly allocated completely as a quantity cost in contrast to the latter item of $1,580,000 (item 53) which is allocated as partly a quantity cost and partly a quality cost. The division of this sewage treatment cost is indicated by items 37 and 46, respectively, as 55% of the debt service and 45% of the operating expenses to quantity cost and the remainder of 45% debt service and 55% operating expenses to quality cost.

A summary statement of the division between quantity cost and quality cost may now be made as follows:

Annual Costs Chargeable to Quantity of Sewage—

Item	Description	Dollars	
54	Annual costs of collecting system	3,420,000	
55	55% of annual treatment plant debt service (55% × $730,000) (item 51)	401,500	
56	45% of annual treatment plant operating expenses (45% × $850,000) (item 52)	382,500	
57	Subtotal (84% of item 61)		4,204,000

Annual Costs Chargeable to Quality of Sewage—

Item	Description	Dollars	
58	45% of annual treatment plant debt service (45% × $730,000)	328,500	
59	55% of annual treatment plant operating expenses (55% × $850,000)	467,500	
60	Subtotal (16% of item 61)		796,000
61	Total annual costs		5,000,000

It may be argued that the debt service from the sludge digestion plant should be counted as a quantity cost, because it develops power that is used for the pumping of sewage quantity. At the same time it may be argued that the allocation of operating cost is too heavy on the quantity side. A change of both would leave the final answer approximately the same as indicated by items 54 to 61.

Unit Cost of Sewerage Service for a Large Office Building

Detailed data for the nine office buildings included in the study are as follows:

Item	Name of office or building	Feet of sewer in street	Gallons per year
62	Law and Finance Building	220	5,478,000
63	Grant Building	500	16,454,000
64	Gulf Building	600	22,379,000
65	Koppers Building	560	20,220,000
66	Oliver Building	570	74,122,000
67	Park Building	300	22,287,000
68	Pittsburgher Hotel	210	21,842,000
69	Union Trust Building	1,160	24,162,000
70	William Penn Hotel	1,100	119,529,000
71	Totals	5,220	326,473,000

If the ground-water infiltration is allocated equally to both sides of the street, at the rate of 3.8 gal per day per ft of sewer, the total allocation against these nine office buildings is at the rate of 9,918 gal per day or about 3,620,000 gal per yr, as 1.1% of the water sales.

The addition of this quantity for infiltration and a similar quantity for storm water capacity would produce a total flow per year from these nine office buildings of about 333,740,000 gal, which, at 5.7¢ per thousand gal, represents a cost of $19,023.18. Unit costs, in dollars per thousand gallons, are as follows:

Item	Description	Quantity
72	Quantity cost: $\frac{19{,}023.18}{326{,}500{,}000}$	5.8
73	Quality cost	1.9
74	Total (items 72 and 73)	7.7

A fair comparison can be made between the 18.5¢ per thousand gal, as the top rate for domestic service, and the 7.7¢ per thousand gal as an equitable rate for a large office building, or an equivalent industry.

Basis of Cost Data

The extension of the foregoing analysis to the component parts of the whole, by reduction ratios, is treated in this section. As basic data, consider the following sources of sewage quantities (in schedule units of thousand gallons per day) for the year 2000:

Item	Description	Quantity	
75	Average sewage from customer properties		165,039
76	The peak of sewage flow from customer properties	226,118	
77	Ground-water infiltration	80,775	
78	Provision for storm water flow	66,688	
79	Computed peak rate of plant inflow		373,581

In item 77 the ground-water infiltration is computed at the rate of 25,000 gal per day per mile of sewer; and, in item 78, the storm water flow is presented as the difference between the dry weather peak flow and the peak of flow during storm. The corresponding allocation as to class of customer is as follows:

Item	Group items	Domestic	Commercial	Industrial	Total
80	Customer sewage	78,661	46,274	40,104	165,039
81	Infiltration	64,620	12,116	4,039	80,775
82	Subtotals	143,281	58,390	44,143	245,814
83	Storm water	56,324	10,990	1,374	68,688
84	Total allocation	199,605	69,380	45,517	314,502
85	Reserve capacity for daily peak flow at the foregoing composite ratio for all classes				61,079
86	Computed peak rate of plant inflow (items 84 and 85)				375,581

In item 81 the infiltration is reported at a certain ratio for each class, dermined by observation, experience, and judgment regarding the length of sewer. Item 83 (storm water) also is determined on a judgment basis. The ratio for industry, in this case, is further decreased because industries are located along a river and therefore most of the storm water can drain directly to it over the ground surface. Item 86 was computed[4] as 579.6 cu ft per sec of maximum inflow at the sewage plant, for the year 2000.

A weighted average cost allocation is evolved by a comparison of quantity ratios and quality ratios, as follows:

Item	Group items	Domestic	Commercial	Industrial	Total
87	Quantity ratio (from item 84)...	64%	22%	14%	100%
88	Quality ratio[5].................	43%	26%	31%	100%
89	Weighted average cost allocation.	61%	23%	16%	100%

The quality ratio in item 88 has been established by dividing the cost estimates between the quality and quantity function. For instance, the function of the digesters is related to concentration or quality and the function of the interceptors is related only to the quantity of sewage flow.

DIVISION OF ANNUAL BURDEN BETWEEN CUSTOMER CLASSES

The annual burden is divided between customer classes as follows:

Item	Group	Domestic	Commercial	Industrial	Total
90	Annual burden, 1954, at the cost ratio (item 18)..........	$3,032,000	$1,132,000	$836,000	$5,000,000
	Gain in Special Revenue Items of Direct Allocation				
91	"Minimum charges"	40,000	....	....	40,000
92	Allocated base by surcharge estimate....	....	....	60,000	60,000
93	Possible extra by surcharge estimate....	....	....	50,000	50,000
94	Remainder of 1954 annual burden for revenue sewage....	$2,992,000	$1,132,000	$726,000	$4,850,000
95	Revenue sewage, 1954, in units of thousand gallons (items 10 to 13)..............	16,500,000	11,000,000	9,000,000	36,500,000

Dividing item 94 by item 95, the average unit costs by customer classes for the year 1954 are:

Item	Unit	Domestic	Commercial	Industrial	Total
96	Cents per thousand gallons....	18+	10+	8+	13+

The adopted schedule of sewerage charges will allow for a range of rates (in cents per thousand gallons of sewage) from 18¢ at the top to 7¢ at the bottom, as compared to the average cost for each group of service as given in item 96.

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[4] "Preliminary Report," Allegheny County San. Authority, Pittsburgh, Pa., 1949, Appendix VII, Pt. 9, p. 168.

[5] *Ibid.*, Appendix XXI, p. 67.

INDUSTRIAL WASTES

The regular schedule of sewerage charges would apply for the quantity of industrial wastes except that a surcharge would also apply for concentrations that are higher than the average.

Let: F equal a factor to be applied to the basic rate; $R = 0.15$ be a ratio of quality cost to total annual cost; S be the suspended solids in sewage, in parts per million—S_i denoting a particular industry and $S_a = 275$ denoting the average for all sewage; and B equal the B.O.D. of sewage—B_i and $B_a = 300$, respectively, being that of a particular industry and the average of all sewage. Then:

$$F = 1 + R\left[\frac{0.75\,(S_i - S_a)}{S_a} + \frac{0.25\,(B_i - B_a)}{B_a}\right]\dots\dots\dots\dots(1)$$

Eq. 1 contains an arbitrary division between the solids content and the organic matter. An alternate formula has therefore been considered, without the arbitrary feature, as follows:

$$P = Q\,C\,X\,(S_i - S_a) + Q\,C\,Y\,(B_i - B_a)\dots\dots\dots\dots(2)$$

in which P is the surcharge, in dollars, for the billing period; Q is the quantity of sewage, in million gallons, for the billing period; $C = 8.34$ is a constant for converting parts per million to pounds per million gallons; X equals about one third of a cent is the cost of treating one pound of suspended solids; and Y equals about one tenth of a cent is the cost of treating one pound of B.O.D.

Comparing Eqs. 1 and 2, it may be noted that the resulting surcharge for one famous industry would be \$4,791 per yr by the first formula and \$5,126 by the second. For another famous industry, the surcharge would be \$14,321 per yr by the first formula and \$12,831 by the second.

SUMMARY

Through item 74 the analysis is illustrated by an example of an average household, for collection and treatment costs of domestic sewage, and by an example of nine office buildings for the quantity cost of high demand within a small area. In contrast, items 75 to 96 represent a composite analysis for unit costs on design quantities, without illustration by example or type. It will then be noted that the two methods produce similar results.

For domestic use of about 30 gal per capita, the infiltration will roughly equal the water sales. A further inclusion of storm flow capacity, at a cost about equal to the water sales, will require an interceptor capacity, for domestic sewage (on customer premises), of about three times the domestic water sales as compared to slightly more than 2% for the case of very large office or hotel buildings.

As limited to interceptor sewer collection and primary sewage treatment, the slide in this particular rate schedule of sewage service charges is found to range from 18¢ per thousand gal at the top to an average of 7.7¢ per thousand gal at the bottom (or 8¢ + by the composite study). This unit (7.7¢ per thousand gal) is the average cost for concentrated high water use. The low block rate, or charge, will be lower than 7.7¢ because of the higher rates or charges on the first blocks or steps of quantity service. Inclusion of lateral and trunk sewers, or secondary treatment and other differences, would affect the ratio between this extreme spread of service cost.

The total of those costs that relate to the quality of normal sewage flow is 1.9¢ per thousand gal and that relating to quantity of sewage at the treatment plant is 5.7¢ per thousand gal, or approximately a total of 17¢ per thousand gal on domestic water sales.

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TRANSACTIONS

Paper No. 2430

ANALYTICAL METHOD OF DETERMINING THE LENGTH OF TRANSITION SPIRAL

By MICHAEL V. SMIRNOFF,[1] ASSOC. M. ASCE

WITH DISCUSSION BY MESSRS. T. F. HICKERSON, THOMAS R. KLINGEL, CARL F. MEYER, JOSEPH BARNETT, J. J. LEEMING, ALLEN G. TYSON, DONALD THOMPSON, RALPH A. MOYER, AND MICHAEL V. SMIRNOFF.

SYNOPSIS

There is no intention on the part of the writer to advance a new theory for the transition spiral design, but rather to consolidate what has already been known into some well defined expressions that could be used to better advantage in design. This paper is a step-by-step approach to the problem of expressing the length of transition spiral as a function of three quantities—the speed, the radius of the circular curve, and the maximum rate of superelevation on the curve.

Let: a be the acceleration, in feet per second per second; v be the velocity, in feet per second; t be time, in seconds; and ρ be the radius of curvature at any point on a space curve. The general expression for the acceleration of a particle moving along a space curve is

$$\bar{a} = \frac{dv}{dt}\bar{T} + \frac{v^2}{\rho}\bar{N} \quad \text{(1)}$$

in which acceleration $\bar{a}$, itself a vector, is represented as the sum of two vectors comprising tangential and normal accelerations, $\bar{T}$ being a unit vector in the direction of the tangent at any point of the curve and $\bar{N}$ being a unit vector in the direction of a normal at the same point. In the case of a uniform motion along this, as yet, undefined curve (that is, if v is a constant), $\frac{dv}{dt} = 0$, and Eq. 1 becomes

$$\bar{a} = \frac{v^2}{\rho}\bar{N} \quad \text{(2)}$$

If the curve is a "plane" curve, the acceleration vector will lie in the plane of the curve and in the direction of the radius of curvature. This direction

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NOTE.—Published in November, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or the discussion was received for publication.

[1] Lecturer, Dept. of Eng., Univ. of California, Los Angeles, Calif.

will also be the direction of the force acting on the particle, since $F = m\,a$; thus,

$$\bar{F} = \frac{m\,v^2}{\rho}\bar{N} \qquad (3)$$

in which F is a force and m is a mass, the bar over any symbol signifying a vector.

If a body, such as an automobile, moves along the pavement which may be superelevated, there will be another force acting on the center of gravity of the body also directed along the radius of curvature of the curve which the center of gravity describes. This force is a radial component of the weight of the body and is equal to $W \sin\theta$, in which $W = m\,g$ is the weight of the body, g being the acceleration of gravity. The angle of superelevation, θ, is usually small, so that $\theta = \sin\theta = \tan\theta$.

The net force acting radially on the body is

$$F = \frac{m\,v^2}{\rho} - m\,g\,\theta \qquad (4)$$

Considering a uniformly increasing rate of superelevation from zero at the beginning of a curve to $e_{\max}$ per foot of width of pavement at the end of the curve, the rate of superelevation $e = \theta$ at a distance S from the beginning of the curve will be $e = S\dfrac{de}{dS}$. Thus, Eq. 4 becomes

$$F = \frac{m\,v^2}{\rho} - m\,g\,S\frac{de}{dS} \qquad (5)$$

and

$$a = \frac{v^2}{\rho} - g\,S\frac{de}{dS} \qquad (6)$$

A transition spiral now will be defined in a usual manner, as a curve of uniformly changing curvature along the curve, thus,

$$\frac{\kappa}{S} = \frac{1}{\alpha} \qquad (7)$$

in which κ is the curvature at a distance S along the curve and α is a constant. Since $\kappa = \dfrac{1}{\rho} = \dfrac{S}{\alpha}$, Eq. 6 becomes

$$a = \frac{v^2}{\alpha}S - g\,S\frac{de}{dS} \qquad (8)$$

The rate of change of this radial acceleration with respect to the distance S along the curve is

$$\frac{da}{dS} = \frac{v^2}{\alpha} - g\frac{de}{dS} - g\,S\frac{d^2e}{dS^2} \qquad (9)$$

and, since the rate of superelevation along the curve was assumed to be constant $\left(\text{that is, } \dfrac{de}{dS} \text{ is a constant}\right)$, $\dfrac{d^2e}{dS^2} = 0$, or

$$\frac{da}{dS} = \frac{v^2}{\alpha} - g\frac{de}{dS} \qquad (10)$$

As all terms on the right side of Eq. 10 are constants, $\frac{da}{dS}$ is also a constant; and, as a is also a function of t, $\frac{da}{dS} = \frac{da}{dt}\frac{dt}{dS} = \frac{da}{dt}\frac{1}{v}$, in which $\frac{da}{dt}$ is a constant. Designating this constant by C and substituting $\frac{da}{dS} = \frac{C}{v}$ in Eq. 10: $\frac{C}{v} = \frac{v^2}{\alpha} - g\frac{de}{dS}$, from which $\left(\frac{v^3}{C\,\alpha} - 1\right) dS = \frac{g\,v}{C}\,de$; $\frac{v^3 - C\,\alpha}{C\,\alpha}\int_0^{S_s} dS = \frac{g\,v}{C}\int_0^{e(\max)} de$; and

$$(v^3 - C\,\alpha)\,S_s = g\,v\,\alpha\,e_{\max} \dots\dots\dots\dots\dots\dots (11)$$

in which S_s is the length of transition spiral.

At the point where the transition spiral joins a circular curve of radius R, $\rho = R = \frac{1}{\kappa}$; and, from Eq. 7, $\alpha = \frac{S_s}{\kappa} = R\,S_s$. Eq. 11 then becomes: $(v^3 - C\,R\,S_s)\,S_s = g\,v\,R\,S_s\,e_{\max}$; or $v^3 - C\,R\,S_s = g\,v\,R\,e_{\max}$; and

$$S_s = \frac{v^3}{R\,C} - \frac{v\,g\,e_{\max}}{C} \dots\dots\dots\dots\dots\dots (12)$$

which is the final expression for the length of the transition spiral.

If the effect of superelevation were neglected,

$$S_s = \frac{v^3}{R\,C} \dots\dots\dots\dots\dots\dots (13)$$

Eq. 13 is the formula widely used for the length of transition spiral. The constant C has received much attention in engineering literature but quite too often with no consideration given to the effect produced by the superelevation. The fact is that the term representing the effect of superelevation should be included in the formula. This term will be of quite sizable magnitude in most cases and thus will help to reduce the lengths of otherwise unnecessarily long spirals, especially on flatter curves. Considering, for instance, a speed of 60 miles per hr (or 88 ft per sec) and a maximum rate of superelevation of 0.10 ft per ft; and, using $C = 2$, which is one of the values used in Eq. 13—$\frac{v\,g\,e_{\max}}{C} = \frac{88 \times 32.2 \times 0.10}{2} = 142$ ft.

Considering, further, a 6° circular curve, such that $\frac{v^3}{R\,C} = \frac{88^3}{955 \times 2} = 357$ ft; Eq. 12 yields $S_s = 357 - 142 = 214$ ft. This length is rather low, at least much lower than the generally recommended value for the same conditions. It appears that a lower value than $C = 2$ should be used in Eq. 12. The entire question of the length of spiral thus reduces to a proper value of C in Eq. 12.

By definition, $C = \frac{da}{dt}$; that is, C is a constant rate of change of acceleration with respect to time. Such a quantity is a little hard to visualize, and a better conception of it may be represented as follows: $C = \frac{da}{dt} = \frac{1}{m}\frac{dF}{dt} = \frac{g}{W}\frac{dF}{dt}$; or

$$C = g\,\frac{dF/dt}{W} \dots\dots\dots\dots\dots\dots (14)$$

In Eq. 14, C is a constant rate of change of force, in pounds per second per pound weight of a driver or a rider, multiplied by the acceleration of gravity, at which the driver will be pushed radially when riding around the curve.

It appears that a theoretical investigation can go no further, and that the most desirable value for the magnitude of this rate of change of force should be determined by some reliable practical method. The papers that have appeared in engineering literature indicate that several investigations of this kind have been made with rather inconclusive results because, apparently, it was necessary to place too much reliance on the personal reactions of the performers in the experiment. What may be considered as a desirable condition by one driver may appear uncomfortable to another.

Without any further experimental investigation of the relative validity of various C-values, it seems that some fairly conservative value should not be difficult to agree upon in view of the fact that C, as used in Eq. 13 (that is, without considering superelevation) appears to vary only over a small range of values—between 1.25 and 3. In railroad practice, C seems to approximate the value of 1.25, whereas some highway departments seem to favor $C = 3$. To obtain corresponding values of S_s from Eq. 12 (that is, considering superelevation), C-values should be somewhat lower.

TABLE 1.—COMPARATIVE VALUES OF THE LENGTH OF SPIRALS FROM EQ. 13, GIVEN BY $\frac{v^3}{RC}$, AND EQ. 12, GIVEN BY S_s, FOR VARIOUS VALUES OF C

C, in feet per second^3	$D = 3°$; $e_{\max} = 0.07$			$D = 6°$; $e_{\max} = 0.10$		
	$\frac{v^3}{RC}$	$\frac{v\,g\,e_{\max}}{C}$	S_s, in feet	$\frac{v^3}{RC}$	$\frac{v\,g\,e_{\max}}{C}$	S_s, in feet
0.80	446	248	198	892	354	538
0.90	396	220	176	793	314	479
1.00	357	198	159	713	283	430
1.25	285	159	126	571	226	345
1.50	238	132	106	476	189	287
2.00	178	....	....	357	....	....
3.00	119	....	....	238	....	....

Table 1 shows comparative values of S_s as computed by Eqs. 12 and 13 for a speed of 60 miles per hr for various values of C. It remains to reduce speed v, in feet per second, to V, in miles per hour, in Eq. 12. Since 44 ft per sec = 30 miles per hr and $g = 32.16$,

$$S_s = 3.15\,\frac{V^3}{RC} - 47.2\,\frac{V\,e_{\max}}{C} \qquad (15a)$$

or

$$S_s = \frac{3.15\,V}{C}\left(\frac{V^2}{R} - 15\,e_{\max}\right) \qquad (15b)$$

From previous discussions of the subject in engineering literature it seems reasonable to assume that the value of C in Eqs. 15 should be about 1 ft per sec^3.

In conclusion it may be pointed out that with ever increasing speed of traffic the uniformity of practice in highway design is essential to safe driving, since it will eliminate, to a great extent, the element of surprise in driving along various highways.

DISCUSSION

T. F. HICKERSON,[2] M. ASCE.—Dividing Eq. 4 by m,

$$a = \frac{v^2}{\rho} - g\,e \qquad (16)$$

which is the net acceleration that develops when rounding a superelevated curve. When $\rho = R$, Eq. 16 becomes

$$a_{\max} = \frac{v^2}{R} - g\,e_{\max} \qquad (17)$$

Dividing Eq. 17 by C yields the time, in seconds, in which this acceleration can take place without discomfort. Meanwhile the car has traveled a distance that is herein assumed equal to the length of the spiral S_s. Therefore,

$$t = t_1 - t_2 \qquad (18)$$

in which $t\left(= \frac{S_s}{v}\right)$ is the time for traveling at a constant speed the distance S_s; $t_1\left(= \frac{v^2}{R\,C}\right)$ is the time for acquiring the radial acceleration proportional to the centrifugal force; and $t_2\left(= \frac{g\,e_{\max}}{C}\right)$ is the theoretical time allowed for rotating the car through an angle $e_{\max}$.

Multiplying both sides of Eq. 18 by v, and restoring equivalent quantities,

$$S_s = \frac{v^3}{R\,C} - \frac{v\,g\,e_{\max}}{C} \qquad (19)$$

which is the same as Eq. 12.

Using the values in Table 1—$D = 3°$; $R = 1{,}910$ ft; $v = 88$ ft per sec (60 miles per hr); $e_{\max} = 0.07$; and $C = 1$—Eq. 18 gives: $1.81 = 4.06 - 2.25$, from which it is seen that the time for changing from the tangent to the fully superelevated curve is only 1.81 sec.

Eq. 19 shows that $S_s = 0$ if—

$$R = \frac{v^2}{g\,e_{\max}} = \frac{V^2}{15\,e_{\max}} \qquad (20)$$

—that is, no spiral is needed as far as the conditions of the formula are concerned. Even in such cases, however, no harm is done by the use of a spiral. In fact, the appearance and the ease of approach would be improved thereby.

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[2] Prof. of Applied Math., Univ. of North Carolina, Chapel Hill, N. C.

The question now arises: Is Eq. 12 (or Eq. 19) a reliable formula? Certainly it is more comprehensive than the widely used formula:

$$S_s = \frac{v^3}{R\,C} \quad \text{or} \quad \frac{3.155\,V^3}{R\,C} \qquad (21)$$

in which the effect of superelevation is omitted; however, both contain the factor C—which is assumed to be constant—but actually is quite variable. Thus, rearranging Eq. 12,

$$C = \frac{v^3}{R\,S_s} - \frac{v\,g\,e_{\max}}{S_s} \qquad (22)$$

in which it is seen that C varies with speed, length of spiral, and the value of the superelevation ratio, $e_{\max}$, reached on the curve, and with cross products of these functions, combined with the cube of the speed.

Recent actual tests on a number of curved roads in England by John Joseph Leeming and Archibald Niel Black[3] have shown "* * * that the comfort criterion is an approximate guide to the minimum radius to be used at any design speed, but not to the length of transition."

Thomas R. Klingel,[4] Assoc. M. ASCE.—The development of the spiral easement curve, and particularly the formulas for the determination of the length of curve, have been based on rule-of-thumb methods in many cases. The necessary refinement in highway design brought about by steadily increasing vehicle speeds has made it necessary to develop a more fundamental approach to many of the problems and to reanalyze the basic principles upon which many present-day formulas are based. Mr. Smirnoff takes a step in this direction, and it should encourage engineers to take a fresh look at the basic principles upon which many of their present design formulas are based.

When a car enters a curve, a centrifugal force is set up by the radial acceleration; and the superelevation sets up an opposing force which is the radial component of the force as a result of the acceleration of gravity. An ideal situation would exist if these two opposing forces were equal so that there would be no residual side force acting on the car. This situation is seldom realized since curves must ordinarily be designed for cars operating at all speeds from zero miles per hour to the maximum permissible speed on the highway. As a result, an unbalanced radial force will exist for all speeds except one. This force is given by Eq. 4.

The length of the easement curve as determined by Mr. Smirnoff is based on the premise that the increase in this force at maximum velocity must be at a uniform and gradual rate throughout the length of the spiral, so that the greater the unbalanced force, the longer the spiral, and the smaller the unbalanced force, the shorter the spiral. This leads to the entirely false conclusion that, if the speed and the superelevation are so adjusted as to eliminate the unbalanced force, no spiral is necessary. To illustrate, assume a 6° curve

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[3] "Road Curvature and Superelevation: A Final Report on Experiments on Comfort and Driving Practice," by John Joseph Leeming and Archibald Niel Black, *Journal*, Institution of Municipal Engrs., February, 1950, p. 522.

[4] Associate Prof. of Civ. Eng., Univ. of Minnesota, Minneapolis, Minn.

($\rho = 955$ ft) and a speed of 60 miles per hr ($V = 88$ ft per sec). The elevation required to balance this centrifugal force is 0.25 ft per ft of roadway width. With these values and Eq. 12 as derived by Mr. Smirnoff, the spiral length is found to be zero. This implies that the alinement can change from a tangent to a 6° curve and the superelevation from 0 to 0.25 ft per ft of width, instantaneously.

One of the primary functions of an easement curve is to provide for a gradual increase in the superelevation. The magnitude of the unbalanced force is determined entirely by the spread between the balancing velocity (V_B) and the maximum velocity (V_M), and is independent of the magnitude of the superelevation. It can be computed by the equation:

$$F = \frac{m}{\rho}(V^2_M - V^2_B) \quad \text{.....................(23)}$$

The length of the spiral, as determined by Mr. Smirnoff, is directly proportional to the magnitude of the unbalanced force, and is, therefore, entirely independent of the magnitude of the superelevation. It is dependent only on the spread between the maximum velocity and the balancing velocity. The spread is entirely arbitrary in accordance with current design practice and, therefore, should not be the sole basis for determining the length of the curve, since it completely ignores the prime function of the spiral, to provide room within which to "run in" the superelevation.

Imagine a superexpress highway, in which the various traffic lanes are designed for definite increments of velocity. There might be a lane on a 6° curve on which all cars operate within the limits of a minimum velocity of 45 miles per hr, and a maximum velocity of 60 miles per hr. The balancing speed would probably be selected at 53 miles per hr so that the maximum unbalanced force would be 0.0553 lb per lb of weight in an outward direction at 60 miles per hr, and the same value in an inward direction at 45 miles per hr. The superelevation for this one lane would then be 0.197 ft per ft of width. The length of the easement curve for this lane, based on Eq. 12, with a value of C of 1.25 would be 125 ft. The same curve designed for speeds from 0 miles per hr to 60 miles per hr instead of from 45 miles per hr to 60 miles per hr would have a superelevation of only 0.10 ft per ft of width, and an easement curve 345 ft long, as taken from Table 1. In the first case, with a high average speed and a high superelevation, the spiral is only 125 ft long and the superelevation is "run in" at the rate of 0.158 ft per station along the spiral; in the second case, with a lower average speed and a lower superelevation, the spiral is 345 ft long and the superelevation is "run in" at the rate of 0.029 ft per station. These rates can be varied at will, depending only on how close the balancing velocity can be taken to the maximum velocity. In the first case, the spread between the balancing velocity and the maximum velocity is from 53 miles per hr to 60 miles per hr, and in the second case the spread is from 38 miles per hr to 60 miles per hr. This situation becomes more aggravated in designing traffic lanes for higher vehicle speeds.

The limitation of current practice in designing for all speeds on a single lane may keep the length of spirals, as designed by Eq. 12, within safe and comfortable limits since there is generally quite a spread between the balancing velocity and the maximum velocity. A theoretical approach to any problem, however, should provide for all the possibilities that may arise in future design problems, and if the formulas break down for other than current practice they are little better than rule-of-thumb methods.

It can be concluded that the length of easement curve is controlled by two independent factors: One is the maximum rate at which the superelevation can be "run in." This is the factor used in railroad design and for which the American Railway Engineering Association recommends a maximum rise in superelevation of 1¼ in. per sec. The second factor is the maximum rate at which the unbalanced force can be allowed to build up. This can be expressed as an increase in rate of radial acceleration per second, which is the value C in the paper, or as an increase in force in pounds per second per pound of weight of car. This latter value would be 0.031 lb per sec per pound of weight for $C = 1$ and 0.039 lb per sec per pound of weight for $C = 1.25$. In problems involving high superelevations with a relatively low spread between the maximum velocity and the balancing velocity, the first factor would apply; and in cases of relatively low superelevation and a high spread between the maximum velocity and the balancing velocity, the second factor would apply. It may be necessary to develop two separate and independent formulas for length of easement curve, one based on each of the foregoing factors. The length to be used in any case would be the larger value given by these two formulas. The use of a formula for the length of spiral with complete disregard for the magnitude of the superelevation or the rate at which it is to be "run in" can lead to some very erroneous conclusions, and it is probably equally true that the determination of the length of spiral, without regard for the rate at which the unbalanced force will build up, is equally false.

Carl F. Meyer,[5] M. ASCE.—To the best of the writer's knowledge Mr. Smirnoff has presented, for the first time in an American publication, a mathematical expression for the length of a transition spiral which includes the effect of superelevation.

Charles M. Noble,[6] M. ASCE, has pointed out that the use of superelevation cancels out part or all the centrifugal acceleration, depending on the speed of the vehicle. This effect was also discussed at length by Mr. Leeming.[7] In neither of these instances, however, was a mathematical relation given; yet in an earlier paper[8] Mr. Leeming presented a relation of substantially the same form as Eq. 15*b*. Although this equation appears to be theoretically correct, it produces peculiar results under some conditions. For example, at "equilibrium speed" (that speed at which the centrifugal force is exactly balanced by the component of the weight of the vehicle toward the center of the superelevated curve) the value of e_{max} must be equal to $V^2/(15\ R)$. The length of transition required by Eq. 15*b* then becomes zero. This is an absurd result, yet it is required by the theory.

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[5] Prof. of Civ. Eng., Worcester Polytechnic Inst., Worcester, Mass.

[6] "Thoughts on Highway Design Research As Related to Safety of Vehicle Operation," by Charles M. Noble, *Proceedings*, Highway Research Board, National Research Council, Vol. 17, 1937, p. 247.

[7] "The General Principles of Highway Transition Curve Design," by John Joseph Leeming, *Transactions*, ASCE, Vol. 113, 1948, p. 868.

[8] "Road Curvature and Superelevation," by John Joseph Leeming, *Road Paper No. 7*, Inst. C. E., October, 1942.

A simpler form of Eq. 15*b* for the usual case in which V is greater than the equilibrium value is found by substituting $15(e + f))$ for V^2/R, f being the familiar "side friction factor" or "cornering ratio." The result is

$$S_s = \frac{47\ V f}{C} \quad \text{(24)}$$

A relation of this basic form was given by Elmer R. Haile, Jr.,[9] Assoc. M. ASCE, although he recommended that it should not be used because of the short spiral lengths produced when f is small. There is no necessity to discard the theoretically correct relation given by Eq. 15*b* or Eq. 24 solely for this reason. It is important, however, to consider the rotational effect. Although an extremely short spiral to a superelevated curve will result in no centrifugal sensation if traversed at equilibrium speed, yet the rotational change about the axis of the car may be too rapid for comfort or safety.

Long years of operating experience on spiraled superelevated railroad track have led the American Railway Engineering Association to recommend that minimum spiral lengths be based on attaining superelevation across standard gage track at a desirable maximum rate of 1¼ in. per sec. Highway and railroad operations are by no means analogous; but, in the absence of research on the motor vehicle rotational rate at which discomfort begins, there is logic in the tentative adoption of the same rate used successfully on railroads. (Both Mr. Haile and T. T. Wiley, M. ASCE, have advocated[9,10] consideration of the rotational effect.)

A rise of 1.25 in. per sec across the track gage of 4.71 ft is equivalent to about 0.022 ft per ft per sec. Consequently, $S_s = 1.47\ V(e/0.022)$, or

$$S_s = 67\ V e \quad \text{(25)}$$

The two formulas for minimum desirable spiral length are separately inconsistent for certain conditions. Eq. 24 gives values which are too short when f is small ($S_s = 0$ at equilibrium speed), whereas Eq. 25 gives values which are too short when e is small ($S_s = 0$ for nonsuperelevated curves). However, by using each formula within its proper range, the resulting minimum spiral lengths will produce neither unsafe values of f nor an uncomfortable rate of angular rotation.

The spiral length at which the change from one equation to the other occurs is found by equating the two values of S_s. Thus, $47\ V f/C$ governs the choice of minimum S_s when $e < 0.7 f/C$; and $67\ V e$ governs S_s when $e > 0.7 f/C$.

Rotational change, as represented by Eq. 25, governs minimum spiral lengths for practically all values of superelevation normally used on the open highway. If the superelevation rate is less than one half the usual maximum for a curve of the given radius, Eq. 24 will usually govern.

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[9] Discussion by Elmer R. Haile, Jr., of "The Modern Express Highway," by Charles M. Noble, *Transactions*, ASCE, Vol. 102, 1937, p. 1091.

[10] Discussion by T. T. Wiley of "The Modern Express Highway," by Charles M. Noble, *ibid.*, p. 1103.

The practical value of the rotational change method of choosing minimum spiral lengths is well illustrated in the case of the Pennsylvania Turnpike. The spirals on this highway were designed on the basis of a rotational change of about 0.02 ft per ft per sec—practically the value used in obtaining Eq. 25. The generally favorable operating characteristics of these spirals, as observed on high-speed tests,[11] are undoubtedly a result, in no small measure, of the constant (and comfortable) rate of rotational change experienced on successive curves.

Until recently, valid objection to this method of selecting spiral lengths might have been made on the ground that it does not necessarily produce a constant value of C. The assumption that C must be constant on a highway spiral (as it must, of necessity, be in the case of a spiral on railroad track) has never been conclusively demonstrated; it has been merely taken for granted. On the other hand, the careful field tests and statistical analyses made by Messrs. Leeming and Black[3,12] (which were described in part by Mr. Leeming in another paper[7]) show that C varies between such wide limits that there is no natural rate of turning the wheel, such as is represented by $C = 2$ or by any other constant value.

From Mr. Leeming's experiments it appears that safety and comfort depend mainly on the value of f, and that a driver slows down to reduce f, not to reduce C. Relatively large values of C (far in excess of 2 ft per sec^3) were frequently recorded as producing an "imperceptible" degree of discomfort, as long as the accompanying values of f were moderate (not greater than approximately 0.15).

In view of the foregoing research, it is felt that Mr. Smirnoff has placed too much emphasis on the term C, to the exclusion of the more important term f. The writer cannot agree with the author (as stated in his discussion of Eq. 12) that: "The entire question of the length of spiral thus reduces to a proper value of C * * *," nor that (as stated following Eq. 15*b*):

> "From previous discussions of the subject in engineering literature it seems reasonable to assume that the value of C in Eqs. 15 should be about 1 ft per sec^3."

Neither of these statements is borne out by the most extensive research made thus far—namely, that of Messrs. Leeming and Black.

Moreover, the author did not point out that Eqs. 15*a* and 15*b* do not necessarily yield the most desirable length of transition, but merely the minimum length required if an assumed rate of change of centrifugal acceleration (C) is not to be exceeded.

The paper seems to have as its purpose, proof that spirals may be unnecessarily long owing to neglect of the superelevation factor $v\,g\,e/C$. However, it should not be overlooked that practical considerations—such as suitable gutter grades, or pleasing appearance—may require even longer spirals than determined by Eq. 15*b* or Eq. 25.

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[11] "Curve Design and Tests on the Pennsylvania Turnpike," by Kenneth A. Stonex and Charles M. Noble, *Proceedings*, Highway Research Board, National Research Council, Vol. 20, 1940, p. 429.

[12] "Road Curvature and Superelevation: Experiments on Comfort and Driving Practice," by John Joseph Leeming and Archibald Niel Black, *Journal*, Institution of Municipal and County Engrs., December, 1944, p. 137.

Engineers interested in this problem should be indebted to Mr. Smirnoff for his careful mathematical analysis, and particularly for pointing out the fallacy (and its magnitude) of neglecting the effect of superelevation on the length of the transition curve.

Joseph Barnett,[13] M. ASCE.—It is important to be reminded that superelevation should be provided for when using the rate of change of centripetal acceleration in computing the length of the transition curve. To be sure, the author counterbalances the reduction in the length of transition due to superelevation by suggesting a lower value for the constant than is generally used in highway practice; but there is no doubt that Mr. Smirnoff's approach is correct and to be preferred to the use of formulas that omit the effect of superelevation. The value of the constant should be determined experimentally, as the author suggests, but the writer feels that the effort has a low priority. There is no need for determining the length of transition precisely, and research in driver behavior and other aspects of highway engineering have a much higher priority.

Transitions should be used in all cases except on very flat highway curves. Length can vary over a wide range and any curve that varies in sharpness with the length is satisfactory. Mr. Smirnoff was wise not to advocate a particular shape of curve as has been done so often for the clothoid, cubic parabola, the A.R.E.A. (American Railway Engineering Association) curve in the United States, and for the lemniscate in England. It makes little practical difference how a transition curve is obtained or what exact curve is used.

A system of highways that has been admired for its appearance and smoothness of operation is the parkway system of Westchester County, New York; yet the transitions were obtained without using a mathematical curve. Instead, the line was laid down on topographic contour maps by the use of a spline, which is a flexible strip of wood or plastic held on the drawing board with weights. (The general location and alinement, of course, were first determined by reconnaissance in the field.) The stiffness of a spline causes the line to form transitioned curves naturally. The line was then computed as a sequence of compound curves, frequently three in number, the central curve being sharper than the outer ones, to fit the spline line best. When located in the field the line was considered satisfactory for grading purposes only. When forms were laid for paving, the fact that the line did not adequately conform to transitioned curves became evident; lining in by eye was the final step. Lateral shifts of the forms amounted to several inches and in many cases more than a foot. Considering that the computed line already was a form of transition, these shifts were appreciable. This method had the advantage of direct observation of the relation of horizontal alinement to vertical alinement, but the method was laborious. It is much more satisfactory to locate transitioned curves in the first place; and, with adequate tables, the work of computation and location is no more laborious than that required for compound curves. The method of computing transitioned curves to fit spline lines has been used by the United States Bureau of Public Roads on all its park and forest roads, and the results are just as satisfactory, if not more satisfactory, than the method used in Westchester County.

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[13] Chf., Urban Road Branch, Bureau of Public Roads, U. S. Dept. of Commerce, Washington, D. C.

Recognizing that precision in length of spiral is not necessary, the writer, in 1938, developed a set of tables[14,15] which makes it just as easy to design and locate curves with transitions as to design and locate simple curves, provided the design is made initially as transitioned curves and not as simple curves to be adjusted later for the addition of transitions.

J. J. LEEMING.[16]—In 1942 the writer proposed[8] a formula—

$$S_s = \frac{1}{C}\left[\frac{v^3}{R} - v\,g\,(e_{\max} \pm b)\right] \ldots\ldots\ldots\ldots\ldots\ldots (26)$$

—which differs from Eq. 12 only by the addition of a symbol *b* to account for the camber. With traffic passing on the right, Eq. 26 applies to the left-hand curve when *b* is positive. In a strict sense, *C* should be *Q*, the rate of change of lateral force.

On being first introduced, much criticism was directed at Eq. 26 because, if camber is neglected, as in Eq. 12, the length of transition becomes zero at the "hands-off" speed. At first, it is difficult to account for the anomaly—although the difficulty is no greater than a similar one in the conventional formula—that the same length of curve is given no matter what the superelevation may be. Both of these anomalies were explained by experiments[12] in 1944, which showed that the rate of change of force is not a valid criterion for the design of transition curves. There is no such thing as a "safe" or "comfortable" rate of change of force so the equation is meaningless. In 1950, the writer and Mr. Black have proposed[3] an empirical method for the design of transition curves which includes an alinement chart for the solution of the rather complicated regression equation.

ALLEN G. TYSON,[17] M. ASCE.—An improvement in the equations for the length of the transition, developed independently by the writer, has the form:

$$S = \frac{3.15\ V^3}{C_o\,R}\,(1 - k) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (27)$$

in which *k* denotes the superelevation factor defined as the ratio of actual superelevation to the centrifugal ratio; and C_o is the allowable value of *C* on a roadway that is not superelevated, with $k = 0$. Since the practice of superelevating a curved pavement is more common than that of inserting transitions,

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[14] "Transition Curves for Highways," by Joseph Barnett, Bureau of Public Roads, U. S. Dept. of Commerce, U. S. Govt. Printing Office, Washington, D. C., 1940.

[15] Discussion by Joseph Barnett of "Mathematical Examination of the Highway Transition Spiral," by John Oran Eichler and Howard W. Eves, *Transactions*, ASCE, Vol. 111, 1946, pp. 978–979.

[16] County Surveyor, Dorsetshire, England.

[17] Asst. County Surveyor Flintshire County Council, North Wales, Great Britain.

a more important relationship is

$$C_k = \frac{3.15\ V^3}{S\ R} = \frac{C_o}{1 - k} \qquad (28)$$

Eqs. 27 and 28 show (in theory, at least) how transitions should be shortened as a higher superelevation factor is used; and they may also explain why the British practice, with a usual k-value of 0.33 to 0.40, tends toward a lower value of C_k than the American practice, with a k-value as great as 0.56. It is also interesting to recall that W. H. Shortt[18] states: "A curve which cannot be given superelevation is more in need of easing at the ends than one that can be superelevated." This statement clearly anticipates most subsequent conclusions, without putting them on a mathematical basis.

It is gratifying to the writer to read Mr. Smirnoff's opinion that the range of C (or rather, C_k)—namely, between 1.25 ft per sec^3 and 3 ft per sec^3 is small. In Great Britain this same range is considered by some highway engineers to be so unreasonably large that the use of C for design purposes should be abandoned entirely. One of these men has in fact expressed the opinion[19] that it is

> "* * * distressing to find the attitude that because a certain basis of design (i.e. the use of C) has been used in the past, its use should be perpetuated irrespective of the fact that evidence has been produced showing it to be fundamentally unsound."

The evidence referred to consists of admittedly extensive experiments, probably of the type which Mr. Smirnoff considers to have led to inconclusive results, and, as an unashamed reactionary, the writer shares Mr. Smirnoff's view that it should not be difficult to agree on some fairly conservative value for C_o to be used in design, say, in the range of from 0.5 ft per sec^3 to 1.00 ft per sec^3.

As long ago as 1945, the writer[20] suggested the following arbitrary sliding-scale values for C_o:

$$C_o = 1.05 - 0.0045\ V \qquad (29)$$

Substituting Eq. 29, with the British superelevation factor $k = 0.40$, in Eq. 28,

$$C_k = 1.75 - 0.0075\ V \qquad (30a)$$

and, with the value $k = 0.5625$ in use in the United States,

$$C_k = 2.40 - 0.01\ V \qquad (30b)$$

The object of the sliding scales is to shorten the spiral lengths on slow speed roads where the topography often limits the tangent lengths available and to lengthen the spirals for high design speeds which may well increase in the future.

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[18] "A Practical Method for the Improvement of Existing Railway-Curves," by W. H. Shortt, *Proceedings*, Inst. C. E., Vol. 176, 1909, pp. 97–99.

[19] *Roads and Road Construction*, March, 1950, p. 85.

[20] "The Design of Partly Transitional Curves," by A. G. Tyson, *Roads and Road Construction*, November, 1945, pp. 357–363.

Donald Thompson.[21]—The formula for the length of a transition curve, as presented in this paper, is a real contribution to engineering literature. The formula is theoretically correct since it includes the effect of superelevation, and it should prove to be extremely useful not only in determining standards for transition curve design, but also, by its application to the investigation of existing transition curves, in arriving at a safe value for the maximum allowable rate of increase of the unbalanced centrifugal acceleration. The rate of increase of the unbalanced centrifugal acceleration, usually designated as C in transition curve literature (designated Q by Messrs. Leeming and Black[3]) has been the subject of much controversy among highway engineers. Values ranging from 1 to 3 have been used in the formula—

$$S = \frac{3.15\,V^3}{C\,R} \qquad \text{(31)}$$

—but, since Eq. 31 is incorrect because it ignores superelevation, the values of C experienced when traveling a transition curve did not correspond at all with the values assumed by the designer in calculating the length of the transition curve. With the formula given by Mr. Smirnoff, however, true values of C can be computed from tests of existing curves, and design standards can be made to conform more closely to actual driving practice.

In fact, present reports of actual driving experience on curves supply considerable information as to the comfortable and maximum values of C. Messrs. Leeming and Black, in reporting the results of many observations of driving practice on curves,[3] found that of the curves graded E (sensations of lateral acceleration imperceptible to observers) 77% had a C-value greater than 1.00, 36% had a C-value greater than 2.00, and 10% had a C-value greater than 4.00. As a result of the high speed tests on the Pennsylvania Turnpike,[11] Kenneth A. Stonex and Mr. Noble recommended an absolute maximum value of $C = 3.54$. These tests were run at speeds of 100 miles per hr and more, whereas the design speed of the Pennsylvania Turnpike for sections with radii larger than 1,432 ft was 70 miles per hr. The computed value of C for the turnpike curves at the design speed of 70 miles per hr is 1.20. In view of the foregoing tests, the author's recommendation for using a maximum value of $C = 1.00$ should certainly be adopted by highway engineers.

For those engineers who might have found the derivation of Mr. Smirnoff's formula unduly complicated, a more simple derivation, using the notation of the paper, but without the use of calculus, follows:

A vehicle traveling a superelevated curve at a speed greater than that for which the curve is superelevated experiences a centrifugal acceleration of

$$a = \left(\frac{v^2}{g\,R} - e\right) g \qquad \text{(32)}$$

Letting C equal the rate of increase of a, in feet per second cubed,

$$C = \frac{a}{t} \qquad \text{(33)}$$

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[21] Civ. Engr., Bureau of Eng., Los Angeles, Calif.

in which t is the time, in seconds, required to traverse the transition curve. Also,

$$t = \frac{S}{v} \qquad (34)$$

Therefore,

$$C = \frac{a}{S/v} = \left(\frac{\frac{v^2}{g\,R} - e}{S/v}\right) g = \frac{v^3}{R\,S} - \frac{e\,g\,v}{S} \qquad (35)$$

Converting Eq. 35 to miles per hour by substituting 1.467 V for v,

$$C = \frac{47.2\,V}{S}\left(\frac{V^2}{15\,R} - e\right) \qquad (36)$$

and $S = \dfrac{3.15\,V^3}{R\,C} - \dfrac{47.2\,e\,V}{C}$, which is the same as Eq. 15*a*.

Current practice regarding the length of transition curves will not be greatly affected by the use of Eq. 15*a*. There are two methods in general use at the present time for calculating the length of a transition curve. The federal government standards for interregional highways[22] state that transition curves shall be of such length that the difference in slope between longitudinal profiles, separated by the width of one traffic lane, shall not be greater than 1 in 200. These standards also state that all curves sharper than 1° shall be superelevated and that the superelevation shall be such as to counterbalance completely the centrifugal force of a vehicle traveling at 0.75 of the design speed of the road, except that the superelevation shall not exceed 0.12 ft per ft under normal conditions and 0.08 ft per ft where snow or ice may be encountered. If traffic lanes are assumed to be 12 ft wide, the foregoing conditions regarding maximum slope and superelevation result in the following expression for the length of a transition curve:

$$S = \frac{200 \times 12 \times (0.75\,V)^2}{15\,R} = \frac{90\,V^2}{R} \qquad (37)$$

Substituting Eq. 37 in Eq. 36 for S and also substituting $\dfrac{(0.75\,V)^2}{15\,R}$ for e in the same equation,

$$C = 0.0153\,V \qquad (38)$$

With this method of determining the length of a transition curve, C would vary from 0.765 at 50 miles per hr to 1.071 at 70 miles per hr.

The formula for the length of a transition curve which has had perhaps the widest use in highway design is

$$S = \frac{1.58\,V^3}{R} \qquad (39)$$

Substituting this expression in Eq. 36 and using the amount of superelevation recommended for interregional highways, $C = 0.875$. The use of Eq. 39 for transition curve lengths results in a constant value for C at all design speeds.

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[22] "Interregional Highways," *House Document No. 379*, U. S. Govt. Printing Office, Washington, D. C., 1944.

Eqs. 37 and 39 give equal values for S at 57 miles per hr. At velocities greater than 57 miles per hr, Eq. 39 will give the greater value and should be used for design speeds of 60 miles per hr or greater. For velocities less than 57 miles per hr, Eq. 37 gives the greater length and should be used for design speeds from 30 miles per hr to 55 miles per hr. Under the foregoing conditions both Eq. 37 and Eq. 39, now in general use, will give values of C well below the maximum value of 1.00 recommended by the author, and their continued use is recommended by the writer.

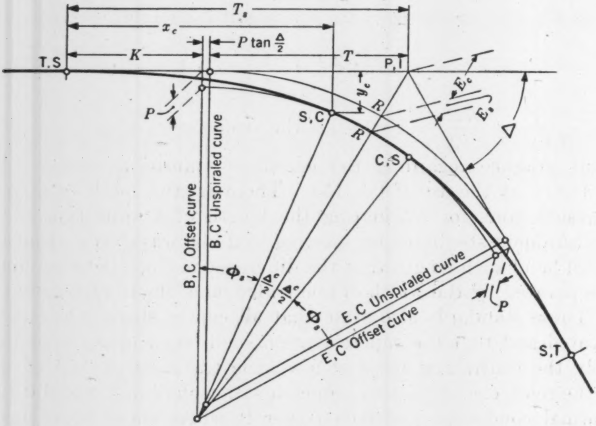

FIG. 1.—ELEMENTS OF THE TRANSITION CURVE

Since the tendency of the formula presented by Mr. Smirnoff is to permit shorter transition curves than were previously thought desirable, and since the shorter the transition curve, the less shift there will be in the path of a vehicle, it may be desirable to eliminate transition curves completely when the shift in the vehicle path is small enough to be taken up in one traffic lane. If transition curves can be eliminated without any sacrifice of safety, it would certainly be desirable, since there can be no question but that a spiraled alinement is more complicated to design and construct than one containing only circular curves. The interregional highway standards state that all curves with radii less than 2,865 ft shall be designed with transition curves. In the writer's opinion this rule is too general. The true criterion as to whether or not transition curves should be used is the amount the circular curve must be shifted to insert the transition curves. In Fig. 1 this shift is designated as P. Also,

$$\theta_s = \frac{S}{2R} \qquad (40)$$

$$K = \frac{S}{2} - \frac{S^3}{240 R^2} \qquad (41)$$

$$y_c = \frac{S^2}{6R} - \frac{S^4}{336 R^3} \qquad (42)$$

and

$$x_c = S - \frac{S^3}{40\,R^2} \qquad (43)$$

The maximum shift in the path of a vehicle if it follows the spiraled alinement is

$$E_s = P\left(\sec\frac{\Delta}{2}\right) \qquad (44)$$

Assuming that the maximum space available for maneuvering laterally in a 12-ft traffic lane is about 2.00 ft, and substituting this value for E_s in Eq. 44,

$$P = 2.00\left(\cos\frac{\Delta}{2}\right) \qquad (45)$$

For a 90° central angle, P would be 1.41 ft; that is, when P is less than 1.41 ft a transition curve would be unnecessary since there is sufficient width in the traffic lane for the vehicle to make its own transition. Therefore, it would seem to involve no compromise with safety to state that, when P is less than 1.00 ft, transition curves can be eliminated completely and the superelevation transition can be run out on the tangent and the circular curve.

It can be shown that

$$P = \frac{S^2}{24\,R} - \frac{S^4}{2{,}688\,R^3} + \frac{S^6}{506{,}880\,R^5} - \cdots \qquad (46)$$

The first term of Eq. 46 will give values accurate enough for determining whether or not transition curves should be used. When P is less than 1.00 ft, in the writer's opinion, radii do not require transition curves. Drivers will be able to create their own transitions, maintain sufficient side clearances between adjacent vehicles, and still remain in one traffic lane without the use of transition curves.

Ralph A. Moyer,[23] **Assoc. M. ASCE.**—The author is to be commended for developing a term representing the effect of superelevation as a refinement of the widely used formula given in Eq. 13 for the length of the highway transition spiral. In 1934 the writer published a comprehensive report[24] in which the mathematical analysis for the formula in Eq. 13 was given, together with experimental data covering the various phases of vehicle operation on highway curves and their application to highway design. In that report the writer presented considerable data in support of the well-known fact that superelevation makes it easier to steer a vehicle on a curve, and that it makes the ride more comfortable. The data also indicated that the length of the transition spiral could be made shorter on a curve with superelevation than on a curve with the same radius but without superelevation. The writer, however, did not develop a term in the formula in Eq. 13 to express, on a mathematical basis, the shortening in the length of the spiral for various values of superelevation. This refinement has now been provided for in Eq. 15.

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[23] Prof. of Civ. Eng. and Research Engr., Inst. of Transportation and Traffic Eng., Univ. of California, Berkeley, Calif.

[24] "Skidding Characteristics of Automobile Tires on Roadway Surfaces and Their Relation to Highway Safety," by R. A. Moyer, *Bulletin No. 120*, Eng. Experiment Station, Iowa State College, Ames, Iowa, 1934.

The analysis by Mr. Smirnoff is made only on a force basis. From the discussion by the author and the data presented, one gains the impression that the transition spiral is required solely to define the curved path of the vehicle that will provide a constant rate of change of radial acceleration in terms of the unbalanced centrifugal force developed by the vehicle on the curve, and that, presumably, a transition spiral is not required for the purpose of developing the maximum superelevation on the curve.

According to Eq. 15*a*, the computed length of spiral, S_s, for a 6° curve (with $e_{\max} = 0.10$, $C = 1$, and $V = 60$ miles per hr) is 177 ft, whereas the length required to develop the maximum superelevation as computed by the term $\frac{v\,g\,e_{\max}}{C}$ is 236 ft. It is the writer's contention that the S_s-value applies only when it is greater than the $\left(\frac{v\,g\,e_{\max}}{C}\right)$-value, and that the latter value should be used when it is greater than the S_s-value; that is, the transition spiral should always have a length equal to, or greater than, the length required to develop the maximum superelevation; and it is on this basis that the length should be great enough to permit the driver to steer the vehicle in such a way as to develop, at a safe and comfortable rate, the radial acceleration provided by (1) the unbalanced centrifugal force, and (2) the centripetal force obtained by the use of superelevation on the curve.

Although the writer recognizes the validity of the author's mathematical analysis on a force basis, the writer's study and experimentation on the subject of vehicle behavior on curves have led him to believe that a better understanding of the transition curve length requirements can be obtained by an examination of the steering requirements on curves, with and without superelevation, rather than by using only a mathematical analysis on a force basis.

The principal concern of the driver and the highway designer is that the driver should be able to steer the car around the curve safely at a reasonable speed consistent with the speed design standard for the given highway. The driver is not concerned and has little knowledge of the centrifugal, frictional, and superelevation forces involved in steering the car around the curve. He should only be concerned with the control and movement of the steering wheel to keep the car in the desired curved path.

The actual forces at the rim of the steering wheel within the normal driving speeds rarely exceed 5 lb and usually are of such a small magnitude that they are difficult to measure. The centrifugal and frictional forces that may be developed by the vehicle on the curve, as the result of turning the steering wheel, may exceed 1,000 lb; and it is evident, therefore, that the driver with little effort can set into motion very large forces that can be controlled only by the skilful manipulation of the steering wheel. It is for this reason that highway curves should be designed to permit a gradual change when the vehicle is being steered from the tangent to the curve, where large forces are required to hold the car on the given curved path at high speeds.

In Fig. 2 the average variations in the turning or steering angles at the steering wheel and the resulting coefficients of friction are shown for three popular make passenger cars driven at various uniform speeds on 5° curves and

10° curves, with and without superelevation. The turning angles at the steering wheel increase for each curve as the speed is increased to some power approaching the square of the speed. At slow speeds on the superelevated curves, negative steering angles are required to steer the car along the desired path of the curve since, at these speeds, the superelevation will pull the car toward the center of the curve. At a speed of 30 miles per hr on both superelevated curves, the gravity force provided by the superelevation is equal to the centrifugal force and the steering angles required are the same as the wheel base angles. When measured at the front wheels, the steering angle is the angular deviation of the front wheels from the car axis; and, at the speed where the lateral forces acting on the car are balanced, it is equal to the angle subtended by the wheel base of the car on the given curve.

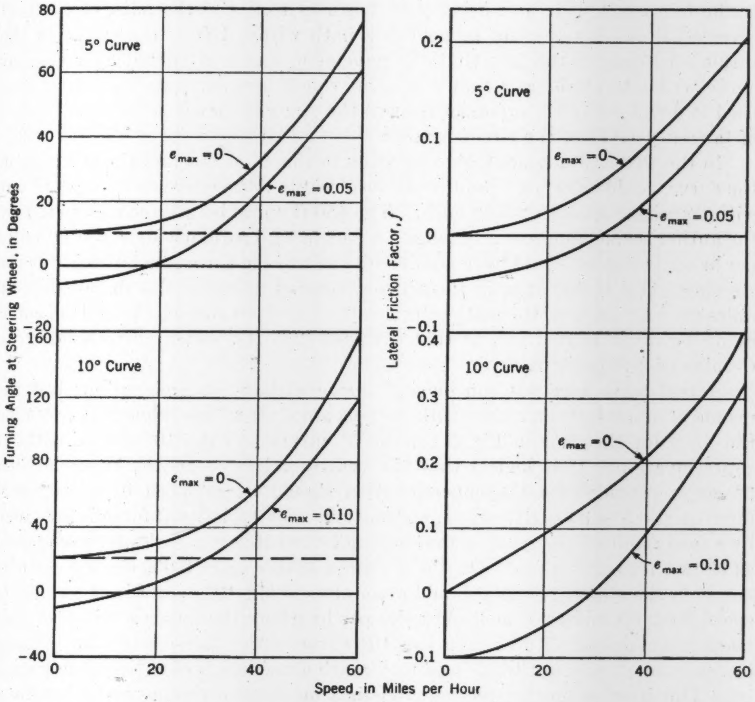

FIG. 2.—VARIATIONS IN THE TURNING ANGLE AT THE STEERING WHEEL AND THE LATERAL FRICTION FACTOR WHEN DRIVING PASSENGER CAR AT VARIOUS UNIFORM SPEEDS ON 5° AND 10° CURVES WITH AND WITHOUT SUPERELEVATION

For speeds greater than 30 miles per hr on these curves, the steering angles increase rather sharply since the tires at both the front and rear must be turned toward the center of the curve by increasing amounts to develop the slip angles required to produce the tire friction and road friction which are necessary to hold the car in the desired curved path at the given speed.

Actually, at 60 miles per hr on the 5° curve, the steering angle (amounting to about 60°) should be considered as an easy turn. The spiral curve length for this curve, using Eq. 15 (with the recommended value of $C = 1$ and a design speed of 60 miles per hr), is 451 ft. Neglecting superelevation, and using $C = 2$, the corresponding spiral length according to the widely used Eq. 13 is 296 ft. Tests conducted by the writer on a two-lane concrete pavement (with traffic lanes 9 ft wide and using path markers to define the path of the car) revealed that the car could be held in a path within 1 ft of the center of the traffic lane despite the fact that the pavement was constructed as a circular curve without spirals, and that a tangent runoff section, only 80 ft long, was used to develop the full superelevation at the point of curvature (PC) and point of tangency (PT) of the circular curve.

In the writer's judgment, the approach to this 5° curve, and the steering on the curve, could have been improved considerably if the curve had been laid out with transition spirals at both ends. The 450-ft spiral length recommended by the author seems unnecessarily long. A person who regularly drives at 60 miles per hr on a tangent would have no difficulty in turning through a 60° angle at the steering wheel in 450 ft or in the 5.1 sec required to travel this distance at 60 miles per hr. In fact, the tests on this curve indicated that the transition could easily be made in one half of that distance (225 ft), which corresponds to a C-value of 2 ft per sec^3.

Actual tests were not run on a 5° curve without superelevation; but the results of many tests on curves with varying amounts of superelevation provided the basis for the data in Fig. 2 for the 5° curves and the 10° curves without superelevation. It is logical that the centripetal force on the superelevated 5° curve provided by the superelevation should be replaced by a frictional force on the 5° curve without superelevation. This frictional force is obtained by a fairly uniform increase in the slip angles and steering angles for each speed as shown for each curve. On the 5° curve with $e_{\max} = 0.05$, the average increase in the steering angles should amount to about 10° except in the range of speed from 50 miles per hr to 60 miles per hr where the increase will probably reach a maximum of 15°. On the 10° curve with $e_{\max} = 0.10$, the average increase should amount to 20° and may reach a maximum of 40° at 60 miles per hr. This increase in the steering angle is a measure of the increased length of transition required on a curve without superelevation as compared to the length required at the same speed on a curve of the same radius with superelevation. Thus, with a 25% increase in the steering angle at 60 miles per hr for the 5° curve, the length of the transition curve should be 25% longer for the curve without superelevation than for the curve with a superelevation of $e_{\max} = 0.05$.

If the value of $C = 1$ is used for both terms in the formula in Eq. 15*a*, the S_s-value for the 5° curve ($V = 60$ miles per hr) is 451 ft when $e_{\max} = 0.05$, and 593 ft when $e_{\max} = 0$; that is, the increase in spiral curve length, according to the formula, is about 32% as compared with a 25% increase based on the steering angle requirements. The analysis of the length requirements of a spiral curve, in terms of steering angles, indicates that higher C-values may be used in computing the length requirements for developing superelevation than for developing the radial acceleration produced by the unbalanced centrifugal force. However, the writer believes that the difference in the *C*-values and in the length requirements obtained by this method of analysis is not large enough to require the use of two *C*-values in Eq. 15*a*. The author's mathematical analysis indicates that the same *C*-value should be used for both terms in Eq. 15*a*. At least that seems to be his interpretation inasmuch as he uses only one *C*-value in Eq. 15*b*.

Although the author did not discuss the length requirements of transition curves to develop the maximum superelevation, the factors that make this an important consideration in computing the length of the transition spiral should be emphasized as follows: Superelevation is the most important factor that railroad engineers have to consider in determining the length of transition for railroad curves because, in railroad practice, a large part of the centrifugal force on a curve is balanced by a gravity force provided by superelevation. It is unsafe to depend on a large amount of friction or flange pressure force to hold the trains on a curved track. Since superelevation is the dominant factor in the design of railroad curves, the length of the railroad spiral has generally been determined by using a uniform rate of rotational change in the cross slope of the track to develop the maximum superelevation. The maximum rate of rotation used in determining the length of railroad spirals has been established by the American Railway Engineering Association (A.R.E.A.) as the rate at which the superelevation may be attained without discomfort to passengers. A rate of 1¼ in. per sec, which is equivalent to a rotational change of 2% of cross slope per sec at the assumed speed, is recommended by the A.R.E.A.[25]

The *C*-value corresponding to a rotational change of 2% of cross slope per sec is equal to 32.2 (0.02) or 0.644 ft per sec^3. This is a very low value for *C*; and, although it may be a desirable value for providing a comfortable train ride, it is far below the values used or required in highway practice. It has been almost a universal highway practice in the United States to lay out the center line using circular curves without spirals and to develop the maximum superelevation, in part or entirely, on the tangent. Obviously, the lengths of the transition sections used to develop the maximum superelevation have been short. The writer is familiar with the curve standards in Iowa, and it is on these curves that practically all his test driving was done. The Iowa standard for developing superelevation for many years called for an 80-ft transition section on the tangent ahead of the PC and following the PT for all curves on which $e_{\max} = 0.05$ or less; and a 125-ft transition section was used when $e_{\max} = 0.062$ or more. Although the Iowa standard did not indicate a speed factor or time value, the application of a suitable speed value (as, for example, 60 miles per hr on a 5° curve and 45 miles per hr on a 10° curve, for which test data are given in Fig. 2) will yield *C*-values using the author's formula, $\frac{v\,g\,e_{\max}}{C}$, of 1.76 ft per sec^3 and 1.70 ft per sec^3, respectively, for the 5° curves and the 10° curves. The rotational change on these curves was three times greater than the usual railway standard, but the higher rate was neither hazardous nor uncomfortable in the tests at the accepted design speeds for these curves.

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[25] *Bulletin No. 108*, A.R.E.A., February, 1911.

In the writer's opinion, the requirements for obtaining a comfortable ride in a railroad train are quite different from those for an automobile on a paved highway; and the needs are likewise different. The shock absorbing rubber tires of the automobile are a major factor contributing to a comfortable ride; and, at the same time, the passengers in the automobile do not require the smooth ride demanded in a railroad dining car. It is not uncommon on a highway to have a change in pavement crown of 2 in. in 30 ft; whereas, according to railway curve standards, at least 80 ft would be required to make such a change. On the basis of these tests and observations, the writer recommends $C = 2$ as the maximum value for design purposes and $C = 1.5$ as a desirable standard using Eq. 15.

Transition spirals have been used on some notable projects, such as the Pennsylvania Turnpike and the new highways in national parks and national forests, with gratifying success. The writer has observed widespread reluctance on the part of state highway departments, however, to adopt transition spirals as standard practice in the design of new highways, or in making adequate provision on existing highways to permit the driver of a vehicle to make the transition from the tangent to the circular curve with the same factor of safety and comfort as is possible on curves designed with transition spirals. There is little doubt that in the construction of more than 90% of the rural highways in the United States since the 1920's, the horizontal alinement has consisted of tangents and circular curves without spirals. Many design engineers have insisted that transition spirals are not necessary because they have observed that, since drivers are able to steer all types of vehicles with reasonable safety within the 10-ft traffic lanes on old paved highways built without transition spirals, they surely should be able to do so on the wider lanes on new highways. The new formula developed by the author strengthens this belief, since it introduces a term which reduces the length of spiral when compared with the length computed according to the formula now in general use.

The writer concurs with the general belief that transition spirals are not required up to a certain point, but he has a suspicion that few engineers actually know the range in curvature and speed within which it is reasonably safe to design curves without spirals and beyond which the spiral curve requirements increase at an alarming rate. Reference to Fig. 3 will clarify this important factor in spiral curve design. The P-value is an important one in an analysis to determine the spiral curve length requirement, since it is a measure of the amount by which drivers tend to "cut in" toward the center of a curve when making the transition from the tangent to the circular curve (see curves A and B, Fig. 3). Actually the highway is usually laid out with a center-line alinement consisting of a tangent and a circular curve, such as curve A; but the natural path of the vehicle (which both the theory and tests indicate is the easiest to follow) is that shown by curve B, Fig. 3. If P is only 0.14 ft, it should not be necessary to use transition spirals to hold the vehicle within the traffic lane; but if P is 16.15 ft, the vehicle will be forced outside the traffic lane unless transition spirals are built into the pavement, or unless an exceptionally skilful driver is at the wheel developing large C-values. As a practical consideration, on all curves and at all speeds where the P-value is less than 1.0 ft, the driver should have no difficulty holding the vehicle within a 12-ft, or even a 10-ft, traffic lane on highways built with unspiraled circular curves. Thus, under this specification (for a design speed of 70 miles per hr and using the writer's recommended value of $C = 1.5$) transition spirals should not be necessary on 2° curves or less; and, if a value of $C = 2.0$ is accepted (as tests indicate that it can be), spirals should not be necessary on 3° curves or less in so far as lane width requirements are concerned.

Although transition spirals are not necessary on 3° curves or less at 70 miles per hr, as far as the lane width requirements are concerned, they may be necessary to meet the superelevation requirements. Thus, the length of spiral required to effect the rotational change that will develop the maximum superelevation of 0.10 ft per ft, based on the new term proposed by the author $\left(\frac{v\,g\,e_{\max}}{C}\right)$, and using $C = 2.0$, is 165 ft; and, using $C = 1.5$, the length is 220 ft. Even the superelevation requirement can be met on an unspiraled circular curve if one half of the required length to develop superelevation is on the tangent curve and the other half is on the circular curve. The superelevation should be started on the tangent at a point corresponding to the tangent to spiral point (TS) in spiral curve design; it should then be increased uniformly over the computed lengths of 165 ft or 220 ft, to a point on the circular curve corresponding to the curve to spiral point (CS) where the maximum superelevation is reached. A pavement built in the foregoing manner, where the P-value is less than 1.0 ft, would permit identically the same operation with equal comfort on curves without transition spirals as on curves with transition spirals. However, on 4° curves, the P-values range from 1.82 ft to 4.65 ft, and transition spirals

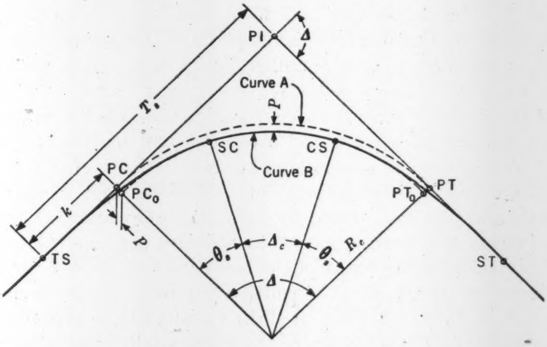

FIG. 3.—CIRCULAR CURVE WITH SPIRAL TRANSITIONS AT BOTH ENDS

from 250 ft to 400 ft in length are required for the 70-mile speed. On 6° curves the P-values range from 6.97 ft to 10.88 ft, and spiral lengths of from 400 ft to 500 ft are required.

At a speed of 70 miles per hr, on a 6° curve, with $e_{\max} = 0.10$, the writer has found the lateral friction factor, f, to be 0.24; and, with $e_{\max} = 0.06, f = 0.28$. The recommended maximum friction value for a design speed of 70 miles per hr is $f = 0.10$. Therefore, a 6° curve should not be considered acceptable on a highway built for a design speed of 70 miles per hr. However, on the Pennsylvania Turnpike it was found necessary to include 6° curves and vehicles are frequently operated at 70 miles per hr on these curves. These curves were designed with spiral lengths of 320 ft and 360 ft which Mr. Noble, special highway engineer for the Pennsylvania Turnpike, reported as a subnormal length.[26] Tests on the 6° curves as reported by Messrs. Stonex and Noble were run at speeds up to an average uniform value of 85 miles per hr which is well above the usual maximum 60-mile design speed for 6° curves. It is significant that in these tests a maximum value of $C = 3.54$ was measured and reported by Messrs. Stonex and Noble on the very same basis as the C-value according to the formula developed by the author. Although the drivers in these tests were highly skilled, and the C-values are close to the maximum that may be developed when entering a curve, these test results indicate that the author's recommended value of $C = 1$ is too low and that the writer's recommended values of $C = 1.5$ and $C = 2.0$ provide an ample margin of safety. The measured and calculated lateral friction factor, f, in these tests averaged 0.3869 and 0.3955, respectively. This is well above the f-value considered safe and was so reported by Messrs. Stonex and Noble. The most positive evidence of the steering hazard at 85 miles per hr on the 6° curve was the marked deviation in the paths of the test cars with deviations of 12 ft and 16 ft measured over the 24-ft paved width available. These deviations in the paths of the test cars are an indication of the P-values to be expected when driving at high speed on a 6° curve where the drivers follow spiral curve paths in making steering adjustments at various points along the curve very similar to those required in going from the tangent to the circular curve.

The spiral curve data compiled by the writer[25a] indicate that, on the sharper curves permitted for a speed of 50 miles per hr, larger values of P are encountered for the corresponding C-values and f-values than in the curve design at 70 miles per hr. At 50 miles per hr the accepted maximum lateral friction factor (f) in curve design is 0.16, and a 9° curve is the recommended maximum curve using $e_{\max} = 0.10$. For a 9° curve the P-value is 4.09 ft when $C = 1.5$ and 2.62 ft when $C = 2.0$. These P-values are so large that many drivers will find it impossible to maintain control of the vehicles within 10-ft, or even 12-ft, traffic lanes if the curves are designed as circular curves without spirals. The data for speeds of 50 miles per hr indicate that on a 9° curve spiral lengths of from 200 ft to 300 ft are required. The corresponding data for the 6° curve show that a P-value of 0.98 ft is required. Therefore, transitional spirals are not necessary on 6° curves or less, designed for 50 miles per hr, for the same reasons as those given previously in the discussion of the 2° curves and 3° curves. However, provision should be made in the design of all superelevated curves without spirals to introduce the superelevation in about the same way that it would be introduced if transition spirals were used.

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[26] "Curve Design and Tests on the Pennsylvania Turnpike," by Kenneth A. Stonex and Charles M. Noble, *Proceedings*, Highway Research Board, National Research Council, Vol. 20, 1940, p. 440.

The use of the author's formulas for computing the length of transition spirals requires the acceptance of the spiral as a curve of uniformly changing curvature and a constant rate of change of radial acceleration. The argument has frequently been advanced in discussions of transition spirals that drivers approach curves along many different paths and that it is impossible to design a curved path which all drivers will follow. It is further contended that different makes of vehicles have different steering characteristics which influence the driver's control of the vehicle on curves.

The writer is fully aware of the many variations in the paths of cars when entering a curve due to the driver and the car; but it is his contention that control of the car is greatly simplified if the curves are designed with adequate superelevation and with transition spirals where needed, using a substantial factor of safety of about 3 at moderate speeds and 4 at high speeds. With a factor of safety of 3 or 4, the driver has considerable latitude as to the path he can follow and still stay within the 10-ft to 12-ft traffic lanes. At speeds below the design value, steering is futher simplified and the factor of safety increased. It is when drivers operate at speeds greater than the design speed that the unsafe steering characteristics of certain vehicles will be evident and exceptional skill on the part of the driver may be required to control the car and keep it within the traffic lane. The writer, therefore, favors the spiral curve path with its constant rate of change of curvature as the easiest and least hazardous path along which the average driver can steer a vehicle.

Messrs. Stonex and Noble[27] report results which show that in the Pennsylvania Turnpike tests the friction values (*f*) increased uniformly from zero at the TS of the test curve to a maximum value at the beginning of the circular curve. This supports the validity of the spiral curve theory in which the rate of change of radial acceleration is constant.

The author has made an important contribution in the development of a new term in the formula for determining the length of the transition spiral. This discussion has been offered in an attempt to clarify the author's analysis and its application to highway design in the hope that reasonable standards for the use of transition spirals will be developed and adopted by all highway departments.

MICHAEL V. SMIRNOFF[28] ASSOC. M. ASCE.—With the very interesting discussions presented on the question of transition spirals, it will be possible to make summaries and draw certain conclusions. The discussions show that the question of the length of transition curves has been approached from many different angles which are just as important as the unbalanced centrifugal force used in the derivation of Eq. 12 by the writer.

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[27] "Curve Design and Tests on the Pennsylvania Turnpike," by Kenneth A. Stonex and Charles M. Noble, *Proceedings*, Highway Research Board, National Research Council, Vol. 20, 1940, p. 443.

[28] Lecturer, Dept. of Eng., Univ. of California, Los Angeles, Calif.

The questions raised were: Whether the rate of change of radial acceleration, C, should be considered by highway designers as being constant; and how the length of transition spiral may be affected by such factors as the lateral friction factor, f, the rotational effect of the automobiles, the superelevation runout, the value of p on spirals, and the steering and the slip angles developed by automobiles on curves.

The Rate of Change of Radial Acceleration.—The important factor in Eq. 14 is the rate of change of the unbalanced centrifugal force which is certain to affect the drivers. Obviously there must be a definite limit of this rate of change of force that could be considered safe and comfortable by all drivers and riders. It is difficult to arrive at such a limiting value merely because its magnitude is much more a matter of personal reaction and judgment than the subject of an exact mathematical analysis; but, once a certain value of C is agreed on, there seems to be no reason to suppose that it should vary under different design conditions. The value of C may be variable as far as the rate of change of any acceleration may be variable; but, to a highway designer, it is no more variable than the speed which he considers constant for the particular highway he is designing.

Mr. Tyson presents arbitrary sliding scales for C in Eqs. 29 and 30 to make, as he points out, the spirals longer for higher speeds and to provide for the possible future increase in speeds. Such provisions are scarcely necessary, since the length of spiral, as shown by Eq. 12, is directly proportional to the speed and, thus, will be longer for higher speeds, whereas a possible future increase in speed will affect not only the length of transition spiral but also the circular curves, the superelevation, and other factors which will not be safe for higher speeds.

In regard to the proper value of C in Eq. 12, the question might well be asked as to how such a value as 1.5 ft per sec^3 will affect a man standing in a bus while the latter is moving along a transition curve, say, in 5 sec. If the unbalanced centrifugal force is the controlling factor in the design and the man weighs about 200 lb, $\frac{dF}{dt} = 9.3$ lb per sec. Thus, the man will be pushed by a gradually increasing force which will accumulate to an additional 9.3 lb at the end of each second, until finally the man will feel a thrust of 46.5 lb on the circular curve. To give this man 3 sec or 5 sec to brace himself against that force may spell the difference between an accident and escape, and it certainly will be more comfortable for him to brace himself in 5 sec than in 3 sec. The writer finds it impossible to support Mr. Leeming's contention that there is no such thing as a safe or a comfortable rate of change of force.

A word of caution should be offered about Mr. Thompson's suggestion to substitute Eq. 39 in Eq. 36 in order to determine the value of C. Eq. 39 was originally derived from Eq. 13 by using $C = 2$. Thus it will be possible to substitute only Eq. 13 in Eq. 36—giving a relation between C_{12} as used in Eqs. 36 or 12 and C_{13} as used in Eq. 13. The relation is

$$C_{12} = \left(1 - \frac{15\,R\,e}{V^2}\right) C_{13} \quad \ldots\ldots\ldots\ldots\ldots\ldots (47)$$

For a specific example of a 6° curve, with $V = 60$ miles per hr and $e = 0.10$, $C_{12} = 0.6\ C_{13}$. Although this equation does not hold for other conditions, it gives a good idea of the corresponding values of C as used in two equations. Mr. Moyer states that he considers $C_{12} = 1.5$ as a desirable value, whereas Mr. Barnett[14] recommends $C_{13} = 2.0$, which corresponds approximately to $C = 1.2$. The writer is inclined to take the average of 1.2 and 1.5 which will give $C = 1.35$ for Eq. 12 or Eq. 15, as discussed subsequently.

The Lateral Friction Factor.—The desirable maximum value of the lateral friction factor, f, has been well investigated by Messrs. Moyer, Stonex and Noble, Barnett, and others. As reported by Mr. Moyer and D. S. Berry,[29] Assoc. M. ASCE, most states use $f = 0.15$ to determine a safe driving speed on curves for posting. To quote Messrs. Moyer and Berry,[30]

> "The general acceptance of this value is rather surprising because it is after all an arbitrary value at which the driver of a car senses some discomfort and where the hazard of skidding off the curve becomes apparent."

The authors also express the opinion that, for speeds greater than 60 miles per hr, a lower value should be used. Messrs. Stonex and Noble[26] came to the conclusion that $f = 0.10$ should be used for speeds of 70 miles per hr and greater-

The lateral friction factor enters a well-known equation,

$$e + f = \frac{V^2}{15\ R} \quad \text{(48)}$$

For a given design speed, Eq. 48 gives two parameters from which a third unknown may be determined. This equation can be best employed to determine the maximum radius of a circular curve for a maximum rate of superelevation, e, and a desirable value of f or, else, to determine on a given curve the rate of superelevation for a desirable value of f.

Substitution of e from Eq. 48 in Eq. 15a gives Eq. 24, $S_s = \frac{47.2\ V f}{C}$, cited by Mr. Meyer. The value of superelevation, which may be found from Eq. 48, is usually rounded to the nearest one sixteenth of an inch; and, to find the final value of f for Eq. 24, Eq. 48 must be used. Eq. 24 is no more applicable than Eq. 15a, since either one can be used with the same results.

In case of recommendations for interregional highways, cited by Mr. Thompson, the balanced superelevation on curves would be designed for 75% of design speed, provided that e does not exceed 0.12. Thus, $f = 0$ when $0.75\ V$ is used instead of the V in Eq. 48. The proper substitution gives $e = \frac{0.5625\ V^2}{15\ R}$, and

$$\frac{e}{e + f} = 0.5625 \quad \text{(49)}$$

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[29] "Marking Highway Curves with Safe Speed Indications," by R. A. Moyer and D. S. Berry, *Proceedings*, Highway Research Board, National Research Council, Vol. 20, 1940, p. 399.

[30] *Ibid.*, p. 409.

Eq. 49 gives a definite relation between e and f, or $f = 0.778\, e$. When this value is substituted in Eq. 24, it becomes

$$S_s = \frac{36.7\, V\, e}{C} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (50)$$

Eq. 50, like Eq. 15*a*, is based on an unbalanced centrifugal force. Unfortunately, it will have no practical application, since, as it will be evident from further discussion, the unbalanced force will not be a controlling factor in the length of transition spiral when the superelevation is designed for 75% of design speed, and Eq. 50 will not hold for any other conditions.

It is interesting to note that Mr. Tyson gives $k = 0.40$ as the value for $\frac{e}{e+f}$ in Eq. 49, which is used in Great Britain. This value would make the superelevation of roads in Great Britain designed for about 63% of the design speed. Under these conditions, the unbalanced centrifugal force will be always a controlling factor in the length of transition spiral.

The Rotational Effect of Automobiles.—It was brought out by Messrs. Klingel, Meyer, and Moyer that the American Railway Engineering Association recommends a rise of 1¼ in. per sec for cars on superelevation runouts for a 4-ft 8½-in. gage of track—which is equivalent to a rate of rotation of 0.0221 ft per ft per sec. In railroad practice the main consideration in the rotational effect of the cars is attached to the fact that on superelevation runouts the rail gradients remain unparallel, which results in one diagonal pair of wheels of a car truck having an uneven bearing pressure as compared with the other pair. This fact leads to an uneven compression of the springs, to the danger of the leading wheel climbing the rail, and to other complications which are absent in case of an automobile. Besides, automobile chassis are more flexible than are railroad car trucks and the rate of rotation for automobiles can be considerably greater than that for the railroad cars before any discomfort will be noticed by the riders.

If the rate of rotation of automobiles is designated by ϕ feet per foot per second and if any length of spiral S is considered to have a uniform rate of gradient $\frac{de}{dS}$, the time that an automobile will travel this length S at a speed of v feet per second will be S/v seconds. Thus, $\frac{S}{v}\phi = S\frac{de}{dS}$, or $\phi\, dS = v\, de$, and for the entire spiral $\phi \int_0^{S_s} dS = v \int_0^{e_{\max}} de$, which gives

$$S_s = v\,\frac{e_{\max}}{\phi} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (51)$$

Eq. 51, which reduces to Mr. Meyer's Eq. 25 when $\phi = 0.0221$, shows that the length of spiral needed on the basis of rotational effect of cars is the same function of v and $e_{\max}$ as the second term in Eq. 12—that is, $\frac{v\, g\, e_{\max}}{C}$. If K instead of C is used as a constant of proportionality and the two terms are equated,

$$K = g\,\phi \qquad (52)$$

and Eq. 51 becomes

$$S_s = \frac{v\,g\,e_{\max}}{K} \qquad (53)$$

In order to obtain a dimension for S_s, in feet, K must be expressed in feet per second.[3]

The interregional standards, cited by Mr. Thompson for the minimum length of transition spiral, to be such as not to exceed the difference in longitudinal profiles of 1 ft in 200 ft in one traffic lane are the requirements that limit the rate of rotation of cars to a certain maximum. Substituting in Eq. 51, $S_s = 200$ and $e = \frac{1}{w}$ (in which w is the width of one traffic lane) and solving for ϕ, $\phi = 0.00733\,\frac{V}{w}$ (in which V is in miles per hour). The section of interregional standards cited applies for speeds greater than 50 miles per hr, which means that, according to its provisions, the rate of rotation will be satisfactory even at high speeds. Thus, if $V = 70$ miles per hr is used on, say, an 11-ft pavement, $\phi = 0.0467$. This value is slightly more than twice the rate of rotation recommended on railroads—that is, 0.0442. If it is assumed then that $\phi = 0.0442$ is within the limits set by interregional standard requirements, $K = 1.42$ from Eq. 52.

The desirable value for K can also be verified by the standards of various highway departments. For instance, if those cited by Mr. Moyer for Iowa are used, a transition section of 125 ft is required for an $e_{\max}$ of 0.062 or greater, and one of 80 ft for an $e_{\max}$ of 0.05 or less. Substituting these values in Eq. 53 for a speed of 60 miles per hr, or 88 ft per sec, one obtains $K = 1.40$ or greater in the first case, and $K = 1.77$ or less in the second case. Similar values for K may be computed by using the standards of other highway departments. The foregoing examples show that $K = 1.35$ is within the limits provided by the existing highway standards.

The close correlation which obviously exists between the K-values in Eq. 53 and the recommended values for C in Eq. 12 really has a deeper significance. The second term in Eq. 12 represents the reduction, due to superelevation, in the length of transition spiral based on centrifugal force without superelevation. In such a case, the second term itself should be adequate for the lengths of spiral due to superelevation. It develops, then, that K in Eq. 53 and C in the second term of Eq. 12 are one and the same constant—which result gives a new basis for judging the proper value of C in Eq. 12. When the second term in Eq. 12 is greater than the length obtained by the entire equation, that term should be used for the length of transition curves. Mr. Moyer, no doubt, came to the same conclusion when he made his statement to this same effect. Thus, instead of using more or less arbitrary distances for superelevation runouts, either on a spiral or on a tangent, a more consistent expression of the second term of Eq. 12 can be used for such purposes. It will provide better designed highways.

The Effect of p-Values on Spirals.—Mr. Moyer presents valuable experimental data regarding p-values. According to these data, drivers have no difficulties in negotiating a circular curve—that is, keeping in lane—without a transition spiral if the latter has a p-value equal to 1 ft or less. Mr. Thompson supports this contention by mathematical analysis.

On curves flatter than 2.5°, p-values greater than 1 ft are possible only at speeds greater than 70 miles per hr; and, on curves flatter than 5.5°, p-values greater than 1 ft are possible only at speeds greater than 40 miles per hr. Thus, within these speed limits, the degrees of curvature between 2° 30′ and 5° 30′ will offer a basis to judge such minimum lengths of transition curves beyond which the curves will apparently serve no useful purpose.

Using the first term in the series given by Eq. 46, substituting $5{,}730/D$ for R, and solving for the length of spiral, one obtains, $S = 370 \sqrt{p/D}$. For $p = 1$ and $D = 2° 30'$ and $5° 30'$, S is equal to 234 ft and 158 ft, respectively. Thus, relying on the presented data regarding p-values, it may be concluded that, from the standpoint of keeping a car in lane, the spiral will not be necessary on any circular curve if the computed length of spiral is 150 ft or less.

It has been also stated in discussions that on such curves a spiral may be necessary for the superelevation runout. The superelevation runout will be a controlling factor in the design of spirals when twice the value of the second term in Eq. 15*a* is greater than the first term. Setting such inequality and substituting e from Eq. 48, one can easily verify the fact that the superelevation runout will be the controlling factor when $e > f$. Such conditions may occur on relatively flat curves with high speeds or on sharper curves with lower speeds. In either case it would appear adequate to run the superelevation out on the tangent. For instance, the question may arise as to whether it would be necessary to introduce a spiral on a 3° curve designed for 60 miles per hr with $e = 0.07$. With $C = 1.35$ in Eq. 15*a*, $S_s = 263 - 147$, which shows that a 150-ft spiral is required for the superelevation runout. Although the design would be better if such a spiral were introduced, there could not be much objection to running the superelevation out on a tangent on such a curve. Thus, a minimum limit of 150 ft for spirals may serve the purpose in either case.

The Effect of Steering and Slip Angles on Spirals.—The steering and slip angles have been thoroughly investigated by Messrs. Moyer and Stonex and Noble.[26] Mr. Moyer presents an excellent discussion of the question and emphasizes that the C-value in Eq. 15*a* can be greater than 1 ft per sec^3 because of such requirements. Using his analysis, one can check the previously suggested value of $C = 1.35$ to meet the requirements for steering angles. On a 5° curve with $V = 60$ miles per hr and with $e = 0.05$ (which data are used by Mr. Moyer for illustration) the length of spiral will be $440 - 105 = 335$ ft, and 440 ft when $e = 0$—an increase of 31% as compared with the 32% estimated by Mr. Moyer for $C = 1$. The increase based on the steering angle requirement is about 25% from Fig. 2. It appears that the C-values do not appreciably affect the percentage increase in spiral lengths because of omission of superelevation, and that with either *C*-value the length of spiral will fall within the limits needed for the increase in steering angles.

It is worthwhile to mention one additional useful purpose served by transition curves, which is not only to provide the highways with such alinement that will make it easy to keep the cars in lane, but also to eliminate the tendency for cars to leave a lane. A spiral that is too short will not eliminate a "so-called" tendency to "cut the corners," whereas a spiral that is too long unnecessarily increases the amount of field and office work and, thus, adds to the cost of construction.[31]

Conclusions.—From the paper and its discussions it can be concluded that:

1. Although individual opinion of engineers may differ somewhat regarding the proper value of *C* in Eq. 15*a*, the discussions indicate that it should not deviate appreciably from $C = 1.35$ ft per sec.^{3} With this value, Eq. 15*a* becomes

$$S_s = 2.33 \frac{V^3}{R} - 35\ V\ e_{\max} \qquad (54)$$

2. When the second term in Eq. 54 is greater than the length computed by the entire equation, the value of that term should be used for the length of transition curve.

3. The length of transition curve will be the same in the foregoing two cases when the superelevation is computed for 70.7% of design speed. Such a balanced condition occurs when $e = f$.

4. If the required length of a transition curve is less than 150 ft, no serious defect in design will result in omitting the spiral.

5. The second term in Eq. 54 can also be used to determine the length of superelevation runouts on tangents when the spiral is omitted.

6. A better design in any case will result in attaining the maximum superelevation used on curves at the point of connection with a circular curve.

Many thanks are due to those who contributed their discussions and useful experimental data to clarify the purposes and requirements for certain lengths of transition curves. These ends have been ably served.

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[31] "Transition Curves on Transportation Routes," by M. V. Smirnoff, thesis presented to the Dept. of Eng., Univ. of California, at Los Angeles, in January, 1950, in partial fulfilment of the requirements for the degree of Master of Science in Engineering.

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DIVERSION TUNNEL AND POWER CONDUIT OF NANTAHALA HYDROELECTRIC DEVELOPMENT

By D. J. Bleifuss,[1] Assoc. M. ASCE

With Discussion by Messrs. Joseph R. Bowman, and D. J. Bleifuss.

Synopsis

The salient features of the diversion tunnel and power conduit of the Nantahala Hydroelectric Development, in North Carolina, constructed in 1940, 1941, and 1942, are described in this paper. The tunnel was used to carry the flow of the river during construction, and in the finished project provides emergency release from the reservoir. A self-removing temporary closure gate was used. Permanent control is attained by narrow and long fixed-wheel gates operating in a rigid arrangement of guides and liner. An energy dissipator prevents damage to the unlined tunnel below the control gates.

The power conduit carries water from the reservoir to the powerhouse. There are lined and unlined sections of pressure tunnel, a pipe line 9 ft in diameter crossing Dick's Creek, a fully welded penstock carrying a static head of slightly more than 1,000 ft, a surge tank with a restricted orifice, a rock catcher, and an intake. The tunnel was located so as to avoid water loss by seepage.

General Description of Plant

The Nantahala Hydroelectric Plant is situated in the Great Smoky Mountains of Macon County, North Carolina, about 18 miles west of Franklin, the county seat, and 50 miles south-southeast of Knoxville, Tenn. It was designed and built in 1940 to 1942, although preliminary designs had been made years before. Approximately 1 mile of tunnel had been completed in 1931. The development takes its name from the small river on which it is located. The valley is narrow and deep, so the Cherokees christened it "Nantahala," meaning the "valley of the noonday sun."

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[1] Chf. Engr. and Vice-Pres., International Eng. Co., Inc., San Francisco, Calif.

Fig. 1 shows the general layout of the development. A rock and earth-fill dam creates the reservoir. In order to build the dam, it was necessary to drive a diversion tunnel. The power conduit extends from the intake in the reservoir to the powerhouse and has a total length of 29,499.02 ft, or 5.58 miles, horizontal measurement. Full reservoir water surface is at El. 2890.0, and normal tailwater, at El. 1882.0, so that the maximum static head on the plant is 1,008.0 ft. The powerhouse contains a single Francis-type reaction turbine. The dam and the turbine installation[2] have been described elsewhere.

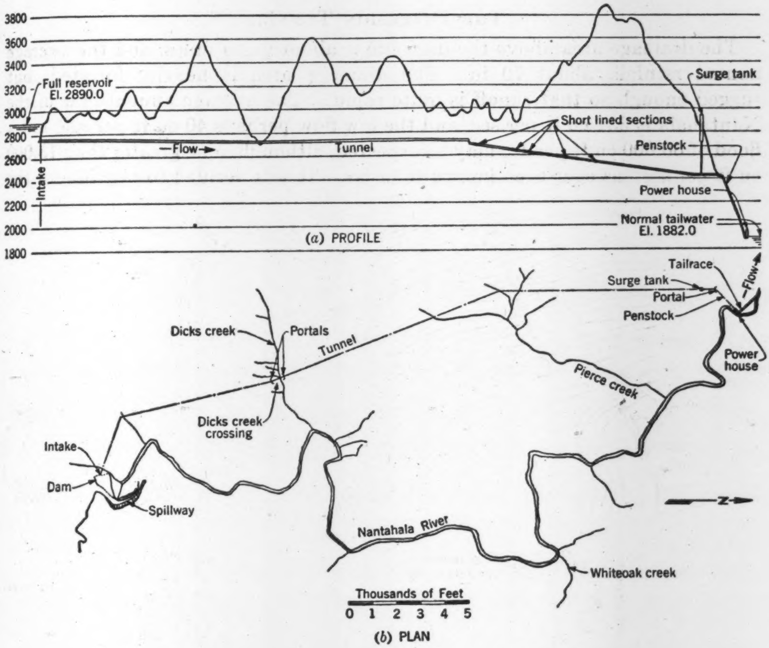
3800
3600
3400
3200
3000
2800
2600
2400
2200
2000
1800
Full reservoir El. 2890.0
Intake
—Flow→
Tunnel
Short lined sections
Surge tank
Penstock
Power house
Normal tailwater El. 1882.0
(a) PROFILE
Tailrace
Surge tank
Portal
Penstock
—Flow→
Power house
Dicks creek
Portals
Tunnel
Pierce creek
Dicks creek crossing
Intake
Dam
Spillway
N
Nantahala River
Whiteoak creek
Thousands of Feet
0 1 2 3 4 5
(b) PLAN

FIG. 1.—GENERAL PLAN AND PROFILE OF NANTAHALA DEVELOPMENT, NORTH CAROLINA

GEOLOGY

The Great Smoky Mountains form part of the Appalachian system and are relatively old. In the area of the Nantahala development, the rock is nearly all arkose, a variety of sandstone containing much feldspar. Feldspars are silicates of aluminum with potassium, sodium, and calcium present in varying amounts, and the sands were presumably derived from the weathering of feld-

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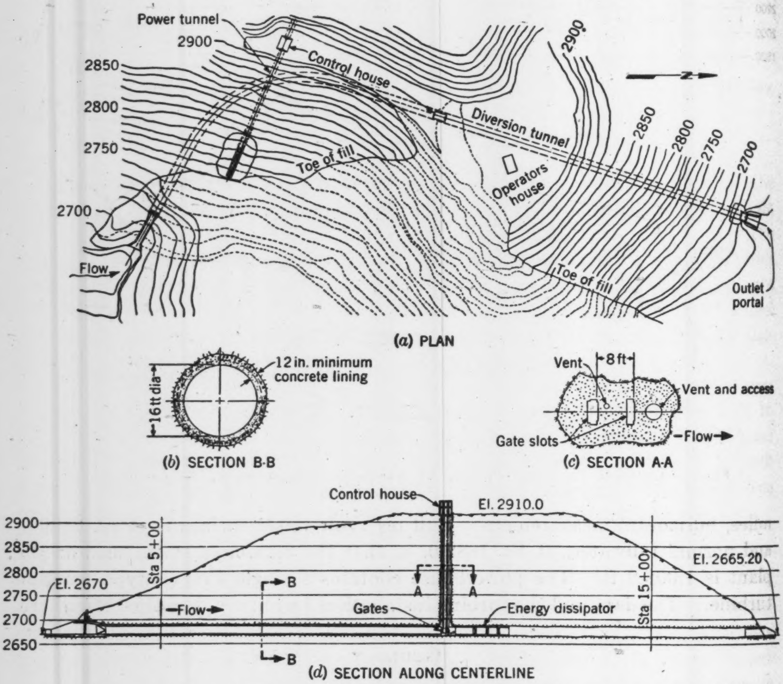
[2] "Nantahala Turbine," by J. P. Growdon, R. V. Terry, and H. H. Gnuse, Jr., *Transactions*, A.S.M.E., Vol. 68, 1946, p. 687.

spathic rocks such as granite. The rock is generally sound, very abrasive, and blocky. When work was begun, no faults were known in the area.

The overburden is a red clay, varying widely in depth, which stands well in its natural condition. Previous experience on another project in similar material had demonstrated no slides need be expected on reservoir slopes.

THE DIVERSION TUNNEL

The drainage area above the dam site is about 91 sq miles, and the average annual rainfall, about 70 in. The drainage area is heavily forested, but rugged enough so that runoff is quite rapid. The average annual flow of the Nantahala is 331 cu ft per sec, and the low flow perhaps 40 cu ft per sec; but a flood of 60,000 cu ft per sec may be expected, although none greater than 15,000 cu ft per sec has ever been known to occur. It was decided to provide for the diversion of a maximum of 7,500 cu ft per sec. Fig. 2 shows the general arrangement of the diversion tunnel.


Power tunnel
2900
2850
2800
2750
2700
Control house
Diversion tunnel
Toe of fill
Operators house
Flow
Toe of fill
2900
2850
2800
2750
2700
Outlet portal
N
16 ft dia
12 in. minimum concrete lining
Vent
8 ft
Vent and access
Gate slots
Flow
Control house
El. 2910.0
2900
2850
2800
2750
2700
2650
El. 2670
Sta 5+00
B
A
A
Flow
Gates
Energy dissipator
Sta 15+00
El. 2665.20
B

(*a*) PLAN

(*b*) SECTION B-B

(*c*) SECTION A-A

(*d*) SECTION ALONG CENTERLINE

FIG. 2.—GENERAL ARRANGEMENT OF DIVERSION TUNNEL

Four main factors influence the discharge capacity of such a tunnel: (1) The tunnel length; (2) the tunnel area; (3) the head available to force water through the tunnel; and (4) the roughness of the tunnel walls.

The tunnel length is determined by alinement and grade, and these depend on the terrain. A diversion tunnel should be as low as possible to reduce the height of the upstream cofferdam. The entrance must be far enough upstream so that ample room is available for cofferdam construction, and so that currents will not disturb the cofferdam. The exit must be downstream a sufficient distance to prevent the discharge from interfering with the work or from damaging any structure. If possible, the axis of the tunnel, at both entrance and exit, should be at right angles to the surface contours. There should also be enough rock cover so that in driving the abutment of the dam is not weakened.

The remaining three factors are interdependent. The head available depends on the height to which the upstream cofferdam is built, and must equal the sum of velocity head and losses. Relative economics of various combinations of cofferdam height and tunnel diameter must be considered. The Manning formula was used to determine losses; thus,

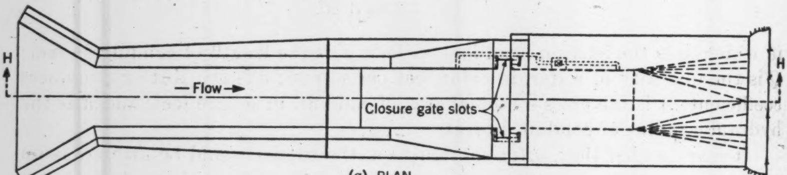
$$h_f = L \frac{(Q\,n)^2}{1.486\,A\,R^{2/3}} \qquad (1)$$

in which h_f is the friction head loss, in feet; L is the length of conduit, in feet; Q is the quantity of water, in cubic feet per second; n is the Kutter roughness coefficient; A is the cross-sectional area of conduit, in square feet; and R is the hydraulic radius of conduit, in feet.

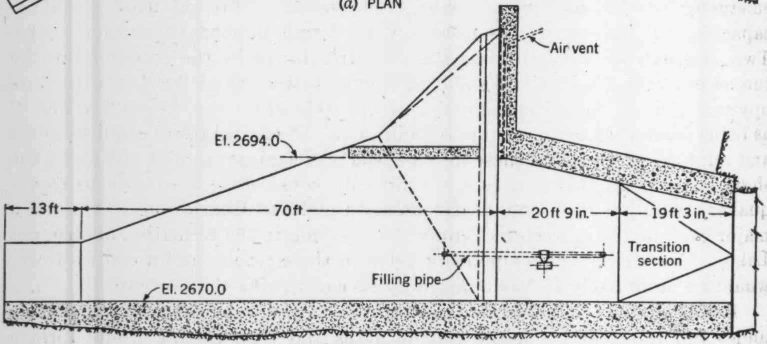
It was decided that, after construction, the tunnel would be utilized as an emergency outlet, and as a means of providing additional flood discharge capacity. Therefore, it was necessary to design permanent control works. Two alternatives were considered—a control tower in the reservoir at the tunnel entrance, and a shaft about on line with the axis of the dam extending upward from the tunnel to the rock surface. The shaft was chosen (see Fig. 2) as being somewhat more economical and safer. A control tower requires forms and reinforcing steel, and these may be balanced against the cost of sinking the shaft. Although this area is not ordinarily considered as subject to earthquakes, there has been one disastrous earthquake at Charleston, S. C., and a major earthquake in western Tennessee (resulting in the formation of Reelfoot Lake). Natahala is about midway between these places, and a control tower would be more likely to be damaged by an earthquake than a shaft.

It was obvious that since the control would have to be capable of operating under full reservoir head, the control gates and their setting would form a constriction that would reduce the carrying capacity of the tunnel during diversion. Therefore, it was decided that the gates would be installed after diversion, while the reservoir was filling, and that, otherwise, as much work as possible should be completed before diversion was started. Accordingly—the tunnel was driven, the concrete portals were constructed, the tunnel was lined upstream from the control gates, and the control shaft was sunk and lined before diversion.

An interesting arrangement was made for closure of the tunnel after diversion to begin filling the reservoir. A closure gate was required which would resist the head while the permanent control gates at the base of the shaft were being installed, and which could easily be removed thereafter. The opening at the intake portal was made high and narrow (10 ft by 24 ft) to accommodate a sliding gate. Fig. 3 shows the design of this structure, and Fig. 4 shows the closure gate. The latter had a skin plate on both faces; it was in fact a caisson. It was determined that not more than 150 ft of head could accumulate while the permanent gates were being placed, and the gate was designed accordingly, using a unit stress of 24,000 lb per sq in. It was placed by a portable crane, and water pressure held it firmly in place as long as the tunnel behind it was empty. The concrete portal was provided with a by-pass valve for filling the tunnel after the permanent gates were installed and closed, and with a 3-in. pipe in the roof to allow the trapped air to escape. The theory was that, when the pressures on the two sides of the gate balanced, it would simply float to the surface. The scheme was successful, but the gate, as fabricated, did not have a strictly planar sealing surface, and permanent gate installation was somewhat hampered by leakage into the tunnel.



(*a*) PLAN



(*b*) SECTION H-H

FIG. 3.—INTAKE PORTAL, DIVERSION TUNNEL

Fixed-wheel gates, with cable hoists, were chosen as being the simplest and most economical control. There are two gates, in series; the downstream gate is considered the operating gate, and the upstream one, the emergency gate, although the two are identical. The first installation step was to place the gate guide and liner assembly, shown in Fig. 5. This was a very rigid arrangement, designed in four pieces for easy shipment and handling. However, it was completely shop assembled and very carefully checked. The field joints were bolted, as it was feared that any welding might disturb the exact alinement of roller paths and seals. The assembly comprised a bell mouth, the pocket for the upstream gate, a tapered section, the pocket for the downstream gate, and another tapered section. Air vents were provided downstream from each gate, since at partial gate openings considerable air is required to prevent irregularities in flow. Seal and roller paths are of stainless steel.

Fig. 6 shows the details of concrete in the shaft. The air vents are extended to the top of the shaft, the downstream vent being made 3.5 ft in diameter so that it may be used as a manhole. The vents terminate in an air passage under the gatehouse floor, and open to the outer air, in order that no partial vacuum can ever be created within the gatehouse. Wood water stops were provided at all horizontal lift joints.

The fixed-wheel control gates, shown in Fig. 7, were made long and narrow. There are two main advantages in this shape—the span between wheels is short, and the gate has less tendency to chatter. The gates were filled solidly with concrete, and will close by their own weight under full head. The rubber seals were of a new design and, when the gates were first closed, these seals were absolutely watertight. However, it has been found that when the gates are operated under full head, the flow over the top of the gate and around the sides above the ends of the seal plates damages the rubber seals. Apparently, the seal plates should have been extended upward far enough so that no flow could pass the side seals until the gate was fully opened. The top seal should have been protected by suitable baffle plates to prevent water under high pressure and velocity from passing the seal.

The gatehouse at the head of the shaft has two stories. The upper story carries the gate hoists, and the lower story is high enough so that a completely hoisted gate will hang clear of the floor and is therefore entirely accessible. Each gate is lifted by two wire ropes 1⅝ in. in diameter. Hoists are electrically driven, and as no permanent power line to the dam was contemplated, a prime mover was provided in the shape of a 10-kw, gasoline-driven generator set. In this mountainous country, subject to intense thunderstorms, a transmission line is likely to give trouble, and the generator set would have been required as a stand-by source of power in any event.

It was known that only infrequent use would be made of this tunnel for emergency discharge. However, it was feared that a jet of water issuing from the gate under a 213.5-ft head would tear rock out of the walls of the unlined tunnel. Experiments were made, and an energy dissipator was designed to destroy the energy of this jet. Details are shown in Fig. 8. Just downstream from the gate is a steel-armored deflector to split the jet and throw the halves upward and outward. This deflector is in a cylindrical chamber, at the lower end of which is a baffle having a circular hole 9 ft in diameter. Downstream is a second cylindrical chamber, baffled in turn, but the hole is 11.5 ft in diameter. A third chamber follows, the hole this time being 14 ft in diameter. The energy dissipator has had one severe test. It was operated continuously

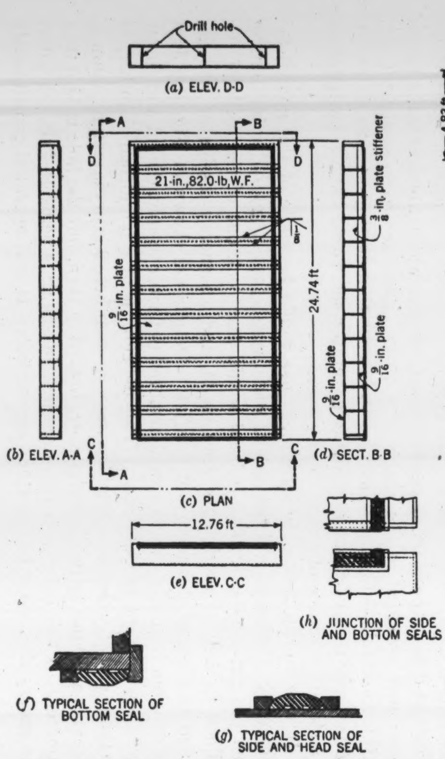

Drill hole
(a) ELEV. D-D
(b) ELEV. A-A
A
B
D
21-in., 82.0-lb, W.F.
$\frac{9}{16}$-in. plate
$\frac{1}{8}$
24.74 ft
$\frac{3}{8}$-in. plate stiffener
$\frac{9}{16}$-in. plate
$\frac{9}{16}$-in. plate
C
(c) PLAN
(d) SECT. B-B
12.76 ft
(e) ELEV. C-C
(h) JUNCTION OF SIDE AND BOTTOM SEALS
(f) TYPICAL SECTION OF BOTTOM SEAL
(g) TYPICAL SECTION OF SIDE AND HEAD SEAL

FIG. 4.—CLOSURE GATE, DIVERSION TUNNEL

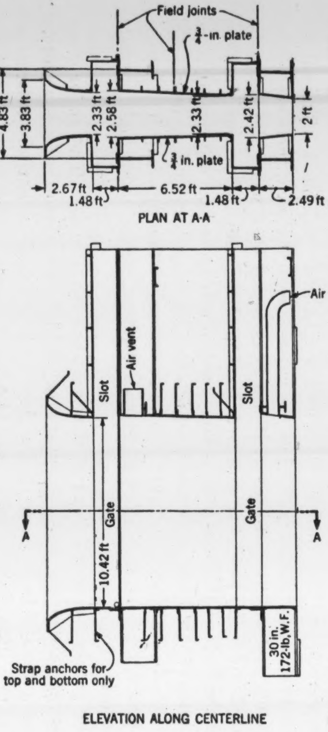

Field joints
$\frac{3}{4}$-in. plate
4.83 ft
3.83 ft
2.33 ft
2.58 ft
2.33 ft
2.42 ft
2 ft
$\frac{3}{4}$ in. plate
2.67 ft
1.48 ft
6.52 ft
1.48 ft
2.49 ft
PLAN AT A-A
Air vent
Air vent
Slot
Slot
Gate
Gate
A
A
10.42 ft
30 in. 172-lb, W.F.
Strap anchors for top and bottom only
ELEVATION ALONG CENTERLINE

FIG. 5.—CONTROL GATE GUIDES AND LINER

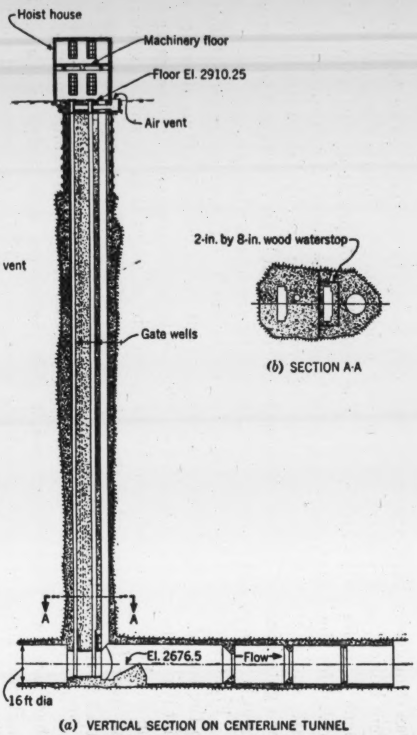

Hoist house
Machinery floor
Floor El. 2910.25
Air vent
Gate wells
2-in. by 8-in. wood waterstop
(b) SECTION A-A
A
A
El. 2676.5
Flow
16 ft dia
(a) VERTICAL SECTION ON CENTERLINE TUNNEL

FIG. 6.—CONCRETE, DIVERSION TUNNEL SHAFT

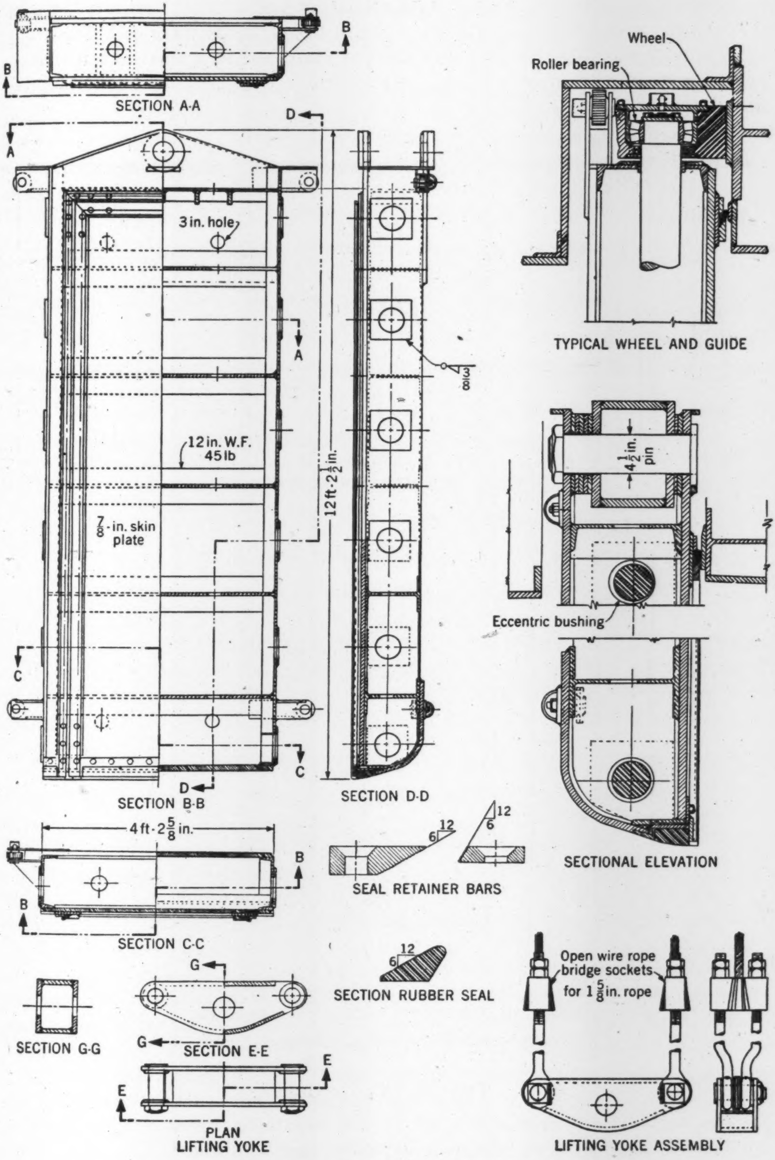
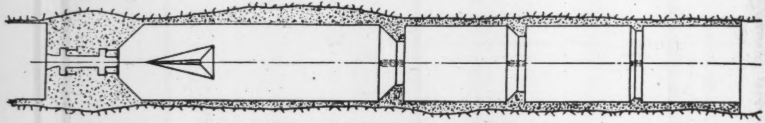

SECTION A-A
B
B
D
A
3 in. hole
A
12 in. W.F. 45 lb
$\frac{7}{8}$-in. skin plate
12 ft-2$\frac{1}{2}$ in.
C
C
D
SECTION B-B
SECTION D-D
$\frac{3}{8}$
Wheel
Roller bearing
TYPICAL WHEEL AND GUIDE
4$\frac{1}{2}$ in. pin
Eccentric bushing
SECTIONAL ELEVATION
4 ft-2$\frac{5}{8}$ in.
B
B
SECTION C-C
12
6
SEAL RETAINER BARS
SECTION RUBBER SEAL
SECTION G-G
G
G
SECTION E-E
E
E
PLAN
LIFTING YOKE
Open wire rope bridge sockets for 1$\frac{5}{8}$ in. rope
LIFTING YOKE ASSEMBLY

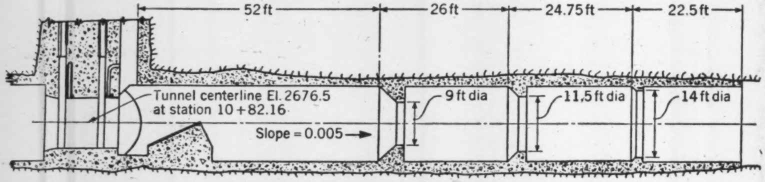
FIG. 7.—CONTROL GATES, DIVERSION TUNNEL

for more than 30 days. Water issued smoothly and quitely, and later inspection showed no evidence of attack on any of the concrete walls or baffles.

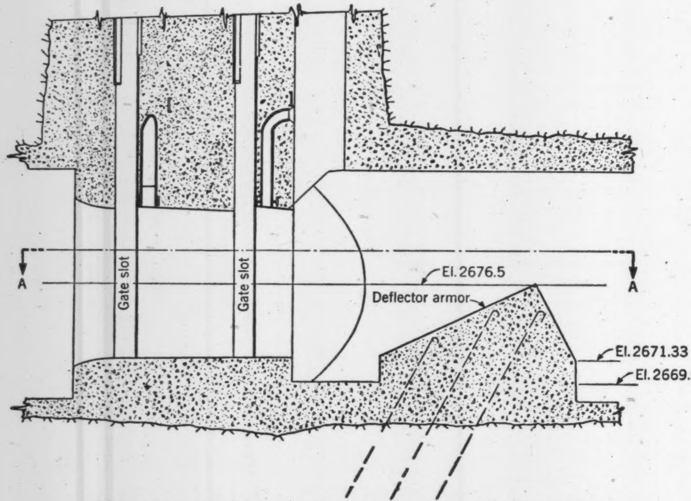
The rock was pressure grouted from the shaft, and in a vertical fan-shaped pattern extending downward from the area where the permanent control gates were later set.



(*a*) HORIZONTAL SECTION - TUNNEL CENTERLINE


52 ft
26 ft
24.75 ft
22.5 ft
Tunnel centerline El. 2676.5
at station 10+82.16
9 ft dia
11,5 ft dia
14 ft dia
Slope = 0.005

(*b*) VERTICAL SECTION - TUNNEL CENTERLINE


A
Gate slot
Gate slot
El. 2676.5
Deflector armor
A
El. 2671.33
El. 2669.67

(*c*) VERTICAL SECTION CENTERLINE CONTROL SECTION AND DEFLECTOR

FIG. 8.—ENERGY DISSIPATOR, DIVERSION TUNNEL

THE POWER CONDUIT

General.—The dam was located at the most economical site, considering all factors, such as cost, reservoir capacity, etc. Its foundation is at about El. 2670.0 and full reservoir level is at El. 2890.0. For approximately 10 miles downstream from the dam site, measured along its course, the river flows in a very deep, narrow, and crooked valley, falling in this distance to El. 1882.0. It then flows out into a wider valley, and its gradient flattens. The mouth of the narrow valley is about 5.5 miles from the dam site, as the crow flies. Preliminary investigation disclosed that the head between the dam site and the valley mouth could be developed by placing the powerhouse at the mouth, and providing a pressure conduit from the dam. The pressure conduit had to be either a steel pipe line or a tunnel, and it was easily demonstrated that it should be mostly tunnel. A pipe line would have been very long, and difficult and expensive to construct and to maintain.

Beginning at the face of the concrete intake, the power conduit comprises:

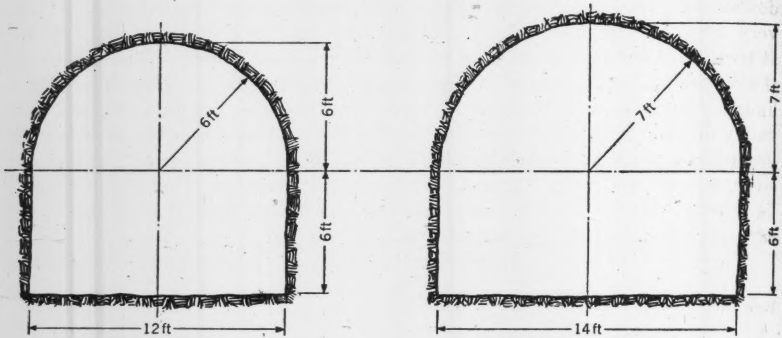
Item	Description	Length, in feet
1.	A tunnel	8,922.10
2.	A steel pipe line, across the valley of Dick's Creek, a tributary of the Nantahala (length between portals)	573.07
3.	A tunnel	18,706.62
4.	A steel penstock, terminating at the upstream face of the butterfly valve above the turbine	1,247.23
	Total length	29,449.02

(There is a surge tank 1,847.23 ft from the face of the butterfly valve.)

The Tunnel.—When the Nantahala development was started for the first time in 1930 it was decided that the character of the rock would permit the use of unlined pressure tunnels, and in general this proved to be true. It was realized that the rock was not absolutely watertight, and it was obviously more desirable to have water leak into the tunnels than leak out of them. The tunnels were therefore so located that, at any point in unlined sections, the static head on them is less than the vertical distance up to the water table. No loss of water has been observed. The area around the tunnels has been carefully examined for any augmentation of springs or brooks, but nothing of the kind has been found, and it is believed that instead of losing water in the tunnel, actually a little is gained. Alinement and grade have been shown in Fig. 1.

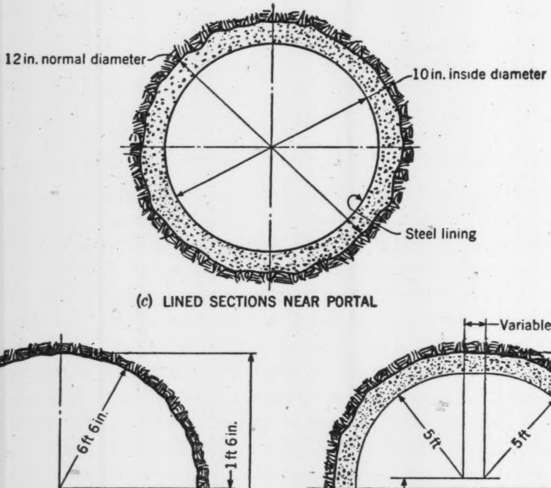
There is a sharp irregularity in the tunnel grade at Station 102 + 92.94. Originally, it was planned to drive from Portal No. 2 on an adverse grade of 0.25% as far as Station 119 + 02.9, and from Portal No. 1 on an ascending grade to intersect at this station. A well hole was sunk at the planned high point to provide a 9-in.-diameter air vent. As the job progressed, it was found advisable to stop driving from Portal No. 1 at Station 102 + 92.94, and complete the tunnel from Portal No. 2. A final grade was adopted as shown. Since water velocities are low, the maximum being only 3.66 ft per sec, the irregularity has a negligible effect on the hydraulic properties of the tunnel, but does make it a little inconvenient to inspect.

Four different cross sections of tunnel were used, as shown in Fig. 9. Work by force account was started at Portal No. 1 in 1930, using a section, 12 ft by 12 ft, with a flat invert and a semicircular roof. It was found that a shot

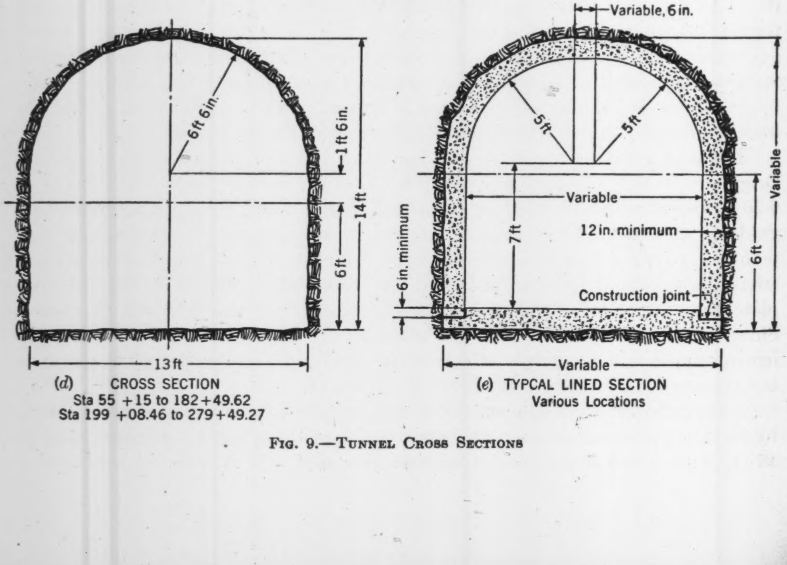

6 ft
6 ft
6 ft
12 ft
7 ft
7 ft
6 ft
14 ft

(*a*) CROSS SECTION
Sta 14+82 to 35+22

(*b*) CROSS SECTION
Sta 35+22 to 55+15


12 in. normal diameter
10 in. inside diameter
Steel lining

(*c*) LINED SECTIONS NEAR PORTAL


6 ft 6 in.
1 ft 6 in.
14 ft
6 ft
13 ft
Variable, 6 in.
5 ft
5 ft
Variable
Variable
7 ft
12 in. minimum
6 in. minimum
6 ft
Construction joint
Variable

(*d*) CROSS SECTION
Sta 55 +15 to 182+49.62
Sta 199 +08.46 to 279+49.27

(*e*) TYPCAL LINED SECTION
Various Locations

FIG. 9.—TUNNEL CROSS SECTIONS

would only "pull" from 6 ft to 7.5 ft. A section, 14 ft by 13 ft, with a 14-ft flat invert and a semicircular roof was tried, and it was found actually to be cheaper per linear foot than the smaller section. This section was used until work was discontinued because of the economic depression in 1931. When work was resumed in 1940, the contractor elected to drive a section, 13 ft by 14 ft, with a 13-ft flat invert and a semicircular roof. The fourth type of cross section was a 12-ft-diameter circle, used where the tunnels were to be steel lined.

Tunnel headings were driven full face, with no pilot tunnel, except at the one point where difficult ground was encountered. At Station 100 + 42 an apparent fault was encountered (extending to Station 100 + 86) which was filled with an assortment of sand, calcite, talc, and disintegrating arkose. A flow of about 150 gal per min of water developed. It was necessary to use a top heading and bench through this zone, and afterward a heavily reinforced concrete lining was placed for a total length of 215 ft centering on the bad spot.

Although no other bad driving was encountered, it was necessary to place plain concrete linings for short lengths in four other places where narrow zones of poor rock were found.

It has been previously stated that the guiding principle of design was the desirability of having water tend to seep into the conduit, instead of out of it. This tendency is reversed near all portals except at the intake. To prevent water losses, the tunnels were therefore steel lined for varying distances back in from Portals Nos. 1, 2, and 3. At Portals Nos. 2 and 3, the linings were extended back to points where the ground surfaces directly above are at elevations slightly higher than full reservoir level. At Portal No. 1 the ground surface directly over the end of the steel lining is about 400 ft above full reservoir level; but the mountainside to the right of the tunnel is so steep that the minimum distance to the surface is much less. Another reason for extending the lining well back in this portal is that the rock here is generally inferior to that elsewhere on the project.

The Rock Catcher.—The reaction turbine has a speed of 450 rpm, and any solid object passing through with the water would probably cause quite a little damage to the buckets. Also, any sand would damage the turbine seals. It is necessary, therefore, to take all practicable precautions to the end that no solids, large or small, are carried down the power conduit.

When all work in the tunnel had been completed, the walls and crown were inspected carefully and loose material was scaled and, finally, they were hosed down. The invert was then very carefully cleaned. The entire conduit was finally inspected carefully, from end to end, to make certain that nothing loose had been left in it. Then, as the last step, the conduit was flushed out by "cracking" the intake gate, as described subsequently. All these precautions were designed to take care of any solids or debris resulting from construction.

There remained provision to be made for catching any solids originating in the tunnel or in any way entering it during operation. As previously mentioned, normal water velocity is low, and any material moved which is not buoyant must be washed along the invert; but the invert is fairly rough, and the thousands of pockets in it will ordinarily catch and hold such material. However, whenever the tunnel has been emptied and is refilled, there will be a tendency for any loose material to move along the invert. Fig. 10 shows the rock catcher built to intercept such possible loose material just at the entrance to the steel lining within Portal No. 1. The tunnel was widened on one side, and its invert made 3 ft lower for a length of 100 ft. The invert was smoothed up with concrete, and concrete walls 10 ft apart were built up to the level of the invert of the steel lining. Wood baffles 3 ft apart were placed from wall to wall at right angles to the tunnel axis, forming a series of pockets of dead water, over which any solid material would have to pass to enter the steel-lined section and the penstock. The baffles were made of planks, 3 in. by 12 in., with keeper bars to prevent their floating up out of the retaining slots. In more than 5 years of operation, apparently no solid material has ever reached the turbine, and almost no material has accumulated in the rock catcher.

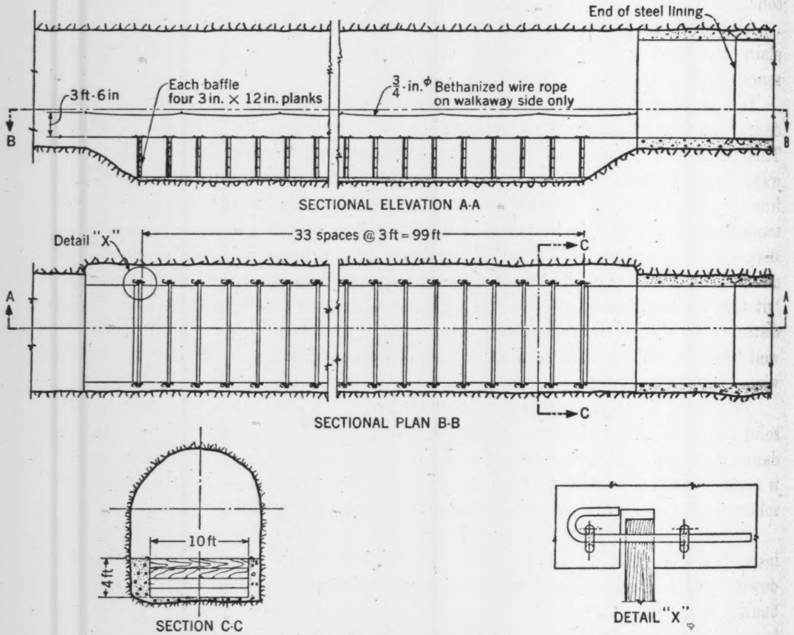
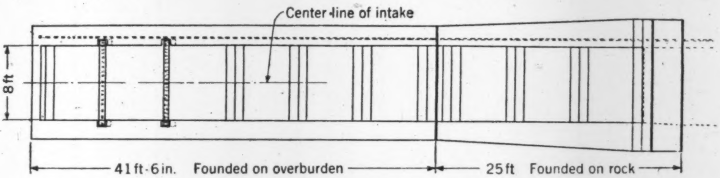
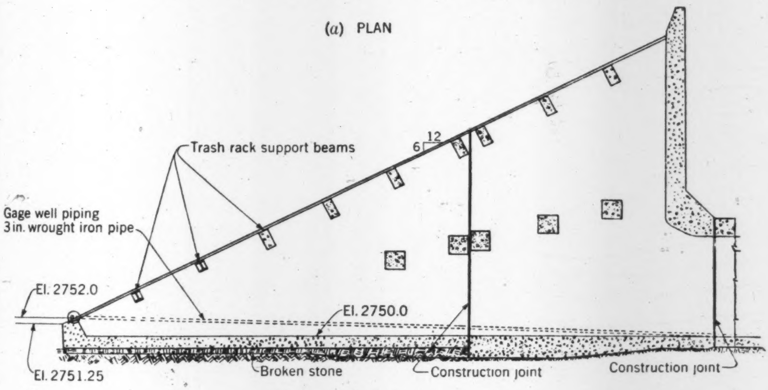

End of steel lining
3 ft-6 in
Each baffle four 3 in. × 12 in. planks
$\frac{3}{4}$-in.φ Bethanized wire rope on walkaway side only
B
B
SECTIONAL ELEVATION A-A
Detail "X"
33 spaces @ 3 ft = 99 ft
C
A
A
C
SECTIONAL PLAN B-B
10 ft
4 ft
SECTION C-C
DETAIL "X"

Fig. 10.—Rock Catcher

The Intake.—The upstream entrance to the power tunnel is on the side hill, about 70 ft above the bottom of the original river channel. The side hill here is covered with soil and decayed rock. Previous experience at the Santeetlah Reservoir in North Carolina had demonstrated that this material would not slide after submergence, and it was therefore decided that it should be left in place as a matter of economy, and disturbed as little as possible. The control scheme is generally similar to that of the diversion tunnel, and was chosen for the same reasons. The control shaft is located about 240 ft from the entrance. It became necessary, therefore, to design an intake structure at the entrance. To disturb the overburden as little as possible, this concrete structure was so designed that when completed only a long narrow slot would show on the slope of the mountain. The structure is in two distinct pieces—the one nearest the portal being founded on solid rock and the outer one on the talus slope. Details are shown in Fig. 11. The construction adopted permitted bracing the walls against each other, and heavy retaining walls were obviated. The tunnel is lined for a distance of 40 ft in from the parapet face, to a point where the crown is sound rock.


Center line of intake
8 ft
41 ft-6 in. Founded on overburden
25 ft Founded on rock

(*a*) PLAN

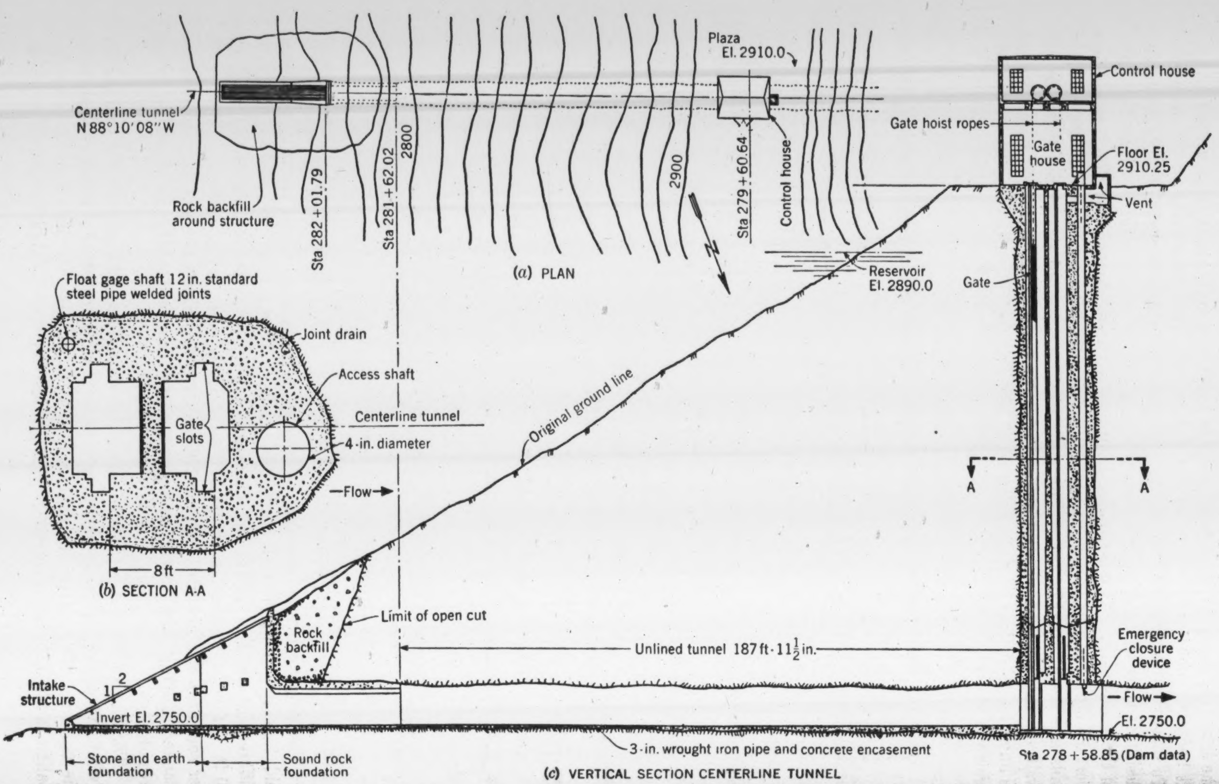

Trash rack support beams
12
6
Gage well piping
3 in. wrought iron pipe
El. 2752.0
El. 2750.0
El. 2751.25
Broken stone
Construction joint
Construction joint

(*b*) SECTION

FIG. 11.—POWER TUNNEL INTAKE

The sloping intake opening is covered with a steel trash rack, in sections, laid loosely on steel and concrete beams. The racks were designed for a differential head of 10 ft, and the supporting beams for a differential head of 20 ft. The gross area is about 532 sq ft for a contemplated flow of 600 cu ft per sec, and the net area, about 70% as great. Since the racks can often be reached during low water seasons, no devices for raking or hoisting were provided.

The Intake Control.—General arrangement of the intake is shown in Fig. 12. Two fixed-wheel gates, in series, are placed at the foot of a shaft intersecting the tunnel. The gates themselves are very similar to the control gates installed in the diversion tunnel. The guides were not as rigidly made, however,


(a) PLAN

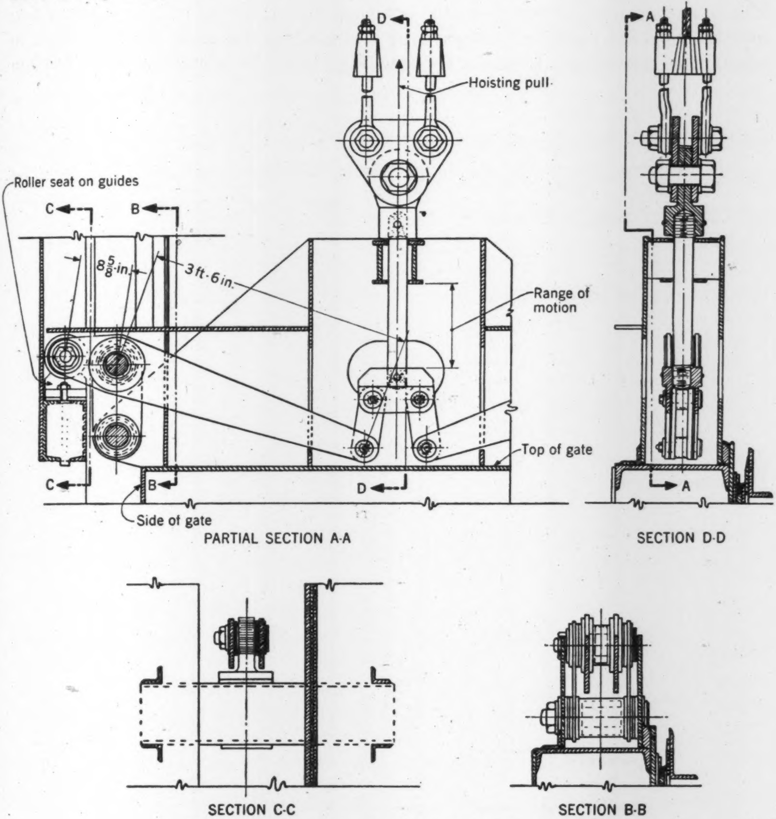
(b) SECTION A-A

(c) VERTICAL SECTION CENTERLINE TUNNEL

Centerline tunnel
N 88°10'08"W
Rock backfill around structure
Sta 282 + 01.79
Sta 281 + 62.02
2800
2900
N
Plaza
El. 2910.0
Sta 279 + 60.64
Control house
Reservoir
El. 2890.0
Float gage shaft 12 in. standard steel pipe welded joints
Joint drain
Access shaft
Centerline tunnel
Gate slots
4-in. diameter
Flow
8 ft
Original ground line
Control house
Gate hoist ropes
Gate house
Floor El. 2910.25
Vent
Gate
A
A
Limit of open cut
Rock backfill
Unlined tunnel 187 ft-11½ in.
Emergency closure device
Flow
El. 2750.0
Intake structure
1
2
Invert El. 2750.0
3-in. wrought iron pipe and concrete encasement
Sta 278 + 58.85 (Dam data)
Stone and earth foundation
Sound rock foundation

FIG. 12.—GENERAL ARRANGEMENTS, POWER TUNNEL INTAKE

and were slightly warped somehow in the installation, and it was necessary to do a very expensive field grinding job on the seal and roller paths. The rubber seals, similar to those on the diversion control gates, have proved entirely satisfactory here.



PARTIAL SECTION A-A SECTION D-D

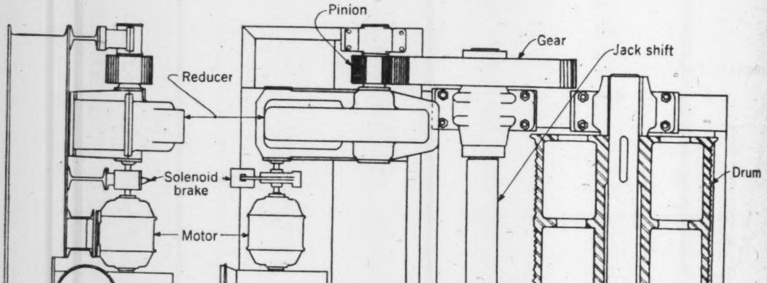
SECTION C-C SECTION B-B

FIG. 13.—HOISTING ATTACHMENT ON INTAKE GATES

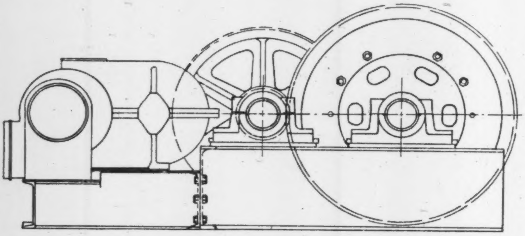
The concrete wall between the gate wells was made too thin, and therefore leaked considerably. It has been necessary to repair it.

An unusual feature is the method of "cracking" the gates—that is, opening them only a little. When the power conduit has been drained for inspection or repair, it is desirable to refill it slowly. This may be done by a small by-pass gate or by "cracking" the regular gate. The danger in using the latter method is that it is not foolproof; the gate may be opened too wide, and the conduit filled too rapidly. The arrangement adopted guards against this possibility.

Each of the gates is hoisted by two $1\frac{1}{8}$-in. ropes, and will close under its own weight. The hoist provided is strong enough to lift the weight of the gate, but not strong enough to overcome gate weight plus friction under pressure, and if the hoist ropes were fixedly connected to the gate, it could not be opened. The hoisting force is multiplied so that it is sufficient for "cracking," by connecting the hoist ropes to a pair of levers having a limited range of motion. The levers are hinged on the gate, but their fulcrums are on the gate guides, and the hoisting force is multiplied by the ratio of distances from fulcrum to hoist rope and hinge, respectively (see Fig. 13). This ratio is 5.67. When hoisting tension is put on the wire ropes, the long ends of the levers move through a vertical distance of 14 in. before they hit a rigid stop on the gate, and, in so doing, lift or "crack" the gate 2.47 in. Once the levers hit the stop, the hoist ropes in effect become fixedly attached to the gate, and the hoist is not strong enough to lift farther until the pressures on both sides of the gate are the same.



(a) PLAN · PARTLY SECTIONAL



(b) ELEVATION

FIG. 14.—INTAKE GATE HOIST

Another noteworthy feature is a simple and foolproof brake on the hoist for lowering gates. It is automatic in operation, and made of standard stock parts. It consists of an ordinary air blower direct connected to the motor shaft, in such fashion that the blower runs backward when hoisting. Fig. 14 shows the arrangement, and its operation is explained as follows: Assume that the gate is hoisted, the hoist is held stationary by a solenoid brake, and that no power is available. Release the brake, and the hoist starts to overhaul, the gate descending. As the gate picks up speed, the hoist runs faster, and the blower begins to absorb power. The gate accelerates until it develops power at the same rate as the blower absorbs it, and thereafter the speed of descent is constant. Since the power that a blower requires varies as the cube of its speed, it is easy to design so that the speed of free descent is always less than three times the hoisting speed, which is usually about all the motor will stand. There are only two precautions to be taken—the motor must be checked for the maximum speed, and the blower discharge and intake must be so screened that they cannot be closed accidentally, since a closed blower acts in the same way as a centrifugal pump, and requires no power when it does no work.

All parts of the hoist are made strong enough to stand the stalling torque of the motor, or to withstand the maximum pull on the ropes.

To illustrate the action of the "cracking" device, and the blower, the following data are given:

A. Roller and Roller Bearing Friction.—In the formula—

$$F = \frac{P}{R}(f_r + f_a r) \ldots\ldots\ldots\ldots (2)$$

—the following values applying to rollers were:

Symbol	Description	Value
F	Vertical force, in pounds, found necessary to overcome friction	6,280
P	Horizontal pressure on gate, or load on wheels, in pounds	735,000
R	Radius of wheels, in feet	0.645
f_r	Coefficient of rolling friction for roller bearings	0.003
f_a	Coefficient of axle friction	0.01
r	Radius to center of roller bearings, in feet	0.25

B. Lift Necessary to "Crack" Gate.—The total weight required to "crack" the gates is as follows:

Item	Description	Pounds
1	Weight of gate, filled with concrete, in air	38,700
2	Weight of wire rope	650
3	Downward hydraulic pressure, on top of gate	85,000
4	Roller and roller bearing friction	6,280
5	Seal friction, caused by mechanical deflection of seals, assuming a friction coefficient of 1.0	9,130
6	Seal friction, caused by water pressure on seals	10,900
	Total	150,660

Dividing 150,660 lb by the lever arm ratio of 5.67, the rope pull is 26,600 lb. Thus, the hoist exerts somewhat less force in "cracking" than in lifting the dead weight of the gate.

C. Tension in Hoist Ropes, Gate Closing Under Pressure, Before "Cracking" Device Functions.—The components of force to be considered are:

Item	Description	Pounds
1	Downward forces exerted on gate—	
	a. Weight of gate	38,700
	b. Downward hydraulic pressure	83,000
	Total	121,700
2	Forces resisting downward motion—	
	a. Roller and roller bearing friction	6,280
	b. Seal friction, caused by mechanical deflection	9,130
	c. Seal friction, caused by water pressure	10,900
	Total	26,310
3	Tension in hoist ropes is the difference between items 1 and 2	95,390

D. Tension in Hoist Ropes, Gate Closing Under Pressure, Almost Closed, and "Cracking" Device Functioning.—The component forces for this case are:

Item	Description	Pounds
1	Downward forces exerted on gate—	
	a. Weight of gate	38,700
	b. Pressure on overhang of head seal	5,370
	Total	44,070
2	Forces resisting downward motion (as in item C-2)	26,310
3	Tension in hoist ropes is the difference between items 1 and 2, or 17,760 lb; divided by 5.67 the rope pull is	3,100

The size of the hoist ropes was determined by item C-3. Two 1⅛-in. wire ropes were used, with a total breaking strength of 205,000 lb. Thus, the factor of safety under emergency closing conditions is $\frac{205,000}{95,390}$ or 2.15.

The hoist motor must be able to lift the gate in the air, and drive the blower backward at the same time. Given: Motor speed, 870 rpm; a reducer ratio of 230 to 1; gear and pinion ratio, 65 to 15; second gear and pinion ratio, 54 to 13; and total reduction, thus, 4,140 to 1.

The drum speed is then 0.21 rpm, and with a pitch diameter of 42 in., the hoisting speed is 2.31 ft per min. The motor must put out, merely to hoist the gate in the air, 3.28 hp, computed as follows—$P = \frac{2.31 \times 39.350}{33,000 \times (0.95)^2 \times 0.93} = 3.28$. This value takes account of the fact that the reducer has an efficiency of 93%, and each gear and pinion combination, 95%.

The blower requires 2.0 hp at 825 rpm running forward, and one eighth as much running backward. At 870 rpm it requires 0.29 hp running backward, computed as follows: $P = \frac{2}{8} \times \frac{(870)^3}{825} = 0.29$. Thus, the motor must put out a total of 3.57 hp.

The speed at which the gate will descend when the solenoid brake is released is computed as follows:

Step Description

1 Let N equal revolutions per minute of motor, and therefore of blower.

2 Speed of descent can then be expressed, in feet per minute, as $\frac{3.5\,\pi N}{4,140}$ or $\frac{N}{377}$.

3 Power generated by the falling gate is speed of descent times tension in hoist ropes, divided by 33,000; thus,

$$P = \frac{N}{377} \times \frac{95,390}{33,000} = \frac{N}{130.5} \quad \text{(3)}$$

4 The power computed by step 3 is reduced by the gear and reducer efficiencies to

$$P = \frac{N}{155.5} \quad \text{(4)}$$

5 The power required to run the blower is

$$P = \left(\frac{N}{825}\right)^3 \times 2 \quad \text{(5)}$$

6 Equating Eq. 4 to Eq. 5, and solving for N, $N/155.5 = 2\,N^3/(825)^3$; and $N = 1,344$ rpm. This speed is conservative, since some friction in the hoist has been neglected.

The control house at the top of the shaft is similar to that at the top of the diversion control shaft. Power for the hoist is furnished by a small gasoline engine-driven generating set. Since the two control houses are only a short distance apart a power line was run between them, so that the generating set in either house may be used to operate either set of gates. Thus, each generating set is a stand-by for the other set.

PIPE LINE CROSSING DICK'S CREEK

To preserve reasonable tunnel alinement and grade, it was necessary to come out in the open with the power conduit and carry a steel pipe line across a ravine at Dick's Creek, and Fig. 15 shows the arrangement adopted. This provided the further advantage of additional portals for tunnel driving. Surface rock in this area was poor, and overlaid by a considerable depth of soft ground. Sufficient floodway for Dick's Creek had to be provided between pipe supports. It was decided to make the pipe 9 ft in diameter in the open,

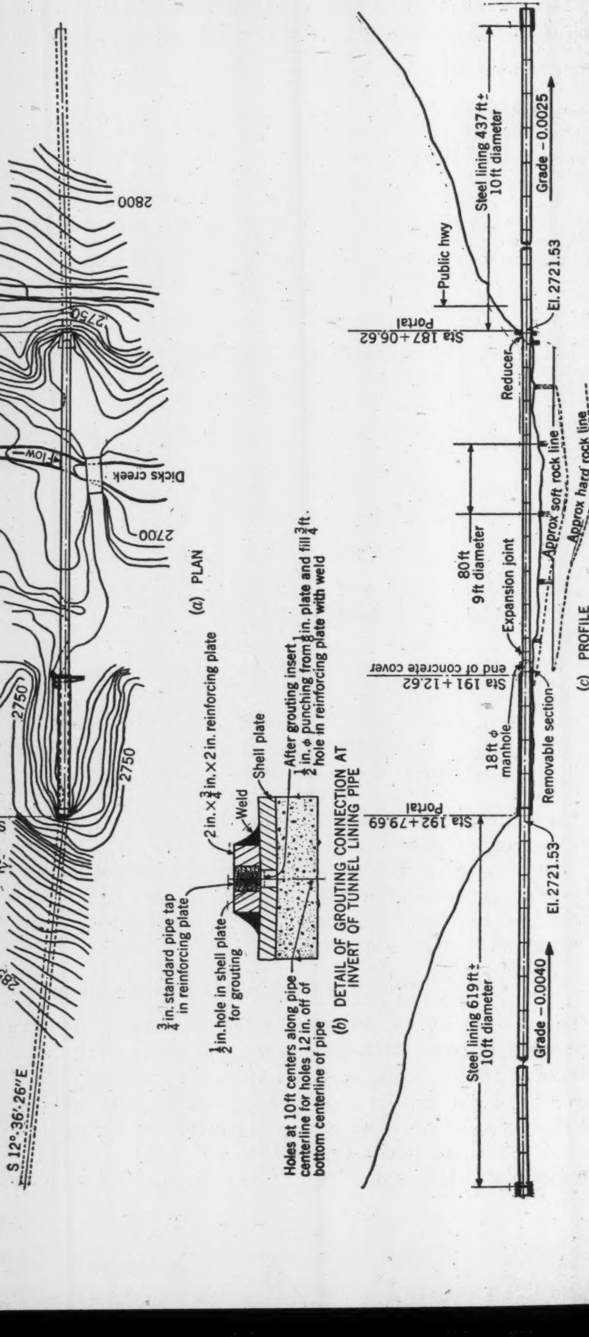

S 12°-36'-26"E
2850
2800
Sta Pi 192+79.69
2750
2750
Dicks creek
Flow
2700
Sta 187+06.62 Pi
2750
2800
(a) PLAN
$\frac{3}{4}$ in. standard pipe tap in reinforcing plate
2 in.×$\frac{3}{4}$ in.×2 in. reinforcing plate
Weld
Shell plate
After grouting insert $\frac{1}{2}$ in. ϕ punching from $\frac{3}{8}$ in. plate and fill $\frac{3}{4}$ ft. hole in reinforcing plate with weld
$\frac{1}{2}$ in. hole in shell plate for grouting
Holes at 10 ft centers along pipe centerline for holes 12 in. off of bottom centerline of pipe
(b) DETAIL OF GROUTING CONNECTION AT INVERT OF TUNNEL LINING PIPE
Steel lining 619 ft± 10 ft diameter
Grade -0.0040
El. 2721.53
Sta 192+79.69 Portal
Removable section
18 ft ϕ manhole
Sta 191+12.62 end of concrete cover
Expansion joint
80 ft 9 ft diameter
Approx soft rock line
Approx hard rock line
Reducer
Sta 187+06.62 Portal
El. 2721.53
Public hwy
Steel lining 437 ft± 10 ft diameter
Grade -0.0025
(c) PROFILE

FIG. 15.—DICK'S CREEK PIPE LINE IN NORTH CAROLINA

and it was designed as a continuous beam over six supports, the longest span being 80 ft. Four of the six supports were founded on concrete caps over steel H-beam piling.

Immediately outside each portal was a concentric reducer between 10-ft-diameter tunnel linings and the 9-ft-diameter pipe in the open. A drain valve was provided in the pipe span over the creek, but because of the difference in diameter the tunnel sections do not drain completely, and this is objectionable. It probably would have been better to make all this pipe 10 ft in diameter.

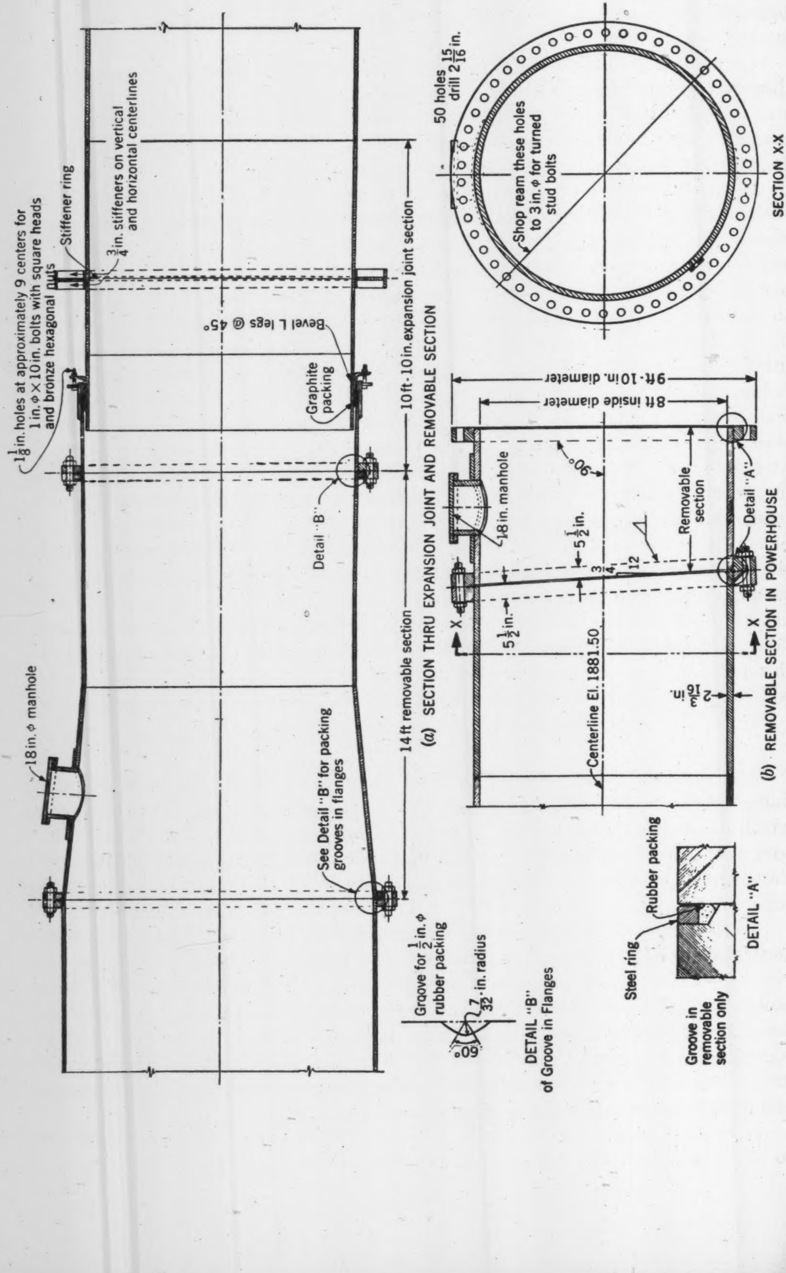
In the last stages of constructing steel pipe lines and penstocks, access to their interiors becomes a matter of first importance. They must be thoroughly cleaned and painted. After a plant has been put in operation, such access is still very important. Repairs are necessary such as repainting, or cleaning out a rockfall in a tunnel. In either case, it is both inconvenient and expensive to be forced to pass men and materials through manholes, and easier access is highly desirable, although manholes should be provided for the purpose of inspection. At Dick's Creek a 14-ft length of pipe was made removable. The reducer at the stream portal forms part of the removable section. At each end the section is flanged (see Fig. 16) and the flanges drilled for 2½-in.-diameter stud bolts about 8 in. apart on the pitch circle. A corresponding flange is placed on the end of the 10-ft-diameter pipe upstream. All flanges are grooved for round rubber packing. It is the detail at the downstream end of this removable section which makes it easily workable. The section of flanged pipe adjacent is only 25 in. long, and terminates in an expansion joint. When the expansion joint is loosened, and the flange bolts are removed, the short section of pipe can be slid away from the removable section, so that the latter has ample room to come out. It is only necessary to put blocking under it, take a few turns of cable around it, and roll it to one side. It was easy to make this provision; unquestionably, it paid for itself during construction, and it will probably pay for itself several times over in the course of one repair job.

The Penstock

The penstock is considered to extend from a point in the tunnel where steel lining begins to the upstream flange of the butterfly valve in the powerhouse, which lies at the foot of a spur jutting out from Rattlesnake Mountain. That part of the penstock in the open runs down along the narrow spine of this spur. Pertinent data on steel pipe is given in Table 1, distances given being along the penstock and not horizontal, and Fig. 17 shows the general arrangement.

Table 1 shows the location of three removable sections. The upper one is generally similar to the one at Dick's Creek previously described, and is provided for the primary purpose of access. The middle one is much the same, except that it is not a reducer, and was provided essentially for a different reason. Although it may afford access, it was essential to the scheme adopted (described subsequently) for flushing out the tunnel invert. The third and lowest removable section is adjacent to the butterfly valve in the powerhouse, and is quite different from the other two (see Fig. 16).

It was not practicable to put in an expansion joint here, but it was necessary to provide some room for possible removal of the butterfly valve. The last


1 1/8 in. holes at approximately 9 centers for 1 in. φ × 10 in. bolts with square heads and bronze hexagonal nuts
Stiffener ring
3/4 in. stiffeners on vertical and horizontal centerlines
Bevel L legs @ 45°
Graphite packing
Detail "B"
18 in. φ manhole
See Detail "B" for packing grooves in flanges
14 ft removable section
10 ft-10 in. expansion joint section
Groove for 1/2 in. φ rubber packing
7/32 in. radius
60°
DETAIL "B" of Groove in Flanges
Steel ring
Rubber packing
Groove in removable section only
DETAIL "A"
Centerline El. 1881.50
2 3/16 in.
5 1/2 in.
5 1/2 in.
X
X
3
4
12
18 in. manhole
Removable section
Detail "A"
90°
8 ft inside diameter
9 ft-10 in. diameter
50 holes drill 2 15/16 in.
Shop ream these holes to 3 in. φ for turned stud bolts
SECTION X-X

(a) SECTION THRU EXPANSION JOINT AND REMOVABLE SECTION

(b) REMOVABLE SECTION IN POWERHOUSE

FIG. 16—EXPANSION JOINTS AND REMOVABLE PIPE LINE SECTIONS

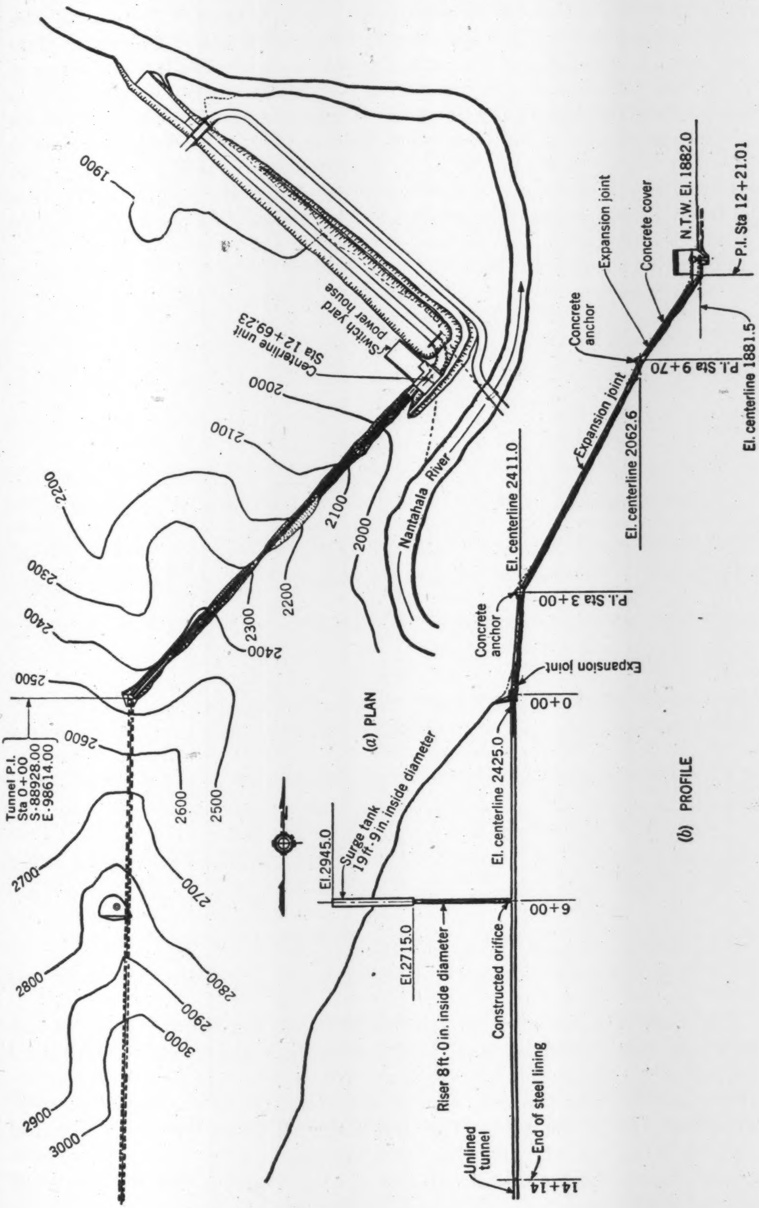

1900
Centerline unit
Sta 12+69.23
Switch yard
power house
2000
2100
2200
2300
2400
2500
2600
2700
2800
2900
3000
2100
2000
2200
2300
2400
Nantahala River
Tunnel P.I.
Sta 0+00
S: 88928.00
E: 98614.00
2600
2500
2700
2800
2900
3000
2900
3000
(a) PLAN
El. 2945.0
Surge tank
19 ft-9 in. inside diameter
El. 2715.0
Riser 8 ft-0 in. inside diameter
Constructed orifice
El. centerline 2425.0
6+00
0+00
Expansion joint
Concrete anchor
P.I. Sta 3+00
El. centerline 2411.0
Expansion joint
El. centerline 2062.6
P.I. Sta 9+70
Concrete anchor
Expansion joint
Concrete cover
N.T.W. El. 1882.0
P.I. Sta 12+21.01
El. centerline 1881.5
Unlined tunnel
End of steel lining
14+14
(b) PROFILE

FIG. 17.—PROFILE OF PENSTOCK AND SURGE TANK

section of rigidly held penstock was provided with a flange, 5 ft from the valve flange, inclined at a bevel of $\frac{3}{4}$ to 12 to a line at right angles to the axis of the penstock. The removable section is flanged to match this inclined flange on one end and the valve flange on the other, and is thus slightly wedge shaped, as shown in Fig. 16, the wide part of the wedge being on top. It is under the powerhouse crane, and can be lifted straight out. It is carefully machined to a close fit; it will be shrunk by using dry ice when removal is necessary, and was originally placed using dry ice. The packing at the joints is placed from the interior of the pipe after the removable section is in place. The flanges are recessed as shown; the packing is placed in the recesses and held with a retaining ring sprung into place and tack welded. This type of joint has been very satisfactory.

TABLE 1.—DIMENSIONS OF PENSTOCK (SEE FIG. 17)

Wall thickness (inches) (1)	Length (feet) (2)	Center-line elevation[a] (feet) (3)	Maximum static head[a] (feet) (4)
(*a*) INSIDE DIAMETER, 10 FT			
0.41	204	2,439.35	450.65
0.55	210	....	....
1/2	128	2,427.50	462.50
9/16	24	....	....
5/8	24	....	....
11/16	24	....	....
3/4	384	....	....
13/16	42	....	....
7/8	18	....	....
15/16	18	....	....
1	18	....	....
1 1/16	16	....	....
1 1/8	16	....	....
1 3/16	14	....	....
1 1/4	14	....	....
1 5/16	260	....	....
1 5/16	12.91	2,425.00[b]	465.00
1 5/16	14.00[c,d]	....	....
(*b*) INSIDE DIAMETER, 8.5 FT			
1 3/16	250.83	....	....
1 1/4	22.58	....	....
1 1/4	60.58	2,411.00[e]	479.00
(*b*) INSIDE DIAMETER, 8.5 FT (*Cont'd*)			
1 5/16	64	2,383.05	506.95
1 3/8	64	2,353.52	536.48
1 7/16	70	2,324.00	566.00
1 1/2	63	2,291.70	598.30
1 9/16	56	2,262.64	627.36
1 9/16	7[d]	2,236.80	653.20
(*c*) INSIDE DIAMETER, 8 FT			
1 9/16	101.83	2,233.57	656.43
1 5/8	70	2,186.59	703.41
1 11/16	70	2,154.30	735.70
1 3/4	65	2,122.00	768.00
1 13/16	63.76	2,092.02	797.98
1 7/8	6	2,062.60[e]	827.40
1 7/8	54[c]	2,059.09	830.91
1 15/16	54	2,027.49	862.51
2	57.5	1,995.90	894.10
2 1/16	55	1,962.26	927.64
2 1/8	60	1,930.07	959.93
2 3/16	23.02	1,894.07	995.93
2 3/16	21.22	1,881.50[f]	1008.50
2 3/16	5.00[c]	1,881.50	1008.50

[a] At upper end of run. [b] Anchor at Portal No. 1; vertical and horizontal point of intersection (P.I.). [c] Removable section. [d] Reducer. [e] Anchor at vertical P.I. [f] Vertical P.I. inside powerhouse.

The middle removable section in the penstock was left out when the penstock was built. After all work in the surge tank and power conduit above this point (including painting) was completed, and—so far as was known—the interior had been cleaned scrupulously, the tunnel invert was flushed out after "cracking" the intake gate. The tunnel then filled as far as the apex, and since the water velocity upstream from the apex was low, the flushing did no good in this part of the tunnel so far as moving anything heavier than water was concerned. It was expected, however, that this procedure would float out any loose pieces of wood overlooked in cleaning; but, from the apex downstream, the water flowed as a turbulent stream down the invert.

Where the removable section had been left out, a temporary funnel section was attached to the penstock, and the discharge was piped over the side of the mountain where it could do no damage. Water was allowed to run in this way for 2 days, and the effluent was carefully observed. At first, this effluent was quite murky because of the rock dust it carried. Any solid object could be heard striking the funnel. From these sounds, it was obvious that the solid particles were all small, with one exception. In spite of the rigorous inspection, one piece of wood, 2 in. by 4 in., and several feet long, came down. When the water became clear, flushing was discontinued, the funnel removed, the removable section inserted, and the power conduit was ready for service.

The penstock was anchored carefully at four points—at the portal, at each of the two vertical curves, and at the lower end. The anchors were designed to resist hydraulic thrust, and to carry the dead weight of pipe. On a steep slope, the latter becomes an appreciable item, since all longitudinal forces are carried by the anchors. For example, the anchor at Station 3 + 00 P (see Fig. 15) resists any longitudinal force because of the weight of pipe between the expansion joints near the portal and about 370 ft below the anchor. At the lower end, just above the powerhouse, the pipe was incased in concrete. The incasement provided a continuous anchor about 238 ft long. Its purpose was to insure that no movement of the penstock could possibly take place which might conceivably disturb the alinement of the generating unit.

Ordinarily the penstock will be drained through the turbine, but it was deemed necessary to provide for draining in case the turbine cannot be run. The penstock is tapped just inside the powerhouse, and a 12-in. pipe, controlled by a gate valve, is taken off and connected to a manifold mounted on a corner of the powerhouse. The manifold feeds four nozzles of 2¾-in. diameter, the nozzles are inclined at 45° with the horizontal, and point up the Nantahala Valley to the right of, and upstream from, the powerhouse. Each nozzle is controlled by a gate valve. When operated, the jets from these nozzles tear themselves to pieces in the air, and the discharged water falls as a heavy mist over an area perhaps 200 ft wide and 600 ft long. It is determined that more than 6,000 hp is dissipated in this way and, since it is all done by air resistance, it sets up a considerable windstorm in the valley.

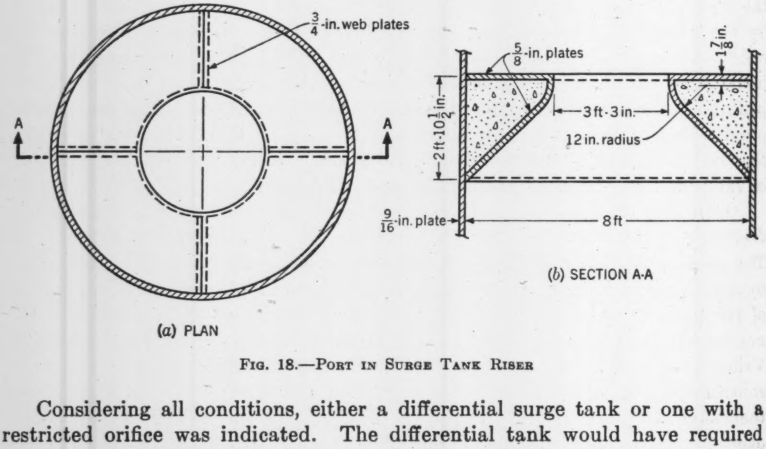
THE SURGE TANK

The surge tank was placed as close as possible to the powerhouse, consistent with not having too tall a tank projecting above ground surface. Because of the surface contours, it was not advisable to locate it directly above the tunnel, and it was actually slightly more than 30 ft to the left of the tunnel axis. The riser was inclined with an elbow at the bottom, as shown in Fig. 17. This arrangement permitted the use of a rock bin at the bottom of the riser, so that footings for the riser and surge tank could be excavated without interfering with work in the tunnel.

The surge tank is 600 ft from Portal No. 1, and the most important elevations are as follows:

Item	Location	Elevation (feet)
1	Center line of tunnel at this point	2,426.50
2	Constricted orifice in riser	2,461.80
3	Top of riser, and bottom of surge tank	2,715.00
4	Rock surface, after excavating a flat area	2,840.00
5	Top of surge tank	2,945.00
6	Full reservoir water surface	2,890.00

The riser is steel lined, with an inside diameter of 8 ft. The surge tank has an inside diameter of 19.75 ft, and has a total height of 230 ft, the lower 125 ft of its height being in a shaft excavated in the mountain. Because of the distance between the surge tank and the powerhouse, speed and load regulation cannot be delicate.



(*b*) SECTION A-A

(*a*) PLAN

FIG. 18.—PORT IN SURGE TANK RISER

Considering all conditions, either a differential surge tank or one with a restricted orifice was indicated. The differential tank would have required a bottom at El. 2630.0 and, of course, an internal riser. The other type of tank required the development of a restricted orifice with a coefficient of discharge for water passing upward about 50% greater than for water passing downward. Experiments were made on various types of orifices and one finally was designed with discharge coefficients of 0.85 and 0.60. Fig. 18 shows this orifice or port. Computations indicated that, with this orifice, a tank could be built with its bottom at a much higher elevation. The savings in excavation and steel made the orifice type tank much cheaper, and therefore it was used.

DESIGN AND PLACING OF WELDED STEEL CONDUIT AND SURGE TANK

The steel for all tunnel linings, the Dick's Creek pipe line, the surge tank and its riser, and the penstock, conformed to the Specification for Flange Steel, Serial No. A-70-39, American Society for Testing Materials. This steel was chosen because of its suitability for all welded construction. All welding conformed to the Code for Unfired Pressure Vessels (1938 edition) of the American Petroleum Institute and the American Society of Mechanical Engineers.

This code permits a working stress of 25% of the minimum tensile stress of steel, multiplied by four factors, as follows:

Item	Material	Factor
1	For flange steel	0.97
2	For double-welded butt joints	0.80
3	For stress relieving	1.06
4	For radiographing	1.12

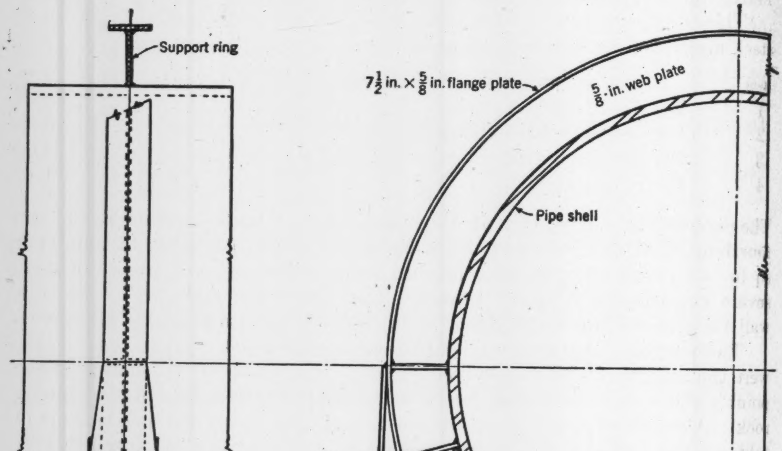
The permissible working stress is thus the product of these four factors and one fourth of 55,000, and proves to be 12,667 lb per sq in. Actually, 12,600 lb per sq in. was used, designing for instantaneous stress resulting from the most severe combination of static pressure, water hammer, and surge. To pipe wall thicknesses thus determined, $\frac{1}{16}$ in. was added as a corrosion allowance.

Two purposes were kept in mind in designing the steel tunnel linings. These were the resistance to stress and the prevention of leakage. Obviously, at the point a pipe enters a tunnel it derives no added strength from any surrounding rock. Also, at some point within the tunnel, good rock becomes capable of taking all the stress. Theoretically, then, the pipe at the entrance must be capable, by itself, of resisting all bursting pressure, and may progressively be made thinner until zero thickness is required. Practical considerations preclude theoretical design. No one knows exactly how to compute the resistance that rock offers to bursting pressure, and pipe must have a certain minimum thickness to make it stiff enough to handle.

Conservative design assumptions were therefore adopted. It was assumed that all bursting pressure would be safely taken by rock at any point in the tunnel where upward pressure was balanced by the sheer weight of rock included between two diverging lines. These lines originated at the ends of the horizontal pipe diameter, extending upward and outward at an angle of 10° from the vertical. Between such points and the portals, linings were designed capable by themselves of resisting all pressures, and to thicknesses so determined the corrosion allowance of $\frac{1}{16}$ in. was added. Beyond these points, pipe wall thickness was decreased by $\frac{1}{16}$ in. per course to a minimum of $\frac{1}{2}$ in. Lengths of such courses varied from 14 ft to 24 ft. The minimum thickness of lining was originally intended to be carried to points at which the vertical distance to the water table would exceed the static head, with the idea that, in parts of the tunnel thus left unlined, water would tend to seep in rather than out, as previously stated. However, in all three places where steel linings were used, they were extended farther than originally intended. At Portal No. 1, the lining was so extended 414 ft; the decision to extend was made during construction and $\frac{1}{2}$-in. plate was not readily available. Plates of 0.41-in. thickness and 0.55-in. thickness were used.

For pipe in the open air, the design of spans and supports was guided by the results of tests reported by R. L. Templin and R. G. Sturm, Members

ASCE, in 1933.[3] These tests indicated that the greatest stresses in a pipe span occurred at supports when it was just full of water under no additional head, and that, if a pipe is stiffened adequately at supports for this condition, it may otherwise be considered as an ordinary cylindrical beam. The tests also indicated a special arrangement of supports that would cause minimum stresses. Fig. 19 shows typical details. The pipe has stiffeners only at sup-


Support ring
$7\frac{1}{2}$ in. × $\frac{5}{8}$ in. flange plate
$\frac{5}{8}$-in. web plate
Pipe shell
Bracket on pipe
$\frac{3}{4}$-in. plate
$1\frac{1}{2}$-in. plate
Rocker bearing
$\frac{1}{2}$-in. plate
Saddle
14-in., 68-lb wide flange
14-in., 68-lb, wide flange
1-in. plate
$1\frac{1}{2}$-in. plate
Grillage for concrete foundation

END VIEW TYPICAL

SADDLE AND SUPPORT

FIG. 19.—PIPE LINE SUPPORTS

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[3] *Transactions*, ASCE, Vol. 98, 1933, p. 154.

ports. Pipe was assumed to act as a continuous beam with no break at expansion joints. Longitudinal stresses were computed to be, in general, quite low. The maximum in the penstock was only 4,000 lb per sq in. and in the Dick's Creek crossing only 6,500 lb per sq in. Strain gage measurements made in the field, with and without water in the pipe, indicated stresses in satisfactory agreement with computed stresses.

Every precaution was taken to insure good welding in the shop and in the field. Pipe was shop assembled in sections not exceeding 32 ft in length or 12 tons in weight. Longitudinal welds, which are the most highly stressed, were thus generally made in the shop (the exception being those in the surge tank) and were stress relieved and radiographed. Circumferential welds were stress relieved. Longitudinal shop welds were machine made in double V-butt joints. Circumferential shop welds were hand welded in double V-butt joints.

Pipe sections were shipped to the job with interior spreaders to preserve their shape. It is especially important that sections be kept round and true at the ends, where they butt up against adjacent sections for welding.

Circumferential field joints on pipe in the open were double V-butt welds. Welds on all plate 1¼ in. thick and greater were radiographed. Where the pipe shell was 1½ in. thick or more, it was electrically heated to 300° F before welding. To radiograph a joint, radium was placed at the center line of the pipe, and photographic film was placed outside, and in contact with, the joint of which a picture was desired. Lead shields were hung inside the pipe on either side of the radium to protect workmen from the dangerous radiations.

On circumferential field joints in tunnel linings, backing-up strips on single V-butt joints were used, as they could only be welded from the inside of the pipe.

Interior surfaces of all pipe steel were given one coat of bitumastic solution, applied cold. Exterior surfaces were not shop painted. After steel had been placed and welded, interior surfaces were given an additional cold coat of bitumastic solution, and a final coat of bitumastic enamel was applied hot. Exterior surfaces against which concrete was to be placed were left unpainted and exterior surfaces exposed to the air were given one coat of an alkyd base red lead primer, followed by two coats of aluminum in an alkyd vehicle. After the development had been in operation 5 years, the paint was still "standing up" well.

Steel tunnel lining was placed in sections of various lengths, the maximum being 32 ft. The pipe sections were shop equipped with grout connections at the top and at the invert. Those at the top were spaced about 16 ft on centers, and 2-in.-diameter pipes were inserted to make a contact with rock. Those at the bottom were spaced about 10 ft on centers, and were for ¾-in. pipe. After concrete had been placed around the steel lining, and had set thoroughly, the 2-in. pipes at the top were grouted, under pressure rising gradually to 200 lb per sq in. Quite often the concrete placed under the invert does not make good contact with the steel, probably because of air bubbles, and the ¾-in. pipes were grouted at low pressure to insure contact. High pressure bulges the steel very quickly.

At each of the three places where steel tunnel linings ended, concrete lining was placed for 6 ft beyond the steel. Eight radial holes at 45° intervals were drilled 28 ft into rock through this lining, and the holes grouted to 500 lb per sq in. When this grout had set, a second set of eight holes, intermediately spaced, was drilled and grouted to the same pressure. Thus, a barrier was created against seepage along the tunnel.

The steel tee connecting the surge tank riser and the tunnel lining was shop fabricated, with all welds stress relieved and radiographed. It is known that, under pressure, a tee will develop stress concentrations, and it was considered inadvisable to accentuate these by field welding.

The surge tank riser and the surge tank lining were grouted after concrete backfill had set to insure that no hollows existed which would permit the lining to "work" under variations in water level.

All pipe anchors were designed for a safety factor of 2 against overturning, and were dimensioned so that the force tending to slide the anchor was not more than 0.8 of its weight, considering the anchor to be resisting (by gravity alone) the most severe combination of forces that could be imposed. In addition, all but one were doweled into rock. One anchor was founded on a soft brownish shale and was not doweled; and the maximum foundation pressure was limited, in this case, to 2.5 tons per sq ft.

Surge tank and penstock designs were based on the assumption that a turbine discharge of 700 cu ft per sec would be shut off in 5 sec. The effect of water hammer was computed by the arithmetic integration method introduced by Norman R. Gibson,[4] M. ASCE. Computations for surge were also made by a method of arithmetic integration. Efficiency tests were run on the turbine in August, 1942, using the Gibson method, the action of the surge tank being observed at the same time. Results indicated that the action is in accordance with design assumptions, and may be predicted for any operating condition with a reasonable degree of accuracy.

Observations of friction head loss at all rates of flow indicate that Kutter's *n* has values of 0.011 for lined tunnel or steel pipe, and 0.0295 for unlined tunnel, respectively, as determined by the Manning formula.

ACKNOWLEDGMENTS

The Nantahala Hydroelectric Plant is owned by the Nantahala Power and Light Company, a subsidiary of the Aluminum Company of America. It was designed and built under the supervision of J. P. Growdon, M. ASCE, chief hydraulic engineer of the Aluminum Company of America, the writer being then principal assistant hydraulic engineer. The general contractor was the Utah Construction Company. All steel pipe was furnished and erected by the Chicago Bridge & Iron Company, and all field paint was applied by the Wailes Dove-Hermiston Corporation.

The design of the pipe line, penstock, and surge tank was the particular province of Frederick W. Ely, M. ASCE, and the writer wishes to acknowledge indebtedness to him for many of the data required in the paper.

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[4] "Pressures in Penstocks Caused by the Gradual Closing of Turbine Gates," by Norman R. Gibson, *Transactions*, ASCE, Vol. LXXXIII, 1919–1920, p. 707.

DISCUSSION

Joseph R. Bowman,[5] **Assoc. M. ASCE.**—The Nantahala development has become particularly noteworthy as the subject of several excellent papers describing in enlightening detail the solution of the many design, construction, and operational problems encountered. The project itself is outstanding in its utilization of many features which, at the time of their conception, were considered somewhat advanced over conventional practice.

Selection of Control Works.—The relative merits and economics of the control tower versus the control shaft are well presented. The writer has observed that in virtually every case in which the likelihood of earthquakes is anticipated the control shaft in rock has been found to be more economical than the free-standing control tower. Another type of control works occasionally investigated in conjunction with rock-fill dams or earth-fill dams is the inclined type, wherein the gate guide structure is constructed to conform to the natural slope of the mountainside. Under favorable circumstances this type possesses adequate stability against earthquakes at a worthwhile saving over the shaft in rock. If, however, the overburden is deep, the natural slope is too flat, or a heavy trash burden is anticipated, the inclined control works can prove to be rather expensive.

Outlet Tunnel Closure Bulkhead.—Although the "floating out" of the closure bulkhead is not a new idea, it has been somewhat less successful in a few cases in the past. The writer believes that the key to the successful use of this idea lies in selecting proper proportions of the buklhead. For a given head and closure area, a high and narrow bulkhead ordinarily weighs less than one having more nearly equal proportions, and, inasmuch as the seal friction and cubical content are very nearly the same in either case, the net result is that the high and narrow bulkhead is more buoyant. The writer should like to inquire as to whether the bulkhead was designed for use in future closures for inspection or for other purposes. He also ponders the fate of the intrepid soul who opened the by-pass valve and then dashed for the nearest exit 200 yd away.

Outlet Tunnel Energy Dissipator.—A rather well-defined trend toward the elimination of expensive stilling basins at tunnel outlets is seen in the location of more compact energy dissipators immediately downstream from tunnel control works. The writer recently attempted to adapt free discharge types of valves to subterranean control works for outlet tunnels, wherein the valves were arranged to discharge into a combination deflector and baffle-weir type of energy dissipator. It was found that conditions requiring the admission of large volumes of air through a conduit entirely separate from either the tunnel or the operating chamber frequently result in abandonment of the scheme. The writer suspects that the large round shaft immediatley downstream from the service gate slot in the Nantahala control shaft was not selected for reasons of access alone. A summary of the studies that guided the design of the Nantahala energy dissipator should be of considerable interest to engineers engaged in this branch of the profession.

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[5] Civ. and Hydr. Engr., Harza Eng. Co., Chicago, Ill.

Control Gates.—The choice of the type of gate used in both the outlet tunnel and the power conduit is of considerable interest to the writer. Preliminary considerations of the head and closure area alone could have resulted in the selection of a caterpillar-type gate, because the heads encountered at Nantahala are considerably higher than those heretofore imposed on fixed-wheel gates. Closer examination, however, again reveals the importance of proper gate proportions. The writer has made studies for the selection of gates in which hydraulic thrust on the gate is viewed in terms of a uniform load on the gate guides, with the result that the product of opening width (L) and the head on the center line of opening (h) forms a convenient parameter for the selection of vertical-lift gates. The application of this parameter ($L\,h$) to a large number of caterpillar and fixed-wheel gates installed over the past quarter century has led to the following generalizations as to submerged vertical-lift gates:

1. For gates operated in enclosed slots, such as the Nantahala gates—
    a. Fixed-wheel gates can be used economically for values of ($L\,h$) to 1,000; and
    b. Caterpillar-type gates should be used for values of ($L\,h$) exceeding 1,000.
2. For gates operated in exposed locations, such as on the face of a dam, fixed-wheel gates can be used for values of ($L\,h$) as high as 2,500.

In general, as the ($L\,h$)-value of a fixed-wheel gate is increased, the axle-mounted wheels gradually assume the proportions of rollers, and it becomes more economical to allow the gate leaf to bear directly on a larger number of smaller rollers.

Mr. Bleifuss undoubtedly is aware that the difficulties experienced with the top seals are not peculiar to the Nantahala gates alone. In spite of the objections to its use for this type of service, the J-seal is one of the most versatile types of seals yet devised. In some recent top seal applications the J-seals are so mounted that full water pressure is not applied until the gates are nearly seated, and the seals are guarded to prevent excessive deformation as the gates are opened.

The writer is interested to note the use of stainless steel for both the seal paths and wheel tracks. Corrosion resisting metals have been used as seal paths for many years, but only recently has it become practical to use such metals for wheel tracks. The vast improvements in the techniques of handling stainless steels have resulted in weldability and surface hardening without appreciable loss of corrosion resistant properties. A series of tests conducted by The Panama Canal[6] from 1941 to 1946 shows the corrosion resistance of the chrome-nickel steels to be far superior to those of other metals, whether exposed alternately or continuously to fresh, brackish, or salt water.

Gate Guide Alinement.—In mentioning certain difficulties encountered in the alinement of guides for the power conduit control gates, the author has revived the writer's interest in a design problem that has been the object of widely divergent opinions, but which has been the subject of very little published discussion.

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[6] "Long Term Metals Corrosion Investigation," prepared by The Panama Canal, ***First Interim Report***, Balboa Heights, Canal Zone, January, 1947, Appendix I, Tables 3 to 7, inclusive.

The failure of gate guides to assume true final alinement can be traced to one or more sources, the more fundamental of which include: (1) Insufficient rigidity of the guides; (2) inadequate provision for precise alinement prior to final embeddment; and (3) inadequate supervision or inspection of fabrication, handling, or field erection. Failure to comply with plans or specifications usually is caused by conditions beyond the control of the designer, but in most cases the provision of well designed guides and the choice of proper construction procedures are the responsibility of the design engineer.

Occasionally, however, the engineer is required to make the fabrication and erection costs of his designs conform to predetermined minimum costs. Recently, the writer was afforded the opportunity of comparing the effects on alinement of the different types of gate guide construction used on two similar projects. In the first case, the guides themselves were of substantial construction, and provision was made for precise adjustment prior to final grouting. These practices resulted in nearly perfect guide alinement. In the second development, it was decided that both the construction of the guides and the provisions for alinement used on the first project were too expensive. Consequently, the guides were made considerably lighter than originally contemplated, and provisions for precise final adjustment were omitted so that the guides could be poured in place. These practices, together with an unsubstantial method of bracing the guides prior to concreting, resulted in a very unsatisfactory job of alinement. In addition to sags and humps, the final check revealed considerable warping of the wheel tracks. Fortunately, in anticipation of just such misalinement, the designers equipped the gate wheels with spherical roller bearings and other adjustments so that the inadequacies in guide design could be partly counteracted.

The theory (shared by many engineers) that structural steel erection tolerances are sufficiently accurate for gate guide erection should be reappraised. Thorough analyses made in connection with the Bonneville development[7] on the Columbia River show how guide deviations from "true," as small as a few thousandths of an inch, can result in seriously overloading the wheels of a gate. Furthermore, where seal clearances are very small, as in the case of the Nantahala gates, sagging or twisting of the guides can cause the gate seals to function improperly and may lead to the eventual destruction of the seals.

Gate-Opening Device.—The problem of reducing initial gate-opening forces has challenged the ingenuity of vertical-lift gate designers for several decades. By-pass valves and small filler gates have been used rather extensively to fill comparatively short conduits and thus obtain balanced head conditions prior to opening the intake gates. A valved by-pass would have been quite impractical at Nantahala, for it not only would have been well-nigh inaccessible, but also would have required a week to fill the 5½ miles of conduit. The lever-operated opening device installed on the power conduit control gates offers a most ingenious solution to this problem. Also, as the gate is lowered, this feature introduces the highly desirable characteristic of retarding the rate of gate movement shortly before the gate is seated. The writer would like to know whether this device has been patented.

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[7] "Structural-Steel Tolerances at Bonneville," by T. M. Ober, *Transactions*, A.S.M.E., Vol. 64, 1942, pp. 721–729.

Power Intake Gate Hoist.—Evidence of properly planning the gate and hoist as an intergral unit is seen in the simple and compact hoist arrangement. The writer will be interested to learn what arrangements were made for handling the emergency gate, as this point is not clearly shown in the paper. Also, the use of an automatic closure device is indicated in Fig. 12, but no description of the manner in which it functions is given.

The blower used to brake the descent of the gate during an emergency closure is one of the more reliable devices developed for this type of service, and many similar arrangements have been in successful operation for several years. The author states that in an emergency the gate is released simply by releasing the solenoid brake. Inasmuch as the principal recommendation for this type of brake is that it is applied when the power is cut off, it is questionable whether the brake could be released automatically in a "power-off" emergency. However, in view of the provision of interconnected sources of power for the gate hoists, it is highly improbable that an absolute power-off emergency would exist.

Pipe Line, Penstock, and Surge Tank.—It is encouraging to note the willingness on the part of a privately owned utility to accept the higher costs of field radiographing of pipe line welds in exchange for the benefits of an appreciable saving in materials, greater safety, and generally longer life of pipe line. Other recent improvements, notably the introduction of the ring-girder support and the development of more reliable field welding techniques, have led to the increased exploitation of the advantages of welded pipe lines.

In striking contrast with Pacific Coast practice is the omission of emergency works, such as a butterfly valve, at the head of the Nantahala penstock. The similarity in methods of penstock construction used recently in both sections of the United States seems to exclude the possibility of penstock rupture due to excessive pressure rise as the reason for this difference. Rather, it appears that greater susceptibility to earthquakes and exposure to falling boulders, in contrast to geological conditions at Nantahala, are responsible for the extensive use of emergency penstock closure works in the west.

The restricted orifice in the surge tank riser is to be admired for its simplicity. It is fairly well understood how the use of an orifice can effect marked economies in surge tank design, but the reasons for selecting a smaller discharge coefficient for the "load-on" condition could have been more clearly explained. It has been stated elsewhere[3] that design problems associated with the high head imposed on the Francis-type turbine do not permit the plant to be operated either as a peaking plant or a stand-by plant. Therefore, it was not necessary to design the surge tank for water supply demands caused by rapid increases in load.

Summary.—It is fitting and proper that a project embracing so many interesting features should be described in several papers, for in this manner many features otherwise overshadowed by the more spectacular can receive the individual treatment they so well deserve. The excellent form in which this paper is presented may well serve as a guide for many to follow.

D. J. Bleifuss,[8] Assoc. M. ASCE.—

Outlet Tunnel Closure Bulkhead.—Mr. Bowman mentions that the successful use of the idea of floating out the bulkhead depends on its proportions. While holding back the water, the bulkhead is held in place by friction developed between it and the guides. When the water pressure is balanced this friction disappears, and if the bulkhead is buoyant it must rise to the water surface, provided it cannot "cock" in the guides and so become jammed in place. Bulkheads of varying proportions can be made buoyant, but a high and narrow bulkhead cannot "cock" and jam, where a low and wide bulkhead might. It is essential that the bulkhead and guides be so designed that no other force but friction caused by unbalanced water pressure can come into play to hold the bulkhead down.

The bulkhead was used only once, and after it floated up, it was scrapped. There was no practicable way to use the bulkhead for inspection or maintenance purposes.

There was no "intrepid soul" to open the by-pass valve. When the bulkhead was welded together, it proved to have a slight warp—that is, its downstream surface was not planar and consequently, when placed, it leaked. It was necessary only to close the gates downstream, and the leakage around the bulkhead filled the tunnel. The by-pass valve, provided for filling the tunnel, was never used.

Outlet Tunnel Energy Dissipator.—The correct design for the dissipator was determined by model tests. Mr. Bowman is correct in assuming that the round shaft was not intended for access alone; it was intended to supply air. Although the top of this shaft is in the gatehouse, air is not drawn from within the gatehouse. It was thought that there might be a diminution of pressure which might break windows or cause even greater damage, and provision was made for the entrance of air to the shaft directly from outdoors.

Gate Guide Alinement.—As a result of his experience at the Nantahala plant and elsewhere, the writer is most emphatically of the opinion that ordinary structural steel erection tolerances do not give sufficient accuracy for gate guide erection. Guides should be designed with screw supports permitting extremely accurate alinement before they are concreted in place. If the guides are not alined accurately, it becomes very difficult to build a gate that will not be overstressed somewhere at some point of its travel. It can be done, but too much machinery is required.

Gate-Opening Device.—This type of device, which automatically provides for "cracking" a gate, has been used repeatedly by designers of the Aluminum Company of America. The writer does not know whether it ever was patented.

Power Intake Gate Hoist.—The two intake gates are in series, and are handled in identically the same fashion. As far as the writer knows, the automatic closure device was never installed.

In the writer's opinion, the blower is the most reliable braking device that could possibly be employed for this service. Probably the first gate hoists using blowers as brakes were the eight built for the Chute à Caron development on the

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[8] Chf. Engr., International Eng. Co., San Francisco, Calif.

Saguenay River in Quebec, Canada, about 1930. Ordinarily, the gates are lowered under power. When all power is off, the solenoid brakes at Nantahala must be released manually. At Chute à Caron, the solenoid brakes were operated by either alternating current or direct current and were so arranged that, when operated by direct current, they could be released by pushbuttons in the control room.

Pipe Line, Penstock, and Surge Tank.—The penstock was designed very carefully, and great care was taken in its construction. The possibility that the penstock would fail is extremely remote, and failure would be likely to occur only in the case of a natural disaster, such as a bad storm or an earthquake which might prevent the operation of a butterfly or other valve at the head of the penstock. Such a valve, therefore, appeared to represent a useless expenditure.

The writer felt that a complete discussion of the surge-tank design would have required too much space for inclusion in his paper and, as Mr. Bowman points out, some of the considerations involved have already been discussed elsewhere.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2432

A STUDY OF END CONNECTIONS FOR STRUTS

By Marshall Holt,[1] Assoc. M. ASCE, and J. W. Clark,[2]
Jun. ASCE

With Discussion by Messrs. Jack R. Benjamin, James W. Harland, F. P. Shearwood, S. K. Ghaswala, and Marshall Holt and J. W. Clark.

Synopsis

Results of tests to evaluate the effect of different types of end connections upon the static and fatigue strengths of struts are described in this paper. Tubes, single angles, and double angles were tested with the usual types of end connections employed for these members as well as some types not in common use. Strain measurements were made to determine the distribution of stress in the members and their connections.

It was found that the rigidity of the connection, degree of eccentricity, and the amount of stress concentration caused by changes of section at the column ends had a marked influence on the static and fatigue strengths.

Although the tests were made on aluminum alloy members, it is believed that the comparisons drawn apply to members of other materials also.

Introduction

One of the difficult problems in the design of latticed structures concerns the end connections of the web members. In lightweight structures such as towers, booms, and relatively light roof trusses, structural angles are generally used, and the members are connected by bolts or rivets through one leg. It is generally recognized that this method of attachment produces an eccentricity of loading so that: (1) The deformations are larger than estimated on the basis of uniform stress in the member; (2) the maximum stress is somewhat larger than the average stress; and (3) the ultimate strength is less than that of a similar member with axial load.[3,4] The performance of the structure is influenced by the performance of the individual members, so it is reasonable

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Note.—Published in December, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or the discussion was received for publication.

[1] Senior Research Engr., Aluminum Research Laboratories, New Kensington, Pa.

[2] Research Engr., Aluminum Research Laboratories, New Kensington, Pa.

[3] "Model Tests of Latticed Structural Frames," by E. C. Hartmann, R. L. Moore, and Marshall Holt, *Technical Paper No. 2*, Aluminum Research Laboratories, New Kensington, Pa., 1938.

[4] "Tests of 28-Foot Span Aluminum Alloy Trusses," by R. L. Templin, E. C. Hartmann, and H. N. Hill, *Technical Paper No. 3*, Aluminum Research Laboratories, New Kensington, Pa., 1940.

to believe that the analysis of the structure will be simpler and more satisfactory if the behavior of the individual members is more thoroughly understood. In addition, the performance of the structure might be improved if the eccentricity of loading of the individual members is eliminated or at least decreased.

The round tube with symmetrical connections appears to be a desirable type of member, but the difficulties of making suitable end connections have discouraged the use of this type of member in riveted construction. The use of tubular members in welded construction, however, has been discussed and advocated.[5,6]

An experimental investigation involving both static and repeated loadings was undertaken at the Aluminum Research Laboratories in New Kensington, Pa., to study the relative merits of various types of end connections for round-tubular struts, single-angle struts, and double-angle struts. Although aluminum alloy members were used in the tests, it is believed that the comparisons drawn apply to other metals also.

Description of Specimens

The specimens used in this investigation are listed in Tables 1 and 2. Types A, B, C, and D were $2\frac{1}{4}$ in. in outside diameter, with walls 0.134 in. thick, and were seamless, drawn round tubes of aluminum alloy 52S in the H32 or stabilized quarter-hard temper. Types E, F, and G were single $3\frac{1}{2}$-in. by $3\frac{1}{2}$-in. by $\frac{1}{4}$-in. structural angles of extruded aluminum alloy 17S in the T4 or fully heat treated and naturally aged temper. (Abbreviations such as 17S and 52S denote the trade designations by which the alloy is commonly recognized.[7]) Type H was composed of two extruded 17S-T4 unequal-leg angles $2\frac{1}{2}$ in. by 2 in. by $\frac{3}{16}$ in. separated by $\frac{1}{4}$-in.-thick washers. The specimens were loaded through a gusset plate connection.

The tubes of types A and B had flattened ends to facilitate making a riveted or bolted connection. In type A, the ends were flattened against a tube having a wall thickness of 0.138 in. and a length of 10 in. which was telescoped into the end of the member to serve as a filler. The flattening was done cold with two blows by a steam hammer. The resulting thickness of the flattened end with the filler tube left in was about $\frac{1}{2}$ in. For the static tests, the specimens were attached, by turned studs, directly to a steel loading fixture. In type B, there was no filler tube, but a filler plate was inserted temporarily and the end of the specimen was flattened against this plate. The paddle-shaped end was split along the edges, and a gusset plate fitted snugly into the slot thus formed.

The tubes of types C and D had slotted ends but were not flattened. In type C, the ends of the tubes were fitted with segmental plugs to prevent flattening against the gusset plate when the bolts were tightened. In type D, the tube and gussets were joined by welding on both sides of the gusset plates.

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[5] "Tubing Saves Steel in Long Trusses," *Engineering News-Record*, July 2, 1943, p. 37.

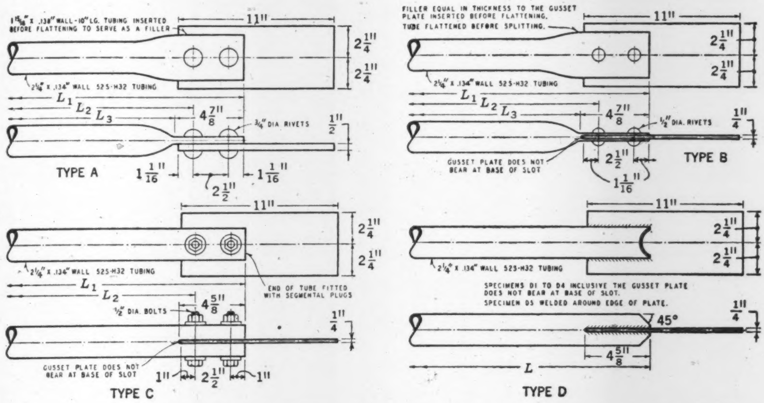
[6] "The Use of Tubular Steel Sections in Structures," by A. Quisling and C. M. Cosman (rept. prepared under the auspices of the Literature Div. of the Eng. Foundation Welding Research Committee), Supplement to *Journal*, Am. Welding Soc., March, 1943.

[7] *Transactions*, ASCE, Vol. 102, 1937, p. 1279, Table 7.

The welds were made in two passes with an oxyhydrogen flame. In specimen D5, which was tested under repeated load, the weld was continuous around the end of the gusset plate whereas in all the other specimens of type D the weld was shorter than the overlap of the tube and gusset plate.

The specimens of type E, Table 2, were simply equal-leg structural angles with the connections through only one leg, the conventional type of single-angle connection.

TABLE 1.—TUBULAR STRUTS LOADED THROUGH GUSSET PLATES


1 15/16" x .138" WALL-10" LG. TUBING INSERTED BEFORE FLATTENING TO SERVE AS A FILLER
11"
2 1/4"
2 1/4"
2 1/4" x .134" WALL 52S-H32 TUBING
L1
L2
L3
4 7/8"
3/4" DIA. RIVETS
1/2"
TYPE A
1 1/16"
2 1/2"
1 1/16"
FILLER EQUAL IN THICKNESS TO THE GUSSET PLATE INSERTED BEFORE FLATTENING.
TUBE FLATTENED BEFORE SPLITTING.
11"
2 1/4"
2 1/4"
2 1/4" x .134" WALL 52S-H32 TUBING
L1
L2
L3
4 7/8"
1/2" DIA. RIVETS
1/4"
GUSSET PLATE DOES NOT BEAR AT BASE OF SLOT
2 1/2"
TYPE B
1 1/16"
11"
2 1/4"
2 1/4"
2 1/4" x .134" WALL 52S-H32 TUBING
END OF TUBE FITTED WITH SEGMENTAL PLUGS
L1
L2
4 5/8"
1/2" DIA. BOLTS
1/4"
GUSSET PLATE DOES NOT BEAR AT BASE OF SLOT
1"
2 1/2"
1"
TYPE C
11"
2 1/4"
2 1/4"
2 1/4" x .134" WALL 52S-H32 TUBING
SPECIMENS D1 TO D4 INCLUSIVE THE GUSSET PLATE DOES NOT BEAR AT BASE OF SLOT.
SPECIMEN D5 WELDED AROUND EDGE OF PLATE.
45°
1/4"
4 5/8"
L
TYPE D

Specimen	TYPE A[a]			TYPE B[a]			TYPE C[a]		TYPE D[a]
	L_1	L_2	L_3	L_1	L_2	L_3	L_1	L_2	L
1[b]........	78	70⅞	68½	68	60⅞	58½	68	61	68
2[c]........	78	70⅞	68½	68	60⅞	58½	68	61	68
3[c]........	38⅝	31½	28⅞	28⅝	21½	18⅞	28⅝	21⅝	28⅝
4[d]........	61	53⅞	51½	61	53⅞	51½	61	54	61
5[d]........	....	....	....	....	....	....	....	....	61

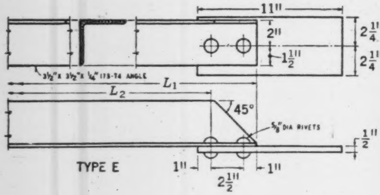
[a] Lengths, in inches; see diagrams. [b] Tested as columns with round ends. [c] Tested as columns with flat ends. [d] Fatigue tested.

The specimens of type F were equal-leg angles with flattened ends. To prevent cracking along the fillet during the flattening operation, the heel of the angle was rounded so that the material was of nearly constant thickness for a length of about 8 in. The flattening was done with a hammer while the metal was at a temperature of about 700° F. The specimens were then reheat-treated and straightened on a "gag" straightener.

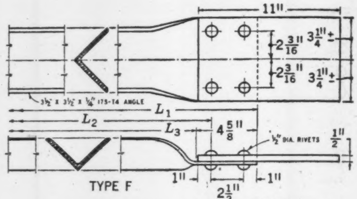
The ends of the equal-leg angle specimens of type G were only partly flattened. This operation was done with the metal at room temperature using special dies in a hydraulic press.

The specimens of type H were double-angle members. The long legs of the individual angles were adjacent and were separated by $\frac{1}{4}$-in.-thick washers spaced about 8 in. apart. The spacing of the stitch rivets is somewhat less than the spacing of about 11 in. allowed by the accepted method of design.[8] The gusset plates were fitted between the two angles forming a member symmetrical about one axis.

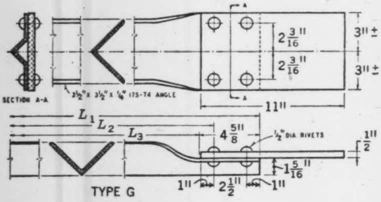
TABLE 2.—ANGLE STRUTS LOADED THROUGH GUSSET PLATES


11"
2"
$2\frac{1}{4}$"
$2\frac{1}{4}$"
$1\frac{1}{2}$"
$3\frac{1}{2}$" x $3\frac{1}{2}$" x $\frac{1}{4}$" 17S-T4 ANGLE
L_1
L_2
45°
$\frac{5}{8}$" DIA. RIVETS
$\frac{1}{2}$"
1"
$2\frac{1}{2}$"
1"

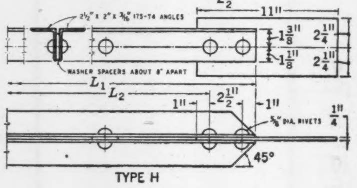
TYPE E


11"
$2\frac{3}{16}$"
$3\frac{1}{4}$" ±
$2\frac{3}{16}$"
$3\frac{1}{4}$" ±
$3\frac{1}{2}$" x $3\frac{1}{2}$" x $\frac{1}{4}$" 17S-T4 ANGLE
L_1
L_2
L_3
$4\frac{5}{8}$"
$\frac{1}{2}$" DIA. RIVETS
$\frac{1}{2}$"
1"
$2\frac{1}{2}$"
1"

TYPE F


A
$2\frac{3}{16}$"
$2\frac{3}{16}$"
3" ±
3" ±
SECTION A-A
$3\frac{1}{2}$" x $3\frac{1}{2}$" x $\frac{1}{4}$" 17S-T4 ANGLE
A
11"
L_1
L_2
L_3
$4\frac{5}{8}$"
$\frac{1}{2}$" DIA. RIVETS
$\frac{1}{2}$"
$1\frac{5}{16}$"
1"
$2\frac{1}{2}$"
1"

TYPE G


11"
$2\frac{1}{2}$" x 2" x $\frac{3}{16}$" 17S-T4 ANGLES
$1\frac{3}{8}$"
$2\frac{1}{4}$"
$1\frac{1}{8}$"
$2\frac{1}{4}$"
WASHER SPACERS ABOUT 8" APART
L_1
L_2
1"
$2\frac{1}{2}$"
1"
$\frac{5}{8}$" DIA. RIVETS
$\frac{1}{4}$"
45°

TYPE H

Specimen	Type E[a]		Type F[a]			Type G[a]			Type H[a]	
	L_1	L_2	L_1	L_2	L_3	L_1	L_2	L_3	L_1	L_2
1[b]	68	61	68	61	$58\frac{3}{4}$	68	61	$58\frac{3}{4}$	68	61
2[c]	68	61	68	61	$58\frac{3}{4}$	68	61	$58\frac{3}{4}$	68	61
3[c]	$28\frac{5}{8}$	$21\frac{5}{8}$	$28\frac{5}{8}$	$21\frac{5}{8}$	$19\frac{3}{8}$	$28\frac{5}{8}$	$21\frac{5}{8}$	$19\frac{3}{8}$	78	71
4	78[c]	71[c]	61[d]	54[d]	$51\frac{3}{4}$[d]	61[d]	54[d]	$51\frac{3}{4}$[d]	61[d]	54[d]
5	$38\frac{5}{8}$[c]	$31\frac{5}{8}$[c]	61[d]	54[d]	$51\frac{3}{4}$[d]	61[d]	54[d]	$51\frac{3}{4}$[d]	61[d]	54[d]
6[d]	61	54	....	....	....	....	....	....	....	....
7[d]	61	54	....	....	....	....	....	....	....	....

[a] Lengths, in inches; see diagrams. [b] Tested as columns with round ends. [c] Tested as columns with flat ends. [d] Fatigue tested.

The elements of the cross sections are given in Table 3(*a*). The particular sections used were chosen so that the least radii of gyration of the tubes, single angles, and double angles would be approximately equal. The mechanical properties of the materials are given in Table 3(*b*). Both the tensile and compressive tests on the tubing were made on specimens of the full cross section, and the strains were measured with type F Huggenberger tensometers (gage length of 0.5 in. and mechanical multiplication of about 2,000). The tensile tests on the angles were made on standard sheet metal specimens 0.5 in. wide,[9] and the strains were measured with an Ewing extensometer having a gage length of 2 in. The compression tests were made on specimens of the full cross section having a slenderness ratio of about 10. The compressive yield strengths were determined from stress-strain relations obtained by measuring the relative movement of the platens of the testing machine by dial gages at the corners of the platens. This procedure has been found satisfactory by comparisons of such values of yield strength with values determined from precise stress-strain curves. The values given in Table 3(*b*) meet the requirements of federal specification QQ-A-351 for 17S shapes or the United States Navy Department specification 44T32 for 52S tubing.

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[8] "Alcoa Structural Handbook," Aluminum Co. of America, Pittsburgh, Pa., 1948, p. 72.

[9] "Standard Methods of Tension Testing of Metallic Materials" (E8-46), *A.S.T.M. Standards*, Philadelphia, Pa., 1946, Pt. 1B, p. 297, Fig. 2.

TABLE 3.—STUDY OF END CONNECTIONS FOR STRUTS

Shape	Size (in.)	(*a*) ELEMENTS OF SECTIONS				(*b*) MECHANICAL PROPERTIES OF MATERIALS				
		Actual thickness (in.)	AREA (SQ IN.)		Nominal radius of gyration (in.)	Alloy and temper[b]	STRENGTH (LB PER SQ IN.)			Elongation in 2 in. (%)
			Nominal	Actual[a]			Compressive yield[c]	Tensile	Tensile yield[c]	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Round tube...	2¼ × 0.134[d]	0.136	0.891	0.904	0.750	52S–H32	24,200	32,700	23,500	17.3
Angles—										
Equal leg...	3½ × 3½ × ¼	0.252	1.691	1.714	0.686[e]	17S–T4	35,800	56,700	39,100	29.5
Unequal leg.	2½ × 2 × 3/16[f]	0.198	1.630	1.726	0.871[g]	17S–T4	39,000	61,400	42,250	21.0

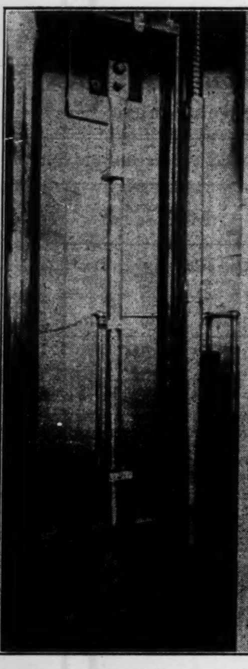
[a] Computed from the length and weight of representative samples and the nominal specific gravity of the material. [b] Abbreviations in Col. 7 denote the trade designation by which the alloy is commonly recognized. [c] 0.2% offset. [d] Outside diameter and wall thickness, respectively. [e] About the axis of least stiffness. [f] Two angles. [g] About the axis parallel to the long legs; angles are separated ¼ in.

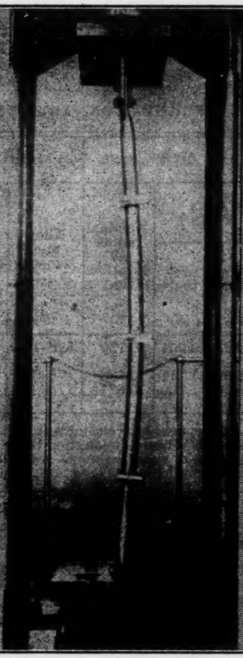
The nominal composition of alloy 52S includes 2.5% magnesium, 0.25% chromium, and the remainder aluminum. The various tempers of this material are obtained by introducing certain amounts of cold work after an annealing treatment and then stabilizing by aging for 4 hours at 320° F or for 1.5 hours at 350° F. The nominal composition of alloy 17S includes 4% of copper and 0.5% each of manganese and magnesium. The stronger tempers of this alloy are obtained by a solution heat treatment consisting of heating at 945° F for a short period depending on the thickness of the material and immediately quenching in cold water. The material ages naturally at room temperature so that the full T4 temper is reached in about 4 days.

METHODS OF TESTING

Round End Tests.—Specimen 1 of each type was tested under static load as a column with round ends. The steel end fixtures bore against testing machine platens that were supported on knife-edge bearings as shown in Fig. 1. This loading condition corresponds to the attachment of a web member to a chord of zero torsional resistance. The tubing specimens and the specimens of type H were centered on the platens so that the axis of the specimen was in the plane of the knife-edge bearings of the platens. Thus, the shafts of the members were loaded axially although the end connection for specimen A1 (specimen 1, type A) was subjected to some bending. The single-angle specimens were placed so that the plane of shear between the angle and the gusset plate was in the plane of the knife-edges.

Deflections at the center were measured with a scale and taut wire. In some cases the deflections at points near the gussets were also measured.





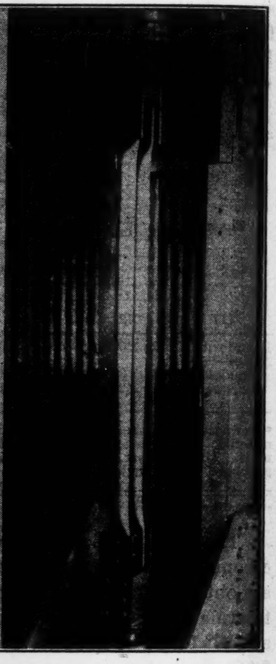


FIG. 1.—SPECIMEN A1; SETUP FOR TESTING A COLUMN WITH ROUND ENDS (KNIFE-EDGED BEARING FIXTURES)

FIG. 2.—SPECIMEN A3 UNDER LOAD AS A COLUMN WITH FLAT ENDS

FIG. 3.—SPECIMEN A5; TWO-PART SPECIMEN AND ADAPTERS FOR TEST UNDER REPEATED LOAD

The static load tests were made in an Amsler testing machine of 300,000-lb maximum capacity and intermediate load ranges. The load range of 30,000 lb was used in these tests. The machine (shown in Figs. 1 and 2) is of the four-column type and, being relatively rigid, is well adapted for column testing. Two specimens of 52S–H32 round tubing (specimen A1 with round ends and specimen A3 with flat ends) are shown under load. The periodic calibrations indicate that the loads are correct within 0.5% for the load range used.

Flat End Tests.—Two specimens of each type but of different lengths were tested under static load as columns with flat ends. The steel end fixtures bore against testing machine platens that were fixed against tipping and rotation (see Fig. 2). This condition corresponded to the attachment of a web member to a chord of great torsional stiffness. Deflections at the center were measured with a scale and taut wire, and strains were measured at a number of points by type A Huggenberger tensometers (mechanical multiplication of about 1,200). At points of rapidly changing stress, a gage length of 0.5 in. was used; at other places a gage length of 1 in. was used.

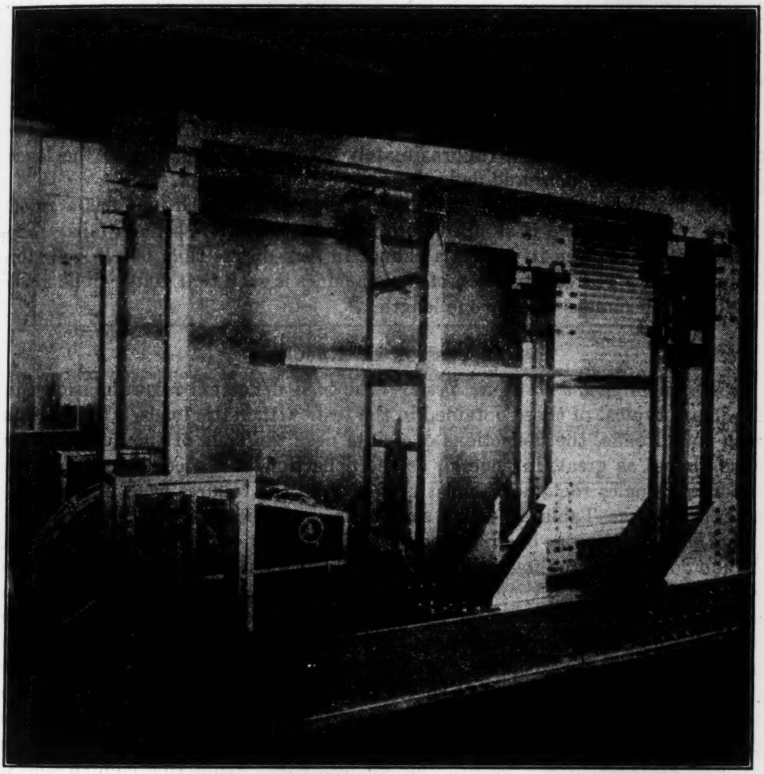


Fig. 4.—Laboratory Setup for Testing Column Fatigue

Fatigue Tests.—Units consisting of a pair of tubes or angles 61 in. long were tested under repeated load ranging from zero to a maximum in compression. The two-part specimen was attached to the adapter of the testing machine as shown in Fig. 3. Using plate fulcra within the adapters the total load was divided equally between the two members. After the load range had been adjusted, the machine was operated by hand while strains were measured at the center and near one end, again using the type A Huggenberger tensometers.

The relative deflection of the two members was measured with calipers and a scale.

The fatigue tests were made in units 3 and 4 of the structural fatigue testing machines[10] of the Aluminum Research Laboratories shown in Fig. 4. Essentially each unit consists of two parallel beams, one the foundation beam and the other the loading beam. They are hinged together by plate fulcra at one end and by an adjustable-throw motor-driven crank at the other. The specimen is inserted between the two beams near the hinge end and undergoes a cyclic loading as the crank rotates. The loading beam is used as the load weighing device and the deflection of the beam is measured while the crank is turned slowly by hand. The load on the specimen is then obtained from the calibration curve previously determined. The inertia effects for the machine operating at normal speed (210 cycles per min) have been determined so that the necessary small correction (approximately 2%) can be applied to the loads obtained from the measured deflections of the loading beam.

The machine is equipped with a cycle counter and each unit has a cutoff switch. The machine is stopped automatically when a specimen fails. The arrangement is so sensitive that a change in load of less than 600 lb will stop the machine. The construction of the machines is such that tests can be made readily on specimens of only one length.

Discussion of Results

Column Strengths in Static Tests.—The load deflection relations shown in Fig. 5 are typical of those obtained in the tests with static loads. In the tests with round ends, the deflections of points near the ends of the gusset plates were almost as great, in general, as the deflection at the center, so that the tubes and angles remained almost straight and the failures were of a double-hinge type (see Table 4). The specimens appeared to be articulated with joints at the knife-edge bearings and at the gusset plate or flattening of the shaft. As should be expected for members with this characteristic, the ultimate loads were relatively low. A comparison of the test results with the Euler column curve is of interest, but it should be borne in mind that with such discontinuities along the length and eccentricities of loading these specimens do not fulfil the conditions of an Euler column. The comparison is shown in Fig. 6, where the data points are plotted at slenderness ratios corresponding to a length equal to the distance between knife-edges and the radius of gyration about the axis parallel to the gusset plate in the case of the symmetrical sections or the axis of least stiffness in the case of the single angle. From this comparison, it appears that the type A specimen was most nearly like an Euler column since it supported 76% of the computed load. The connection at the end fixture was probably much stiffer in this specimen than in any of the others since it was attached directly to the steel end fixtures. Specimen E1 which had the greatest eccentricity on the angle was supporting only 34% of the Euler load (gross area) when the test was stopped because of the large deflections. The test on specimen G1 was also stopped for this reason, but in this case the load was 51% of the Euler load. Although specimen H1 has a symmetrical section,

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[10] "Fatigue Machines for Testing Structural Units," by R. L. Templin, *Proceedings*, A.S.T.M., Vol. 39, 1939, pp. 711–722.

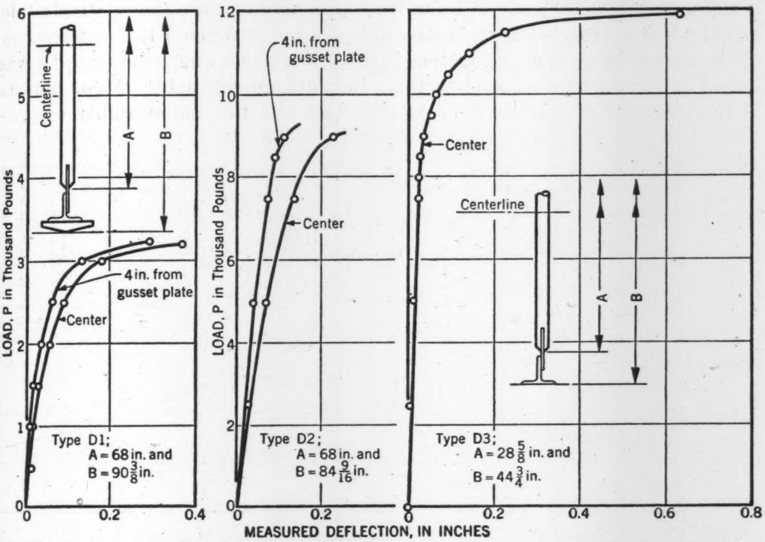

FIG. 5.—LOAD DEFLECTION CURVES OF STRUTS LOADED THROUGH GUSSET PLATES (TUBING DIMENSIONS: OUTSIDE DIAMETER, 2.5 IN.,; WALL THICKNESS, 0.134 IN.,; AND SECTION AREA, 0.904 SQ IN.)

TABLE 4.—RESULTS OF STATIC TESTS ON STRUTS LOADED THROUGH GUSSET PLATES

Specimen (1)	Description of specimen (2)	Over-all length (in.) (3)	Maximum load, P (lb) (4)	Type of failure and test condition (5)
A1	Flattened end tube	88$\frac{11}{16}$[a]	4,920	Double hinge at ends.[b,c]
A2		82$\frac{3}{8}$	11,600	Lateral deflection.[b,d]
A3		43$\frac{9}{16}$	19,390	Lateral deflection.[b,d]
B1	Tube flattened and split	89$\frac{1}{4}$[a]	1,540	Double hinge at ends.[c]
B2		84$\frac{9}{16}$	9,550	Double hinge at ends.[d]
B3		43$\frac{15}{16}$	10,930	Double hinge at ends.[d]
C1	Tube split but not flattened	90$\frac{3}{8}$[a]	2,800	Double hinge at ends.[c]
C2		84$\frac{9}{16}$	12,250	Lateral deflection.[d]
C3		44$\frac{1}{8}$	18,600	Lateral deflection.[d]
D1	Tube split, welded, but not flattened	90$\frac{3}{8}$[a]	3,190	Double hinge at ends.[c]
D2		84$\frac{9}{16}$	9,400	Lateral deflection, hinged at gussets.[d]
D3		44$\frac{3}{4}$	11,900	Lateral deflection, hinged at gussets.[d]
E1	Plain angle	90$\frac{3}{8}$[a]	3,500[e]	Double hinges at ends.[c]
E2		84$\frac{9}{16}$	16,050	Lateral deflection and twisting.[d]
E3		45	16,400	Rivets sheared.[d]
E4		82$\frac{13}{16}$	18,400	Lateral deflection and twisting.[b,d]
E5		43$\frac{9}{16}$	29,425	Lateral deflection and twisting.[b,d]
F1	Flattened angle	90$\frac{11}{16}$[a]	1,500	Double hinge at ends.[c]
F2		84$\frac{1}{2}$	20,150	Lateral deflection.[d]
F3		45$\frac{1}{8}$	17,075	Lateral deflection.[d]
G1	Half-flattened angle	90$\frac{5}{16}$[a]	5,300[e]	Double hinge at ends.[c]
G2		84$\frac{1}{4}$	18,570	Lateral deflection.[d]
G3		44$\frac{3}{4}$	24,850	Lateral deflection.[d]
H1	Double angle	90[a]	2.000	Double hinge.[c]
H2		84$\frac{1}{2}$	20,800	Gusset plate buckled.[d]
H3		82$\frac{13}{16}$	23,500	Lateral deflection.[b,d]

[a] Between knife-edges. [b] Attached directly (by turned studs) to steel end fixture. [c] Tested as a column with round ends, with knife-edge fixtures. [d] Tested as a column with flat ends. [e] Test stopped because of large deflections.

it supported only 15% of the Euler load, probably because the relatively thin gusset plate made a rather effective hinge. In specimen F1, the secondary hinge was at the ends of the flattened part of the angle where the material was only ¼ in. thick, as compared to the ½-in.-thick gusset plate. A comparison of the strengths of specimens F1 and G1 indicates the undesirability of completely flattening the angle. In other words, it is desirable to maintain some of the stiffness of the member in the region of the connection.

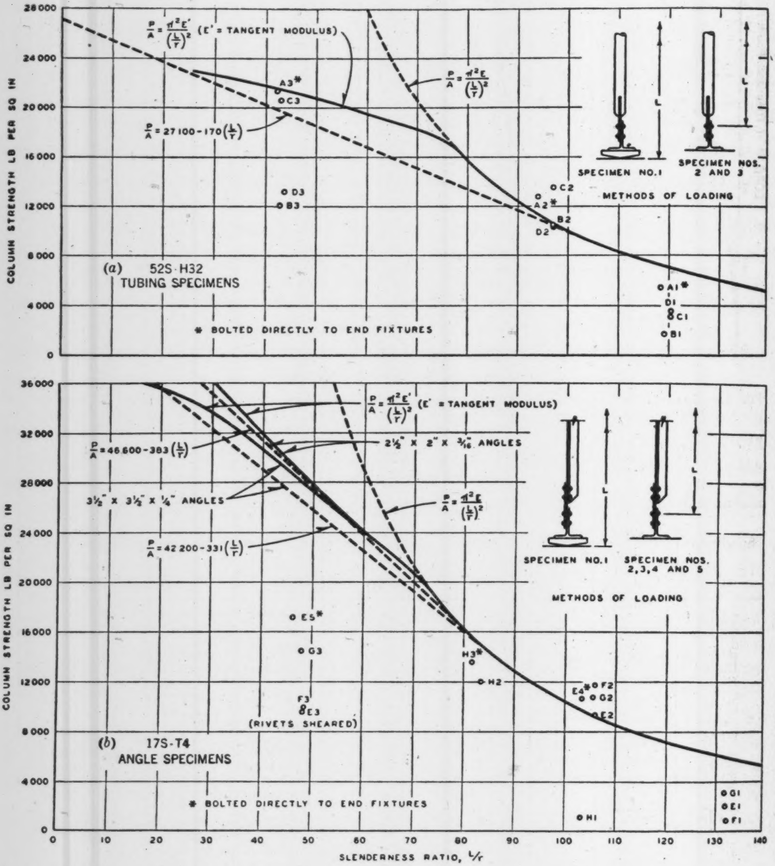

FIG. 6.—COMPARISON OF TEST RESULTS WITH COMPUTED CURVES OF COLUMN STRENGTH

Figs. 2 and 7 show typical failures of the longer specimens tested as columns with flat ends and Fig. 8 shows the failure of the short specimens of types C and E.

In Fig. 2 it will be noticed that the tube appears to be hinged near the end of the flattened part. The round end Euler column load for a tube with the length equal to the distance between the inside studs (70.5 in.) is 10,300 lb. The test result is 11,600 lb—showing some restraint, as should be expected.

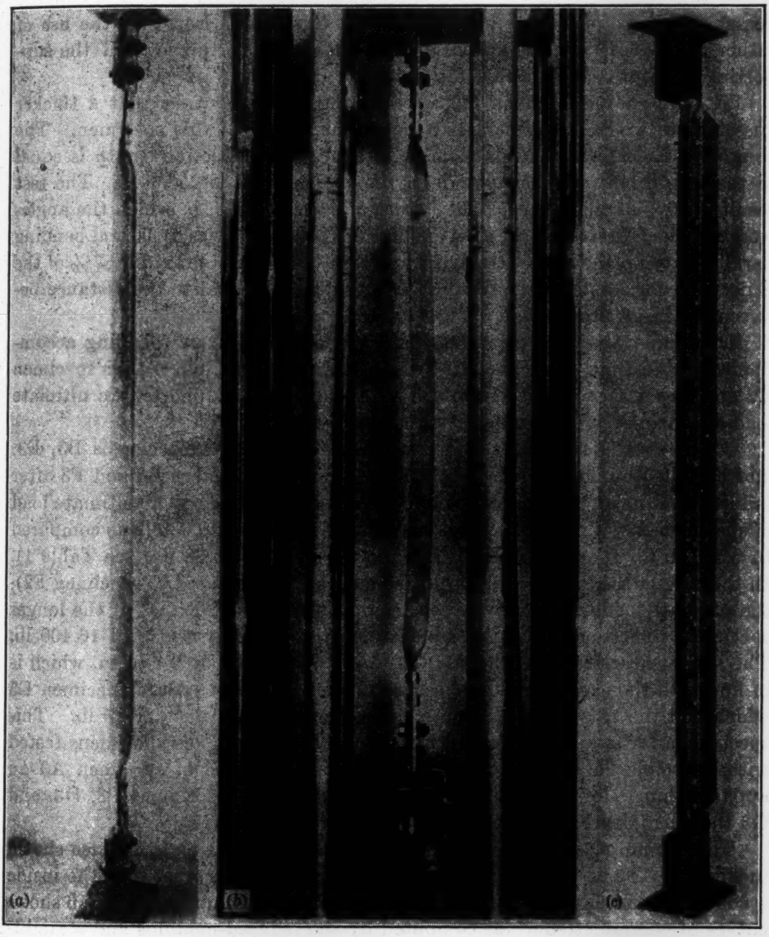

FIG. 7.—LONG SPECIMENS, AFTER FAILURE AS COLUMNS WITH FLAT ENDS: (a) SPECIMEN B2 (52S–H32 TUBING); (b) SPECIMEN F2 (17S–T4, EQUAL-LEG ANGLE); AND (c) SPECIMEN H2 (17S–T4, DOUBLE ANGLES)

The indicated fixity coefficient K is about 0.94. With less rigid end fixtures the degree of restraint, of course, would have been less.

Specimen B2 shown in Fig. 7(a) exhibits a double-hinge type of failure similar to that exhibited by the specimens tested with the knife-edge fixtures.

Of course, in this case both hinges developed within the specimen, one at the end of the flattening and one in the gusset plate. Specimen F2 (see Fig. 7(*b*)) with a thicker gusset plate ($\frac{1}{2}$ in. as compared to $\frac{1}{4}$ in.) did not develop a double-hinge effect although the same two points of weakness were present. Fig. 7(*b*) shows this specimen under a load near the maximum. The bending of the gusset plates is apparent. It appears from this comparison that the use of rather thick gusset plates is advisable with members of this type, if the supporting structure can furnish some restraint.

Fig. 7(*c*) shows specimen H2 after testing. It is obvious that a thicker gusset plate would have increased the ultimate load of this specimen. The computed Euler load for this member, assuming the effective length is equal to the length between inside studs of the end fixtures, is 25,900 lb. The test result is 80% of this value. The failure of specimen H3, in which the angles were attached directly to the steel end fixtures (Table 2), was by lateral bending parallel to the plane of the steel gusset plates. The ultimate load is 87% of the Euler load for a specimen having an effective length equal to the distance between the inside studs.

The plain-angle specimen (E2, Table 2) failed by lateral bending accompanied by twisting. Specimen E4 (which was about 9 in. longer than specimen E2 and was attached directly to the steel end fixture) supported an ultimate load approximately 15% greater than that of specimen E2.

Of the shorter specimens tested with flat ends all but specimens B3, E3, and F3 failed by lateral deflection. The shapes of specimens B3 and F3 after failure gave the appearance of the double hinge at one end. The ultimate load of specimen B3 (which had a tube length of 28$\frac{5}{8}$ in.) was 10,930 lb as compared to 9,550 lb for specimen B2 which had a tube length of 68 in. (see Table 1). In the case of the flattened angles (specimen F3 as compared to specimen F2), the ultimate load of the shorter specimen was only 85% of that of the longer specimen. Specimen E3 failed by shearing the rivets at a load of 16,400 lb. The single shear stress on the 53S–T4 rivets was 26,800 lb per sq in. which is in line with the expected value for such rivets. The strength of specimen E5 which had the same over-all length but no gusset plate was 29,425 lb. This specimen after failure is shown in Fig. 8(*b*). The other short specimens tested with flat ends failed by lateral deflection. In the case of specimen A3 an apparent hinge developed in the flattened part, and in specimens C3, D3, and G3 the hinge developed in the gusset plates.

The column strengths developed in the tests with flat ends are also shown in Fig. 6. The effective length was taken as the distance between the inside studs of the end fixtures. In addition to the Euler column curve, Fig. 6 shows curves of column strength as computed by the reduced-modulus and straight-line formulas for the tubing and angles, respectively. The reduced-modulus formula is the same as the Euler formula, except that in this comparison the tangent modulus is used as the reduced modulus for values of average stress above the proportional limit. The straight-line formula for the 52S–H32 tubing, based on the yield strength given in Table 3(*b*) and determined by a

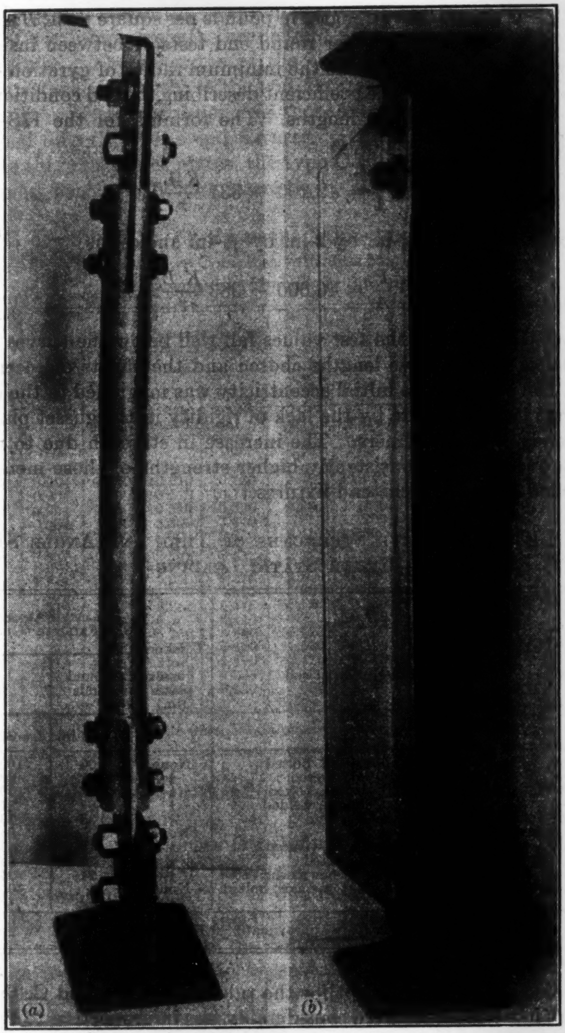

FIG. 8.—SHORT SPECIMENS, AFTER FAILURE AS COLUMNS WITH FLAT ENDS: (*a*) SPECIMEN C3 (52S–H32 ROUND TUBING) AND (*b*) SPECIMEN E5 (17S–T4, EQUAL-LEG ANGLE)

procedure developed in 1938, is as follows:[11]

$$\frac{P}{A} = 27{,}100 - 170\,\frac{KL}{r} \qquad \text{(1)}$$

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[11] "Column Strength of Various Aluminum Alloys," by R. L. Templin, R. G. Sturm, E. C. Hartmann, and Marshall Holt, *Technical Paper No. 1*, Aluminum Research Laboratories, New Kensington, Pa., 1938.

in which P/A is the column strength, in pounds per square inch; L is the length, in inches (between knife-edges for round end tests or between inside studs of end fixtures for flat end tests); r is the minimum radius of gyration of the cross section, in inches; and K is the coefficient describing the end conditions, taken as unity for the aforementioned lengths. The formula for the 17S–T4 alloy,[11] $3\frac{1}{2}$-in. by $3\frac{1}{2}$-in. by $\frac{1}{4}$-in. angles, is

$$\frac{P}{A} = 42{,}200 - 331\,\frac{K\,L}{r} \qquad (2)$$

and the formula for the $2\frac{1}{2}$-in. by 2-in. by $\frac{3}{16}$-in. angles[11] is

$$\frac{P}{A} = 46{,}600 - 383\,\frac{K\,L}{r} \qquad (3)$$

The fact that most of the test values fall well below the curves reflects the uncertainty of the effective lengths chosen and the effects of eccentricities in loading. The effect of the initial eccentricity was magnified in the cases of the double-hinge type of failure by the lack of rigidity in the gusset plates and the flattened ends of the members. The increase in strength due to rigid gusset plates is proved by the consistently higher strengths of those members which were fastened directly to the end fixtures.

TABLE 5.—RELATIVE STRENGTHS OF TUBE AND ANGLE STRUTS UNDER STATIC LOADING

Type	Form	Description of end	Thickness of gusset plates (in.)	Relative Average Stress at Failure (Gross Area)[a]		
				Round Ends	Flat Ends	
				L = 90 in.	L = 90 in.	L = 31 in.
A.....	Tube	Flattened, not slotted, bolted	None	100	100	100
B.....	Tube	Flattened, slotted, riveted	$\frac{1}{4}$	31	82	57
C.....	Tube	Not flattened, slotted, bolted	$\frac{1}{4}$	57	106	96
D.....	Tube	Not flattened, slotted, welded	$\frac{1}{4}$	65	81	62
E.....	Single angle	Riveted by one leg	$\frac{3}{8}$	38	73	45[b]
	Single angle	Bolted by one leg	None	....	84	80
F.....	Single angle	Completely flattened, riveted	$\frac{1}{4}$	16	92	47
G.....	Single	Partly flattened, riveted	$\frac{1}{2}$	57	84	68
H.....	Double angle	Symmetrical about one axis, riveted	$\frac{1}{4}$	21	94	....
	Double angle	Symmetrical about one axis, bolted	None	....	106	....

[a] Values of L are approximate. [b] Rivets sheared.

Fig. 6 also shows that, except for the tubular types A and C, the strengths of the short flat-end struts were considerably less in proportion to the strengths predicted by the formulas than were the strengths of the longer specimens. This fact can also be explained partly by the effect of eccentric loading. The initial eccentricity of a given type of connection is a constant, of course, regardless of the length of the member. If the member then fails by the formation of the double hinges at the ends with the central part of the member remaining comparatively straight (as in the case of specimens B3 and F3), the strength is essentially independent of the length.

A further comparison of the test results can be found in Table 5, in which the relative static column strengths are listed. The values are based on average stress at failure. The strengths of the two, type E, plain angle, specimens which were attached directly to the end fixtures, are seen to be 84% and 80% of the strengths of the corresponding type A tubes flattened over filler tubes and attached directly to the end fixtures. Of the specimens attached to the end fixtures through gusset plates, the type C tubes with segmental filler plugs show the most consistently high strength.

TABLE 6.—SUMMARY OF STRESSES MEASURED IN FLAT END TESTS
(Stresses Are Compressive Unless Marked +; All
Stresses Are in Pounds per Square Inch)

Specimen	Over-all length (in.)	Gross area (sq in.)	Load[a] (lb)	AVERAGE STRESS (LB PER SQ IN.)		RANGE OF STRESS AT:				Lateral deflection at center line (in.)	RATIOS	
						Center Line		End				
				Computed P/A	Measured at center line	Minimum	Maximum f_m	Minimum	Maximum f_e		$\frac{f_e}{f_{avg}}$	$\frac{f_e}{f_m}$
A3....	$43\frac{9}{16}$	0.904	9,500	10,510	10,520	9,380	11,640	+ 2,850	18,460	0.034	1.75	1.58
B2....	$84\frac{9}{16}$	0.904	9,500	10,510	10,740	7,600	14,160	+ 1,440	20,690	0.294	1.92	1.46
B3....	$43\frac{15}{16}$	0.904	9,500	10,510	10,320	9,380	11,750	+ 2,030	28,380	....	2.74	2.41
C2....	$84\frac{9}{16}$	0.904	7,500	8,300	7,640	250	15,200	5,540	10,560	....	1.38	0.70
C3....	$44\frac{1}{8}$	0.904	7,500	8,300	8,670	8,410	8,820	6,210	10,300	....	1.19	1.17
D2....	$84\frac{9}{16}$	0.904	7,500	8,300	8,245	6,290	9,980	4,730	9,970	0.130	1.21	1.00
D3....	$44\frac{3}{4}$	0.904	7,500	8,300	8,075	7,130	9,230	4,560	9,970	0.024	1.24	1.08
E2....	$84\frac{9}{16}$	1.714	9,000	5,250	5,275	+9,800	11,490	+ 5,240	12,330	0.40	2.34	1.07
E3....	45	1.714	9,000	5,250	4,985	+5,860	12,740	+ 6,230	15,590	0.096	3.12	1.22
E4....	$82\frac{13}{16}$	1.714	9,000	5,250	5,440	+3,120	9,780	+ 1,990	9,870	0.260	1.81	1.01
E5....	$43\frac{9}{16}$	1.714	9,000	5,250	5,415	+4,350	10,470	+ 4,170	11,860	0.090	2.19	1.13
F2....	$84\frac{1}{2}$	1.714	9,000	5,250	5,340	2,520	8,300	+13,950	17,220	0.032	3.22	2.35
F3....	$45\frac{1}{8}$	1.714	9,000	5,250	4,990	1,530	8,190	+ 9,390	12,950	0.021	2.60	1.58
G2....	$84\frac{1}{4}$	1.714	9,000	5,250	5,490	2,470	8,360	+10,360	16,740	0.134	3.05	2.00
G3....	$44\frac{3}{4}$	1.714	9,000	5,250	4,335	2,100	7,650	+11,060	18,040	0.042	4.16	2.36
H3....	$82\frac{15}{16}$	1.726	12,000	6,950	6,920	6,080	7,760	4,260	9,540	0.030	1.38	1.23

[a] Load at which strain measurements were made.

It should be remembered that the gusset plates for the single-angle specimens were $\frac{1}{2}$ in. thick, whereas $\frac{1}{4}$-in. gussets were used for the tubular and double-angle struts. Obviously the thinner gusset plate should be the more effective hinge. Weakness in the gusset plate probably accounts for the low comparative strength of the type H double-angle specimen in the round end test. On the other hand, in the flat end tests double angles proved to be superior to single angles, despite the failure of the gusset plate of specimen H2.

The data in Table 5 indicate a generally greater structural efficiency for the tubular and double-angle types of struts with relatively rigid end connections when compared with the single-angle struts.

Measured Stresses, Static Tests.—The measured strains were interpreted into stresses by simply multiplying by the modulus of elasticity;[12] thus, for 52S, $E = 10{,}200{,}000$ lb per sq in.; and, for 17S, $E = 10{,}600{,}000$ lb per sq in. The measured stresses thus obtained are summarized in Table 6 and specific examples are shown in Fig. 9. The dotted lines indicate the value of the average stress on the gross section of a strut. Compressive stress is plotted to the right and tensile stress to the left. The loads at which the measurements were made corresponded to average stresses ranging from 5,250 lb per sq in. for the single angles to 10, 510 lb per sq in. for the tubes of types A and B. Although there is considerable variation in measured stresses, the averages of the stresses measured at the center agree very well with the simple (P/A)-stresses. In thirteen of the seventeen specimens the average measured stress is within 5% of the computed average stress and only in the case of specimen G3 is the difference greater than 8%. In general the range of stress at the end is greater than that at the center, and for the specimens with flattened ends it is much greater. The maximum compressive stresses are also generally greater at the end than at the center. For example, in the case of specimen A3 under a load of 9,500 lb, the stresses at the center range from 9,380 lb per sq in. to 11,640 lb per sq in. in compression whereas at the end the stresses range from 2,850 lb per sq in. in tension to 18,460 lb per sq in. in compression. The maximum compressive stress at the end is 1.6 times the maximum compressive stress at the center. Because of large lateral deflections, the range

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[12] "The Elastic Constants for Wrought Aluminum Alloys," by R. L. Templin and E. C. Hartmann, *Technical Report No. 966*, National Advisory Committee for Aeronautics, Washington, D. C., January, 1945.

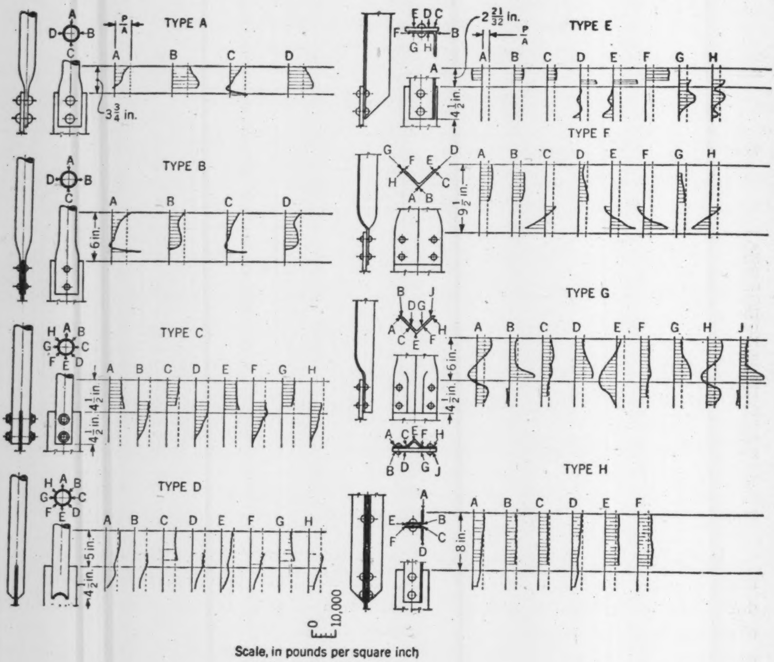
TYPE A
$\frac{P}{A}$
A B C D
$3\frac{3}{4}$ in.
TYPE B
6 in.
TYPE C
A B C D E F G H
$4\frac{1}{2}$ in.
TYPE D
5 in.
$4\frac{1}{2}$ in.
TYPE E
$2\frac{21}{32}$ in.
$4\frac{1}{2}$ in.
TYPE F
$9\frac{1}{2}$ in.
TYPE G
A B C D E F G H J
6 in.
TYPE H
A B C D E F
8 in.
0
10,000
Scale, in pounds per square inch

FIG. 9.—TYPICAL MEASURED STRESS DISTRIBUTIONS IN STRUTS

of measured stresses at the center was greater than at the end for specimens C2, E2, and E4.

A study of Fig. 9 indicates that, in general, the stress distribution in the flattened ends of the tubes is alike, despite the difference in the attachment to the gusset plate. It can be noted that a tensile stress was developed in the tapered part of the tube near the flattening. The stress gradients in this region are rather high. For specimens of types C and D the stress gradients are relatively flat.

In the plain-angle specimens (type E) the load is eccentrically applied, but in addition to the eccentricity there are certain restraining moments that decrease the effects of the eccentricity. The restraining moment in one direction rises by virtue of the bending stiffness of the gusset plate and in the other direction by the use of more than one rivet in the connection. By working backward from the measured stresses, the actual eccentricities in the flat end columns were found to be, approximately: e_1 normal to the gusset plate = 0.75 in.; and e_2 parallel to the gusset plate = 0.31 in. to 0.60 in. The nominal eccentricity taken equal to the distance from the back of the angle to the centroid of the section is 0.94 in. The stiffness of the gusset plate accounts for the difference.

The eccentricities of single-angle connections were greatly reduced by the flattened ends of types F and G. The measured stresses indicated an effective eccentricity normal to the gusset plate of approximately 0.09 in. toward the heel of the angle in type F, and of 0.16 toward the toes of the angle in type G with the ends only partly flattened. Thus, stresses measured at the center of the specimens of types F and G were all compressive stresses, whereas at the center of specimen E2 the stresses varied from 11,490 lb per sq in. in compression to 9,800 lb per sq in. in tension. It appears that a greater amount of flattening of the G-type specimens would have reduced the eccentricity.

As in the case of the flattened end tubes, deforming the ends of the angles produced high stress gradients near the connections. Both maximum compressive and maximum tensile stresses measured at the ends of the specimens of types F and G are higher than those found in the specimens of type E.

Placing two angles back to back to make the strut symmetrical about one axis greatly reduced the stress variation. The range of stress at the center of specimen H3 was only from 6,080 lb per sq in. to 7,760 lb per sq in. in compression.

Results of Fatigue Tests.—Stress distributions in the fatigue specimens were found to be quite similar to the distributions found in the static tests of flat end specimens shown in Fig. 9.

In the case of the angles, two specimens of each type were tested in fatigue, one with the same average stress as that imposed on the tubing (approximately 7,500 lb per sq in.) and the other with the same total load. In the second specimen of each type the stress cycle was from zero to approximately 4,000 lb per sq in.

Table 7 summarizes the results of the fatigue tests giving ratios of measured stresses and the number of cycles of compressive loading imposed on

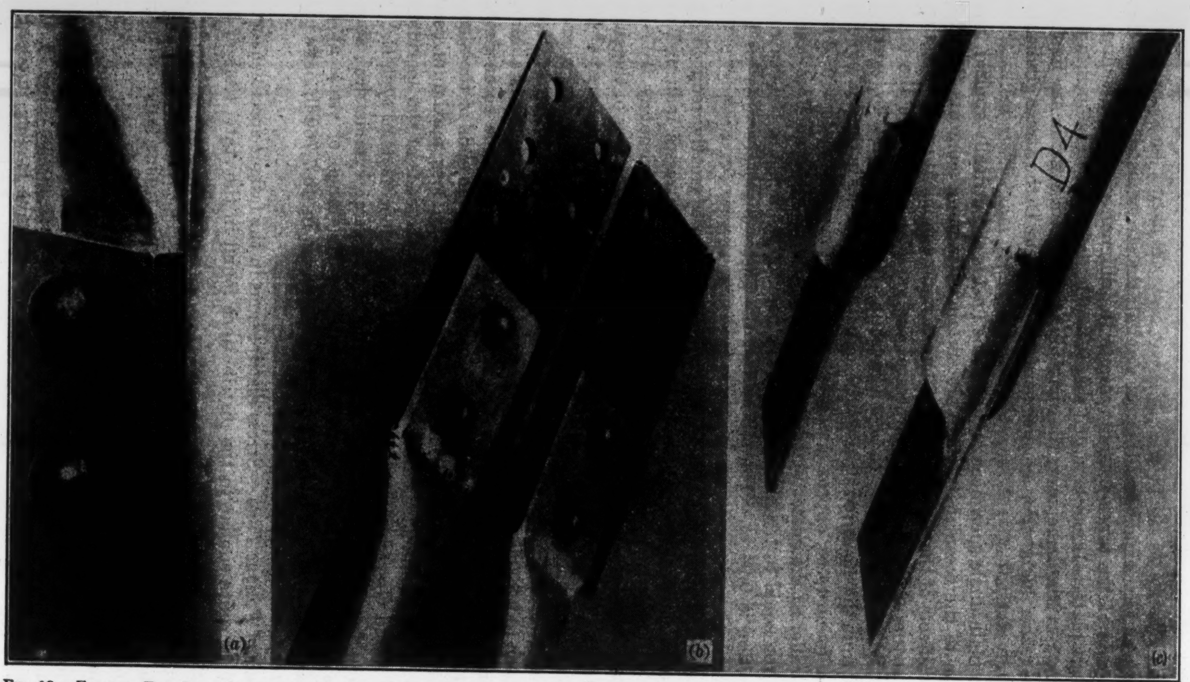

FIG. 10.—ENDS OF TWO-PART SPECIMENS, 52S-H32 ROUND TUBES, AFTER FAILURE UNDER REPEATED COMPRESSIVE LOAD: (*a*) BOTTOM OF SPECIMEN A5; (*b*) TOP OF SPECIMEN B4; AND (*c*) BOTTOM OF SPECIMEN D4

each specimen. Three of the specimens—C4, F5, and H5—withstood approximately 25,000,000 cycles of stress without visible signs of failure. For the other specimens, the number of cycles to produce visible cracks ranged as low as 71,900. In this case (specimen E6), failure occurred by shearing the rivets on which the average shear stress was about 20,400 lb per sq in. The failure of specimen F4 also involved the failure of a rivet. The average shear stress on the rivet was about 15,300 lb per sq in., and the number of cycles to failure was 1,118,600. These two test results lie well above the (*S*-*N*)-curve indicated by the data for joints with cold driven 53S–T4 rivets reported by E. C. Hartmann, M. ASCE, J. O. Lyst, and H. J. Andrews, Jun. ASCE, in 1944.[13] This difference is to be expected since the riveted joints failed by tension in the plates whereas in these specimens the failures were in the rivets.

TABLE 7.—RESULTS OF TESTS ON COLUMN FATIGUE SPECIMENS; STUDY OF END CONNECTIONS FOR STRUTS
(Stresses Are Compressive Unless Marked +)

Specimen	Gross area (sq in.)	Load[a] *P* (lb)	STRESS (LB PER SQ IN.)				No. of cycles	Remarks
			Average	Maximum				
			P/A	Measured at Center Line		Measured at end		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
A4....	0.904[b]	13,650	7,550	8,000	12,700	−16,080 + 3,610	342,200	. ....
B4....	0.904[b]	13,520	7,480	7,710	9,070	10,700	172,000	Removed at 549,800.
C4....	0.904[b]	13,700	7,580	8,110	9,180	9,180	25,004,500	No failure.
D4....	0.904[b]	13,600	7,530	7,610	8,650	8,690	22,891,100	Small cracks at ends of gusset plates; at 25,208,800 cycles crack extended over three fourths of the circumference.
D5....	0.904[b]	13,760	7,620	7,690	8,390	9,130[c]	20,785,400	Small cracks at ends of gusset plates.
E6....	1.714[b]	26,370	7,690	7,550	−17,570 +14,510	−13,930	71,900	Rivets sheared.
E7....	1.714[b]	13,680	3,990	4,060	− 8,140 + 5,110	− 2,300 + 6,130	4,276,700	....
F4....	1.714[b]	25,630	7,480	7,740	14,530	.... +13,700	1,118,600	One rivet sheared, crack in angle.
F5....	1.714[b]	13,530	3,950	4,120	7,450	.... + 6,500	25,489,300	No failure.
G4....	1.714[b]	25,610	7,480	7,000	13,340	−12,000 +12,170	443,800	..
G5....	1.714[b]	13,640	3,980	3,860	5,970	− 6,230 + 7,100	5,508,200	....
H4....	1.726[b]	26,600	7,150	6,910	11,300	8,520	4,216,800	....
H5....	1.726[b]	13,580	3,940	3,820	6,080	4,950	24,899,100	No failure.

[a] Maximum load of stress cycle and load at which strain measurements were made. [b] Two each. [c] Stresses at ends of diameter parallel to the gusset plate average 9,040 lb per sq in. as compared to an average of 7,888 lb per sq in. for the two diameters at an angle of 45° to the gusset plate. In specimen D4, the averages for all the diameters were practically the same.

The test results in Table 7 further indicate the superiority of symmetrical connections over unsymmetrical connections and show that flattening the ends of tubular members decreases their resistance to repeated loads. Photographs of some of the specimens are shown in Figs. 10 and 11. Because of the local nature of the fatigue failures, it is to be expected that the strengths and lives of the specimens were little affected by the length of the specimen.

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[13] "Fatigue Tests of Riveted Joints, Progress Report of Tests of 17S–T and 53S–T Joints," by E. C. Hartmann, J. O. Lyst, and H. J. Andrews, *Advance Restricted Report No. 4115*, National Advisory Committee for Aeronautics, Washington, D. C., September, 1944.

General Remarks.—The results of the tests described in this paper corroborate the general conclusion of J. A. Van den Broek,[14] M. ASCE, based on compressive tests of single angles with different types of connections—namely, that the compressive strength of angles is affected largely by the eccentricity of the applied loads. Although Mr. Van den Broek found that angles with completely flattened ends showed greater eccentricity and less column strength than angles connected by only one leg, the opposite effect was found in the tests described in this paper. Since the eccentricity of the connections of a flattened end angle depends on the manner in which the angle is flattened and the face to which the supporting fixtures are attached, this comparison cannot be further analyzed because the details of the end connections are not discussed in Mr. Van den Broek's paper.

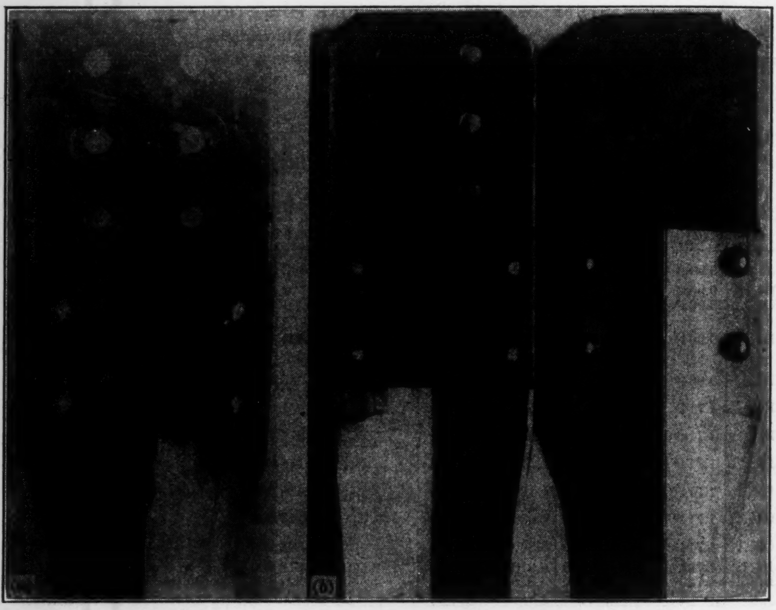

FIG. 11.—TOP ENDS OF TWO-PART SPECIMENS 17S–T4, EQUAL-LEG ANGLE, AFTER FAILURE UNDER REPEATED COMPRESSIVE LOAD: (*a*) PART A, OF SPECIMEN F4 AND (*b*) SPECIMEN G4

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[14] "Effect of Connections and Rivet Holes on Ductility and Strength of Steel Angles," by J. A. Van den Broek, *Civil Engineering*, February, 1940, p. 83.

Various types of welded connections for tubular struts were described by A. Quisling and C. M. Cosman,[6] in 1943. With reference to flattening the ends of tubes they conclude that tubes with flattened ends

> "* * * suffer from the weakness inherent in drastic changes of the shapes of the members, always objectionable. The majority of them are also less rigid laterally, lacking the stabilizing effect of the tubular shape at the joints, where it is most needed."

These statements seem to be borne out by the results of these tests at the Aluminum Research Laboratories. However, the comparatively high static strengths of the type A flattened end tubes (Table 1) bolted directly to the end fixtures indicate that the loss in ridigity due to flattening the ends may be somewhat lessened by first inserting a filler tube to reduce the degree of flattening and by using very rigid gusset plates.

CONCLUDING REMARKS

Although the tests made for evaluating the various types of end connections were made on aluminum alloy members, it is believed that the general conclusions drawn are applicable to members of other materials.

1. Round tubes are superior in static column strength to single angles, provided that the end connections are sufficiently rigid. Of the types of specimens that were fastened to the end fixtures through gusset plates, the type C unflattened tubes, slotted and bolted through segmental filler plugs, showed the most consistently high strength (see Table 5). Based on the average stresses at failure for the types A and E specimens which were attached by turned studs directly to the end fixtures, the static strength of round tubes of type A, with the ends flattened over filler tubes, was about 20% greater than the strength of single angles of the same length, fastened through one leg.
2. Double-angle struts, symmetrical about one axis, are superior in static column strength to single angles, provided that the end connections are sufficiently rigid (see Table 6). The static strengths of the various long single-angle struts tested as columns with flat ends were from about 76% to 96% as great as the average strength of the long double-angle strut tested as a column with flat ends.
3. The stiffness of the gusset plates and the rigidity of the connections are important factors contributing to the static strength of struts.
4. The static strengths of the round tubes tested as columns with round ends were from 24% to 76% of the values computed by the Euler column formula taking the effective length as the length between knife-edges of the supporting fixtures. The corresponding range for the angles was from 15% to 51%.
5. The strengths of long struts of the types tested, loaded as columns with flat ends, ranged from 20% below to 27% above the values computed by the Euler formula using a fixity coefficient equal to unity and an effective length equal to the length between inside studs of the end fixtures.

6. The strengths of the short tubular struts tested as columns with flat ends ranged from 61% to 108% of the values computed by the straight-line column formula taking the effective length equal to the distance between inside studs of the end fixtures and a fixity coefficient equal to unity. The corresponding range for the single-angle struts was from 37% to 55%.

7. Of the types of specimens tested, double angles and round tubes bolted or welded to gusset plates placed in slots in the unflattened ends showed the most favorable distribution of stress near the connections. The ratios of maximum measured stress to average stress for the various types of connections were as follows:

Description of end connection	Ratio: $\frac{\text{Maximum stress}}{\text{Average stress}}$
Slotted tube, unflattened, or double angles	1.38
Flattened tube, not slotted, filler tubes in ends	1.75
Flattened tube, slotted	2.74
Single angle, fastened through one leg	3.12
Completely flattened angle	3.22
Partly flattened angle	4.16

The stresses were measured at loads ranging from 7,500 lb to 12,300 lb depending upon the type of specimen (see Table 6).

8. Of the types of tubular specimens tested, round tubes with the ends unflattened and bolted to gusset plates placed in slots in the ends showed the greatest resistance to repeated loads producing a stress cycle from zero to about 7,500 lb per sq in. in compression. Similar specimens welded rather than bolted to the gusset plates showed slightly less fatigue strength, whereas flattened-end tubes showed the lowest resistance to repeated stress.

The fatigue lives of the various tubing specimens tested were as follows (cycle of average stress from zero to about 7,500 lb per sq in. in compression):

Description of end connections	Number of cycles of stress
Slotted ends, bolts through segmental filler plugs	25,004,500 No failure
Slotted ends, welded to gusset plates	22,891,100 20,785,400
Flattened ends, not slotted, unsymmetrical connection	342,200
Flattened ends, slotted, symmetrical connection	172,000

9. Angles with the ends flattened in order to load the member close to the neutral axis showed greater fatigue strength than angles connected by one leg, but less fatigue strength than double angles symmetrical about one axis. The fatigue lives (number of cycles) of the various angle specimens tested, with

average compressive stresses from zero to about 4,000 lb per sq in. and 7,500 lb per sq in., respectively, were:

Description of end connections	0 to 4,000 lb per sq in.	0 to 7,500 lb per sq in.
Double angle, symmetrical connection	24,899,100 No failure	4,216,800
Completely flattened angle	25,489,300 No failure	1,118,600
Partly flattened angle	5,508,200	443,800
Single angle fastened through one leg	4,276,700	71,900 Rivet failure

10. A desirable combination of static strength and resistance to repeated loads is furnished by round tubes, unflattened, with gusset plates placed in slots in the ends, and by double angles, symmetrical about one axis.

11. Flattened end tubes are roughly comparable in static strength and resistance to fatigue to single angles fastened through one leg.

ACKNOWLEDGMENTS

The work described in this paper was done at the Aluminum Research Laboratories under the direction of R. L. Templin, M. ASCE, assistant director of research and chief engineer of tests and of Mr. Hartmann, chief, Engineering Design Division. The paper was sponsored by the Committee on Design in Lightweight Structural Alloys of the ASCE Structural Division.

DISCUSSION

Jack R. Benjamin,[15] Assoc. M. ASCE.—Excellent as this qualitative investigation is, it is unfortunate that a more quantitative study could not have been made. In all cases the end conditions are unknown. Only a general range is indicated. It is not reasonable to assume that the eccentricities were constant and the writer has found that this item cannot begin to be described by the terms, "round ends" or "flat ends." A much more extensive investigation is necessary before the data can be accepted as being valuable quantitatively.

Extension of strength data to practical structures is impossible on a rational basis. The tests given are only tests of the struts used, with end conditions peculiar to each in a particular testing machine. A much more detailed study must be made before engineers can begin to know what they are doing in the design of struts and their connections.

James W. Harland,[16] Jun. ASCE.—Any investigation which points out possible new uses, or which makes additional information available concerning accepted practices, is to be commended, and the authors are certainly to be complimented for this paper.

It is regrettable that the photographs showing the specimens under load (Figs. 1, 2, and 3), did not show the details of the end connections more clearly.

The authors state that, although these tests were made on various aluminum alloys, the results are thought to be applicable to other materials as well. It is known that the principal structural material, steel, differs from most aluminum alloys in that it possesses much greater ductility. The property of ductility permits a material to yield and distribute load so that stress concentrations in a ductile material are not as severe as in a nonductile material. That is, a stress concentration detail would not be as serious in structural steel as it would be in an aluminum alloy specimen.

Another point at which steel and aluminum differ is in the value of Young's modulus (E). It is notable in Table 4, that 14 of the 26 specimens are listed as having failed wholly or partly as a result of lateral deflection. In a material such as steel, which has a larger value of E, is it not possible that E will limit the deflection so that the failures will occur in some other manner?

F. P. Shearwood,[17] M. ASCE.—The practical structural designer should find this paper of unusual interest and benefit as it calls attention to that very important part of structural framing—namely, the connections of compression members. Connections are often proportioned only to equal the area of the member with disregard to other factors, especially stiffness and stability. In common practice, planes of reduced section are studiously avoided, but reduced stiffness or interference with the even flow of stress is often overlooked. These are of almost equal importance, especially with compression members. These tests are excellent examples to illustrate the weakness of compression members which have planes of reduced stiffness in their connections.

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[15] Asst. Prof. of Civ. Eng., Stanford Univ., Stanford University, Calif.

[16] Louisville, Ky.

[17] Chf. of Tech. Staff, United Shipyards, Ltd., and Cons. Engr., Dominion Bridge Co., Ltd., Montreal, Que., Canada.

The paper also emphasizes that columns are seldom either pin ended or fixed; they vary from that having a point of contraflexure at one or both ends to that having completely rigid end supports.

So-called pin joints—those with no resistance to turning— do not occur except in the testing laboratory, because even a pin joint has turning resistance due to the friction of the pin against its bearing.

Fixed ends, or rather the fixedness of ends, are dependent on the bending resistance of the weakest section in their connections and the flexural length of the strut by the distance between these weak sections.

The section moduli of the struts given in this paper and the weak sections in their connections are approximately as shown in Table 8. These section moduli show that the bending resistance of the connections is much weaker than the main section; therefore it must influence the rigidity of the strut and consequently its compression value.

TABLE 8.—SECTION MODULI OF STRUTS AND CONNECTIONS

Dimension	Section modulus[a]
(*a*) STRUTS	
One—3½ in. by 3½ in. by ⅜ in.	0.79
One—3½ in. by 3½ in. by ¼ in.	0.40
Two—2½ in. by 2 in. by ¼ in.	0.58
One—2½-in. by 0.138-in. tube	0.45
(*b*) CONNECTIONS	
4½ in. by ¼ in. flat	0.047
4½ in. by ½ in. flat	0.187
3½ in. by ½ in. flat	0.146
6½ in. by ¼ in. flat	0.068
6½ in. by ½ in. flat	0.27
6½ in. crimped flat	0.3[b]

[a] Section moduli are for the entire main strut in each case; for example, in the third item of Table 8(*a*) 0.53 is the section modulus of two, 2½-in. by 2-in. by ¼-in. angles. [b] Approximately.

All the tests made with knife-edge bearings—namely, specimens A1 to H1, have at least one other plane of weakness to turning. If these planes had no turning resistance, the connections would be unstable; but all of them have some turning resistance. Therefore, the resistance of these struts is influenced by, and probably limited to, the turning resistance of the plane of weakness of its connections.

The erratic results of these tests are rather confusing; for instance, in comparing the F-tests with the others, why should specimen F1 with knife-edge bearings be so weak, whereas the same make-up with flat bearings gives nearly the highest results? Perhaps there is the possibility of enough difference in the setting of the struts against the bearings to account for these results.

Considering the fixed-ended struts and assuming that the rigidity of the end fixture is fully the equal of the main struts, specimens E4, E5, and H3 have no weak section and therefore should develop the full resistance as fixed end struts. Struts C2, C3, D2, D3, E2, E3, and H2 have one section which has a smaller section modulus than the main member; therefore, the point of contraflexure will be at this weak point and its turning resistance may be the limit of the flexural resistance of the strut, and the distance between these two restricted turning points will be the length on which the strut can deflect.

Struts B2, B3, F2, F3, G2, and G3 have two weak sections, and if these sections had no turning resistance, the struts would be unstable; but both of these weak sections have some turning resistance to overcome before the strut fails, and therefore they provide some strength to the struts.

It is very difficult to visualize clearly or to estimate the resulting effect of the bending resistance of connections which have two or more planes that are weak in turning. For instance, it is hard to follow the reasons for the comparatively high test value of F2 and F3 which have very weak planes where the angle is flattened to $\frac{1}{4}$ in. This may be because all the bending moments from eccentricities happen to balance one another precisely and no flexural stresses are set up; in this particular case the strut has only direct stress to resist as long as an exact balance is not upset.

Specimens G2 and G3 have weak sections at relatively the same location, but in this case the strut angles are not completely flattened so that the section has considerably greater bending resistance.

In comparing specimens B, C, D, and H, which have only $\frac{1}{4}$-in. plates, with specimens A, E, F, and G, which have $\frac{1}{2}$-in. plates, it must be realized that the connection plates of the former are enclosed by the struts whereas the plates of the latter are free to bend between the rivet heads.

The different findings between these tests and those of J. A. Van den Broek,[14] M. ASCE, in regard to the value of the flattened angles and those connected by one leg, may possibly be due to the lower modulus of aluminum as compared with steel, which would exaggerate the flexural deflection as a result of eccentricity.

Tubular sections being symmetrical on all axes are fundamentally preferable to unsymmetrical shapes, such as angles, channels, and even I-beams, but the past and existing construction methods demand connections which do not easily suit tubes. With modern welding and cheaper tubular sections, the ideal compression section may soon come into greater use.

To compute the actual value of such connections, even approximately, as shown in the paper (even if all the unknown inaccuracies of fabrication, assembling, etc., were eliminated) is practically impossible, but this paper and the tests have emphasized, most usefully, the advisability of carrying the full section of compression members right into their reaction points, or with as little interference as possible with the straight path of their stresses.

S. K. Ghaswala,[18] Assoc. M. ASCE.—The paper forms an interesting account of the study of end connections of aluminum struts. In theory, end restraint has a very simple effect—namely, to change the effective column length. In practice, however, the accurate determination of the effects of end connections for any column except a pin-ended column is a difficult problem, especially in lattice structures like towers, booms, and light trusses. The author states that in such structures, members are connected by bolts or rivets through one leg which produces a loading eccentricity. The writer fully agrees with this fact and ventures to condemn this method as bad connection detail. This attachment produces high eccentricity, excessive stress concentration near the rivet hole and close to the edge of the angle leg, and a concentration of ductile flow near the hole. The material therefore does not develop its full strength, but ruptures through the hole before the elastic-limit stress in the main part of the member is reached. This is probably one of the main reasons why most of the test values of column strength found by the author fall well below the theoretical values computed by Eqs. 1, 2, and 3.

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[18] Chartered Engr., Fort Bombay, India.

In evaluating the column strength, P/A, for 52S-H32 tubing (based on the compressive yield strength of 24,200 lb per sq in.), the author has used Eq. 1. This alloy[19] has a compressive yield strength of 27,000 lb per sq in. and, when $\frac{K\,L}{r}$ is less than 100, the column strength is given by

$$\frac{P}{A} = 30{,}600 - 204\,\frac{K\,L}{r} \quad \text{(4a)}$$

For values of $\frac{K\,L}{r}$ greater than 100,

$$\frac{P}{A} = \frac{102{,}000{,}000}{\left(\frac{K\,L}{r}\right)^2} \quad \text{(4b)}$$

In the case of 17S-T4 alloy, this verification is not possible as no specifications are given for this type. However Eq. 3 for 17S-T4 alloy for 2.5-in. by 2-in. by $\frac{3}{16}$-in. angles is the same for 61S-T alloy.[20] These slightly varying expressions require some elucidation to avoid confusion.

The study of design methods for aluminum alloy compression members, with particular emphasis on light triangulated structures, was suggested during 1949 by C. Marsh.[21] The formula of J. Perry widely used in England for steel design is modified to cover all initial imperfections and fixity conditions by an initial curvature, whereas latticed struts are analyzed by providing for distortion due to shear. Comprehensive investigations have also been conducted on the foregoing lines on the strength of aluminum struts by J. F. Baker, Assoc. M. ASCE, and J. W. Roderick of Cambridge University,[22] in Cambridge, England. In evaluating end restraint effects, aircraft structural analysis is worth studying as practically the entire gamut of restraining variables are met within this field. The situation arising here is well surveyed by Paul E. Sandroff,[23] and the subject of column end fixity coefficients has been studied by many.[24] This latter concept has very clearly brought to light (as shown by Theodor von Kármán,[25] Hon. M. ASCE) the facts that efficiently designed structures cannot be made proportionately stronger by just increasing the amount of end restraint and, secondly, that end restraint causes a marked increase in buckling strength only for long slender columns and not for short ones. F. R. Shanley[26] has conducted exhaustive studies and the results of his investigations for 24-ST aluminum alloy columns with varying fixity coefficients is worth noting. An excess of actual fixity over the assumed value can do little good, but lack of fixity (with the same assumed value) can do much harm. End restraint in an eccentric column tends to reduce the effective eccentricity. In the case of columns loaded within the elastic limit this reduction increases with loading and assumes pronounced results in the inelastic range. Small eccentricities, therefore, do not materially affect the structural strength. The combined effects of large eccentricities and end restraint are a subject for further research and experimentation as still many aspects remain unsolved. The authors' investigations on the comparative strength of various sections appear to bridge these gaps to some extent, for light alloy structures. Actual tests can only reveal the validity of the assumption that these findings are also applicable to members of other materials.

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[19] "Alcoa Structural Handbook," Aluminum Co. of America, Pittsburgh, Pa., 1948, p. 37.

[20] "Alcoa Structural Handbook," Aluminum Co. of America, Pittsburgh, Pa., 1945, p. 37.

[21] *Journal*, Soc. of Engrs., Inc., London, April, 1949, p. 15.

[22] "The Strength of Light Alloy Struts," by J. F. Baker and J. W. Roderick, *Research Report No. 3*, Aluminum Development Assn., London, July, 1948, p. 148.

[23] "Notes on Columns," by Paul E. Sandroff, *Journal of the Aeronautical Sciences*, January, 1944, p. 1.

[24] "Notes on Column End-Fixity Coefficients," by G. C. Best, *ibid.*, February, 1949, p. 117.

[25] *Zeitschrift für Architectur und Ingenieurwesen*, 1889.

MARSHALL HOLT,[27] ASSOC. M. ASCE, AND J. W. CLARK,[28] JUN. ASCE.—In helping to clarify the significance of the test results, the discussers have added materially to the paper. Mr. Ghaswala's footnote references are noteworthy.

The analysis by Mr. Shearwood of the variations in stiffness of the struts and their connections is a concise picture of the weaknesses that contributed to the failure of the various members. He raises the question of why specimen F1, with ends completely flattened, was the weakest of the single-angle specimens tested on knife-edge bearings, whereas specimen F2 was one of the strongest of the members loaded as flat-end columns. The comparatively high strength of specimen F2 with flat ends can probably be explained by the fact that the stress measurements indicated less effective eccentricity for the type F connection than for the other single-angle connections. The strength of the round end columns, which failed by the formation of double hinges at the ends, was governed to a large extent by the stiffness of the weak sections that acted as hinges. As stated by Mr. Shearwood, the flattened end of specimen F1 formed a more effective double hinge than the connections of the other round end, single-angle columns. This weakness resulted in a lower ultimate strength.

Some of the discussers question the writers' belief that the general conclusions concerning the relative merits of various types of end connections may be applicable to materials other than aluminum. Although it is true that the strength of a member depends on the material of which it is made, the relative stiffness of various types of columns and connections and the relative degrees of eccentricity and of stress concentration are functions of geometry only. For this reason it seems logical that a comparative evaluation of these factors based on tests of aluminum alloy columns may also be valid for columns of other materials.

Mr. Harland discusses the differences in some of the mechanical properties of steel and aluminum. In this connection it is important that no ill effects resulting from the relatively lower ductility (as measured in the tensile test) of the aluminum alloys as compared with steel were manifest in the tests reported in the paper. Except in the case of specimen E3 where the rivets sheared, there was no parting of material in any of the static tests. The alloys used in these tests were capable of undergoing the elongations consistent with the lateral deflections of the members. As noted in the case of specimens E1 and G1, the loading was stopped because the deformations were so great that there was danger that the specimen would fly out of the testing machine. It is true that the elastic lateral deflections of steel specimens of the same dimensions would be smaller than those encountered in these tests, but the ultimate failures of such steel members would undoubtedly also be accompanied by sidewise bending. The average stresses would probably be above the elastic stress range, as was the case with specimens A3 and C3. The authors know of no series of tests that demonstrates the amount of ductility actually required of the material in structural parts of this type.

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[26] "Applied Column Theory," by F. R. Shanley, *Transactions*, ASCE, Vol. 115, 1940, p. 698.

[27] Senior Research Engr., Aluminum Research Laboratories, New Kensington, Pa.

[28] Research Engr., Aluminum Research Laboratories, New Kensington, Pa.

Mr. Ghaswala suggests that the poor agreement between the test data and the computed curves of Eqs. 1, 2, and 3 is a result of material failure produced by concentration of stress around rivets. This certainly is not the case, but rather the poor agreement results from the eccentricities of loading that are not considered by these equations. Reconciling Eqs. 1 and 4 is quite simple when it is realized that Eq. 1 is based on the yield strength of a certain piece of material and Eq. 4 is based on a typical value obtained for that alloy from production records over a period of time from a variety of shapes produced by a variety of manufacturing processes.

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POLLUTION ABATEMENT POLICY

By Thomas R. Camp,[1] M. ASCE

With Discussion by Messrs. J. E. McKee, Elwood L. Bean, and Thomas R. Camp.

Synopsis

Since the objective of pollution abatement is to reclaim watercourses for appropriate uses, the policy of pollution control agencies should be directed to the maintenance of adequate water quality at least cost and not to the wholesale construction of treatment works. Policy should permit selection of wastes to be treated for greatest over-all economy and should provide for assessment of costs in proportion to amount of wastes produced whether treated or not. Agencies should establish limiting pollution loads allowable at each point of pollution and should reallocate loads when other riparian owners wish to produce liquid wastes.

Corrective Activity

The reclamation of streams and other public waters from pollution by municipal sewage and industrial wastes is now receiving some of the legislative attention which is essential for effective abatement of pollution. For generations municipalities and industries in the United States have dumped their liquid wastes into public waters with little regard for the water users unless brought to task by these users through damage suits. Until recently the federal government had no power to compel abatement of pollution and the power of the state agencies was limited generally to the giving of advice. During the depression years the extent to which the despoliation of water resources had been allowed to progress was brought to public attention by the surveys of the National Resources Planning Board. Since then a number of the states have passed laws "with teeth in them" to regulate pollution, and during the summer of 1948 the Eightieth Congress finally passed a water pollution control act known as *Public Law No. 845*. Moreover, since 1940 a number of interstate water pollution control compacts have been concluded. Among the groups administering these compacts are the Interstate Sanitation Commission comprising New York, New Jersey, and Connecticut; the Interstate Commission on the Delaware River (Incodel); the Ohio River Valley

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Note.—Published in March, 1950, as *Proceedings-Separate No. 10*. Positions and titles given are those in effect when the paper or the discussion was received for publication.

[1] Camp, Dresser & McKee, Cons. Engrs., Boston, Mass.

Water Sanitation Commission; the New England Interstate Water Pollution Control Commission; the Potomac River Commission; and others.

The activity in the planning and construction of sewage and waste treatment works which has resulted from the current interest in pollution abatement is enormous and is on a nation-wide scale. For the first time real progress is being made on the industrial wastes problem. Some of the necessary legislative tools are available and the future is hopeful for the eventual cleanup of streams in the United States. The major part of the task is still ahead, however, and the hardest "nuts" have yet to be "cracked." The current boom in pollution control activities has brought many people and groups into the field who are novices in sanitary engineering problems. At the one extreme, there are strong pressure groups interested in sports and wild life who appear to believe that pollution can be abated completely; and at the other extreme, there are some manufacturers who are fighting to maintain the unhampered use of public waters for the disposal of wastes. The great diversity of interests represented by these newcomers and the lack of understanding on the part of many of them of the technical nature of the problems to be solved are producing a "crazy quilt" of pollution abatement policy. It is necessary to take stock of this situation now and to work for sane and sound policies throughout the nation. If economic, equitable, and effective policies are not developed, the entire program may bog down through unwise and wasteful expenditures.

Forming a Policy

The writer has no general plan to offer for what might be called an ideal pollution abatement policy. In fact, he does not believe that it is possible to have one standard pollution abatement policy, or that it is desirable to attempt to standardize. Each problem is unique and should be studied on its merits and each region should adopt policies best suited to its needs. The purpose of this paper is to point out some of the problems encountered in policy making and to suggest some solutions.

One of the prevailing criticisms of the engineering profession is that too few of its members take an active part in public affairs. No group is more vitally interested than are sanitary engineers in this subject and no group is better qualified to guide policy making. The problems involved present a big opportunity, but they are challenging.

The purpose of water pollution abatement is to reclaim waters for several uses by the public and the riparian owners. As a first step in policy making, therefore, it is necessary for the pollution control agency to designate the appropriate uses of each public watercourse or part thereof and to set standards of suitable water quality. This procedure is analogous to zoning in the preparation of a city plan. A number of pollution control agencies have already classified their public waters as to use, and in other areas classification is now (1950) in progress. This work is extremely important and it must be done judiciously if the cost of pollution abatement is to be kept within reasonable bounds. Some watercourses in sparsely inhabited upland areas may readily be reclaimed and controlled for use as drinking water supplies whereas other grossly polluted watercourses in highly industrialized regions can never be reclaimed to a high degree of purity without economic suicide. The classification policy should be designed to benefit both the public and the riparian owners and should aim at the gradual improvement of all public waters. The policy should be flexible enough to permit changes in classification from time to time as circumstances dictate.

WATER QUALITY STANDARDS

The main uses of public waters in the United States are: (1) Public water supply, (2) fish propagation, (3) recreation and bathing, (4) industrial water supply, (5) agricultural use, (6) water power, (7) navigation, and (8) disposal of sewage and industrial wastes. These purposes are listed in general order of decreasing water quality but not in order of economic importance. The standards of quality of water suitable for some of these uses may be defined in fairly definite terms whereas the quality of water required for some of the other uses is quite variable depending on local circumstances. In any classification program more than one use must be allocated to most of the watercourses, and the standard of water quality for any public water will be determined by the higher uses. Since the conditions of use vary from one region to another, it is understandable that classifications should also vary. A review of the classifications which have already been made by various pollution control agencies reveals a great diversity of opinion regarding the required quality of waters of the various classes. This is a natural result of the diversified use of the waters. If the classifications and standards are soundly drawn up for two adjoining regions, there is no reason why they should be similar, except that agreement must be reached for streams and public waters which are common to both regions.

The two most important characteristics of polluted waters which bear on their use are their contents of coliform bacteria and dissolved oxygen. The coliform count is an index of the concentration of domestic sewage in the water and of the potential hazards from intestinal disease germs. Coliform bacteria emanate almost entirely from the fecal matter in domestic sewage. The dissolved oxygen content of a public watercourse determines its suitability as a habitat for fish life and whether or not it is on the point of becoming septic with the production of black water and foul odors. Oxygen enters the water by solution from the atmosphere. If the dissolved oxygen is not being used up by biochemical reactions in the water, its content will approach the saturation value ranging from about 8 ppm in the summer to about 14 ppm in extremely cold weather. Oxygen contents less than the saturation value indicate that the oxygen is being used up by bacterial decomposition of organic matter in the water. Some decomposable vegetable organic matter enters streams from swampy regions but most of the decomposable organic matter emanates from pollution by municipal sewage and industrial wastes. Hence, the oxygen content in a water is determined primarily by the amount of polluting organic matter.

The coliform count in a water to be used for a public drinking water supply should be less than 1 per 100 ml of sample if the water is to be used without any treatment, less than 50 per 100 ml if the water is to be used after chlorination only, and less than about 5,000 per 100 ml if the water is to be used after rapid sand filtration and postchlorination. More highly polluted waters may be used with additional treatment. The coliform count of sea waters to be used for shellfish propagation should not exceed 70 per 100 ml, to conform with existing standards of the United States Public Health Service. There is a great diversity of opinion regarding the proper coliform standards for public waters to be used for bathing purposes. This diversity of opinion emanates from the fact that there is inadequate epidemiological evidence to demonstrate the extent to which intestinal diseases are transmitted to bathers in sewage polluted waters. Some authorities insist on a maximum coliform count of 100 per 100 ml whereas others will permit counts up to 2,400 per 100 ml for bathing waters. The safe coliform count for waters to be used for irrigation purposes is not well established and research is needed to determine the safe limits. Irrigation waters to be used on crops which require cooking before eating, of course, may have higher bacterial contents than is the case for crops eaten raw.

The oxygen content of public waters should never be completely exhausted if septic conditions and odor nuisances are to be avoided. If a public water is to be a suitable habitat for game fish, the oxygen content should preferably exceed 5 ppm but should not be reduced below about 4 ppm. For hardy fish such as shad and suckers a lower limit of 3 ppm is sufficient.

There are other less important quality characteristics of public waters which influence their suitability for use. The pH-value, for example, should generally be held within the limits of from 5 to 8.5 if the water is to be used for public water supplies or for the propagation of fish life. The temperature of the water should not be unduly increased by the discharge of hot wastes or condenser water if the watercourse is to be maintained as a suitable fish habitat. For most of the higher uses of public waters, the water should contain no objectionable odor, oil, scum, floating solids, or debris except from natural sources; and it should contain no taste producing substances in objectionable concentrations and no substances in harmful concentrations which are themselves toxic or become toxic in combination with other substances. (Note that these qualities are relative and depend primarily on the concentration of the polluting substances.) It is seldom practicable to specify the complete elimination of any substance. The amount and the character of suspended matter should not be such as to produce objectionable sludge deposits. Inordinate color and turbidity when introduced with polluting wastes are of esthetic importance and are particularly noticeable by the general public. Both color and turbidity so introduced may be reduced by the treatment methods employed for the removal of organic matter and bacteria but the extent to which further treatment is justified economically, solely for the removal of color and turbidity, is questionable.

Critical Stream Flows

The flow of streams is quite variable from the low flows during periods of extreme drought to peak flows during major floods—and the range of flows is much greater in some streams than in others depending on size, the nature of the catchment areas, and the extent of low-flow regulation by storage. It is evident therefore that the continuous discharge of a definite quantity of some material into a stream may cause pollution at some flows and may not cause pollution at greater flows. In setting the water quality standards it is necessary to state at what flow in a particular stream a limiting standard is to apply. If complete abatement is expected, the flow selected must be the minimum possible at any time. For many streams, something short of complete abatement must be accepted. A study of flow-duration curves will furnish a basis for the selection of critical flows below which some pollution must be accepted.

Limiting Pollution Loads

After the classification of a public watercourse has been established and the standards of water quality have been set, the limiting quantities of each polluting substance which may be discharged into the watercourse at each point of pollution may be estimated. In making such an estimate, analyses must be made of the quantity and the character of each polluting substance and of the water itself. The stream or tidal estuary must be considered as a dynamic moving entity. Due account must be taken of the dilution available, times of flow, deoxygenation and reaeration rates, sedimentation and benthol decomposition, and bacterial die-off rates; and a particular critical regimen of temperature, flow, wind direction, and velocity must be selected to which the water quality standards are to apply. After the limiting quantities are determined, the amounts of removal may be estimated by deducting the limiting quantities from the total quantities being or expected to be discharged into the watercourse.

The removals required during the critical flow periods will determine the size and the extent of the treatment works, and will fix the capital expenditures required therefor. At flows greater than the critical, and during cold weather, a lesser degree of treatment may suffice to meet the water quality standards. Whether such a lesser degree of treatment is permissible is a question of great importance which depends primarily on the nature and the use of the watercourse. In many streams, organic matter which is not removed by treatment during the winter months deposits in ponds or reservoirs to cause trouble during the following summer.

Optimum Enforcement

In a single watercourse the removal required at the critical flow at one point may be totally different from that required at another point. A high degree of treatment may be necessary at some points of pollution and no treatment at all at other points. Once the standard of water quality is set for a particular public watercourse, any degree of treatment of sewage or industrial wastes in excess of that required to meet the standard will result in economic waste. This matter should be emphasized very strongly. It is wrong to require more treatment than is necessary to produce the desired results and it is wrong to require any polluter to treat his wastes or sewage unless such treatment is necessary to produce the desired results at the least over-all cost.

The problem of pollution abatement for a particular watercourse should be studied as a whole to determine which wastes must be treated and which may be allowed to go untreated for the best economical solution. Pollution control policy should be so developed as to permit the selection of wastes for treatment and to permit the selection of wastes which may be discharged untreated for best over-all economy. This proposal is contradictory to the policies that have been followed by some agencies. Under existing law it is easier for a public agency to enforce a policy that requires treatment by all polluters on a given watercourse, and the same degree of treatment for similar wastes. It seems to the writer that this policy is unreasonable, uneconomical, and dangerous to the over-all program of pollution abatement.

Equitable Assessment of Costs

It is difficult to adopt a policy permitting the selection of wastes for treatment and the selection of wastes that may be discharged untreated. The principal obstacles relate to the assessment of the costs. There is no simple solution. Equitable methods of assessing cost may readily be devised, but legal difficulties must be hurdled before they may be put into operation. It is inequitable, for example, to require one polluter whose wastes are selected for treatment to pay the cost while another, whose wastes are to be discharged untreated, pays nothing. If any polluter is to be assessed, all should be assessed in proportion to the amount of polluting matter each produces, whether their wastes are treated or not.

In a sense, this method of assessment may be considered as a benefit assessment for rights received—the right being the right to use a share of the self-purification capacity of the public watercourse. If the matter is considered in terms of benefits, the question arises as to whether the general public should not share in the cost of manufacturing waste treatment since all who use the watercourse share in the benefits resulting from its improvement. It is not equitable, however, for the public to pay the full cost of abatement because such a method of assessment would relieve the polluters who are principally responsible for the pollution from any share in the cost of its abatement and would thus fail to restrain them from unnecessary additional pollution. It is best that polluters be made to pay for abatement in proportion to the amount of polluting matter produced by each to stimulate reduction of the amount of polluting matter produced.

A method of assessment such as the foregoing is difficult to establish unless all the abatement works are owned and operated by a public agency having over-all supervision of the problem. This agency may be a municipality, a district, or (in some cases) the state. There are other advantages in public ownership and administration of all the abatement works, particularly to the industries whose wastes must be treated. Under present law the manufacturers are usually held wholly responsible for the treatment of their wastes and the cost thereof, the pollution control agency acting in a police capacity to compel action. Under such a system the industry is compelled to pay the full cost of nonrevenue-producing capital improvements on short-term financing and must show this cost as a capital investment on its books. Companion bills, *House of Representatives Bill No. 1047* and Senate *Bill No. 244*, were introduced at the first session of the Eightieth Congress to allow deductions in computing net income for moneys spent by industry for the construction of treatment plants for industrial wastes. No legislation has resulted from these bills.

If works for the collection and treatment of manufacturing wastes are constructed, owned, and operated by a public agency, the construction may be financed by long-term bonds at low interest rates backed by the full credit of the public agency. The costs may then be assessed back against the industries in the form of annual charges all of which may be shown on their books as operating expense and deducted in computing net income. This method of organization, therefore, not only permits the industries to share in long-term financing at low interest rates, but also results in considerable saving to them in income tax. A further advantage to the industries is that they are relieved of the responsibility of going into the business of treatment of their wastes. Except in cases where recovery of valuable by-products is promised, industries have little or no interest in the treatment of their wastes. It is the primary interest, however, of a public agency set up specifically for this purpose.

The methods of assessing the costs of sewerage works which have been developed in the United States over a long period of years are not adequate to meet the problems encountered in pollution abatement. Three methods are in general use which involve: (1) Assessment against the general property tax, (2) betterment assessments against the property served, and (3) sewer rental or service charges. The first two methods were developed for the construction of sewers, and either method or a combination of the two can result in an equitable distribution of the costs of construction and operation of sewers. Payment of the assessments is enforced since the assessments become liens on the properties. The third method, first called a sewer rental charge and now generally referred to as a sewer service charge, was developed primarily to assess the cost of treatment.

Since the cost of treatment of sewage is roughly in proportion to the amount of sewage, the sewer service charges are usually designed for payment in proportion to the amount of sewage. As direct measurement of flow of sewage from each contributor is impracticable, the sewer service charge is usually set up as an additional charge on the water bill. The legality of this method has been questioned and some difficulties have arisen in collections. A recent decision of the Supreme Court of Florida in the case of sewer revenue bonds for the City of Miami validates this method of charge, however, and permits the city to shut off the water for failure to pay sewer service charges. This decision is considered of great importance in that it permits financing of sewerage works by revenue bonds outside the debt limits.

In the writer's opinion this decision is not as important as might be supposed. It does allow the use of revenue bonds for financing outside the debt limit but the method of charging authorized by the decision is not sufficiently flexible to result in equitable assessments of cost against industries. The method does not permit assessing costs of treatment against industries whose wastes are not collected for treatment and it does not permit assessing costs in proportion to the strength of the wastes. In the case of a sewage treatment plant treating only domestic sewage, equitable assessments can result since the cost of treatment is roughly proportional to the flow. In cases where considerable quantities of industrial wastes may be treated, however, the cost of collection and treatment depends on the amount of biochemical oxygen demand (B.O.D.) and suspended solids contributed as well as on the flow. The cost of secondary biological treatment is almost directly proportional to the B.O.D. content of the sewage and the cost of sludge handling and disposal is almost directly proportional to the suspended solids content. For equitable assessments in such cases, therefore, it is most desirable to be able to assess in proportion to the B.O.D. and the suspended solids produced. The simpler method of charging in proportion to the water bill is not equitable. Another difficulty in cases involving industrial wastes is that in many instances the process water used does not come from the municipal supply but from private sources.

ROLE OF POLLUTION CONTROL AGENCIES

Pollution abatement involves processes. The construction of treatment works is only a means to an end and successful abatement requires that operation of the treatment works should be given equal attention. The engineer is primarily interested in the design of processes and in the operation of processes. The works themselves serve only to house these processes. The "processes" that occur in the watercourse itself are just as important as those that take place in the treatment works (and frequently more important). Pollution control agencies should focus their attention on the processes in the watercourses and on the limiting quantities of polluting materials allowed at each point of pollution. Except in cases where the agency is charged with the design and the operation of the treatment processes, the public agency should leave the details of design and operation to the polluter. The control agency should allot to each polluter a definite quantity of polluting substances not to be exceeded at each point of pollution. The procedure by which the required degree of treatment is accomplished should be left to the ingenuity of the polluter. It is wrong to require standard methods of treatment if the polluter can show that he can produce the required quality of effluent by other procedures which he considers less expensive and more appropriate for his use.

In cases where bacterial removal is highly important and B.O.D. removal is of no importance, complete treatment by a biological process should never be required and primary treatment by sedimentation should not be required unless it is absolutely essential to remove suspended and floating material and to make disinfection effective. Where B.O.D. removal is necessary, the process by which this is to be obtained should not be specified by the water pollution control agency. If the polluter is to be required to pay the cost of treatment and to be responsible for the successful operation of the treatment works, he should be left to his own devices as to the type of the treatment. He should be required to demonstrate by laboratory and pilot plant tests, however, that the methods he proposes to use give promise of producing the required results. The activities of the control agency should in such cases be limited to assisting in research and to giving advice.

This paper has advocated that the self-purification capacity of public water be allocated to the various contributors of liquid wastes in such proportions as will produce the required abatement at the least over-all cost. This policy may be interpreted as an allocation of rights to the riparian owners in the self-purification capacity of the public watercourse. Legal difficulties may be introduced in that these rights are not allocated equally or in proportion to the value of the property served. Other legal difficulties arise because all the self-purification capacity of public water will thus be utilized by those who are now discharging wastes. What is to be done for other riparian owners who wish to build industrial plants producing manufacturing wastes? What is to be done for new residential communities which must produce sewage? What is to be done for the growth of existing industries and existing municipalities? There is only one answer: Reallocation must be made from time to time, for every riparian owner has a right to use the stream for the discharge of his wastes. When reallocations are made, higher degrees of treatment will thus be required of existing users.

It is not right to pass legislation forbidding the establishment of new points for the discharge of wastes into a section of a public watercourse which is already similarly used elsewhere. Such legislation deprives the riparian owners who are not now discharging wastes into the stream from a reasonable use of the stream for the discharge of their wastes. The existing users must be compelled to reduce the amount of their discharges if reduction is essential to permit a reasonable use of the watercourse by other riparian owners who wish to produce liquid wastes.

Conclusion

In conclusion the writer wishes to emphasize the fact that one of the most important uses of public watercourses is for the disposal of sewage and industrial wastes. All sewage and most liquid industrial wastes must eventually reach watercourses, in whole or in part. It is practicable only to reduce the quantities of polluting substances and to redistribute the points at which these substances are discharged. The right to discharge wastes must be defended as strongly as the rights for all other appropriate uses of public waters. Policy problems are difficult. Pollution abatement cannot be achieved by the simple expedient of ordering the wholesale construction of treatment works. At some points treatment will be required, at others it will not be. Satisfactory quality of water in public watercourses is the sole objective of pollution abatement.

DISCUSSION

J. E. McKee,[2] M. ASCE.—There can be no serious difference of opinion concerning the basic premises in this paper, inasmuch as the author advocates a sane, sensible approach to the pollution abatement problem in each individual stream. It is well to emphasize again that each stream is an entity unto itself and that each stream must be studied thoroughly to determine its capacity to absorb polluting substances without deleterious effects on the waters and other uses of the stream.

In that part of the United States west of the 100th meridian west longitude, the critical or minimum stream flow is zero for many days of each year and hence the liquid wastes discharged to dry streams percolate into the beds and travel underground. In other instances, industries and municipalities discharge wastes to spreading grounds to enable the liquid to percolate to the ground-water basins. This procedure does not eliminate all problems of pollution abatement; on the contrary, it merely transfers them from the surface streams to well-defined underground streams or to ground-water basins, and it transforms them in the character of their pollution.

As Mr. Camp states, for surface streams:

> "The two most important characteristics of polluted waters which bear on their use are their contents of coliform bacteria and dissolved oxygen."

In underground streams and basins, however, coliform bacteria and pathogenic organisms are seldom a problem inasmuch as they are quickly and effectively diminished by the filtering action of the soil. Dissolved oxygen, although it may be desirable, can seldom be maintained in underground water, and its presence in appreciable concentration is not as critical as it is in surface streams. On the other hand, the end products of organic decomposition, such as nitrates, may accumulate in ground-water basins and may lead to concentrations harmful from the viewpoint of methemoglobinemia and possibly other diseases; or, toxic elements, such as chromium and boron, which may be continually swept away in surface streams, will be given a chance to build up to harmful concentrations in ground-water basins. The decomposition of raw wastes, or of residual organic matter in treated wastes, may not only deplete the dissolved oxygen in the ground water, but may also generate a reducing atmosphere that will allow iron, manganese, calcium, and other substances to be leached from the soil, thereby causing a deterioration of ground-water quality. In considering pollution abatement policy, therefore, engineers in the arid West must reckon with ground-water pollution as seriously as with surface-water pollution. Fortunately, many of the principles outlined by Mr. Camp are sufficiently broad to be applied equally well to surface water and ground water.

The suggestion that all polluters of a given stream "* * * should be assessed in proportion to the amount of polluting matter each produces, whether their wastes are treated or not" is well taken. In effect it means that each polluter should buy a share in the natural purifying capacity of the stream, and that these shares should be used to provide waste treatment and other types of pollution abatement where the most good can be accomplished at the minimum cost. The abatement works may be owned by a public agency, such as a district or authority, as Mr. Camp suggests, and thereby gain certain advantages of financing and tax reduction. However, those persons who object to the encroachment of public ownership on private industry may still prefer to have treatment plants owned and operated by individual firms, especially where industrial wastes rather than municipal sewage contribute most of the pollution. In this case, all industries along a given stream or in a given drainage basin could form an "association" that would act as a financial clearing house. Each industry would pay into the association an annual sum in proportion to the strength and quantity of its pollution, and the association, in turn, would reimburse those industries which have installed and are operating treatment works. Where industries and municipalities must share the purifying capacity of a stream, the state health department, or other water pollution control agency as the case may be, can allot to each municipality and to the association of industries a certain capacity at each point on the stream. The association, in turn, can determine which industries should treat their wastes and to what extent.

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[2] Associate Prof. of San. Eng., California Inst. of Technology, Pasadena, Calif.; Affiliate, Camp, Dresser and McKee, Cons. Engrs., Boston, Mass.

ELWOOD L. BEAN,[3] Assoc. M. ASCE.—The author has very wisely stated (under the heading, "Forming a Policy") that, as the first step in policy making, the control agency should "* * * designate the appropriate uses of each public watercourse or part thereof and to set standards of suitable water quality," indicating a stream zoning plan. He also stated (under the heading "Optimum Enforcement") that "Pollution control policy should be so developed as to permit the selection of wastes for treatment and * * * wastes which may be discharged untreated * * *." These are essential elements of any economical plan, and, although they do introduce hurdles, they must be met, just as the right to discharge waste "must be defended."

In determining the degree of purification required under such plans a mid-course must be followed. It is not right that complete treatment of all wastes be required; on the other hand, in setting up such plans, the treatment required must be such that stream purification capacity is not closely approached at any period. If this state is approached, increased industry (in number or capacity) or residential increases, may quickly necessitate reallocations requiring higher degrees of treatment from existing polluters, thereby burdening them with revisions or expansions.

In connection with the discussion of "Equitable Assessment of Costs," several items may well be emphasized. There are many industries which find that they can recover large percentages of the cost of waste treatment. These are frequently the industries that have been the most objectionable to downstream water users.

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[3] Prin. Asst. Engr., Bureau of Water, City of Philadelphia, Philadelphia, Pa.

Where stream-quality zoning is practiced the downstream polluter is allowed to discharge greater amounts of pollution, therefore reducing his waste treatment costs below those of the upstream polluter; but, on the other hand, a compensating factor frequently exists since this polluter frequently obtains his water supply from the river and therefore is subjected to increased costs on water supply purification, or he obtains his water from other sources at costs much greater than the industries in the upstream locations.

With reference to methods of assessment, the author has indicated that the sewer service charge is inadequate in relation to charges for wastes. He indicates that it would be desirable to "* * * assess in proportion to the B.O.D. and the suspended solids produced." The wording, however, makes no mention of the fact that very complete formulas for charges on such bases are already in use at Columbus, Ohio, Buffalo, N. Y., Elizabeth City, N. J., Philadelphia, Pa., and other municipalities.

The author has suggested that a "public agency" could save industry considerable sums of money by constructing and operating waste treatment works. True, the amortization of construction costs would be reduced; however, individual works would generally have to be established and operated by the public body on lands now owned by the industries. No very appreciable savings could be expected in such individual cases. Where industries are concentrated, generally sewage treatment works are established by the municipality. Such public works are generally justified in accepting and treating industrial wastes after the industry has made such adjustments as are necessary. Encouragement should be given industry by legislation, such as allowing the deduction of money spent for treatment plants in computing net incomes. The advantages and savings that could result from such legislation do not justify the development of far-flung public construction and operating agencies. Such a policy of public works development, in the opinion of the writer, is neither justifiable nor desirable.

Thomas R. Camp,[4] M. ASCE.—Valuable discussions were contributed by Messrs. McKee and Bean. The matter of high level policy is of utmost importance because of the current interest in the formulation of sound national policies for the development of water resources in the United States. The writer is in full agreement with the recommendations of the Committee on Water Resources Policy of the Engineers Joint Council that control and administration of water resources should be at the lowest governmental level consistent with the task. In the case of pollution control, a stream or drainage basin must be considered as a whole if pollution is to be abated at the least over-all cost. Thus districts, states, or interstate agencies are indicated as the proper control agencies. It is not necessary that the federal government act as anything more than a fact-finding agency.

As stated by Mr. McKee, for many days throughout the year, the surface flow in streams in the southwestern part of the United States is zero. This condition results in very severe requirements for pollution abatement works.

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[4] Camp, Dresser & McKee, Cons. Engrs., Boston, Mass.

In these arid or semi-arid regions water is extremely valuable and ground water, wherever attainable, is used to capacity. There is a growing tendency to discharge wastes to spreading grounds for ultimate reclamation from the ground water. Hence, the degree of treatment required in such cases depends on the requirements of the spreading areas and the ultimate reuse of ground water.

To those who object to the idea of ownership of industrial waste treatment works by a public agency such as a district or city, Mr. McKee offers a substitute in the form of an association of industries along a given stream or within a given drainage basin. Mr. Bean objects to ownership of industrial waste treatment works by a public agency because the works might have to be constructed on lands now owned by the industries. Such lands could be purchased, of course, or taken by eminent domain from the industries if they are the only lands suitable for the treatment works. Mr. Bean is of the opinion that the advantages and savings that could result from public ownership and operation of industrial waste treatment works do not justify the development of far-flung public construction and operating agencies. The writer is ready to agree with Mr. Bean if politics is permitted to enter into the organization and operation of the agency. On the other hand, if a coordinated plan for the entire river basin is not worked out, for both construction and operation, there is little chance of obtaining best economy by joint treatment works and by selection of only a part of the wastes for treatment. Without unified control it is most likely that all industries will be required to treat whether treatment is necessary or not. If control by a public agency threatens to be political, perhaps some kind of cooperative undertaking can be devised whereby the industries share in the management in proportion to cost or in proportion to volume or strengths of wastes produced.

Mr. Bean suggests that, in determining the degree of treatment required, a mid-course should be followed so that the stream purification capacity is not closely approached at any period. Such a factor of safety is a desirable ideal, but in many cases it is not practical except at exorbitant expense. In fact, there are some cases where the use classification of the waters must be graded downward because of the economic impractibility of meeting higher standards. In the writer's opinion each case must be studied on its own merits. In some cases a factor of safety can be provided, but in others less than an ideal solution must be accepted.

The writer is aware that assessments in proportion to the biochemical oxygen demand and suspended solids content as well as in proportion to rates of flow are in effect in some municipalities. The practice is not at all general, however, and it is much more common for sewer rental charges to be set up in proportion to flow only. In some states there is no legal authority for sewer service charges on any other basis than rates of sewage flow. Where sewage and industrial wastes are treated jointly, it is essential to consider strength as well as volume if the charges are to be equitable.

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Paper No. 2434

HIGH-VELOCITY FLOW IN OPEN CHANNELS

A SYMPOSIUM

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NOTE.—Published in November, 1949, *Proceedings*. Positions and titles given are those in effect when the Symposium or the discussion was received for publication.

FOREWORD

Although specific problems of high-velocity flow arising from changes in open-channel cross section or alinement had frequently been solved by model tests, not until the early 1930's did the broad applicability of elastic-wave analysis to such gravity-wave phenomena become evident. The resulting principles of wave mechanics were first applied at the California Institute of Technology at Pasadena in 1935, during the study of flow around the curves of the Los Angeles County flood-relief channels in California; shortly thereafter similar applications were made to channel contractions at Lehigh University in Bethlehem, Pa., and the Massachusetts Institute of Technology (M.I.T.) in Cambridge, and to channel expansions at the State University of Iowa in Iowa City.

In distinction to the empirical solution of particular design problems, these investigations involved the use of general principles whereby simple designs could be completed without recourse to experiment and complex designs could be approximated understandingly prior to experimental refinement. Several papers describing certain phases of the investigations were eventually published, but the major part of the material remained in the form of graduate theses and reports to various public and private organizations. Therefore, much of the essential information did not become readily available to the profession, and in no way could the engineer secure a unified treatment of the subject as a whole.

Because of this deficiency in the technical literature, in 1946 the newly formed Fluid Mechanics Committee of the ASCE Hydraulics Division undertook as one of its initial projects the sponsorship of a comprehensive symposium on the design of curves and transitions for high-velocity flow. This was arranged as a series of correlated papers, prepared by those who were responsible for the original investigations and so organized as to include the underlying principles of wave analysis as well as their application to the primary types of transition structure. The high lights of the Symposium were presented to the 1948 Annual Meeting of the Society, and the papers themselves are reproduced for discussion in the following pages. It is hoped both by the authors and by the Committee on Fluid Mechanics that the ultimate result will be a compact yet inclusive treatise on the subject which will prove of practical value to design engineers.

Notation.—The following letter symbols, adopted for the Symposium and for the guidance of discussers, conform substantially with American Standard Letter Symbols for Hydraulics (ASA—Z10.21942), prepared by a committee of the American Standards Association, with ASCE participation, and adopted by the Association in January, 1942:

b = width of channel;
c = celerity of small waves = $\sqrt{g\,h}$;
d = height of a sill;
$\mathbf{F}$ = Froude number;
g = gravitational acceleration;

H = specific head;
h = depth; h_c = critical depth;
K = a correction factor = $\dfrac{\text{actual sin } \beta_1}{\text{theoretical sin } \beta_1}$;
L = distance along a channel;
Q = rate of flow;
r = radius, mean; r_i = radius of a counterdisturbance section;
S = slope; S_c = cross slope;
V = mean velocity = Q/A:
  V_c = critical velocity;
  $\bar{V}$ = dimensionless velocity;
x = longitudinal distance from beginning of expansion;
y = lateral distance from center line;
z = depth of channel drop;
α = angle of a sill in a flume or channel;
β = wave angle;
γ = specific weight;
θ = deflection angle:
  $\Delta\theta$ = small but finite deflection angle;
  θ' = angle of curve of the entire flow;
  θ_o = central angle of the half wave length in the main curve; and
ρ = mass density.

MECHANICS OF SUPERCRITICAL FLOW

By Arthur T. Ippen,[1] M. ASCE

Synopsis

The theory of high-velocity flow—flow defined as shooting, rapid, or supercritical—is presented comprehensively. The principles discussed find practical application in the design of all open-channel structures in which surface disturbances and standing waves appear as a consequence of the geometry of the lateral boundaries. These surface disturbances are subject to systematic analysis on the basis of two distinct methods of approach:

1. Gradual surface changes may be analyzed on the basis of constant specific head; and
2. Standing wave fronts of appreciable height (so-called oblique or slanting hydraulic jumps) can be computed, considering the energy dissipation involved.

Graphical aids for the solution of both types of problems are given in detail, and the characteristic disturbance patterns are developed to illustrate a number of basic cases. Specific verification of the theory by experiment is left to subsequent papers of the Symposium.

Introduction

In recent years the hydraulic designer has been increasingly confronted with problems of high-velocity flow in steep flood channels and spillway chutes. The specific character of such flow results from the fact that the velocities exceed considerably the critical velocity and therefore the velocity at which surface disturbances and waves are transmitted in free surface flow. It was found that similar problems of design are encountered in the field of high-velocity gas dynamics and that, by analogy, a hydraulic theory could be deduced from the concepts and analytic developments already available in that science. Definite findings have resulted from intermittent research conducted since 1934, and the accumulated experimental evidence confirmed essentially the soundness of the theoretical approach. Within the limitations imposed by the fundamental premises of the theory, consistent qualitative and good quantitative results were obtained which justify the comprehensive presentation of the work at this time.

The historical development of research in this field may be summarized briefly. In the United States the first impulse toward work in this field came in the early nineteen thirties when the engineers of the Los Angeles County Flood Control District found the conventional methods of designing flood channels not applicable to the steep gradients employed in their area of service. They therefore approached the Hydraulic Structures Laboratory at the California Institute of Technology, under the direction of R. T. Knapp, M. ASCE, with a proposal to study the flow at supercritical velocities through curved sections of rectangular channels in the laboratory. Extensive tests were then performed by the writer from 1935 to 1938, under a variety of conditions as far as radii, slope, and channel forms were concerned. The test results[2,3,4,5] pointed toward characteristics of such flow, which, according to Theodor von Kármán,[6] M. ASCE, had their counterpart in the supersonic flow of gases and to which the previous findings for such flow could be adapted. The application of these principles resulted in a remarkable correlation of the test data. It also became clear that a general method of analysis for supercritical flow had been found which could be applied to the study of the design characteristics of hydraulic structures.

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[1] Prof. of Hydraulics, Dept. of Civ. and San. Eng., Massachusetts Inst. of Technology, Cambridge, Mass.

The analogy of supercritical flow of water to supersonic gas flow had been pointed out also by D. Riabouchinsky[7] and L. Prandtl[8] without experimental evidence. Ernst Preiswerk[9] worked on the extension of the theory and conducted a number of systematic experiments on a so-called Laval nozzle at Zürich, Switzerland. Although these writers were primarily interested in applications to supersonic flow of gases, work was continued more along hydraulic lines at the California Institute of Technology under Mr. Knapp's direction, at Lehigh University, at M.I.T. under the direction of the writer, and at the Iowa Institute of Hydraulic Research in Iowa City under the direction of Hunter Rouse, M. ASCE. The findings made in these investigations have been embodied in Symposium papers Nos. 2 to 4 in a form adapted to the specific problem of design under discussion. By contrast the task of this paper is an outline of the general principles of supercritical flow and a complete development of the fundamental train of ideas for those interested in the general physical aspects of the various problems.

GENERAL PHYSICAL BACKGROUND

The theory of nonuniform supercritical flow as reflected by its treatment in conventional texts has been concerned mainly with the changes taking place in only two dimensions, length and height. Depth and velocity changes are related in general to the slope and roughness factors by the backwater equation and its basic solutions, giving surface points as a function of distance. Lateral changes of cross section are absorbed into an average depth and the cross slope is always assumed to be zero. Although this method is justified for subcritical flow, it must be rejected for supercritical flow because of the appearance of

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[2] "A Study of High Velocity Flow in Curved Channels of Rectangular Cross Section," by A. T. Ippen and R. T. Knapp, *Transactions*, Am. Geophysical Union, Vol. 17, 1936, p. 516.

[3] "An Analytical and Experimental Study of High Velocity Flow in Curved Sections of Open Channels," by A. T. Ippen, thesis presented to the California Inst. of Technology at Pasadena, in 1936, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

[4] "Experimental Investigations of Flow in Curved Channels" (abstract of results and recommendations), by A. T. Ippen and R. T. Knapp, U. S. Engr. Office, Los Angeles, Calif., 1938.

[5] "Curvilinear Flow of Liquids with Free Surfaces at Velocities Above That of Wave Propagation," by R. T. Knapp and A. T. Ippen, *Proceedings*, 5th International Cong. of Applied Mechanics, Cambridge, Mass., 1938, p. 531.

[6] "Eine praktische Anwendung der Analogie zwischen Überschallströmung in Gasen und über kritischer Strömung in Offenen Gerinnen," by Theodor von Kármán, *Zeitschrift für Angewandte Mathematik und Mechanik*, February, 1938, pp. 49–56.

[7] "Sur l'Analogie Hydraulique des Mouvements d'un Fluide Compressible," by D. Riabouchinsky, *Comptes Rendus de l'Académie des Sciences*, Vol. 195, 1932, p. 998, and Vol. 199, 1934, p. 632.

[8] "Abriss der Strömungslehre," by L. Prandtl, Braunschweig, Vieweg, 1931.

[9] "Application of the Methods of Gas Dynamics to Water Flows with Free Surface," by Ernst Preiswerk, *Technical Memoranda Nos. 934 and 935*, National Advisory Committee for Aeronautics, Washington, D. C., March, 1940.

standing waves as the result of lateral boundary changes for such flow. The different types of backwater curves discussed by the classical theory can be conceived as valid only for rectilinear supercritical flow between parallel walls and for zero cross slope of the bottom. If these conditions are not fulfilled, considerable errors are to be expected from computations in accordance with this theory, although computations for subcritical flow are acceptable.

The physical difference between subcritical and supercritical flow is best revealed by the specific-head diagram (see Fig. 1). A few remarks may summarize its significance with respect to this paper.

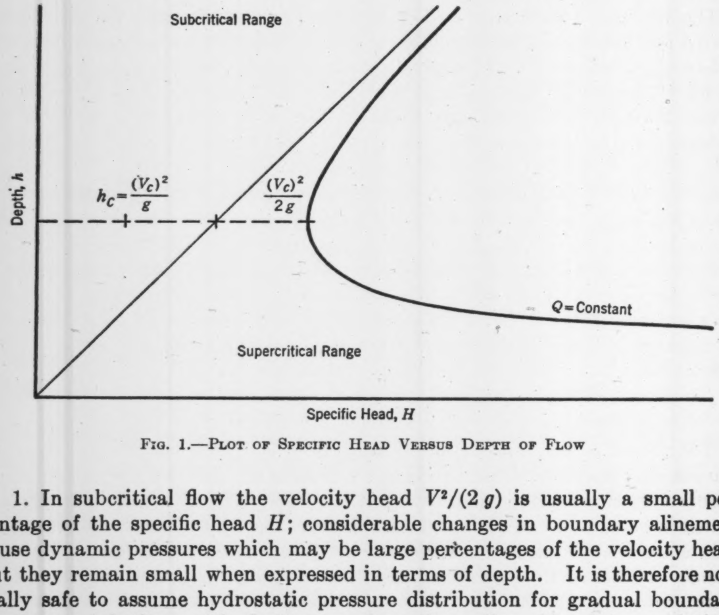

FIG. 1.—PLOT OF SPECIFIC HEAD VERSUS DEPTH OF FLOW

1. In subcritical flow the velocity head $V^2/(2\,g)$ is usually a small percentage of the specific head H; considerable changes in boundary alinement cause dynamic pressures which may be large percentages of the velocity head, but they remain small when expressed in terms of depth. It is therefore normally safe to assume hydrostatic pressure distribution for gradual boundary curvatures, whether on the bottom or along the walls. Also, large variations in H are synonymous with large variations of depth.

2. In supercritical flow the velocity head not only is comparable to depth h in order of magnitude but, in most cases of practical interest, exceeds the depth considerably. In this instance, large variations in specific head H are equivalent to large changes in velocity head. Slight curvatures of the boundaries may cause relatively small dynamic pressures in terms of velocity head, but the changes in depth or surface elevation will be relatively large. Hydrostatic pressure distribution may be altered radically, while the velocity head is changed only a small percentage.

3. Flow near the so-called critical depth, when $V = \sqrt{g\,h}$, results in values of h and $V^2/(2\,g)$ of practically the same order of magnitude. While H remains almost constant, changes in h or $V^2/(2\,g)$ are reflected in mutual changes.

Slight variations of H cause large changes in $V^2/(2g)$ and h. Since slight changes in boundary alinement, because of their influence on the hydrostatic pressure distribution, correspond in effect to slight variations in H, large disturbances result as far as the depth and velocity head are concerned for flow near critical velocity, as evidenced by high undulations. Since the ratio $V/\sqrt{g\,h}$ is generally recognized as the Froude number, **F**, it is used subsequently to characterize these flow conditions. The flow described in item 1 therefore corresponds to Froude numbers smaller than unity, whereas for the flow described in item 2 the Froude number is always larger than unity. For flow at critical depth **F** equals one.

The critical velocity $V = \sqrt{g\,h}$ also has significance as the velocity with which very small disturbances or gravity waves travel in shallow water. However, it must be remembered, that waves of large height travel faster than those of small height and consequently are able to travel upstream even in supercritical flow, whenever the wave velocity exceeds the velocity of flow. If the velocity of flow changes along the line of travel of such a high wave, a position is eventually reached where the velocity of flow equals the wave velocity. Such a stationary surge or wave is well known as the hydraulic jump, the theory of which is assumed to be familiar to the reader.

Gradual changes in supercritical flow can be treated by conventional methods only as long as the flow is confined between parallel, rectilinear walls with zero cross slope of the bottom. Only then will the theoretical surface curves apply as computed for cases of accelerated or decelerated motion in long channels and vertical transitions. The same restrictions hold for the transition from supercritical to subcritical flow in the hydraulic jump, conceived as a standing surge at right angles to the flow.

The theory of nonuniform flow at supercritical velocities is extended in the following text by the additional variants of curvilinear walls and by the abrupt changes of wall alinement considered under specific assumptions, which are outlined as the treatment proceeds. In addition to velocity changes in the original direction of flow, changes in the direction perpendicular to the original one are computed in connection with transverse changes of depth. The surface, therefore, is no longer considered horizontal crosswise or of a definite cross slope; but its configuration may become quite arbitrary. The only restriction remaining is that vertical changes in velocity be neglected, which requires in turn that the pressure distribution remain hydrostatic. It will be shown that satisfactory solutions can be derived for flow through curved sections of open channels and through converging and diverging channels as long as the basic assumptions are fulfilled. The limitations imposed by these shapes are clearly shown in connection with individual applications. The physical features of such flow are characterized by oblique standing wave fronts, originating at the walls of the channels at the points of changing alinement. These waves cross the channel to the opposite wall, where they are reflected, and in this fashion they continue almost undiminished in the downstream direction. Of course this visible evidence of boundary disturbance is made up of a multitude of small disturbances, the characteristics of which may now be taken up in detail.

MECHANICS OF WAVE PROPAGATION IN SUPERCRITICAL FLOW

A uniform rectilinear flow of depth h_1 is assumed with a velocity V_1 in a wide rectangular channel. The streamlines are parallel throughout. If the flow is now disturbed at any point by placing an obstacle in the stream, the surface of the stream will show deformations whose characteristics are dependent on the value of the velocity V_1 with respect to the critical velocity $\sqrt{g\,h_1}$, which is equal to the celerity c of small waves. If the velocity V_1 is smaller than $\sqrt{g\,h_1}$ (that is, if $\mathbf{F}_1 = V_1/\sqrt{g\,h_1} < 1$), the flow upstream from the obstruction will be affected by the presence of the obstacle. The backwater will extend a long distance upstream considering the dimensions of the obstacle, and a typical wake will appear in the rear of the obstacle. The streamlines will be deflected in all directions over a considerable area in conformance with the shape of the body. If the local velocities are expressed as percentages of the initial velocity V_1, a dimensionless streamline pattern may be drawn which remains geometrically similar for a wide range of variation in V_1 and h_1, or better, for values $\mathbf{F}_1 < 1$. Surface depressions or superelevations are not very marked. If the flow conditions are changed, however, so that V_1 approaches the value of $\sqrt{g\,h_1}$ or $\mathbf{F}_1$ approaches unity, a marked change in the flow pattern becomes apparent. Flow conditions at critical velocity would nevertheless still cause a backwater, since in such a more or less hypothetical case the slightest obstruction would cause a wave of sufficient height to travel upstream and the obstruction would remain surrounded by subcritical flow.

Only if the velocity V_1 increases sufficiently above the value of $\sqrt{g\,h_1}$ is it possible to have a sizable obstruction which will not exert any influence upstream, as will be shown in detail subsequently. It may suffice to state that for $\mathbf{F}_1 > 1$, in general, typical standing wave patterns appear as a consequence of any obstacle placed in the stream. These patterns will change with $\mathbf{F}_1$ and therefore the resistance created by flow obstructions will also change as a function of $\mathbf{F}_1$. For supercritical flow, in contrast to subcritical flow, it is no longer possible to describe the streamline patterns independently of $\mathbf{F}_1$, since these patterns do not remain geometrically similar as before in subcritical flow, where they are subject only to Reynolds number distortions or to frictional distortions. Any attempt, therefore, to express head losses due to flow obstructions in terms of the Reynolds number only is impossible unless similarity exists with respect to the Froude number.

Basic Properties of Standing Waves.—If the discussion is confined at first to relatively small disturbances, a very useful analysis may be made which will aid greatly in anticipating the performance of numerous hydraulic structures in supercritical flow, whenever the changes in boundary alinement along a body within the flow or along the side walls are gradual. In such cases angular deflections are considered small, resulting therefore in small changes of depth. For this reason vertical accelerations can be neglected entirely, and consequently hydrostatic pressure distribution will be assumed to exist over the depth of flow at every point. The flow field shown in Fig. 2 conforms with these assumptions. In addition, the velocities are taken as constant over the depth, and the energy dissipation is disregarded along the bottom and within the zone of change from depth h_1 to depth h_2. The assumption of zero shear along the bottom may be modified by stating that the bottom shear equals the component of the gravity force, so that accelerations from this source may be disregarded. This procedure is common in the derivation of the hydraulic jump equation. Thus, one may write the continuity equation and the momentum equation for a unit length of wave front crossing a flow of depth h_1 and velocity V_1 at an angle β_1 as follows:

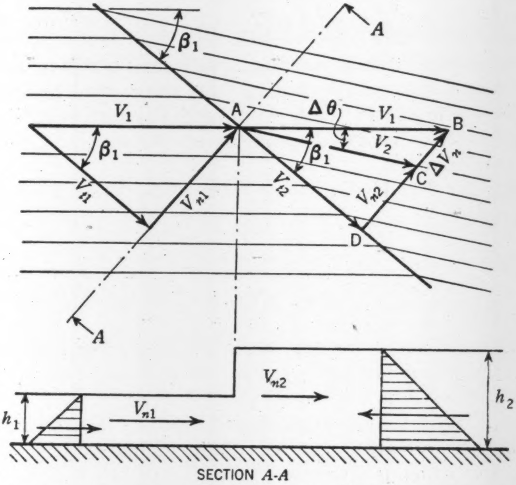

FIG. 2.—PLAN OF WAVE FRONT CROSSING FIELD OF FLOW WITH VECTOR DIAGRAM OF VELOCITIES

$$h_1 V_{n1} = h_2 V_{n2} \qquad (1)$$

and

$$\frac{\gamma (h_1)^2}{2} + \frac{\gamma}{g} h_1 (V_{n1})^2 = \frac{\gamma (h_2)^2}{2} + \frac{\gamma}{g} h_2 (V_{n2})^2 \qquad (2)$$

It is clear that the net pressure force acting to decelerate the flow can affect only the momentum of the stream normal to the wave front; and, since no force component exists parallel to the front, the tangential components V_{t1} and V_{t2} of the velocity must remain unaltered as the flow passes under the wave front.

From Eqs. 1 and 2, the expression for the normal component V_{n1} is obtained in terms of the depths h_1 and h_2:

$$V_{n1} = \sqrt{g\, h_1} \sqrt{\frac{h_2}{h_1} \frac{1}{2} \left(1 + \frac{h_2}{h_1} \right)} \qquad (3)$$

In applying the momentum equation, V_{n1} has been automatically defined as the wave velocity, since the wave front was assumed stationary. In turn any wave of a certain height, $h_2 - h_1$, will assume in supercritical flow such a position that the normal component of V_1 with respect to the front will equal V_{n1} as defined by Eq. 3. The wave front will not be stationary otherwise. It may also be stated that in agreement with the equation there must always be a stationary position for any wave in supercritical flow, until V_{n1} as defined by Eq. 3 exceeds the value of V_1. If $V_{n1} = V_1$, the wave front assumes a position at a right angle to the flow, and becomes the familiar hydraulic jump. For $V_{n1} > V_1$ the jump will start moving upstream, detaching itself from the source of the disturbance, which then remains surrounded by subcritical flow.

Wave Angle.—The relation between V_1 and V_{n1} may best be given by the ratio V_{n1}/V_1 from the vector diagram as $\sin \beta_1 = V_{n1}/V_1$ (in which β_1 is defined as the wave angle).

Substituting for V_{n1} in Eq. 3 the equivalent $V_1 \sin \beta_1$ and solving for $\sin \beta_1$ the expression—

$$\sin \beta_1 = \frac{\sqrt{g\,h_1}}{V_1} \cdot \sqrt{\frac{h_2}{h_1}\frac{1}{2}\left(1 + \frac{h_2}{h_1}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

—is obtained, which holds for any ratio of h_2/h_1. For small wave heights with $h_2 \approx h_1$, the expression under the square root approaches unity and the angle β_1 tends toward a minimum value for any given Froude number $\mathbf{F}_1 = V_1/\sqrt{g\,h_1}$. A continuous small disturbance in a supercritical flow defined by $\mathbf{F}_1 > 1$ will therefore always proceed to an angular position with respect to the oncoming flow which is given by

$$\sin \beta_1 = \frac{1}{\mathbf{F}_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The disturbance cannot travel beyond the line defined by this wave angle β_1 unless the wave height increases materially. It will always assume this position ultimately after a disturbance has been established, since it is the only possible stationary position for it.

Some additional deductions are possible from the geometry of the velocity vectors which are shown in Fig. 2. The law of sines applied to triangle ABC gives: $\dfrac{\Delta V_n}{V_1} = \dfrac{\sin \Delta\theta}{\sin\,(90° - \beta_1 + \Delta\theta)}$ and, for infinitesimal changes of θ,

$$dV_n = \frac{V\,d\theta}{\cos \beta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

in which the subscripts may now be omitted. Rewriting the momentum equation for infinitesimal changes in depth and velocity, a second differential expression for dV_n is obtained: $\gamma\, h\, dh = \dfrac{\gamma}{g}\, h\, V_n\, dV_n$; or

$$dV_n = \frac{dh}{V_n}\, g \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Since V_n may be replaced by $V \sin \beta$, Eqs. 6 and 7 may be combined into $\frac{V\,d\theta}{\cos \beta} = \frac{g\,dh}{V \sin \beta}$; or

$$dh = \frac{V^2}{g} \tan \beta \, d\theta \dots\dots\dots\dots\dots\dots (8)$$

Assumption of Constant Specific Head.—Eq. 8 may be integrated to obtain the gradual change of depth with gradual angular deflections of the stream if the basic assumption made in the beginning of this section is next introduced to relate β and V to h—namely, that energy dissipation may be disregarded for such flow in accordance with the Bernoulli theorem. Therefore: $H = h + \frac{V^2}{2\,g}$ = constant; and, thus, $V = \sqrt{2\,g(H - h)}$. Since $\tan \beta = \frac{V_n}{V_t} = \frac{\sin \beta}{\sqrt{1 - \sin^2 \beta}} = \frac{\sqrt{g\,h}}{\sqrt{V^2 - g\,h}} = \frac{\sqrt{h}}{\sqrt{2\,H - 3\,h}}$. Eq. 8 can be transformed finally into

$$\frac{dh}{d\theta} = \frac{2\,(H - h)\,\sqrt{h}}{\sqrt{2\,H - 3\,h}} = \frac{\sqrt{\frac{2\,h}{H}}\left(1 - \frac{h}{H}\right) H}{\sqrt{1 - \frac{3\,h}{2\,H}}} \dots\dots\dots (9)$$

which was first established by Mr. von Kármán in 1935–1936. The exact integration of Eq. 9 gives

$$\theta = \sqrt{3} \tan^{-1} \sqrt{\frac{\frac{h}{2\,H/3}}{1 - \frac{h}{2\,H/3}}} - \tan^{-1} \frac{1}{\sqrt{3}} \sqrt{\frac{\frac{h}{2\,H/3}}{1 - \frac{h}{2\,H/3}}} - \theta_1 \dots (10a)$$

in which θ_1 constitutes the constant of integration defined by the condition that for $\theta = 0$ the depth h is the initial depth h_1.

Eq. 10*a* may also be written in an alternate form employing the Froude number to express $\frac{h}{2\,H/3} = \frac{3}{2 + \mathbf{F}^2}$. Substitution of this equivalent results in

$$\theta = \sqrt{3} \tan^{-1} \frac{\sqrt{3}}{\sqrt{\mathbf{F}^2 - 1}} - \tan^{-1} \frac{1}{\sqrt{\mathbf{F}^2 - 1}} - \theta_1 \dots\dots\dots (10b)$$

Summary.—Before discussing Eqs. 10 and their implications any further, it is well to summarize the accomplishments of the theory developed up to this point:

1. Assuming an initial flow defined by a definite value of $\mathbf{F}_1 > 1$ and characterized by the assumption that H remains constant, it has been shown that a change in depth Δh due to any disturbance of the stream must in general always occur along lines or wave fronts crossing the flow at a characteristic angle β, and cannot exist for steady conditions at any other angle.

2. This wave angle β and the depth change Δh determine in turn a change in the velocity normal to the wave front. Therefore, a definite change $\Delta\theta$ in the direction of flow must take place under each wave front.

3. As the flow crosses successive wave fronts at arbitrary distances (the only requirement being that they be not too closely arranged to keep vertical accelerations small), the total change in direction may be related directly to the total change in velocity and depth between any two points along any streamline. Since any boundary of arbitrary curvature represents a streamline, the change of depth along such a boundary can be computed. Conversely, for any desirable changes in depth along a boundary the corresponding curvature may be determined.

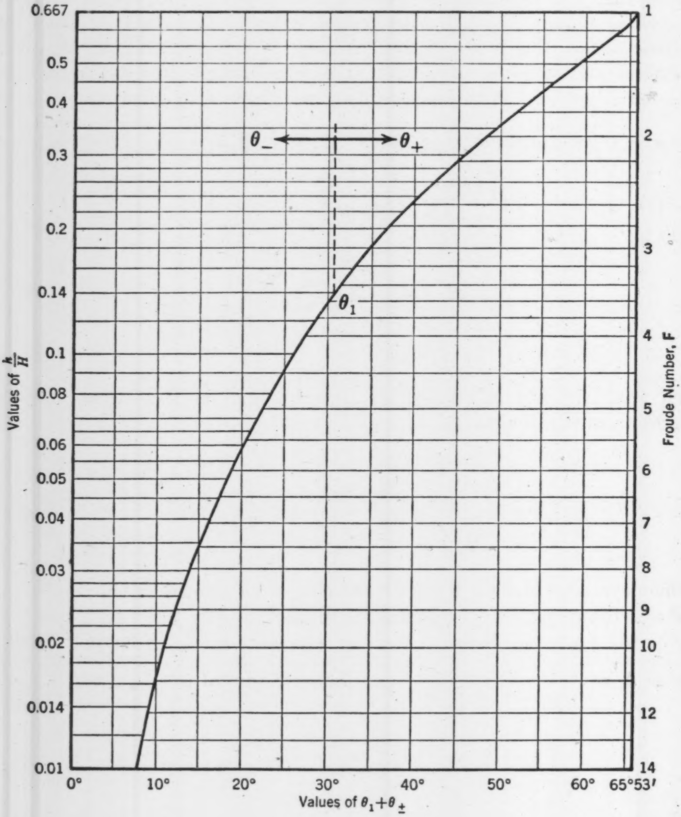

FIG. 3.—PLOT OF EQS. 10

4. Since a curved lateral boundary contains the origins of all disturbance lines, the characteristics of which are given by Eqs. 1 to 10, the entire surface configuration of the supercritical flow is determined by the family of disturbance lines emanating from that boundary or any combination of boundaries. The tools by which such contour surfaces may be determined will be developed subsequently, and a number of applications are given in Symposium papers Nos. 2 to 4.

Graphical Aids.—Returning now to Eq. 10, it is clear that a curve for the maximum range of θ for all values of h/H in the supercritical range of flow will be useful. This range extends from $\theta = 0°$ to $\theta_{\max} = 65° 53'$, if θ_1 is zero and if h/H is varied from a value of two thirds for critical flow to zero. Neither extreme is of practical interest. The curve is plotted in Fig. 3. Since in a specific case the value of θ_1 constitutes the starting point from which the deflection angle θ is measured, it is only necessary to compute the initial value of h_1/H to determine the value of θ_1 from the curve. The values of h/H are then read from the curve as the angles θ, of a boundary or streamline, are added to or subtracted from θ_1. It is important to note that the change in h may be positive or negative, depending on whether the boundary is curving into the stream or away from it. In the first case the surface will rise and in the second case the surface will be lowered along the disturbance lines originating at the boundary. Both possibilities are indicated in Fig. 3 by the arrows pointing to the left and right from an assumed initial value of θ_1. In addition to the $\frac{h}{H}$-scale, a scale of the corresponding Froude numbers is given for convenience.

The general trend of Eq. 10 may easily be recognized from an approximation based on the first term of the series for $\tan^{-1}$ only. The approximate solution, which naturally becomes incorrect for large values of h/H, is given algebraically by

$$\theta = \left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) \sqrt{\frac{\dfrac{h}{2\,H/3}}{1 - \dfrac{h}{2\,H/3}}} - \theta_1 \approx \sqrt{2}\sqrt{\frac{h}{H}} - \theta_1 \ldots\ldots (11a)$$

Since the ratio h/H may be replaced in terms of the Froude number $\mathbf{F}$, the equation may also be written for small values of h/H (that is, for large values of $\mathbf{F}$) in the form:

$$\theta = 2\sqrt{\frac{1}{\mathbf{F}^2 - 1}} - \theta_1 \approx \frac{2}{\mathbf{F}} - \theta_1 \ldots\ldots\ldots\ldots\ldots\ldots (11b)$$

It is obvious that Eqs. 11 could also be solved for h/H or for $\mathbf{F}$ in terms of θ and θ_1. The results given under Eq. 11a for small values of h/H could have been obtained directly by neglecting h as compared to H in Eq. 9, thus simplifying the differential expression to

$$\frac{dh}{d\theta} = \sqrt{2\,h\,H} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12a)$$

which is integrated easily into

$$\theta = \sqrt{2}\sqrt{\frac{h}{H}} - \theta_1 \ldots\ldots\ldots\ldots\ldots\ldots (12b)$$

For small changes in θ the approximations to Eqs. 10 are usually very close.

If the streamlines are given, the changes in h/H or $\mathbf{F}$ along a streamline can be readily determined. Conversely, the angular changes necessary to accomplish desired changes in h/H or $\mathbf{F}$ are also easily obtained. Thus, the application of the equations in this form is restricted to determining the changes taking place along boundaries which are streamlines, as long as the flow process along such boundaries is not subject to disturbances propagated from other boundaries. In the case of most hydraulic structures this latter possibility necessitates the study of the interrelations of various boundaries as sources of disturbance lines and of the corresponding system of wave fronts.

Characteristics of Disturbance Lines.—Any curved boundary can be represented as a series of tangents or chords with small but finite angular changes taking place successively. Every angular change $\Delta\theta$ thus becomes the origin of a line of small finite disturbance or a wave front, which will cause a change in depth as indicated by Eq. 9. These wave fronts will traverse the flow at a typical angle β, which is fixed by the Froude number $\mathbf{F}$ of the flow as given by Eq. 5 for every section of the field of flow. As long as the Froude number remains constant in the undisturbed flow, there cannot be any break or curvature in the line of disturbance and the disturbance itself is constant along its line of propagation.

It has already been mentioned that disturbance lines may indicate positive or negative disturbances. Henceforth, positive disturbances or positive wave or surge fronts will be defined as those which deflect the flow toward the line of disturbance and cause a rise in the water surface. A wall curved into the flow and displacing fluid would be the source of such positive wave fronts. Negative disturbances or depression fronts are caused by boundaries curving away from the flow, providing larger cross sections of flow and therefore producing a lowering of the surface and a deflection of the flow away from the wave front. It is clear that fixed boundaries alone can be the sources of disturbance lines, and that a disturbance line cannot appear or disappear unless by action of a boundary or of another disturbance line. Disturbances once created must be propagated undiminished from one boundary to the other and the effect of the wave front on the flow traversed will be the same as the effect of the boundary itself, the same deflection and the same change in surface elevation being transmitted. This process may now be illustrated schematically in Fig. 4 by a flow between two parallel walls with an initial depth h_1 and a velocity V_1, which at points A and B is subjected to a change in direction through a small angle $\Delta\theta$—exaggerated in the sketch for the sake of clearness.

In agreement with the theory, the disturbances to which the flow is exposed at A and B are communicated along only the fronts BC and AC without effect on the flow in the sector ABC. Along BC and along AC the flow is deflected through an angle $\Delta\theta$. The flow passing under BC is parallel to the wall beyond point B, and its depth has decreased and its velocity has increased in accordance with the equations developed; in other words, it was affected by the negative or depression front BC. Point C, Fig. 4, is also the intersection with a positive or surge front, which deflects the flow passing between points A and C through the same angle $\Delta\theta$. This positive front in continuing beyond C enters a field of flow already turned through $\Delta\theta$, of smaller depth and with a higher value of $\mathbf{F}_{-2}$. It will therefore proceed with respect to the new direction, but at a new angle β smaller than before. As a positive wave, it will produce a further deflection of the stream toward the front and bring the lower value of h in field $\mathbf{F}_{-2}$ back to h_1, thus reestablishing the original value of $\mathbf{F}_1$ beyond CD. In turn, the negative wave front BC enters a field with $\mathbf{F}_{+2}$ as its hydraulic characteristic. Since the deflection $\Delta\theta$ for a negative front is away from the front, it tends to aline the streamlines beyond CE parallel to those beyond CD, so that also beyond CE the normal value of $\mathbf{F}_1$ is restored. Along AE and beyond, the normal depth is raised to a higher value, whereas between points E and G it drops back to below normal under the influence of wave BCE and its reflection. Therefore, it will alternate between values higher than h_1 and lower than h_1 all along this wall. At the reflection points of the positive wave originating first at point A, the depth will be raised; at the reflection points of the negative wave emanating from point B, the depth will always be lowered. In analogy to this, along the wall through points B and D the flow will alternate between zones of lower than normal and higher than normal depth. In zones of normal depth, in the center of the stream, the flow is always at an angle $\Delta\theta$ to the walls. In the other zones adjacent to the walls the flow is always parallel to the walls; but it is also seen that the flow beyond points B and A will always remain disturbed unless further wall-angle changes follow, which may augment the disturbances or may be effective in canceling them, unless negative and positive disturbances of equal strength merge.

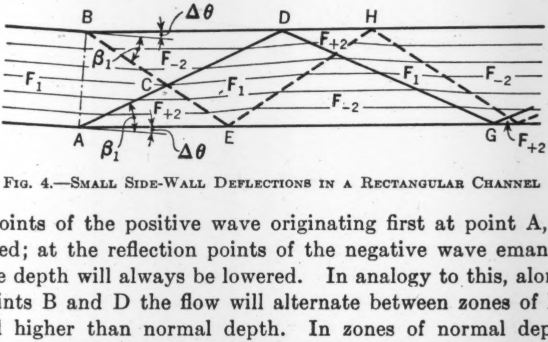

FIG. 4.—SMALL SIDE-WALL DEFLECTIONS IN A RECTANGULAR CHANNEL

Wave Interference.—Of course, by proper alinement of walls, methods of wave interference may be utilized to remove the undesired disturbances. This very possibility will be a challenge to the designer and has already led to certain rules in the design of various structures for high-velocity flow, as shown by subsequent papers in this Symposium.

The fact that a negative wave may be canceled by a positive wave, and vice versa, would lead to the design of a bend as illustrated in Fig. 5. In the first case (Fig. 5(*a*)) the positive wave produced by a wall-angle change at A is not immediately matched by a negative wave opposite A, but the break on the other wall is purposely delayed until the front starting at A arrives at B. Since the flow beyond AB is already deflected through an angle $\Delta\theta$, the wall may be turned through $\Delta\theta$ at B without causing any disturbance at B. This condition may be expressed also by saying that the positive front originating at A is canceled at B by a negative wave, since the latter would coincide with the reflection of AB in B.

The second possibility is shown by Fig. 5(*b*), which indicates that the same turn through an angle $\Delta\theta$ may be accomplished by starting a negative wave first at B and by delaying the break at A until this negative wave has crossed the channel. The possibilities of combining both methods to obtain flow turns with controlled disturbances which will not exceed specified magnitudes are easily recognized. Symposium paper No. 2 is concerned more specifically with such problems, and with other possibilities which follow on the basis of the foregoing principles. It may also be added that a successful analysis must always very clearly be predicated on a fair prediction of the value of the initial Froude number $\mathbf{F}_1$.

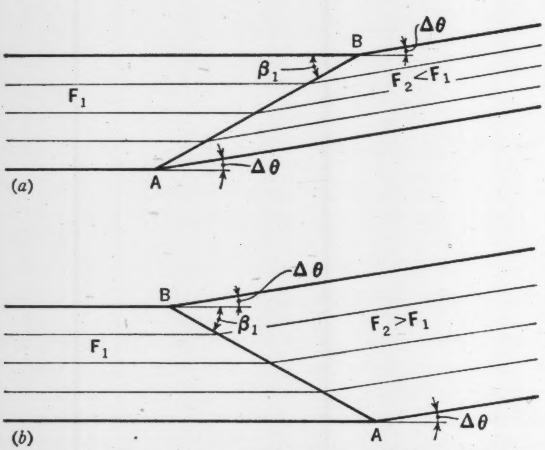

FIG. 5.—SMALL SIDE-WALL DEFLECTIONS DESIGNED TO ELIMINATE DISTURBANCES

This outline was concerned with the principles of wave intersection, reflection, and interference employing an elementary case for illustration. More complicated systems of curved walls may be replaced by a series of short chords or tangents, each one of the latter a source of a surge or of a depression front. Combinations of wave systems are limited only by the mounting difficulties of keeping account of the wave characteristics through numerous intersections and reflections. Except for this fact a useful guide to the design of hydraulic structures is thus available, provided a clear bookkeeping and numbering system can be established and followed.

Successive Waves of Equal Sign.—Before discussion of such a system, a few further properties of these wave fronts may be detailed by use of Fig. 6.[10] If a number of wall sections are arranged so that successive sections as shown send out positive fronts only, it is clear that the wave angles β_1 with respect to the flow between sections must increase more and more in the downstream direction. Ultimately, therefore, these fronts carrying small disturbances only must merge into larger ones imposing such height and angle changes that the angle β_1 can no longer be approximated by Eq. 5 but must be computed from Eq. 4, which shows clearly that the wave velocity of a large wave is not only larger than $\sqrt{g\,h_1}$ but even larger than $\sqrt{g\,h_2}$. Such so-called shock waves are discussed in detail subsequently. These difficulties are not present with negative or depression fronts. With decreasing depth and increasing velocity the wave angles β_1 become smaller in the downstream direction, and angular deflections of the stream are away from the disturbance lines. Therefore, negative fronts will generally diverge, and it can easily be concluded that, within the framework of the basic assumptions made initially, so-called shock waves or disturbances with steeper and steeper fronts cannot be built up from successive negative disturbances of small magnitude.

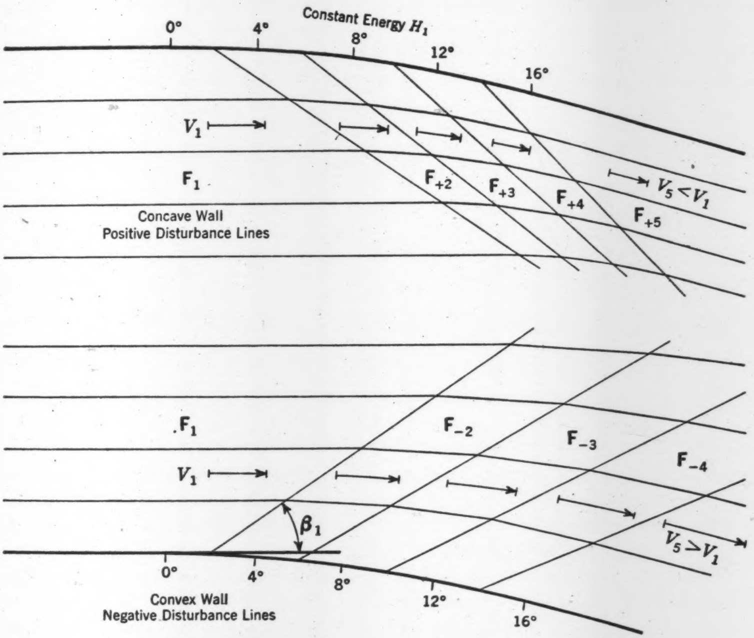

FIG. 6.—FLOW ALONG CURVED WALLS

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[10] "Gas-Wave Analogies in Open-Channel Flow," by A. T. Ippen, *Proceedings*, 2d Hydraulics Conference, *Bulletin No. 27*, Univ. of Iowa Studies in Eng., Iowa City, Iowa. 1942, p. 257, Fig. 3.

METHOD OF CHARACTERISTICS

To facilitate the analysis of wave systems due to continuous disturbances, such as those in curved channels or in channel contractions or expansions, a graphical method has been devised on the basis of the preceding theory with certain ingenious changes. This development was entirely the outcome of an early analysis of supersonic flow of gases by A. Busemann[11] and is known as the "method of characteristics." At the suggestion of Mr. von Kármán this method was adopted in 1935 for use in the supercritical flow of water and was applied to the problem of flow in the curved sections of open channels during the analysis of the test data. Independently, Mr. Preiswerk[9] studied water flow through channel expansions and other problems related by analogy to the high-velocity flow of gases. The basic equation for the graphical method of solving problems of supercritical flow of water is given by Eq. 9. For the purpose of obtaining the deflections of the streamlines as a function of the characteristics of the wave fronts, the ratio (h/H) appearing in that equation will be replaced by its equivalent from the Bernoulli theorem: $1 = \frac{h}{H} + \frac{V^2}{2\,g\,H}$; or with $\bar{V} = \frac{V}{\sqrt{2\,g\,H}}$—

$$\frac{h}{H} = 1 - \bar{V}^2 \qquad (13)$$

—and $\frac{d(h/H)}{d\bar{V}} = -\,2\,\bar{V}$. Eq. 9 is thus transposed into the form:

$$\frac{1}{\bar{V}}\frac{d\bar{V}}{d\theta} = \frac{\sqrt{1 - \bar{V}^2}}{\sqrt{3\,\bar{V}^2 - 1}} \qquad (14)$$

Eq. 14 represents the expression for an epicycloid between the circles of radii $1/\sqrt{3}$ and 1 as the limiting values of $\bar{V}$. Only for supercritical flow has Eq. 14 any physical meaning, since the denominator is zero for $\bar{V}^2 = V^2/(2\,g\,H) = 1/3$ which corresponds to critical flow, whereas the numerator obviously can be zero only for zero depth or $V^2/(2\,g) = H$.

The curve representing Eq. 14 may now be plotted between these limits noting that $d\bar{V}/d\theta = \infty$ for $\bar{V} = 1/\sqrt{3} = 0.577$ and that $d\bar{V}/d\theta = 0$ for $\bar{V} = 1$. The corresponding values (h/H) are two thirds and zero. Values between $\theta = 0°$ and $\theta = 65°\,53'$ corresponding to $\bar{V} = 0.577$ and $\bar{V} = 1.00$ are given in Table 1. For the drawing of the so-called "characteristics diagram" the curve is best transferred to a sheet of celluloid, which is then cut to serve as a templet between the two limiting circular arcs. The diagram may be started and applied as follows: In a rectangular coordinate system the space within a sector of, say, 46° on either side of the $\bar{V}_x$-axis is subdivided into sectors 2° or 4° in central angle, and the intervals are marked by radial lines between the inner and outer limiting circular arcs starting with 0° on the $\bar{V}_x$-axis. From each point on the inner circle the epicycloid may be drawn by the templet in both directions to the outer circle. In this fashion a dense network of intersecting epicycloids is obtained. Since the radial distance between the limit circles comprises all possible values of $\bar{V}$ between 0.577 and 1.00, a corresponding scale of conversion to h/H may be added as shown in Fig. 7. The physical meaning of the diagram may be demonstrated by first assuming a flow with initial values of V_1 and h_1 or $\mathbf{F}_1$ for which $\bar{V}_1$ can be computed. The diagram is next alined with its $\bar{V}_x$-axis parallel to V_1 in the plan of flow, in which the design of the boundaries is laid out. If one of the boundaries is deflected through an angle $\Delta\theta$, the streamlines adjacent to the wall will turn through the same angle and a disturbance line will originate at this point in the flow plan, beyond which the flow will be in a new direction and at a new value of $(h/H)_2$ or $\bar{V}_2$. To find the latter, one must revert to the characteristics diagram (Fig. 7). The initial value of $\bar{V}_1$ laid out along the $\bar{V}_x$-axis will be near an intersection of this axis with the two epicycloids. If the plan of flow indicates a positive disturbance, one of the branches leading inward is followed; in the opposite case one of the branches going outward is followed from the origin until it intersects the radial line belonging to the angle of turn $\Delta\theta$. The radius vector to this point of intersection represents the new value of $\bar{V}$. Its direction is also the direction of the streamlines beyond the disturbance line in the flow plan and its magnitude at once gives the hydraulic conditions of depth and velocity or $\mathbf{F}_2$. One may thus conclude that every point in the characteristics diagram or in the velocity plane determines the hydraulic conditions in a section of the flow plane bounded by walls or disturbance lines. Before proceeding to extend the construction of streamline patterns in the flow plane, a method must be found of constructing the disturbance lines at the same time. For this purpose another drafting device is utilized, which is based on the following theoretical considerations:

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[11] "Gasdynamik," by A. Busemann, *Handbuch der Experimentalphysik*, Vol. IV, 1931.

TABLE 1.—VALUES OF $\bar{V}$ FOR PLOT OF EPICYCLOID BETWEEN CIRCLES OF RADIUS $1/\sqrt{3}$ AND 1.0

0° TO 6°		7° TO 13°		14° TO 20°		21° TO 27°		28° TO 34°		35° TO 40°		41° TO 46°		47° TO 65° 53′	
θ	$\bar{V}$	θ	$\bar{V}$	θ	$\bar{V}$	θ	$\bar{V}$	θ	$\bar{V}$	θ	$\bar{V}$	θ	$\bar{V}$	θ	$\bar{V}$
0	0.577	7	0.709	14	0.782	21	0.840	28	0.886	35	0.925	41	0.952	47	0.972
1	0.613	8	0.720	15	0.791	22	0.848	29	0.893	36	0.930	42	0.956	48	0.975
2	0.635	9	0.731	16	0.799	23	0.855	30	0.900	37	0.935	43	0.960	49	0.978
3	0.651	10	0.742	17	0.808	24	0.861	31	0.905	38	0.939	44	0.963	50	0.980
4	0.666	11	0.753	18	0.817	25	0.868	32	0.910	39	0.943	45	0.966	65°–53′	1.000
5	0.683	12	0.763	19	0.825	26	0.874	33	0.915	40	0.948	46	0.969		
6	0.695	13	0.773	20	0.833	27	0.880	34	0.920	..	....	..	....	....	....

The component of the velocity normal to the wave fronts V_n was assumed to be given by $\sqrt{g\,h}$; therefore $(V_n)^2 = g\,h = g\,H - V^2/2$ and

$$V^2 = 2\,g\,H - 2\,(V_n)^2 \qquad (15a)$$

In accordance with Fig. 2 the velocity V can be related to V_t and V_n by

$$V^2 = (V_t)^2 + (V_n)^2 \qquad (15b)$$

Introducing V^2 from Eq. 15*a* into Eq. 15*b*,

$$2\,g\,H = 3\,(V_n)^2 + (V_t)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

Dividing Eq. 16 by $2\,g\,H$ gives the dimensionless components $\bar{V}_n$ and $\bar{V}_t$ in

$$\frac{(\bar{V}_n)^2}{1/3} + \frac{(\bar{V}_t)^2}{1} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17)$$

Eq. 17 represents an ellipse with major and minor axes of 1 and $1/\sqrt{3}$, respectively, and any radius vector from the center of the ellipse is equal to $\bar{V}$ since it is given by $\sqrt{(V_n)^2 + (V_t)^2}$. This fact is utilized as follows in connection with the characteristics diagram: An ellipse is inscribed on a transparent sheet of plastic to such a scale that the units are identical in both figures.

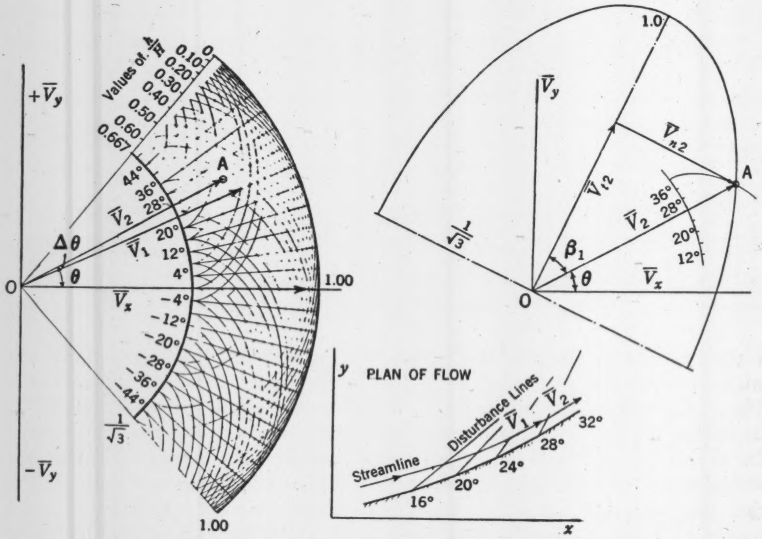
Values of $\frac{h}{H}$
0
0.10
0.20
0.30
0.40
0.50
0.60
0.667
$+\bar{V}_y$
$-\bar{V}_y$
O
A
44°
36°
28°
20°
12°
4°
-4°
-12°
-20°
-28°
-36°
-44°
$\Delta\theta$
θ
$\bar{V}_2$
$\bar{V}_1$
$\bar{V}_x$
1.00
1.00
$\frac{1}{\sqrt{3}}$
1.0
$\bar{V}_y$
$\bar{V}_{n2}$
$\bar{V}_{t2}$
A
36°
28°
20°
12°
$\bar{V}_2$
β_1
θ
$\bar{V}_x$
O
$\frac{1}{\sqrt{3}}$
y
PLAN OF FLOW
Disturbance Lines
$\bar{V}_2$
$\bar{V}_1$
32°
28°
24°
20°
16°
Streamline
x

FIG. 7.—CHARACTERISTICS DIAGRAM; SUPERCRITICAL FLOW WITHOUT ENERGY DISSIPATION

If the ellipse is rotated about the center, O, so that finally the ellipse passes through the point **F**$_1$, which defines $\bar{V}_1$ in the velocity plane, a line drawn parallel to the major axis gives the direction of $\bar{V}_{t1}$, which is identical with the direction of the disturbance line or wave front in the flow plane. Therefore, it may be drawn through the point in Fig. 7 at which the disturbance is said to originate.

An illustrative example is shown in Fig. 7 in the manner just outlined,[12] except that the point chosen for illustration is not the initial point for $\theta = 0$, but one along the wall for $\theta = 24°$ with the change indicated to $\theta = 28°$ and with A marking the position on the epicycloid corresponding to $\mathbf{F}_2$ in the preceding discussion.

---

[12] "Gas-Wave Analogies in Open-Channel Flow," by A. T. Ippen, *Proceedings*, 2d Hydraulics Conference, *Bulletin No. 27*, Univ. of Iowa Studies in Eng., Iowa City, Iowa, 1942, p. 259, Fig. 4.

A detailed example solved by this method for supercritical flow through a channel contraction is given in Symposium paper No. 3. Although the ultimate application of this method permits the rapid and the almost automatic solution of fairly complex problems, anyone applying it must familiarize himself thoroughly with all its aspects and must understand the physical significance of the individual steps at the beginning. Almost any numbering system, methodically applied as the analysis proceeds, will serve satisfactorily, and for this reason no recommendations are made.

High Wave Fronts with Dissipation of Energy

Attention has already been called to the fact that a number of successive positive disturbance lines of small intensity must eventually merge, as distance from the origin increases, into a line representing a surge front of material height. The exact equation for the velocity of such a wave of finite height has been given as Eq. 3 which shows that the wave velocity or celerity is even larger than that for a small disturbance traveling at the larger depth. Consequently the wave angle is larger too in accordance with Eq. 4.

Under the heading, "Method of Characteristics," the wave angle β was approximated by the value of $\sin^{-1} 1/\mathbf{F}$ neglecting the wave height. This section will deal with the modification of the theory due to finite and considerable wave heights, especially with respect to the angle β_1 at which such a large front will assume a steady or "standing" position. Also, the previous assumption of constant energy has to be modified, since energy dissipation is associated with wave fronts of considerable height. However, for many practical cases this modification is still of minor consequence. Basic material for this section was developed by Mr. Rouse and M. P. White,[13] M. ASCE, early in 1937. Mr. Preiswerk[9] adopted Mr. Busemann's shock-polar diagram for use in the hydraulic analogy of supersonic flow in 1938. As before, the problem is the correlation of the wave angle β_1 as defined by Eq. 4 and of the angular change θ accomplished under every standing wave. Fig. 2 may again be referred to except that in keeping with the purpose of this section the angle change $\Delta\theta$ is replaced by θ, thus acknowledging that large deflections are accomplished by large changes in depth under the wave front.

From the geometry of the vectors the angle θ may be related to β_1 by: $V_{t1} = \dfrac{V_{n1}}{\tan \beta_1} = \dfrac{V_{n2}}{\tan(\beta_1 - \theta)}$, in which V_{n2} can be replaced by $V_{n2} = \dfrac{h_1}{h_2} V_{n1}$ from the continuity equation. Therefore,

$$\frac{\tan(\beta_1 - \theta)}{\tan \beta_1} = \frac{h_1}{h_2} \quad \ldots\ldots\ldots\ldots (18)$$

and, solving Eq. 18 for θ,

$$\tan \theta = \frac{\tan \beta_1 (1 - h_1/h_2)}{1 + h_1/h_2 \tan^2 \beta_1} \quad \ldots\ldots\ldots\ldots (19a)$$

---

[13] "Fluid Mechanics for Hydraulic Engineers," by Hunter Rouse, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1938.

Eq. 19*a* can be reduced to a simpler expression if the variations of h_1/h_2 and β_1 occurring in practice are introduced. Thus, for values $h_1/h_2 = 2/3$ to $1/3$ and $\beta_1 = 10°$ to $35°$, only a small error results if Eq. 19*a* is reduced to

$$\tan\theta = \left(1 - \frac{h_1}{h_2}\right)\sin\beta_1 \qquad \text{(19}b\text{)}$$

In agreement with Eqs. 4 and 19*a* the wave angle β_1 and the wave height are fixed for a given Froude number $\mathbf{F}_1$ and a given deflection angle θ. Eqs. 4 and 19*a* can be combined into one by solving for h_1/h_2 as follows:

$$\frac{h_1}{h_2} = \frac{2}{\sqrt{1 + 8\,(\mathbf{F}_1)^2\sin^2\beta_1} - 1} \qquad \text{(20}a\text{)}$$

and

$$\frac{h_1}{h_2} = \frac{\tan\beta_1 - \tan\theta}{\tan\beta_1 + \tan\theta\,\tan^2\beta_1} \qquad \text{(20}b\text{)}$$

Equating Eqs. 20 and solving for $\tan\theta$ gives a very useful expression without h_1/h_2 but containing the initial Froude number $\mathbf{F}_1$:

$$\tan\theta = \frac{\tan\beta_1\,(\sqrt{1 + 8\,(\mathbf{F}_1)^2\sin^2\beta_1} - 3)}{2\tan^2\beta_1 - 1 + \sqrt{1 + 8\,(\mathbf{F}_1)^2\sin^2\beta_1}} \qquad \text{(21)}$$

Normally the deflection angle θ as well as the initial value of the Froude number $\mathbf{F}_1$ would be given by design. For this case it is practical to plot Eq. 21 and to solve graphically for the angle β_1, since the solution of the equation for β_1 in terms of $\mathbf{F}_1$ and θ becomes rather involved. With β_1 known it is an easy matter to determine h_1/h_2 from Eq. 20*a* or from its graphical form. If the latter values are too large for the design in question, the wall or deflection angle θ must be reduced and the process of computation is speedily repeated with graphs such as Fig. 8.

The limiting values of θ for which the flow passes into the subcritical state below the wave front are of considerable interest. For their determination an expression would be needed involving the Froude number $\mathbf{F}_2$ below the wave front. The latter can then be given the value of unity for critical flow and the corresponding quantities for the other variables may be computed. Since a direct relationship between θ and $\mathbf{F}_2$ becomes rather involved algebraically, it is preferable to correlate h_2/h_1, $\mathbf{F}_2$, and $\mathbf{F}_1$ whereupon, by use of Fig. 8, the pertinent values of θ and β_1 are easily read from the curves.

Making use of a geometric condition evident from Fig. 2, since $(V_{t1})^2 = (V_{t2})^2$, one obtains

$$(V_{n1})^2 - (V_{n2})^2 = (V_1)^2 - (V_2)^2 \qquad \text{(22}a\text{)}$$

The continuity equation yields $V_{n2} = \dfrac{h_1}{h_2}V_{n1}$, whereas Eq. 2, based on the momentum relation, results in $V_{n1} = \sqrt{g\,h_1}\sqrt{\dfrac{1}{2}\dfrac{h_2}{h_1}\left(1 + \dfrac{h_2}{h_1}\right)}$. By definition

$\mathbf{F}_1 = V_1/\sqrt{g\,h_1}$ and $\mathbf{F}_2 = V_2/\sqrt{g\,h_2}$, so that, after various substitutions, Eq. 22*a* assumes the following form:

$$(\mathbf{F}_2)^2 = \frac{h_1}{h_2}\left[(\mathbf{F}_1)^2 - \frac{1}{2}\,\frac{h_1}{h_2}\left(\frac{h_2}{h_1} - 1\right)\left(\frac{h_2}{h_1} + 1\right)^2\right] \quad \text{........(22}b\text{)}$$

Eq. 22*b* establishes the relationship between $\mathbf{F}_1$, $\mathbf{F}_2$, and h_2/h_1 for stationary wave fronts, its graphical solution having been added in the fourth quadrant in Fig. 8, in which the critical values for $\mathbf{F}_2 = 1$ are easily located as the intersections of the line for $\mathbf{F}_2 = 1$ with the lines $\mathbf{F}_1$ = constant.

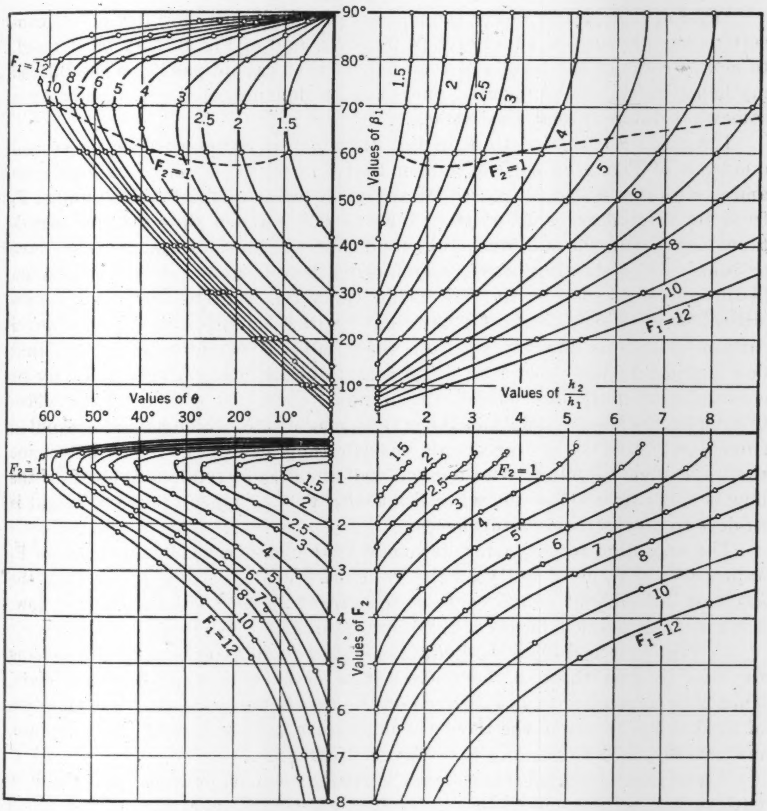

FIG. 8.—GENERAL RELATIONS FOR $\mathbf{F}_1$ VERSUS θ, β, $\frac{h_2}{h_1}$ AND $\mathbf{F}_2$

A second critical condition exists for a wave angle β_1 of 90°, which reduces Eq. 4 to the relation between $\mathbf{F}_1$ and h_2/h_1 familiar from the right hydraulic jump:

$$(\mathbf{F}_1)^2 = \frac{1}{2}\frac{h_2}{h_1}\left(1 + \frac{h_2}{h_1}\right) \qquad \text{(23a)}$$

or

$$\frac{h_2}{h_1} = -\tfrac{1}{2} \pm \sqrt{2\,(\mathbf{F}_1)^2 + \tfrac{1}{4}} = \tfrac{1}{2}\left(\sqrt{1 + 8\,(\mathbf{F}_1)^2} - 1\right) \qquad \text{(23b)}$$

Mr. Rouse has presented[13] a further version of Eqs. 19*a*, 21, and 22*b* giving the deflection angle θ in terms of h_2/h_1 and β_1.

Physical Significance.—For the purpose of this discussion a few points concerning the physical meaning of Eqs. 20 to 23, and of Fig. 8, may be stressed, to clarify some of the results of the analysis that cannot easily be established mathematically. Subsequently, they will be demonstrated graphically by a somewhat different analytical attack.

The angle θ is related to β_1 in Eq. 21, which is represented in the second quadrant of Fig. 8, in such a manner that θ varies from zero to a maximum and again to zero as β_1 increases for any given value of the Froude number $\mathbf{F}_1$ from its minimum value of $\sin^{-1} 1/\mathbf{F}_1$ to 90°. When $\beta_1 = 90°$ and simultaneously $\theta = 0$, the wall deflection cannot be the cause of the standing wave or jump. The standing wave, therefore, is dependent on some downstream disturbances, as soon as the flow below the wave front passes into the subcritical state, which is true for the right hydraulic jump, formed only under certain conditions downstream from the jump. It would seem further that flow parallel to the wall at an angle θ for subcritical conditions will only be accomplished if the downstream disturbance, combined with the wall deflection, produces a wave height of the correct magnitude, which can be computed from Eqs. 20 to 23. In other words, the equations as given no longer determine the wave angle β_1 uniquely for a given wall angle and Froude number $\mathbf{F}_1$ if the flow is subcritical below the wave front with $\mathbf{F}_2 < 1$, since the wave height is subject to downstream disturbances.

The exception to the latter statement is the case in which the value of $\mathbf{F}_1$ approaches a value of unity as the value of θ changes to ± 0, so that the slightest disturbance would cause a wave front at 90° to the oncoming flow; however, this point is purely a philosophical one.

In most practical cases the hydraulic designer will avoid wall angles causing the flow to pass through the critical state or even to approach that state, since little dependence can be placed on the theory when the slightest violation of the basic premises of the theory or inaccurate knowledge of $\mathbf{F}_1$, for instance, may cause a hydraulic jump that travels upstream.

Furthermore, a single wave front is seldom met in practice, and the flow pattern will be determined by intersecting waves and reflections of the waves from opposite walls. These phenomena place severe limitations on practical wall angles, if supercritical flow is to be maintained throughout a structure.

Returning to Fig. 8 it is seen that, for any given hydraulic conditions and geometry of channel walls, the unknown quantities are quickly obtained. For instance, if $\mathbf{F}_1$ and θ are known, β_1 can be read off immediately as can h_2/h_1 and $\mathbf{F}_2$. If the wave front seems too large, then for any desirable value of h_2/h_1 and the given value of $\mathbf{F}_1$, the corresponding values of β_1 and θ are found. If the wave is reflected, the value of $\mathbf{F}_2$ must be known in order that the process may be repeated.

Magnitude of Energy Dissipation.—In contrast to the small changes in depth due to gradual lateral deflections of the stream along walls of small curvature, sudden changes in depth, characterized by the appearance of steep standing wave fronts, are subject to energy losses, which may be computed next. The head loss for the flow through a standing wave or jump is easily expressed by the difference in the specific heads: $H_1 - H_2 = \Delta H$; or $(h_1 - h_2) + \frac{(V_1)^2 - (V_2)^2}{2g} = \Delta H$. Making use of Eq. 22*a* and of the momentum and continuity equations, the velocity-head terms may be eliminated and the head loss may ultimately be expressed as

$$\frac{\Delta H}{h_1} = \frac{1}{4}\frac{\left(\frac{h_2}{h_1} - 1\right)^3}{\frac{h_2}{h_1}} \qquad (24)$$

The head loss remains surprisingly small even for large values of h_2/h_1. Although the head loss is expressed in terms of h_1 (which, for supercritical flow, forms a small part of the specific head), the loss remains a small quantity up to values of $h_2/h_1 = 2$ and for this value it is only one eighth of the initial depth h_1. The ratio of $\Delta H/h_1$ reaches unity for values of $h_2/h_1 = 3$. However, as the possibilities of higher values h_2/h_1 presume higher Froude numbers $\mathbf{F}_1$, the depth h_1 forms a smaller percentage of the total head H_1. Therefore, head losses do not increase as rapidly for larger values of h_2/h_1 as the ratios $\Delta H/h_1$ indicate.

The important fact for practical design is that, in general, head losses up to the ratios $h_2/h_1 = 3$ remain rather small and often may be neglected, since their influence on downstream wave angles remains within the limits of accuracy set by other approximations made in the theory. It seems that in practical design this limit of $h_2/h_1 = 3$ will seldom be exceeded.

SHOCK-WAVE INTERSECTION AND REFLECTIONS

Limiting wall angles, as given in Fig. 8, should never be approached, since the flow in a structure will seldom be determined by the first wave front alone. Reflections on opposite walls and intersections with other wave fronts will have an important role in regard to the character of flow downstream from a disturbance. Various possibilities of basic character will therefore merit discussion. At the outset it may be stated, however, that the following discussion is limited to the range of wave heights mentioned in the preceding paragraph as being little affected by energy losses. Figs. 9 and 10 illustrate, in plan, the following cases to be discussed in order:

1. Reflection of shock wave at opposite wall;
2. Intersection of shock waves:
   - *a.* Of different intensity;
   - *b.* Of equal intensity;
3. Convergence of two shock fronts; and
4. Shock front and negative deflection waves.

1. *Reflection of Shock Wave at Opposite Wall.*—The first case given by Fig. 9(*a*) is the easiest of the four. The flow below the first front AB can be determined from Fig. 8 since the angle of deflection θ is given. As the wave arrives at the opposite wall at point B, the new angle of deflection is the angle of this wall relative to the flow downstream of AB. If this angle happens to be zero, the disturbance is almost wiped out; if the angle is positive, point B will be the source of a wave front determined again from Fig. 8 on the basis of flow parallel to the wall below point B crossing the flow to point C, where the process is then repeated. In the example in Fig. 9(*a*) the opposite wall remains parallel to the original direction of flow. Therefore the flow below front AB is again deflected through the same angle θ and the change from $\mathbf{F}_2$ to $\mathbf{F}_3$ is determined for this angle from Fig. 8. Each reflection brings the flow closer to subcritical conditions and eventually the wave angle will be 90° unless new energy is supplied.

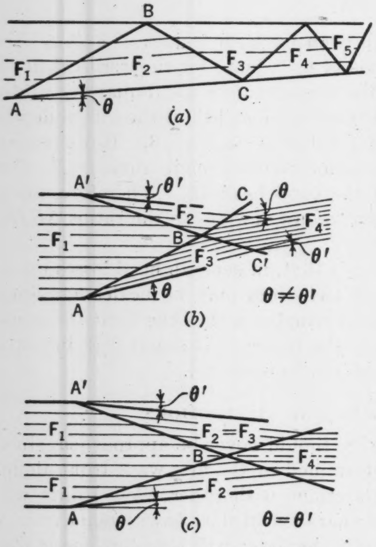

Fig. 9.—Shock-Wave Analysis; Converging Walls

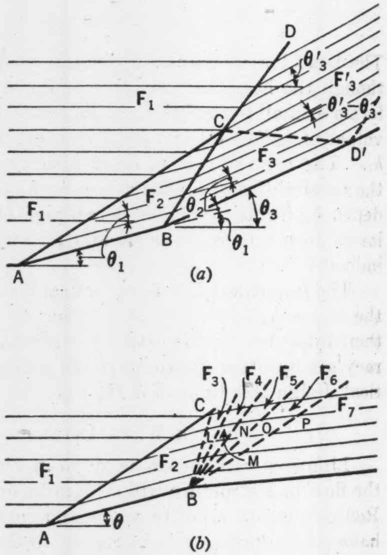

Fig. 10.—Shock-Wave Intersections for Successive Wall Deflections

2a. Intersection of Shock Waves of Different Intensity.—The second case illustrated in Fig. 9(*b*) is somewhat more difficult. When two shock waves meet, each one thereafter will proceed with respect to altered flow conditions. Section AB must be adjusted from $\mathbf{F}_1$ to $\mathbf{F}_2$ and section A′B from $\mathbf{F}_1$ to $\mathbf{F}_3$. The intensities of the two waves in the following sections BC and BC′ are not known and must be determined from the fact that the flow directions below either one are identical. The flow below BC must have the same depth and direction as the flow below BC′. Remembering that for most cases the energy dissipation may be neglected in the first approximation, the new direction of flow in the field $\mathbf{F}_4$ may be determined simply as the difference of the angles θ in fields $\mathbf{F}_2$ and $\mathbf{F}_3$. In most cases the waves BC and BC′ determined on the basis of this first assumption will result in values of $\mathbf{F}_4$ from Fig. 8 which require little adjustment. The approximation becomes better as the two waves differ less in intensity and deflection angle.

2b. Intersection of Waves of Equal Intensity.—In the case of two waves of the same intensity and deflection angle the solution is obvious. The conditions are the same as if a wall were placed along the center line, and the solution is immediate. The shock-wave system on either side of the center line is identical with that illustrated in Fig. 9(*a*).

3. *Convergence of Two Shock Fronts.*—Fig. 10(*a*) presents the case of two converging shock waves, starting on the same side of the wall at points A and B and meeting in point C. The two jumps combine into a single jump CD which produces a head loss higher than the sum of the head losses by wave fronts AC and BC. If no difference in head loss existed, the solution would be simple; the direction of flow would be the same downstream from BC as it is downstream from CD, and the same Froude number $\mathbf{F}_3$ and total deflection angles would exist in this field of flow. This approximation will always be a good one. If more accuracy is warranted, it must be considered that the head below CD must be lower than that below BC and that therefore the intersection point C must be the origin of a depression wave CD′ which forms the boundary between two fields of flow characterized by $\mathbf{F}_3$ and $(\mathbf{F}_3)'$ with $(\mathbf{F}_3)' > \mathbf{F}_3$. The intensity of the depression wave must be such that by trial and error the combined effect of fronts AC and BC and of the depression wave CD′ is identical with the effect of the combined front CD; that is, to either side of a streamline drawn through C the Froude number is essentially $(\mathbf{F}_3)'$ and the deflection angle is $(\theta_3)'$ instead of θ_3 with $(\theta_3)' > \theta_3$. In other words, the effect of the convergence of the jump is a slight additional deflection $((\theta_3)' - \theta_3)$ of the flow away from the wall below the intersection of the depression wave CD′ with the wall.

The depression wave could be treated further according to the methods previously introduced under the heading, "General Physical Background," on the basis of zero energy loss. The effect of CD′ cannot be large since, in agreement with the properties of depression waves discussed previously, negative shock waves are not possible.

4. *Shock Front and Negative Deflection Waves.*—This case involves Fig. 10(*b*) where a positive wall deflection is followed by a negative deflection of finite magnitude. The finite negative deflection of the wall is the origin of a number of depression waves BL, BM, BN, BO, etc., of small intensity as shown in Fig. 10(*b*), some of which will be able to intersect the positive front AC. The faculty of deflecting the flow through a certain small angle $\Delta\theta$, which each depression wave is assumed to have, is not impaired by its intersection with a positive front, since no energy change is involved. In most cases, however, only a few of the negative waves will reach the jump within a finite distance, and usually reflections from opposite walls take place beforehand. Thus, a combination of positive and negative wall deflections will always cause large total disturbances of the water surface, with the exception of the case in which

a positive wave from the opposite wall and of comparable deflection intensity arrives at the point of origin B of the negative disturbances.

In concluding the discussion of problems concerning intersections and reflections, combinations of steep wave fronts and of small positive or negative disturbances may be mentioned very generally. Since small disturbances can be treated without energy dissipation, their influences may be superposed directly on the field of flow subject to the action of steep wave fronts.

POLAR DIAGRAM OF SHOCK-WAVE CHARACTERISTICS

Although Fig. 8 contains, in a convenient form, all possible values of the variables involved in the treatment of steep wave fronts or shock waves, it can be used only indirectly for plotting streamline patterns for flow fields with a shock-wave system. The analysis of shock waves in supersonic gas dynamics has been based, therefore, on the so-called shock-polar diagram as developed first by Mr. Busemann.[11] As its name implies, the shock-polar diagram represents a polar plot of all possible values of the dimensionless velocities upstream and downstream from a shock front similar to the plot of the same velocities on the basis of constant energy. However, since the energy does not remain constant through the finite jumps, the jump-polar diagram cannot be made of such universal character. The basic difference lies in the fact that the dimensionless velocity $\bar{V}_1 = V_1/\sqrt{2\,g\,H_1}$ will change below a jump of deflection intensity θ to the value $\bar{V}_2 = V_2/\sqrt{2\,g\,H_1}$, which is no longer identical with $V_2/\sqrt{2\,g\,H_2}$ since $H_1 > H_2$. Since curves may be plotted only for $\bar{V}_2$ (that is, for dimensionless velocities $\bar{V}_2$ in terms of H_1), the values of $\bar{V}_2$ cannot be used to find $\bar{V}_3$ below a subsequent jump 2–3. The velocity $\bar{V}_2$ must be adjusted below the jump, before the diagram can be used again, by multiplying the value $\bar{V}_2$ by $\sqrt{H_1/H_2}$ which is always larger than unity. A new initial value of $\bar{V}_2\sqrt{H_1/H_2} = (\bar{V}_2)'$ is found, which may then be used to determine $\bar{V}_3$ in terms of H_2. Thus, adjustments must be made continuously in accordance with changing values of H. The second difference appears in the fact that the wave celerity for finite jumps must be considered on the basis of Eq. 3, modifying the wave angles β_1 in conformance with Eq. 4.

The construction of the shock-polar curves requires the transformation of the foregoing basic equations by introducing the x-component and the y-component of the dimensionless velocities $\bar{V}_1$ and $\bar{V}_2$. Following the geometry of the vector diagram of Fig. 11,

$$\bar{V}_{n1}\,(\bar{V}_{n1} - \bar{V}_{n2}) = \bar{V}_{x1}\,(\bar{V}_{x1} - \bar{V}_{x2}) \quad \text{.............}(25a)$$

and

$$(\bar{V}_{n1} - \bar{V}_{n2})^2 = (\bar{V}_y)^2 + (\bar{V}_{x1} - \bar{V}_{x2})^2 \quad \text{............}(25b)$$

All the velocities have been divided by $\sqrt{2\,g\,H_1}$, as indicated by the bar over the velocity components. In addition to Eqs. 25, the momentum relation as stated in Eq. 2 and the continuity equation as given in Eq. 1 are available while, as a fifth equation, the energy equation is used for the definition of the initial energy as $H_1 = h_1 + \dfrac{(V_{x1})^2}{2\,g}$; or

$$1 = \frac{h_1}{H_1} + (\bar{V}_{x1})^2 \quad \text{......................} (26)$$

Thus, all terms containing h_1 and h_2 as well as V_{n1} and V_{n2} can be replaced in terms of V_y and V_x, resulting ultimately in an equation for $\bar{V}_{y2}$ as follows:[14]

$$(\bar{V}_{y2})^2 = (\bar{V}_{x1})^2 \left(1 - \frac{\bar{V}_{x2}}{\bar{V}_{x1}}\right) \times \left[\frac{\bar{V}_{x2}}{\bar{V}_{x1}} - \frac{\sqrt{1 - (\bar{V}_{x1})^2}}{\sqrt{1 - (\bar{V}_{x1})^2 + 4\,(\bar{V}_{x1})^2 \left(1 - \frac{\bar{V}_{x2}}{\bar{V}_{x1}}\right)}}\right] \quad \text{........} (27)$$

The curves of possible values of $\bar{V}_{y2}$ and $\bar{V}_{x2}$ are shown in Fig. 11 for a systematic set of values of $\bar{V}_{x1}$ varying from a maximum of $V_1/\sqrt{2\,g\,H_1} = 1$ to $V_1/\sqrt{2\,g\,H_1} = 1/\sqrt{3}$ as a minimum for critical flow.[15] A radius vector to any intersection with a curve beginning at a certain value of $\bar{V}_1 = \bar{V}_{x1}$ will give, immediately, the only value of $\bar{V}_2$ possible for the assumed direction, whereupon V_2 will be known since H_1 is given initially. Remembering also that $V_{t1} = V_{t2}$ and that the direction of these components represents the direction of the wave front, the latter may be transferred to the plan of flow as was done for the characteristics diagram in the example given. The same ellipse may be used if a graphical method is desired.

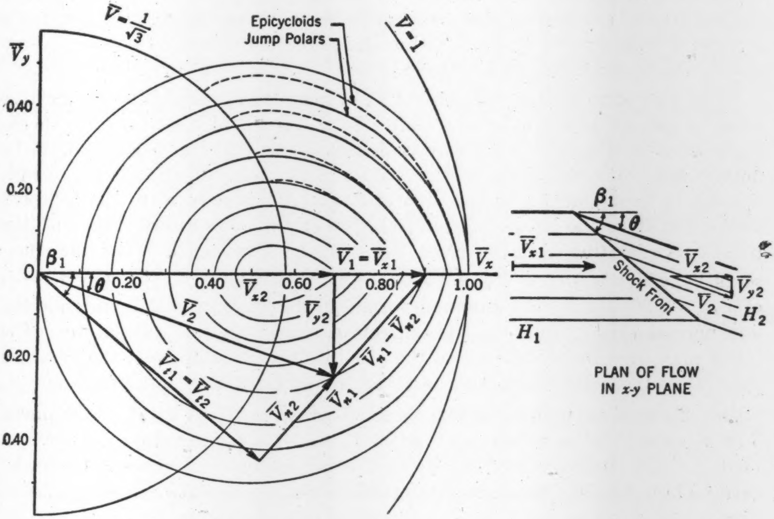

FIG. 11.—EXAMPLE SHOWING THE USE OF THE POLAR DIAGRAM, WITH $V_{x1} = 0.9$ AND $\theta = 20°$

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[14] "Gas-Wave Analogies in Open-Channel Flow," by A. T. Ippen, *Proceedings*, 2d Hydraulics Conference, *Bulletin No. 27*, Univ. of Iowa Studies in Eng., Iowa City, Iowa, 1942, p. 262, Eq. 20.

[15] *Ibid.*, p. 261, Fig. 6.

Only one wave front may be found at one time. With every value of $\bar{V}_2$, of course, belongs a new value of specific head, since $H_2 < H_1$, which has to be computed next as does the corresponding depth h_2. It may be sufficient to remind the reader of Fig. 8 (which relates, in a comprehensive manner, all pertinent variables) to indicate that curves of H_2/H_1 and h_2/h_1 could be plotted also as polar curves similar to the $\bar{V}_2$-curves in Fig. 11. Since shock fronts are usually not very numerous in a plan of flow, it is felt that a completely graphical solution does not offer advantages as great as does the method of characteristics, where the solution requires the drawing of a multitude of disturbance lines.

It may be noted that the polar curves for a wide range of $\bar{V}_1$-values agree quite closely with the epicycloids of the characteristics diagram as indicated in Fig. 11. One outstanding difference is the fact that the polar curves extend into the subcritical range and therefore within the circle of critical velocity with radius $\bar{V} = 1/\sqrt{3}$. As is also evident from Fig. 8, there is a maximum deflection angle θ for every curve (that is, for every initial value $\bar{V}_{x1} = \bar{V}_1$), which falls within the subcritical range of flow below the wave front and is subject to the restrictions discussed earlier in this treatment. The points $\bar{V}_2$ for $\theta = 0$ lying on the $\bar{V}_x$-axis, within the range of subcritical flow, represent the $\bar{V}_x$-values for the right hydraulic jumps.

It is desired to obtain the value of $\bar{V}_2$ in magnitude only, since its direction must be parallel to the deflecting wall. In addition, the location of the shock-wave front (below which all flow is deflected through the angle θ) is to be determined. For this purpose, the $\bar{V}_x$-axis of the polar diagram is alined with the direction of flow $\bar{V}_{x1}$ in the plan of flow. A line parallel to the deflected wall is drawn in the polar diagram from the origin to its intersection with the polar curve starting from the given $\bar{V}_{x1} = 0.90$. The length of the line represents $\bar{V}_2$, which is $0.72 = V_2/\sqrt{2\,g\,H_1}$ so that V_2 is known. Through the ends of $\bar{V}_{x1} = 0.90$ and $\bar{V}_2$, the direction normal to the wave front is found; and the line normal from point O to this direction represents the wave front. The latter may then be transferred to the plan of flow, starting from the break on the wall. If another deflection is to be investigated below the first one, H_2 must be found and an adjusted value of $(\bar{V}_{x2})' = V_2/\sqrt{2\,g\,H_2}$ determined. The $\bar{V}_x$-axis is to be alined again with V_2, and the former steps are then repeated. The treatment of wave front intersections and reflections can be easily adapted to the shock-polar method on the basis of the preceding discussion.

SUMMARY

The fact that boundary-layer influences and nonuniformities of the velocity and pressure distributions are disregarded in the theory imposes certain limitations which are discussed in detail in the Symposium papers dealing with specific practical applications. In general, it may be said that viscous forces do not seriously disturb the basic systems of disturbance lines and of finite wave fronts. Vertical accelerations are usually also of secondary order of magnitude in structures to which the elementary theory applies, since changes in the flow patterns discussed are the result of changes of the lateral boundaries only. The extension of the theory to cases of accelerated motion due to changes of specific head must be reserved for future discussion.

The assumption of a sudden change of flow conditions under a wave front was made for the convenience of analyzing wave systems and does not correspond to physical reality. Shock-wave fronts, especially, represent a finite change in velocity and depth and therefore extend in nature over a considerable distance in terms of depth. The hydraulic jump may be cited to illustrate this point. Nevertheless, the ultimate change accomplished by these standing waves is still as computed by the basic theory. The local phenomena become important only where the dimensions of the hydraulic structure concerned are small compared to the length of the transition zone. As a practical example a bridge pier may be cited, which causes standing waves comparable in height and length of transition to its own dimensions. Information on the transverse extent of hydraulic wave fronts is yet to be obtained experimentally and is tied in with the problems of boundary-layer growth and of vertical accelerations. Another more serious effect of viscous forces may be mentioned: As changes in velocity and depth occur, the balance between gravity and viscous forces existing for uniform flow is disturbed. Minor changes in specific head will result, which are not considered in the elementary theory. The effect of such changes is negligible only for local phenomena, but it is cumulative over a long distance. A small change in wave angle, for instance, will cause only a slight shift of the point of the first reflection of a wave, but the *n*th reflection point is displaced *n* times more than the theoretical one. Thus, longitudinally extended systems of waves may become subject to considerable differences by interference downstream from the original disturbance.

Except for these secondary physical phenomena which the basic analysis neglects, the elementary theory has contributed the following:

1. The primary features of supercritical flow have been rationalized on the basis of accepted principles of mechanics.
2. The general characteristics of the standing wave patterns have been developed systematically.
3. The basic requirements for sound high-velocity structures can be formulated qualitatively for the different types, and the effects of various designs can be visualized. In general, the aim of design will be toward elimination of large surface disturbances.
4. Surface disturbances can be subjected to a detailed quantitative analysis for given designs with respect to local phenonema, and basic formulas have been derived for this purpose.
5. The findings of the theory have been used to develop systematic graphical procedures for the solution of complex problems.

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DESIGN OF CHANNEL CURVES FOR SUPERCRITICAL FLOW

BY ROBERT T. KNAPP,[16] M. ASCE

SYNOPSIS

Characteristics of flows around curved sections of open channels at velocities greater than the wave velocity (that is, $\mathbf{F} > 1$) are discussed in this paper. In simple curves such flows produce cross-wave disturbance patterns which also persist for long distances in the downstream tangent. These disturbance patterns indicate nonequilibrium conditions whose basic cause (when $\mathbf{F} > 1$) is that disturbances cannot be propagated upstream or even directly across the channel. Thus, the turning effect of the curved walls does not act equally on all filaments in a given cross section and equilibrium is destroyed. The paper outlines two basic methods of eliminating these disturbance patterns. Analytical design criteria are developed, and experimental verifications of the analyses are presented. The first method consists of applying a lateral force in such a way that it acts simultaneously on all filaments, causing the flow to turn without disturbing the equilibrium. Bottom banking supplies such a lateral force, and a series of vertical curved vanes across the channel has roughly the same effect. The second method employs interference patterns introduced deliberately at the beginning and at the end of the curve. Compound curves, spiral transitions, and sills all operate on this principle. Rectangular channels are uniquely suited to the interference method of treatment, since for a given channel the wave patterns are substantially independent of the flow. Trapezoidal and other nonrectangular channels should be avoided if possible, unless the flow is invariant. The fields of application of the different treatments are discussed briefly.

DESCRIPTION OF FLOW AROUND SIMPLE CURVES IN RECTANGULAR CHANNELS

One of the common types of curves in normal use for subcritical and supercritical velocities is the simple curve of constant radius in a channel of rectangular cross section. For all subcritical velocities this design is quite satisfactory. For supercritical flows, it is not satisfactory. Fig. 12 shows the surface appearance of such a curve. Very strong cross waves occur within the curve and persist for a considerable distance downstream. This type of behavior could have been predicted from the principles presented in the first Symposium paper. Thus, an examination of the mechanics of the flow shows that the only forces acting on the fluid to cause it to change its direction originate, and are applied, at the walls. At the beginning of the curve the elements next to the outer wall are forced to turn because the wall curves toward them.

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[16] Director, Hydrodynamics Laboratory, California Inst. of Technology, Pasadena, Calif.

The elements next to the inner wall are induced to turn since this wall curves away from them and leaves them without support. However, the elements in the interior of the flow are unaffected at first. They continue to move in the original direction until the wall forces have had time to propagate across the channel far enough to act upon them. These wall forces appear as pressure differences and as such are propagated at the velocity of a surface wave in the channel. Fig. 13 is a diagram of such a curve having a mean radius, r. The point A is the beginning of curvature of the outer wall.[17] The first small disturbance caused by this curvature will be propagated along the line AB. Similarly, the initial disturbance produced by the inner wall is propagated along the line A_1B. All the flow upstream from the boundary ABA_1 is unaffected by the curve and thus continues to move in a direction parallel to the upstream tangent. Beyond point B the disturbance originating at A on the outer wall is no longer propagated in a straight line since the flow in this region is moving in a curved path under the influence of the inner wall. The dotted line BM_1 indicates the path of propagation. Similarly, the disturbance originating at A_1 is propagated on the path BM. Thus, four regions of flow can be delineated as follows:

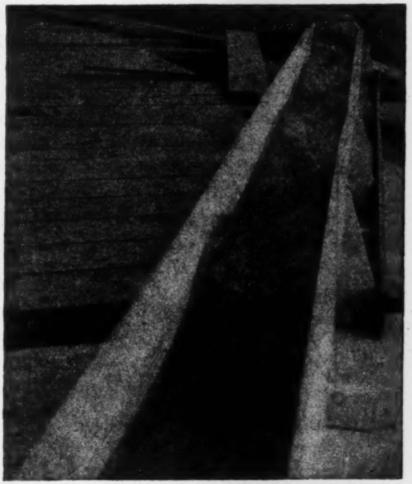

FIG. 12.—CROSS-WAVE DISTURBANCE PATTERN BELOW A SIMPLE CURVE

1. The region unaffected by either wall, lying upstream from ABA_1;
2. The region affected by the outer wall only, within ABM;
3. The region affected by the inner wall only, within A_1BM_1; and
4. The region affected by both walls, downstream from MBM_1

Along the outer wall the fluid surface rises continuously from A to M since an increasing force from the wall has been required to turn the increasing amount of flow acted upon by this wall. At point M, however, the force from the inner wall begins to act. The combined force from the two walls is more than that required to keep the flow turning with the same curvature as that of the channel. Hence, the pressure, and consequently the liquid surface along the outer wall, begin to drop. The reverse condition holds on the inner wall. The liquid surface drops continuously from A_1 to M_1. At the latter point the effect of the outer wall comes into play and therefore the surface begins to rise again. It is evident from Fig. 12 that conditions do not become stabilized when the difference in elevation between the outer and inner walls is again just sufficient to cause the flow to turn with the channel curvature. This equilibrium also is passed through rapidly. The surface continues to drop along the outer wall and to rise along the inner wall to new maxima and minima located at each reflection point of the continuing disturbance paths ABM_1— and A_1BM (Fig. 13). These reflections will conform to the normal laws of wave mechanics. Maxima will occur at all reflection points of the continuation of the path A_1BM and corresponding minima will occur along the continuation of ABM_1.

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[17] "Curvilinear Flow of Liquids with Free Surfaces at Velocities Above That of Wave Propagation," by R. T. Knapp and A. T. Ippen, *Proceedings*, 5th International Cong. for Applied Mechanics, Cambridge, Mass., 1938, p. 533, Fig. 1.

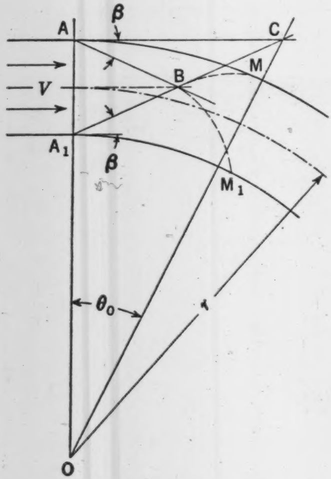

FIG. 13.—DIAGRAM OF FLOW ENTERING A CURVE

CALCULATION OF LIQUID SURFACE PROFILES ALONG THE WALLS

In order to calculate the liquid surface it is necessary to define the assumptions to be used. In the general case, these include: (*a*) Two-dimensional flow, (*b*) constant velocity across the cross section, (*c*) horizontal channel, and (*d*) frictionless flow. Assumptions (*c*) and (*d*) do not exclude the application of the results to inclined channels since, if the flow is in equilibrium, the slope compensates for the friction.

With these assumptions, and the principle of the conservation of energy, Mr. von Kármán[6] derives

$$\theta = \sqrt{3}\,\sin^{-1}\sqrt{\frac{h}{2\,H/3}} - \sin^{-1}\sqrt{\frac{h}{2(H-h)}} + (\text{a constant}) \ldots\ldots (28)$$

from which the change of depth along the walls at the beginning of the curve can be calculated.[18] Angle θ is the angle of turn of the channel wall. Positive values of θ signify a turn toward the channel axis. Eq. 28 is not very convenient to use analytically, although this difficulty may be overcome by using it in a graphical form. A study of the experimental data for a series of flows of this type suggests a simplifying assumption. On the basis of the dual conditions, just specified, of frictionless flow and conservation of energy, the velocity in any filament must change as the depth changes, since the sum of the depth and the square of the velocity must remain constant. The flow around the outer wall, being the deepest, should be the slowest. Measurements indicate, however, that the velocity around the outer wall remains constant, or even increases slightly, whereas along the inner wall it decreases. The deviations are caused by the varying effects of friction on the different elements of the channel. These measurements justify the replacement of the condition of conservation of energy with the approximation that the magnitude of the velocity remains constant. For high-velocity flow the depth represents a comparatively small fraction of the specific head; therefore any possible change in depth can produce only a relatively small change in velocity. On the basis of this assumption the expression for the depth, h, can be derived in the following manner: Consider an open channel with liquid flowing in it at a velocity, V, and a depth, h. If the velocity, V, is supercritical—that is, if V is greater than $\sqrt{g\,h}$—a pressure disturbance can be propagated only at an angle $\beta = \sin^{-1}\frac{\sqrt{g\,h}}{V}$ to the direction of V. If such a pressure wave is propagated across the channel (say, as the result of the liquid encountering a slight bend in the channel), the pressure difference will act normal to the wave front—that is, normal to the angle β. The effect of the pressure difference at the wave front on the velocity, V, will be equal to the change in momentum of the flow normal to the wave front. Thus, if the angle of turn is $d\theta$, the change in depth produced by this change in angle can be expressed as follows:

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[18] "Curvilinear Flow of Liquids with Free Surface at Velocities Above That of Wave Propagation," by R. T. Knapp and A. T. Ippen, *Proceedings*, 5th International Cong. for Applied Mechanics, Cambridge, Mass., 1938, p. 532, Eq. 3.

$$\frac{dh}{d\theta} = \left|\frac{-V_n}{g}\right|\frac{dV_n}{d\theta} \qquad (29)$$

By Fig. 13 $V_n = V \sin\beta$; and, assuming V constant,

$$dV_n = V\cos\beta\, d\theta \qquad (30)$$

Eq. 30 assumes that $d\theta/2$ is small compared to β. Eq. 29 now becomes

$$\frac{dh}{d\theta} = \frac{V^2}{g}\sin\beta\cos\beta \qquad (31)$$

Introducing in Eq. 31 the relationship previously defined that $\sin\beta = V/\sqrt{g\,h}$, and the expression that follows directly from it, that $\tan\beta = \sqrt{g\,h}/\sqrt{V^2 - g\,h}$, Eq. 31 may be transformed to

$$d\theta = \frac{dh}{\sqrt{h\left(\frac{V^2}{g} - h\right)}} \qquad (32a)$$

If Eq. 32*a* is integrated from 0 to θ,

$$h = \frac{V^2}{g}\sin^2\left(\beta_1 + \frac{\theta}{2}\right) \qquad (32b)$$

in which β_1 is the original wave angle corresponding to the depth, h_1, upstream from the disturbance.

Eq. 32*b* is much simpler than Eq. 28 and gives results agreeing equally well with experimental measurements for high velocities. It must be remembered that Eq. 31 is an approximation. However, Eq. 28 is also an approximation, because, among other things, it assumes a constant velocity across the cross section at the entrance to the curve. In the actual case this velocity is not constant. It should be possible to calculate the water-surface profile on the basis of actual velocity distribution instead of utilizing this assumption of constant velocity. However, in most cases the requirements are not rigorous enough to demand this refinement.

There is a close relationship between the wave angle, β, and the Froude number, **F**; and sin β is simply the reciprocal of **F**. Thus either can be used to characterize the state of the flow if the velocity is greater than the critical velocity. The Froude number is more generally used because it retains its significance for subcritical velocities as well as for supercritical ones since physically it is simply the ratio of the flow velocity to the velocity of propagation of a wave with respect to the fluid. On the other hand, the wave angle, β, has no significance for subcritical flow since under these conditions it does not exist.

Neither Eq. 28 nor Eq. 32*b* gives the law locating the maxima and minima along the walls, because they do not contain the factors that determine these locations. Fig. 13 furnishes a basis for estimating their location. As previously stated, the first maximum on the outer wall occurs at point M, which is the reflection point of the disturbance which originates at A_1. The path is straight from point A_1 to point B since the water depth, and hence the wave velocity, is constant. From point B to point M the path is curved, because the water depth (and therefore the wave velocity) is increasing, and also because the direction of flow has been changed by the effect of the outer wall. The path BM may be computed by the methods outlined in the first Symposium paper. Usually it is sufficient to estimate the location of the first maximum and minimum by assuming that they occur at intersections of the line OC with the outer and inner walls. Fig. 13 shows that θ_o represents a half wave length of the disturbance pattern. Its value is given by the expression:

$$\theta_o = \tan^{-1} \frac{b}{\left(r + \frac{b}{2}\right) \tan \beta_o} \dots\dots\dots\dots\dots\dots (33)$$

Therefore, subsequent maxima should occur at $3\,\theta_o$, $5\,\theta_o$, etc., along the outer wall. If by some method the wave pattern could be eliminated and the flow brought to equilibrium within the curve so that the difference in depth between the inner and outer walls would just produce the pressure required to balance the centrifugal force, the difference in depth across the channel would be

$$h_2 - h_1 = \frac{V^2\,b}{r\,g} \dots\dots\dots\dots\dots\dots (34)$$

The depth along the outer wall should be $h_o + \frac{V^2\,b}{2\,r\,g}$. For $\theta = \theta_o$ (first maximum on the outer wall) Eq. 32*b* gives nearly twice this depth. The disturbance pattern that oscillates about the equilibrium depth, therefore, has a wave length of $2\,\theta_o$ and an amplitude of $\dfrac{V^2 b}{2 r g}$.

DISTURBANCE IN DOWNSTREAM TANGENT

The disturbance pattern (that is, the pattern of cross waves superimposed on the equilibrium superelevation) continues into the downstream tangent. The wave length now becomes $\dfrac{2\,b}{\tan \beta_o}$. Furthermore, because of the sudden change of curvature, a new disturbance of amplitude $\dfrac{V^2 b}{2 r g}$ originates at the end of the curve. This new cross-wave pattern has a maximum on the outer wall (at the point of tangency). The wave length is the same as that of the original disturbance, since it is determined by the same factors. The resulting disturbance pattern in the downstream tangent is the sum of these two patterns. In general, they are out of phase and the resultant pattern is dependent on the phase angle. Thus, if a simple curve at equilibrium discharges into the downstream tangent, the resulting disturbance pattern will have an amplitude of $\dfrac{V^2 b}{2 r g}$ with a maximum occurring on the outer wall at the end of curvature. A similar curve having a disturbance pattern of $\dfrac{V^2 b}{2 r g}$ superimposed upon equilibrium conditions will produce no disturbance pattern in the downstream channel if the disturbance within the curve has a minimum on the outer wall at the end of curvature. Curves of lengths θ, $3\,\theta$, $5\,\theta$, etc., will have a maximum on the outer wall at the end of curvature and will produce a disturbance pattern in the downstream channel having maxima of $\dfrac{V^2 b}{r g}$, which is double that of the pattern below a curve operating at equilibrium. It will be noted that the disturbance in the downstream tangent may have a greater amplitude than the corresponding one within the curve but that the actual depth of flow along the wall is never greater than it is on the outer wall within the curve, nor less than it is along the inner wall. The explanation of this anomalous result is that in the tangent the equilibrium condition about which the disturbance oscillates is the constant-depth flow, whereas in the curve the corresponding equilibrium condition is variable flow depth in the cross section, with superelevation on the outer wall, and depression on the inner wall.

CHANNEL WITH MULTIPLE CURVES

In the field most channels are made up of a series of curves separated by tangents of varying lengths. The disturbance pattern in the lower curves and tangents can become very complicated since each curve affects the disturbance in its downstream tangent and the disturbance in each tangent affects the performance of the following curve. It was just shown that the disturbance in the downstream tangent is a maximum when a maximum exists at the end of the outer wall of the preceding curve. The same reasoning will show that the disturbance in a curve will be a maximum whenever a minimum point exists on the outer wall at the beginning of curvature. Thus it is possible by a particularly unfortunate combination of length of curve and of intermediate tangent to build up an extremely high disturbance pattern.

CALCULATION OF THE WATER-SURFACE CONTOURS

In general it is sufficient for design purposes to calculate the water-surface profile along the two walls of the channel. However, it is possible to calculate the entire water surface by the method of characteristics described in the first Symposium paper. The method of characteristics is not applicable when the wave fronts become steep or break. If the vertical accelerations are high some disagreement may be found, but in general this will be confined to a limited zone and thus will not affect the validity of the calculations as a whole.

There are two sources for additional energy losses in curves over those existing in straight channels. Even if equilibrium conditions are established by proper methods of treatment, the cross section of the flow is distorted by the superelevation in such a way as to decrease the effective hydraulic radius and thus increase the frictional losses. Losses are most apparent in the zone of decreased depth. The second major source of loss occurs only if the disturbance pattern is strong enough to produce breaking waves, which correspond to shock waves in supersonic flow of gases. If the wave breaks, an appreciable amount of energy is lost. However, if the surface wave does not break, it represents a very small amount of energy loss. A qualitative proof of this statement is the number of oscillations required for the damping in the downstream channel of a disturbance pattern produced in a curve. Field conditions usually permit the use of radii large enough to avoid breaking waves. Thus, in general, it can be stated that, for supercritical flow, the additional energy losses in curves are of little significance either in the calculation of the disturbance pattern or in the determination of the hydraulic gradient of the channel as a whole.

AVAILABLE METHODS FOR THE REDUCTION OF SUPERELEVATION IN CURVES AND IN THE DOWNSTREAM TANGENTS

Summary of Flow Characteristics Which Affect Method of Treatment.—Before discussing the possible ways of reducing the superelevation in curved channels, it will be well to review briefly the salient physical characteristics of high-velocity flow around curves, as follows:

1. The flow velocity is greater than the velocity of the surface wave (that is, $\mathbf{F} > 1$). Therefore, disturbances cannot travel directly across the channel, but only at the oblique angle determined by the ratio of the wave velocity to the flow velocity.
2. In channels, as normally designed, the side walls are expected to do the turning. This is basically a nonequilibrium process because the effect of a change in wall alinement is not propagated directly across the channel, and therefore cannot immediately affect the entire flow.

3. The form of the disturbance produced when water flows from a straight to a curved channel or vice versa is oscillating and has the properties of a wave train.

Banking.—The most logical method of eliminating the disturbance pattern is to remove its cause. Since it is impossible to act on all the elements of flow at once by a change in direction of the side walls, it would be desirable to employ some better method of applying the lateral force to change the flow direction. The obvious way of applying a lateral force simultaneously to all the fluid elements is by the use of a bottom cross slope—that is, by banking. This cross slope, S_c, can easily be calculated by equating the gravity component along the cross slope to the centrifugal force determined by the radius and velocity. The result is

$$\tan \phi = S_c = \frac{V^2}{r\,g} \qquad (35)$$

It should be emphasized that, with the proper use of banking, all the force necessary to change the direction of the flow is supplied by the cross slope. The walls do nothing except conform to this change of direction. In banking it is not feasible to change the cross slope of the channel instantaneously from a level bottom to the value indicated by Eq. 35. The banking requires an appreciable change in elevation, which can be obtained either by raising the bottom on the outside of the curve, or by lowering it on the inside—or by a combination of the two. To avoid shock, the banking must be introduced gradually and, to follow the path of the stream, the walls must have a decreasing radius of curvature that just matches the increase of cross slope. The most elegant design of a banked bottom would be one in which the center of gravity of the flow followed the mean slope of the channel.

One characteristic of a banked curve is that equilibrium conditions are obtained for only one velocity of flow, and hence for only one depth and one rate of flow. All other flows will show disturbance patterns similar to those in untreated curves. The magnitude of the disturbances will be determined by the degree of departure from equilibrium velocity.

Multiple Curved Vanes.—Eqs. 32*b* and 34 show that for a given depth, velocity, and radius of curvature, the maximum superelevation varies directly with the width of the channel. Therefore, if a given curve is divided into a series of narrower curves by concentric vertical vanes, the superelevation in the subchannels will be correspondingly reduced. Furthermore, the disturbance in the channel below the vanes will dissipate rapidly because of the absence of the vanes that supported the differences in elevation.

Interference Treatments.—The fact that the disturbances produced at the beginning and at the end of a simple curve are wave trains suggests that they might be eliminated by proper interference patterns—that is, by the deliberate introduction of similar disturbances of equal magnitude but of opposite phase. In the flow around a curve of constant radius, the equilibrium condition requires a superelevation of $\frac{V^2 b}{2\,r\,g}$ on the outside wall and a corresponding drop along the inner wall. The disturbance wave pattern oscillates about this equilibrium condition with a wave height practically equal to the equilibrium change in elevation. For complete "interference" the counterdisturbance should have this same wave height. The original disturbance has a minimum point on the outer wall and a maximum point on the inner wall at the beginning of curvature. Therefore, the counterdisturbance should be introduced with a maximum on the outer wall and a minimum on the inner wall at the same point. The opposite conditions exist at the end of curvature. There are several ways of producing the required counterdisturbances, three of which will be discussed briefly—compound curves, spiral transitions, and diagonal sills. All three of these treatments are complete in themselves and do not require auxiliary measures, such as banking of the bottom, to accomplish the desired result. It is possible to combine methods, but such combinations will not be discussed in this paper. It may have been noted that in this discussion of interference treatments, no attention has been paid to the production of a disturbance pattern having the correct wave length. The reason is that the wave length is a function of the velocity and the channel width; hence, the counterdisturbance will automatically have the correct wave length to match the original disturbance. Therefore, care must be taken to obtain a counterdisturbance of the right amplitude and phase. No attention needs to be paid to the wave length except as it is used in establishing the correct phase.

Compound Curves.—A simple curve produces a disturbance pattern having an amplitude and a wave length that can be determined quite accurately. Therefore, it is logical to employ a section of a simple curve to produce the counterdisturbance pattern required to interfere with the one formed by the main curve.

The length of curve most effective in producing a disturbance pattern is a half wave length since this length produces the maximum disturbance for a given radius of curvature. It is also the minimum length in which the disturbance from both inner and outer walls has had time to affect the entire flow. This maximum is on the outer wall, which is where it should be to interfere with the disturbance produced by the main curve. This curve length as measured on the outer wall is $\frac{B}{\tan \beta_o}$. The equivalent central angle is

$$\theta' = \tan^{-1} \frac{b}{\left(r_i + \frac{b}{2}\right) \tan \beta_o} \quad \ldots\ldots\ldots\ldots\ldots\ldots (36)$$

In Eq. 36, r_i is the radius of the counterdisturbance section, not that of the main curve. The required radius for this section is just twice that of the main curve as it is desired to produce a disturbance of one half that caused by the main curve. Thus, the interference pattern must have a wave height equal to $\Delta h/2$, or

$$\Delta h_i = \frac{V^2 b}{2 r g} = \frac{V^2 b}{r_i g} \quad \ldots\ldots\ldots\ldots\ldots\ldots (37a)$$

From Eq. 37*a* it follows directly that

$$r_i = 2\,r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (37b)$$

in which Δh_i is the height of the interference pattern. Therefore, if a simple curve of radius r is preceded by a section of another simple curve, whose length is $\frac{b}{\tan \beta_o}$ and whose radius is $2\,r$, the flow will rise gradually to the equilibrium value and remain in this condition throughout the entire length of the main curve. However, if the main curve terminates abruptly in a downstream tangent, a disturbance pattern will be formed therein having a wave height of $\frac{V^2 b}{2\,r\,g}$. Similar reasoning to that used in developing the design for the upstream counterdisturbance section will show that an identical section applied at the downstream end of the curve will produce the interference pattern required to eliminate the disturbance in the downstream tangent. Fig. 14 shows the construction of such a compound curve which will operate with no disturbance pattern either in the curve itself or in the downstream channel.

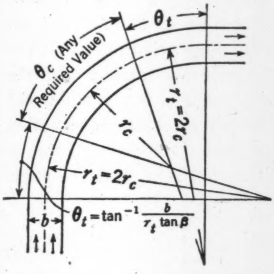
θ_t
θ_c (Any Required Value)
$r_t = 2r_c$
r_c
$r_t = 2r_c$
b
$\theta_t = \tan^{-1} \frac{b}{r_t \tan \beta}$

FIG. 14.—DESIGN CRITERIA FOR A COMPOUND CURVE

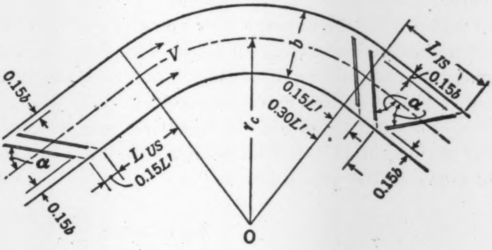

FIG. 15.—PLAN OF SILL INSTALLATION

Spiral Transitions.—Spiral transition curves have been suggested to replace constant-radius transitions. The suggestion is probably traceable to the use of spiral transitions in railroad and highway construction. The spiral can be made to produce satisfactory conditions at the beginning and at the end of the main curve. For that matter, many other designs of transition curves can be found which will be equally satisfactory, provided that the transition curve, whatever its design, must produce a simple disturbance pattern which has a maximum on the outer wall at the beginning of curvature of the main curve and a wave height as given in Eq. 34. One fundamental fact must always be borne in mind: As previously shown, no configuration of curved walls used with a flat bottom channel can produce a change in direction of a high-velocity flow under conditions that maintain equilibrium in all points of the flow. Thus, the primary purpose of the transition curve, whatever its shape, is only the production of the interference wave pattern necessary to cancel out the one produced by the main curve. The fact that the external appearance of the flow shows no evidence of a disturbance pattern is no indication of the absence of the interference phenomenon.

Diagonal Sills.—Diagonal sills on the channel bottom can be used to produce the necessary interference pattern. The effect of a diagonal sill is to produce a change in direction in the lower layer of the flow, which is quite rapidly averaged throughout the entire cross section by the mechanism of momentum exchange. The over-all result can be approximated by calculating the lateral force exerted on the sill by the flow. This force must produce a corresponding lateral change in momentum in the flow. If the force is equated to the product of the mass rate of flow and the change in lateral velocity, the latter can be calculated readily. The vector sum of the lateral and longitudinal velocities gives the resulting direction of the flow below the sill. Since the mass rate of the total flow is used, the direction will be the average for the entire flow.

If skin friction is neglected, the only force that the fluid can exert on a sill will be normal to the face. This force is equal and opposite to the rate of change of momentum of the flow. Its magnitude is obtained by computing the rate of change of momentum of the water that hits the sill. This layer of water has the width of the channel and the height of the sill. It can be assumed to travel with the average flow velocity and to turn through the angle that the sill makes with the channel axis. This layer of water will mix rapidly with the remainder of the flow above it. Therefore, the average angle through which the entire flow is turned can be computed by applying the principle of conservation of momentum to obtain the resulting lateral component of velocity. Thus, if the sill height is d, the angle of sill inclined to the channel is α, and the angle of curve of the entire flow is θ'_s,

$$\theta'_s = \tan^{-1} \frac{d}{h} \frac{\sin 2\alpha}{2} \qquad (38)$$

Eq. 38 shows that the sill has maximum effectiveness when $\alpha = 45°$, and this is confirmed by laboratory tests. Experiments indicate, however, that a smoother disturbance is obtained with a sill angle of 30° with very little loss in effectiveness. The required magnitude of the counterdisturbance produced by the sill is governed by the same consideration as that which governs the magnitude of the counterdisturbance produced by the upstream and downstream sections of the compound curve. The same reasoning, therefore, indicates that θ'_s should be $\theta_o/2$ in which θ_o is the central angle of the half wave length in the main curve, given by Eq. 33. It is not practical to attempt installations for values of θ'_s greater than 10°, as too great a sill height would be required.

On first consideration, the desirable location for the diagonal sills would appear to be with their downstream end at the beginning of curvature on the outer wall. The corresponding position at the end of the curve would be to have the downstream end of the sill at the end of curvature on the inner wall. However, the disturbance produced by a diagonal sill is not as smooth as that resulting from a simple curve. This is particularly true of the first maximum produced. Subsequent maxima become smoother and more similar to those caused by a simple curve. Experiments have shown that the best location for the diagonal sills is as indicated in Fig. 15. The distance L_{US} is given by the expression:

$$L_{US} = \frac{b}{\tan \beta_o} \left[\frac{1.12}{\left(1 + \frac{3\,\Delta h_i}{2\,h_o}\right)^{\frac{1}{2}}} + \frac{0.0313}{\left(\frac{h_i}{h_o} \sin \beta_o\right)^2} \right] \dots\dots\dots\dots (39)$$

The first term in the brackets is a correction for the increase in velocity of the finite disturbance over that of an infinitesimal wave; the second term is an addition for the distance downstream from the toe of the sill at which the maximum disturbance occurs. The symbol h_i denotes the depth of the disturbance produced by the sills and Δh_i is $h_i - h_o$. For normal cases the value of the bracket term will lie between 0.9 and 1.15. Eq. 39 should be used with caution if the correction factor falls outside this range since the experimental verification of the empirical coefficients is rather meager.

Another laboratory finding is also incorporated in Fig. 15. Two sills are shown instead of one at each location. Also the sills end a short distance from the walls. Both modifications are for the purpose of eliminating undesirable local disturbances and of broadening the main disturbance to make it approach the wave form produced by the curved transition section that the sills are replacing. The group of three sills at the end of the curve replace the downstream transition section of the compound curve method of treatment and perform the same necessary function—that is, the elimination of the disturbance in the downstream tangent. All the sills shown in Fig. 15 should have heights one half of that indicated by Eq. 38 since they act in pairs.

FIELDS OF APPLICATION OF DIFFERENT TREATMENTS FOR RECTANGULAR CHANNELS

The treatments described have different fields of use. Banking is most adaptable to major channels that ordinarily operate at or near the design flow. It offers the only method for preventing, completely, the rise in elevation of the water surface in the curve. This requires that the banking be obtained entirely by depressing the inner wall, which usually involves costly excavation. The effect of banking can be decreased or completely nullified by improper wall design. Banking does not produce equilibrium conditions for flows above or below the design capacity. However, for the lower flows, the disturbances produced stay below the design flow line.

The use of multiple curved vanes is rather restricted. They are not desirable for channels carrying debris unless the vane spacing can be made considerably larger than the maximum size of debris to be expected. In case it is necessary to bridge such a channel at the curve, an economical design may be made by using the vanes as bridge piers.

Compound curves offer the most desirable solution for most high-velocity channels. If properly designed, they appear to offer a completely satisfactory solution, not only for the design discharge, but also for all lower discharges.

Spiral or other complicated transition curves are not recommended for any normal application. They simply add to the cost of both the design and the construction with no commensurate improvement in the flow characteristics over that secured by the use of compound curves.

Diagonal sills should be used primarily as a remedial measure in existing channels which have been designed with simple circular curves or other unsatisfactory forms. For such applications, they offer the possibility of changing quickly and economically a completely unsatisfactory condition of flow into one that is at least acceptable. However, diagonal sills operate best at the design rate of flow. For lower flows, the disturbance produced is too large and thus there is some residual pattern both in the curve and in the downstream channel. This is usually not serious for, as in the case of the banked bottom, the superelevations produced by the disturbance pattern fall below the flow line for the design discharge. This disturbance pattern at low flows may have one somewhat intangible by-product. If the channel in question happens to be used for flood-control purposes or for similar uses in which maximum flows are rather unusual, the sight of the disturbance pattern at low flows may cause uneasiness and distrust among the adjacent residents. This is particularly true if overtopping of the channel would be serious. The inexperienced observer is likely to reason that, if a low flow shows such a pronounced disturbance pattern, higher rates are certain to cause increasing superelevations and major flows are sure to cause failure. The engineer may encounter considerable difficulty in explaining that the reverse is true—that the channel performance improves as the flow rate increases.

It must be expected that the maintenance costs for curves with diagonal sills will be higher than those for either banked or compound curves, especially if the flow is debris laden. However, experience has shown that wood timbers bolted to the bottom of concrete channels form quite satisfactory sills even for relatively large channels carrying heavy debris loads. If sills are used with extremely high-velocity flows, cavitation will occur (see subsequently in Fig. 31). Cavitation itself will tend to increase rather than decrease the effectiveness of the sills but the resulting damage may destroy them.

NONRECTANGULAR CHANNELS

A good indication of the problems involved in high-velocity flow in nonrectangular curved channels is obtained by examining two unique advantages of the rectangular cross section which hold for all interference type curve treatments:

1. For a given channel with a given slope, the wave angle remains nearly constant over a wide range of flows; and
2. The channel width is constant.

From these two characteristics it follows that the disturbance pattern in and below a given curve is constant in configuration and location irrespective of the rate of flow. The significance of this statement has already been stressed—that is, designs using interference pattern methods of treatment are satisfactory for all flows within the design maximum.

In a nonrectangular channel the wave angle changes with depth since the wave and flow velocities no longer vary at the same rate. The surface width likewise varies with depth. This means that the wave pattern varies with the flow.

Therefore it must be concluded that curve treatments using the interference method will be much less satisfactory for nonrectangular cross sections than for rectangular cross sections. For example, in nonrectangular channels both compound curves and sills produce interference patterns of the right phase for only one rate of flow. If the channel width varies rapidly with the discharge, this phase shift may even change the interference into a reinforcement and thus increase materially the superelevation in and below the curves. Fig. 34, subsequently, shows such a shift produced by a change in the discharge in a trapezoidal channel. It is suggested, therefore, that the use of nonrectangular cross sections for high-velocity curved channels be avoided wherever possible, especially for applications involving wide ranges of discharge.

EXPERIMENTAL CONFIRMATION

The experimental confirmation of these methods of controlling high-velocity flows in and below curves was obtained during a study conducted in the Hydraulic Structures Section of the Hydrodynamic Laboratories of the California Institute of Technology under the direction of the writer. Much of the experimental work and analysis was done by the author of the first Symposium paper. The study was sponsored by the Los Angeles County Flood Control District, which needed design information for its network of flood-control channels, many of which had steep gradients and high velocities. The first objective was the determination of the physical phenomena involved, and the second objective, the development of methods of treatment for the curved sections of high-velocity channels. Although the Los Angeles County Flood Control District was deeply concerned with the design of a specific group of channels, it was felt that it would be more profitable to conduct the study from this fundamental basis with a view to developing analytical methods of design, rather than to adopt a model study technique and be satisfied with specific solutions for only this particular group of channels.

The equipment consisted primarily of two brass channels, one having a rectangular cross section and one a trapezoidal cross section. They were designed so that under normal flow conditions the sectional areas would be the same. They were both provided with abnormally high walls to permit the investigation of high superelevations. The rectangular channel was 18 in. wide and 14 in. deep; and the trapezoidal channel was 12 in. wide at the bottom, 12 in. deep, and had side slopes of 1 on $1\frac{1}{2}$. The channels were mounted on a platform 100 ft long, which was adjustable to any desired slope up to 1 in 10. A water-circulating system was provided with a maximum rate of flow of 6 cu ft per sec. The flow was measured by a set of three venturi meters having slightly overlapping ranges. Fig. 16 shows the plan and elevation of this equipment,[19] and Figs. 17 and 18 illustrate some of the details of construction. Water depth and surface contours were measured by point gages mounted on rails accurately adjusted to be parallel to the bottom of the

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[19] "Experimental Investigations of Flow in Curved Channels," by A. T. Ippen and R. T. Knapp (abstract of results and recommendations, U. S. Engr. Office), Los Angeles, Calif., 1938, p. 2.

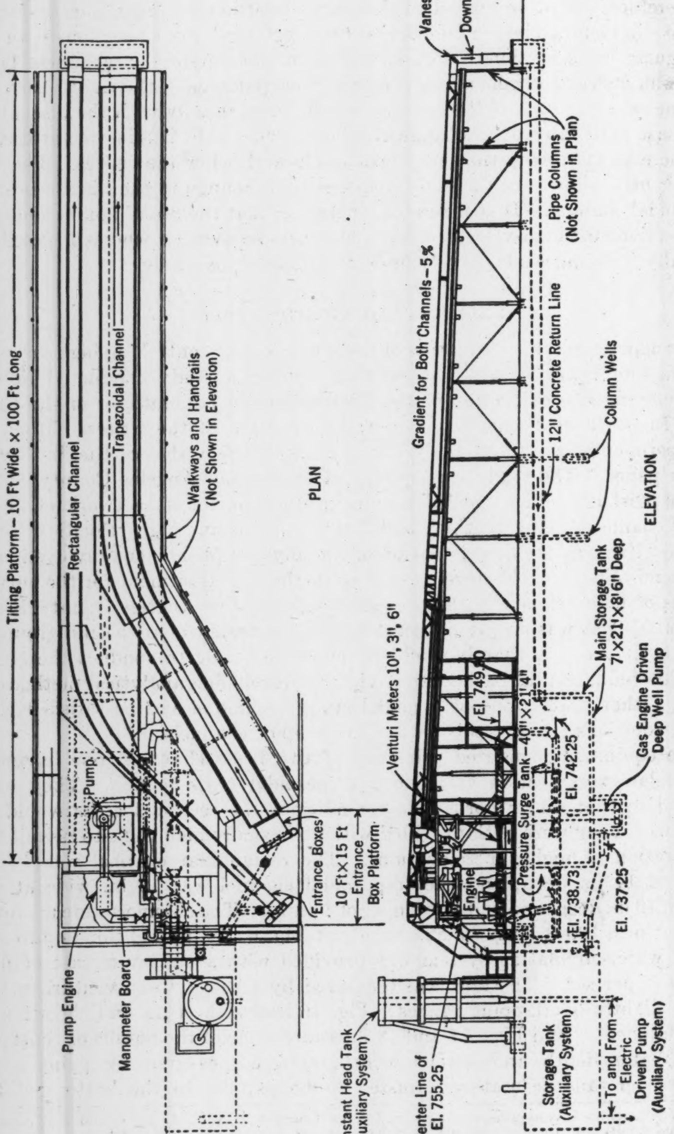

Tilting Platform – 10 Ft Wide × 100 Ft Long
Rectangular Channel
Trapezoidal Channel
Walkways and Handrails (Not Shown in Elevation)
PLAN
Pump
Pump Engine
Manometer Board
Entrance Boxes
10 Ft×15 Ft Entrance Box Platform
Vanes
Down Spouts
Pipe Columns (Not Shown in Plan)
Gradient for Both Channels – 5%
12" Concrete Return Line
Column Wells
ELEVATION
Main Storage Tank 7'×21'×8'6" Deep
Gas Engine Driven Deep Well Pump
Venturi Meters 10", 3", 6"
El. 749.50
Pressure Surge Tank – 40"×21'4"
El. 742.25
El. 738.73
El. 737.25
Engine
Constant Head Tank (Auxiliary System)
Center Line of El. 755.25
Storage Tank (Auxiliary System)
To and From Electric Driven Pump (Auxiliary System)

Fig. 18.—Plan and Elevation of Experimental Channels

FIG. 17.—ADJUSTABLE ENTRANCE NOZZLE

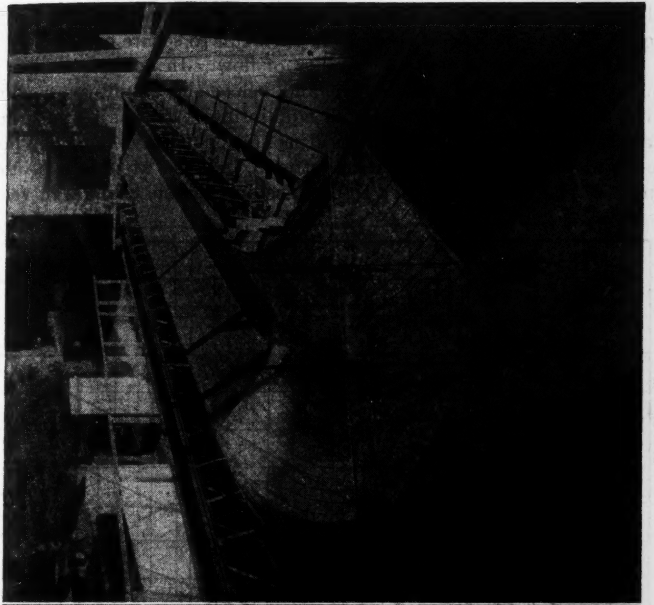


FIG. 18.—CURVE IN TRAPEZOIDAL CHANNEL

channel. Velocity profiles were obtained by pitot tubes mounted on the same carriage. Considerable care was taken in each run to secure equilibrium conditions between the slope and the velocity of flow. In addition, check runs were made with velocities both above and below the equilibrium value to determine the effect of departures from equilibrium. It was found that the equilibrium requirements were not critical; conditions always remained practically constant through the experimental stretch, because, at the slopes used, a long length of channel is necessary to produce appreciable changes in velocity

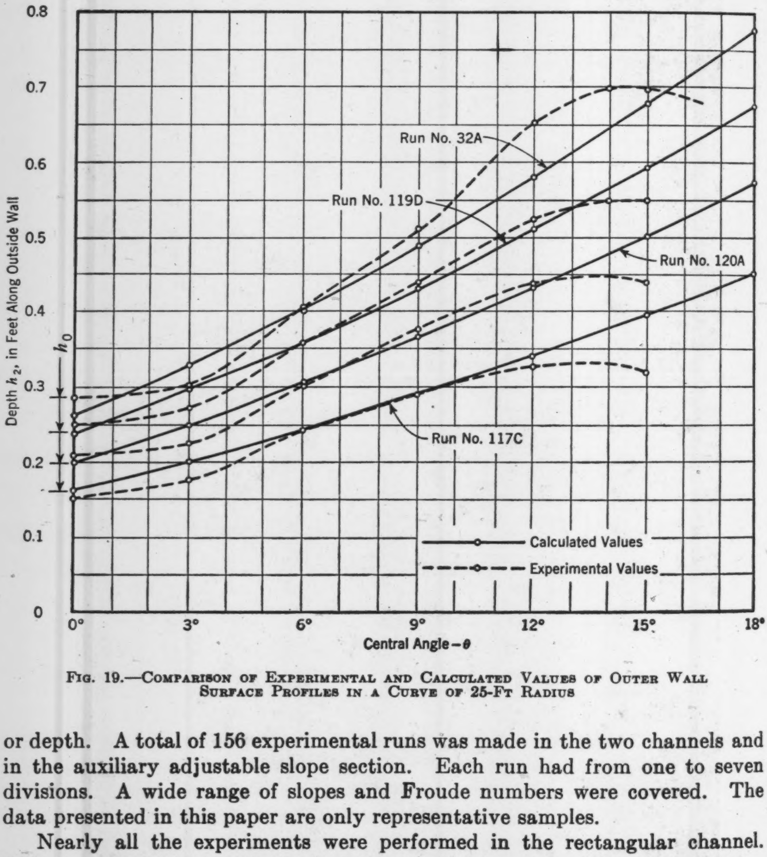


FIG. 19.—COMPARISON OF EXPERIMENTAL AND CALCULATED VALUES OF OUTER WALL SURFACE PROFILES IN A CURVE OF 25-FT RADIUS

or depth. A total of 156 experimental runs was made in the two channels and in the auxiliary adjustable slope section. Each run had from one to seven divisions. A wide range of slopes and Froude numbers were covered. The data presented in this paper are only representative samples.

Nearly all the experiments were performed in the rectangular channel. This was partly due to the limited time available. An additional restriction to the work on the trapezoidal channel resulted from the growing realization of the inherent characteristics of high-velocity flow in curves and the resulting basic limitations imposed upon the use of all cross sections in which the width varied with the depth of flow. A sufficient number of experiments was run in the trapezoidal channel to demonstrate clearly the reality of these limitations.

Constant Radius Curves; Rectangular Channel.—Fig. 19 shows a comparison, for four different rates of flow, of the observed values versus the calculated values for the water-surface profile along the outer wall from the beginning of curvature to the first maximum. These measurements were made in a curve having a mean radius of 25 ft. Fig. 20 shows a parallel set of runs for a 50-ft curve. Table 2 shows a similar comparison for the height and location of all the maxima in the curve and for two sets of maxima in the downstream tangent. These are the same runs shown in Fig. 19. Within the curve the distance between maxima varies from 0.914 to 1.066 of the calculated value, whereas in the downstream tangent, the corresponding ranges are 0.868 to 1.262. Variations of approximately the same magnitude are to be found in the depths. These variations are probably the result of: (*a*) Frictional damping of the disturbance pattern; (*b*) deviation from uniform velocity in the flow; and (*c*) variations in wave velocities caused by variations in the wave heights. Occa-

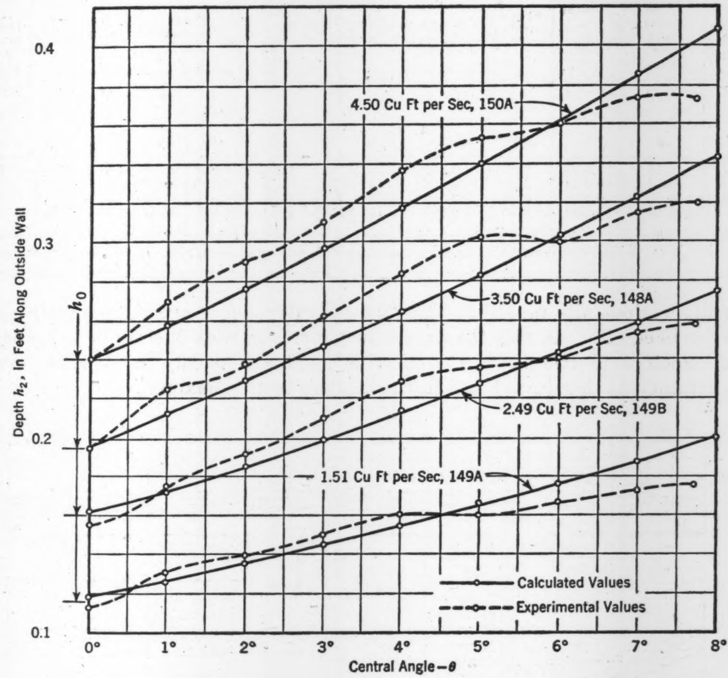


FIG. 20.—COMPARISON OF EXPERIMENTAL AND CALCULATED VALUES OF OUTER WALL SURFACE PROFILES IN A CURVE OF 50-FT RADIUS

sionally it may be desirable to introduce these secondary effects into the analytical treatment. However, in most designs there are other elements of uncertainty, of equal or greater magnitude, which make increased accuracy unnecessary. For example, the slope of steep channels on natural terrain usually varies quite rapidly. Thus the velocity at any given cross section is seldom in equilibrium with the slope, which increases the difficulty in estimating the velocity. Furthermore, the effective roughness usually involves as much uncertainty as that just shown in the location of the maxima.

TABLE 2.—MAXIMUM SUPERELEVATION IN AND BELOW

Run No.	Flow, Q, in cubic feet per second	Depth, h_0, in feet	Froude number, $\mathbf{F}$.	Computed h', in feet	Computed θ_0, by Eq. 37	Computed $L' = \dfrac{b}{\tan\beta}$	IN THE CURVE, AT THE OUTSIDE WALL[a]				
							FIRST MAXIMUM		SECOND MAXIMUM		Distance to end of curve (EC) θ/θ_0
							$\dfrac{h}{h'}$	$\dfrac{\theta}{\theta_0}$	$\dfrac{h}{h'}$	$\dfrac{\theta}{2\,\theta_0}$	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
117C	2.49	0.160	3.29	0.387	14° 33′	6.68	0.868	0.928	0.895	1.014	0.137
120A	3.51	0.199	3.28	0.496	14° 47′	6.79	0.905	0.914	0.935	1.066	0
119D	4.52	0.236	3.25	0.584	14° 46′	6.78	0.942	1.015	0.951	1.016	0
32A	5.51	0.260	3.45	0.693	15° 30′	7.13	1.010	0.967	1.002[c]	1.000	−0.06[a]

[a] Ratios of measured values to computed values. [b] In Cols. 14, 16, 18, and 20, L is the measured

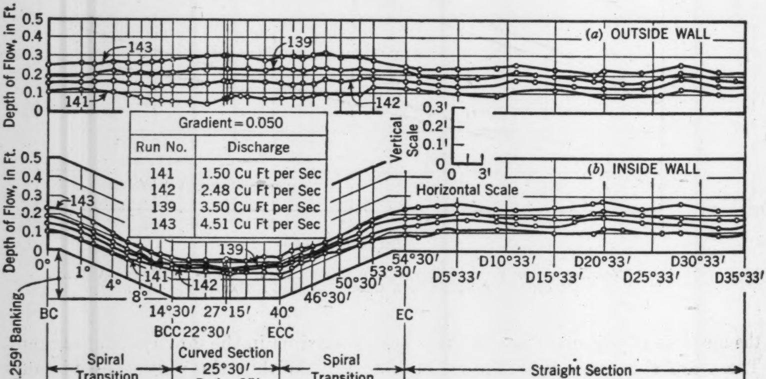


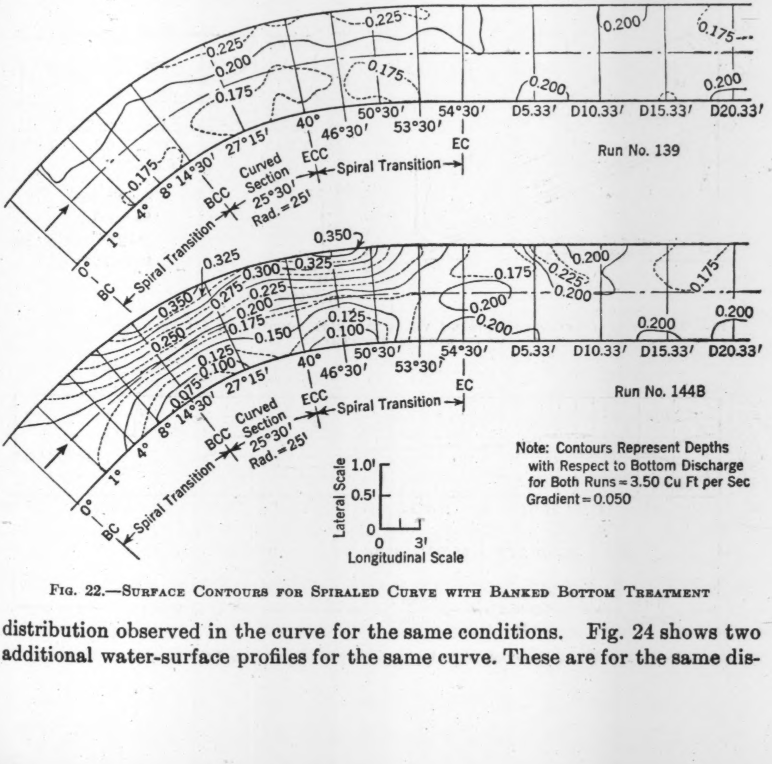
FIG. 21.—SURFACE PROFILES FOR CURVE WITH BANKED BOTTOM TREATMENT

Banked Curves; Rectangular Channel.—Fig. 21 shows a run made with a banked channel having spiral transitions calculated to match the bottom transition. The banking was accomplished entirely by depressing the inner wall, keeping the outer wall at a constant slope throughout the curve. In accordance with the analysis, there is practically no change in the depth on the outer wall in and below the curve. (In the several illustrations, the points of curvature and compound curvature, beginning and end, are designated BC, EC, BCC, and ECC, respectively.) Figs. 22 and 23 show the surface contours and the velocity distribution observed in the curve for the same conditions. Fig. 24 shows two additional water-surface profiles for the same curve. These are for the same dis-

THE CURVE; TESTS WITHOUT SILLS; RECTANGULAR CHANNEL

BELOW THE CURVE[a,b]								DISTANCE, IN FEET, BETWEEN SUCCESSIVE MAXIMA ON THE SAME SIDE			Run No.
FIRST MAXIMUM				SECOND MAXIMUM							
INSIDE WALL		OUTSIDE WALL		INSIDE WALL		OUTSIDE WALL					
$\frac{h}{h'}$ (13)	$\frac{L}{L'}$ (14)	$\frac{h}{h'}$ (15)	$\frac{L}{L'}$ (16)	$\frac{h}{h'}$ (17)	$\frac{L}{L'}$ (18)	$\frac{h}{h'}$ (19)	$\frac{L}{L'}$ (20)	L_{do-1} (21)	L_{di-1} (22)	L_{do-2} (23)	(1)
0.631	0.868	0.762	0.973	0.628	1.197	....	....	1.841	2.170	....	117C
0.688	0.884	0.867	0.884	0.700	1.178	0.617	1.178	1.768	2.062	2.356	120A
0.693	0.958	0.856	0.885	0.757	1.180	0.647	1.180	1.843	2.065	2.360	119D
0.620	0.910	0.815	0.883	0.700	1.262	0.721	1.080	1.793	2.145	2.342	32A

lengths between successive maxima on opposite sides. [c] One half foot below the end of the curve.


0.225
0.225
0.200
0.175
0.175
0.175
0.200
0.175
0.200
0.200
0°
1°
4°
8°
14°30′
27°15′
40°
46°30′
50°30′
53°30′
54°30′
D5.33′
D10.33′
D15.33′
D20.33′
BC
BCC
ECC
EC
Spiral Transition
Curved Section 25°30′ Rad.=25′
Spiral Transition
Run No. 139
0.350
0.325
0.350
0.300
0.325
0.275
0.225
0.250
0.200
0.175
0.125
0.150
0.125
0.100
0.100
0.075
0.175
0.200
0.225
0.200
0.200
0.200
0.175
0.200
0.200
Run No. 144B
Lateral Scale
1.0′
0.5′
0
0
3′
Longitudinal Scale

Note: Contours Represent Depths with Respect to Bottom Discharge for Both Runs = 3.50 Cu Ft per Sec Gradient = 0.050

FIG. 22.—SURFACE CONTOURS FOR SPIRALED CURVE WITH BANKED BOTTOM TREATMENT

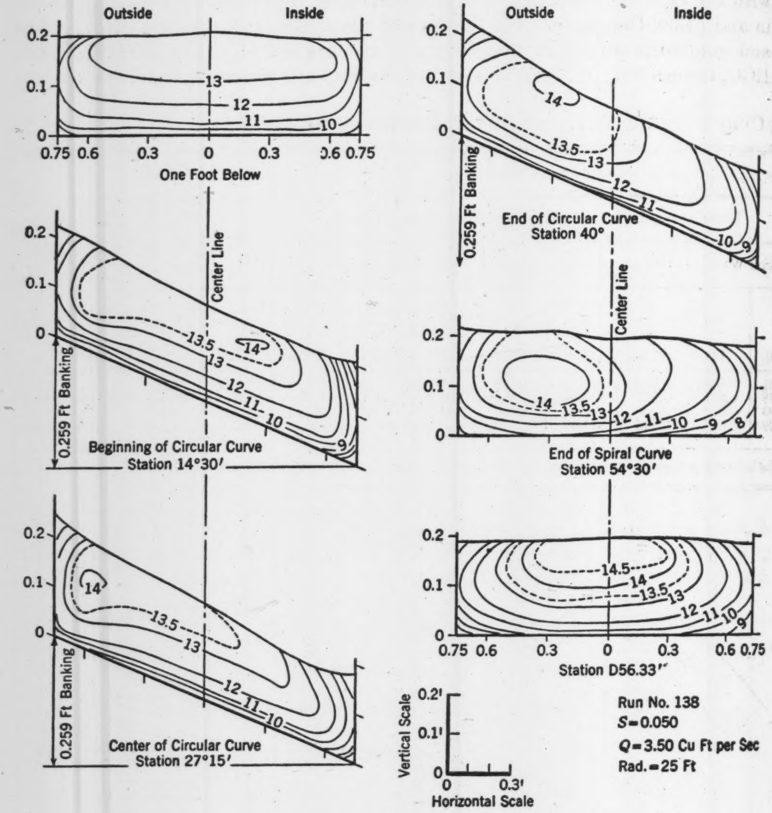


Fig. 23.—Typical Sections Showing Velocity Distribution for a Curve with Banked Bottom Treatment

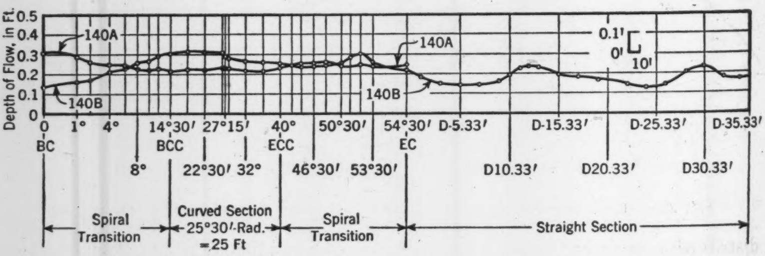


Fig. 24.—Outer Wall Profiles for Banked Curve; High and Low Approach Velocities

charge as that represented in Fig. 21. However, the velocity of approach for run 140A is lower than the equilibrium value for which the angle of banking was computed. The velocity of approach for run 140B was higher than the equilibrium value. The behavior is exactly as would be expected. In the curve the superelevation along the outer wall drops for the low approach velocity because the centrifugal force is not sufficient to overcome the cross slope. Therefore there is a flow toward the inside wall of the channel. The opposite condition is true for the high velocity of approach. Both these runs show small but appreciable disturbance patterns in the downstream channel.

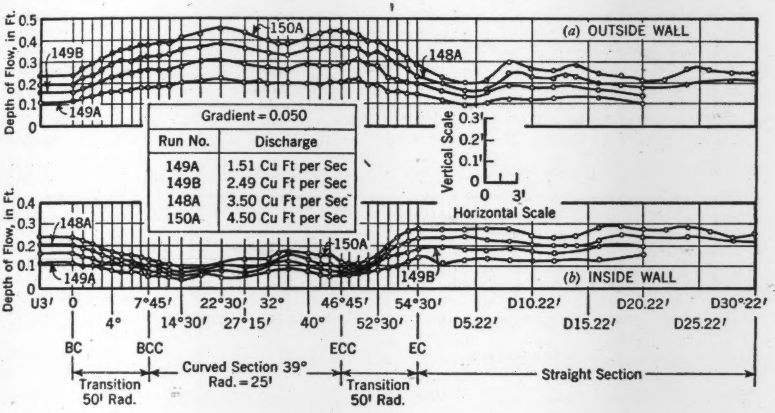


Fig. 25.—Surface Profiles for Compound Curve Treatment

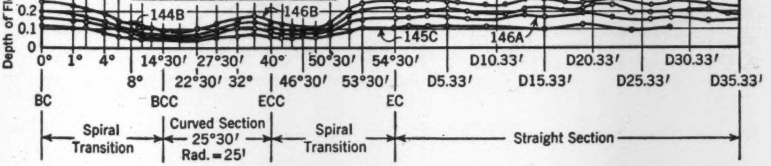


Fig. 26.—Surface Profiles for Spiral Transition Treatment

Compound Curves with Circular Transition Sections; Rectangular Channel.—Fig. 25 shows the surface profile along the outer wall for compound curves designed to have circular transition sections of double the radius of the main curve and with lengths equal to one half of the wave length of the disturbance pattern. The superelevation increases in the transition section, remains nearly constant throughout the main curve, and falls to the initial value in the down-

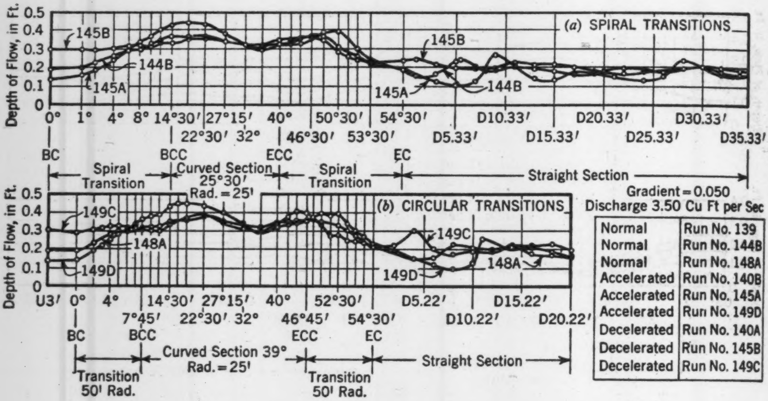


FIG. 27.—OUTER WALL PROFILES FOR SPIRAL AND CIRCULAR TRANSITIONS; HIGH, NORMAL, AND LOW APPROACH VELOCITIES

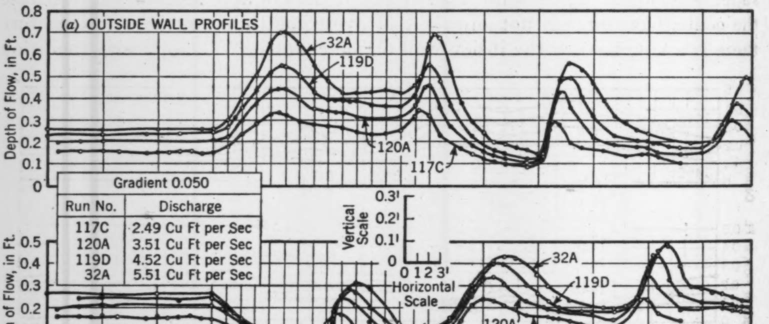
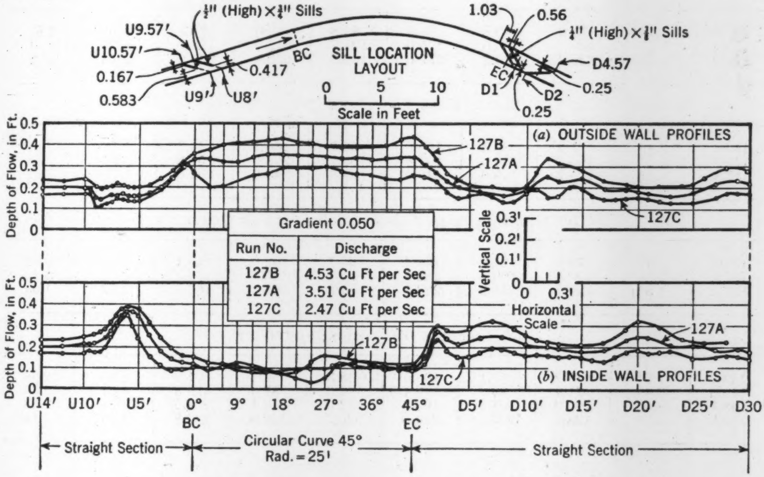


FIG. 28.—SURFACE PROFILES FOR SIMPLE CIRCULAR CURVE WITHOUT TREATMENT

stream transition section. A small disturbance pattern is visible which shows that complete interference was not obtained. The measurements indicate that, if the velocity of approach had been slightly lower, the curve would have behaved exactly in accordance with the predictions. Profiles for different flows are shown in Fig. 25. Although the curve was designed for the discharge of run 148A, the performance was equally satisfactory for the other three flows.

Spiral Transitions; Rectangular Channel.—Fig. 26 shows the surface profiles for similar flows in a curve of 25-ft radius having spiral instead of circular transitions. Its behavior is nearly identical with that of the curve having circular transitions. Both curves were designed for a total turn of 54° 30″. The center-line length of the curve with circular transitions was 30.53 ft, whereas with spiral transitions it was 36.43 ft. Thus, the latter curve is approximately 16% longer, although no better. Fig. 27 shows similar profiles for both types of compound curves for velocities of approach differing from the equilibrium values. Again the performances of the two curves are similar. A comparison of these results with Fig. 24 shows that the sensitivity to deviations from the design velocity is about as low as that of the banked bottom design.



Gradient 0.050

Run No.	Discharge
127B	4.53 Cu Ft per Sec
127A	3.51 Cu Ft per Sec
127C	2.47 Cu Ft per Sec

FIG. 29.—SURFACE PROFILES FOR SILL TREATMENT

Submerged Sills; Rectangular Channel.—The effect of submerged sills can be seen by a comparison of Figs. 28 and 29 which show the wall profiles for the same flow conditions in the same curve with and without sills above and below the curved section. Fig. 30 shows the surface contours for one of the three discharges shown in Fig. 29. It will be seen from Figs. 28 and 29 that the sill treatment produces very acceptable flow conditions.

Experimental Comparison of Effectiveness of Various Designs for Rectangular Channels.—Table 3 shows the comparative effectiveness of various corrective designs in similar rectangular channels having the same slope, the same main radius, and the same total angle of turn. The sets of runs are for nearly identical rates of flow. The curves with sills have slightly higher disturbances in and below the curved section than do other methods of treatment. The

TABLE 3.—RELATIVE EFFECTIVNESS OF CORRECTIVE DESIGNS FOR RECTANGULAR CHANNELS[a]

Run No.	Flow, Q, in cubic feet per second	RELATIVE MEASUREMENTS; PERCENTAGES OF THE DESIGN VALUE OF Q OR OF THE DESIGN VALUE OF h_e (0.198 FT)						DISTANCE FROM END OF CURVE (EC) TO DOWNSTREAM MAXIMUM[b]	
		Flow, Q	Depth, h_e	MAXIMUM DISTURBANCE					
				CURVE SECTION		DOWNSTREAM SECTION			
				Outer wall	Inner wall	Outer wall	Inner wall	Along outer wall	Along inner wall
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
(*a*) NORMAL CURVE									
117*B*	2.49	71.2	80.8	175.6	85.8	149.0	123.2	1.69	2.87
120A	3.51	100.3	100.5	225.0	121.7	217.0	178.7	1.77	2.94
119D	4.52	129.2	119.2	277.6	142.3	250.0	223.0	1.91	3.02
(*b*) SINGLE SILLS ABOVE THE CURVE ONLY[c]									
117B	2.49	71.2	80.3	165.0	68.7	166.1	124.7	1.77	2.87
119C	3.51	100.3	100.0	197.8	69.7	208.6	157.0	1.77	0.88
117A	4.52	129.2	119.6	245.9	92.9	243.0	202.6	1.77	3.02
(*c*) MULTIPLE SILLS ABOVE AND BELOW THE CURVE									
127C	2.47	70.6	84.8	150.4	65.1	102.0	119.1	2.06	0.32
127A	3.51	100.3	100.5	180.1	66.7	125.7	138.3	1.77	0.32
127B	4.53	129.6	118.2	223.0	81.3	169.1	163.0	1.77	1.03
(*d*) CIRCULAR TRANSITION; r = 50 FT									
149A	1.51	43.2	57.1	106.6	35.3	58.1	61.2	1.80	2.98
149B	2.49	71.2	80.3	149.4	52.0	87.4	79.8	1.21	2.39
148A	3.50	100.0	98.5	190.8	102.0	116.1	108.6	1.21	2.68
150A	4.50	128.7	120.2	226.6	126.7	143.3	130.8	1.21	2.68
(*e*) SPIRAL TRANSITION									
145C	1.51	43.2	60.1	103.0	54.0	60.6	61.6	1.37	2.85
146A	2.48	70.8	80.8	151.0	80.8	86.8	90.8	0.78	2.85
144B	3.50	100.0	97.5	187.2	103.5	122.7	111.6	0.79	2.26
146B	4.50	128.7	122.2	220.0	124.7	146.4	128.8	0.86	2.26
(*f*) SPIRAL TRANSITION WITH BANKING									
141	1.50	42.8	58.1	69.2	83.3[d]	61.6	61.6	3.00	3.00
142	2.48	70.8	80.8	92.9	95.0	89.4	92.9	1.67	3.00
139	3.50	100.0	97.5	120.3	95.5	109.6	110.6	1.96	3.00
143	4.51	129.0	122.7	160.5	116.1	129.8	136.9	1.67	3.00

[a] The main radius of all curves used in this series was 25 ft. [b] Location of a point of downstream maximum expressed as half wave lengths from the end of curve for a flow of 3.5 cu ft per sec (6.794 ft). [c] The three runs in Table 3(*b*) are for qualitative comparison only, because sill conditions are not identical. [d] The reference bottom for the banked channel is depressed along the inner wall.

compound curves and the spiral transitions with and without banking all show quite comparable performance as regards disturbance in the downstream channel. Within the curve itself the least disturbance is shown by the banked curve, as would be expected.

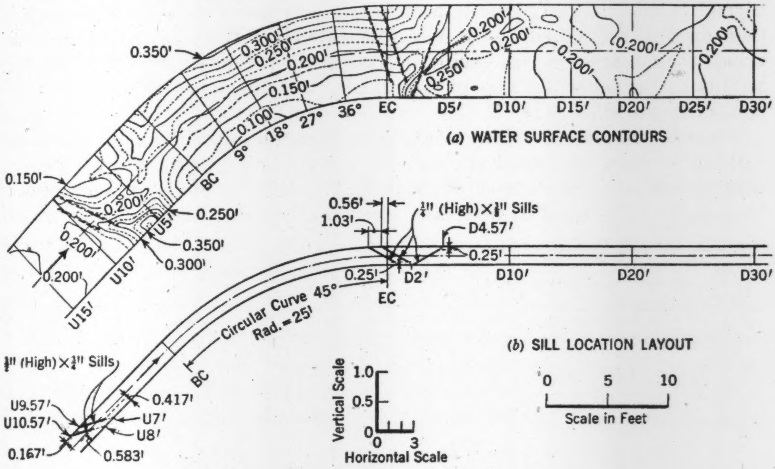

0.350'
0.300'
0.250'
0.200'
0.150'
0.100'
9°
18°
27°
36°
EC
D5'
D10'
D15'
D20'
D25'
D30'
0.200'
0.200'
0.250'
0.200'
0.200'
0.200'
(a) WATER SURFACE CONTOURS
0.150'
0.200'
BC
0.250'
U5'
0.350'
0.300'
0.200'
0.200'
U10'
U15'
0.56'
1.03'
¼" (High)×½" Sills
D4.57'
0.25'
0.25'
D2'
EC
D10'
D20'
D30'
Circular Curve 45°
Rad.=25'
(b) SILL LOCATION LAYOUT
¼" (High)×½" Sills
BC
U9.57'
U10.57'
0.417'
U7'
U8'
0.167'
0.583'
Vertical Scale
1.0
0.5
0
0
3
Horizontal Scale
0
5
10
Scale in Feet

FIG. 30.—SURFACE CONTOURS FOR SILL TREATMENT

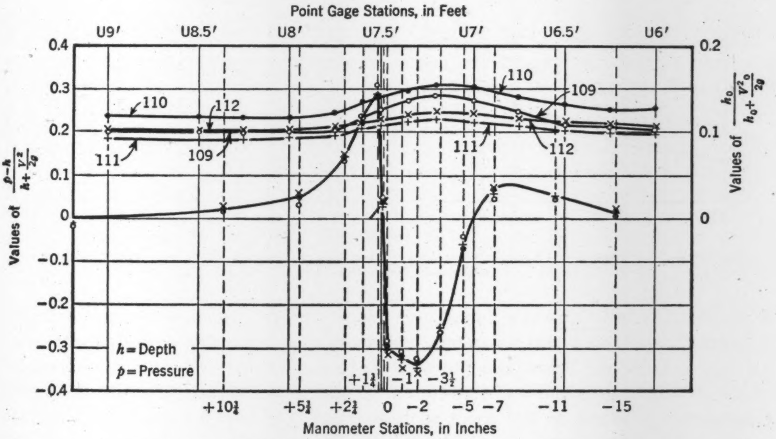


FIG. 31.—BOTTOM PRESSURE DISTRIBUTION ABOVE AND BELOW SILL

In using sills one factor must be remembered. A low pressure region forms on the downstream side of each sill which is very effective in producing the desired deflection of the flow. In channels having extremely high velocities, especially if accompanied by low depths, it is possible for the pressure in this region to drop enough to form an air passage to the surface, with the result that the flow springs clear of the bottom. This action reduces greatly the effectiveness of the sill and causes unsatisfactory operation. Fig. 31 shows the pressure along the bottom above and below the sill installation. This sill was normal to the channel; hence, the pressure differences are somewhat higher than for the inclined sill. The measurements show that, for extremely high velocities and large depths, cavitation may occur below the sill. For all such conditions the other methods of treatment, such as compound curves, are to be recommended.

There are a few additional points to be borne in mind in making the comparison between analytical results and experimental measurements. First, uniform velocity was assumed throughout the cross section. The deviations from uniformity found in real channels will affect the surface configuration. For normal conditions the departures from uniform velocity are not enough to cause significant discrepancies. However, serious departures from uniformity may occur below the junction of two channels or below the entrance of an auxiliary into the main channel. Another departure from the assumptions may be found in the cross section itself. The analysis assumes a true rectangular channel. In the field many rectangular channels have an appreciable invert in the bottom. The effect of this invert may be quite noticeable for low rates of flow but usually disappears for design flows.

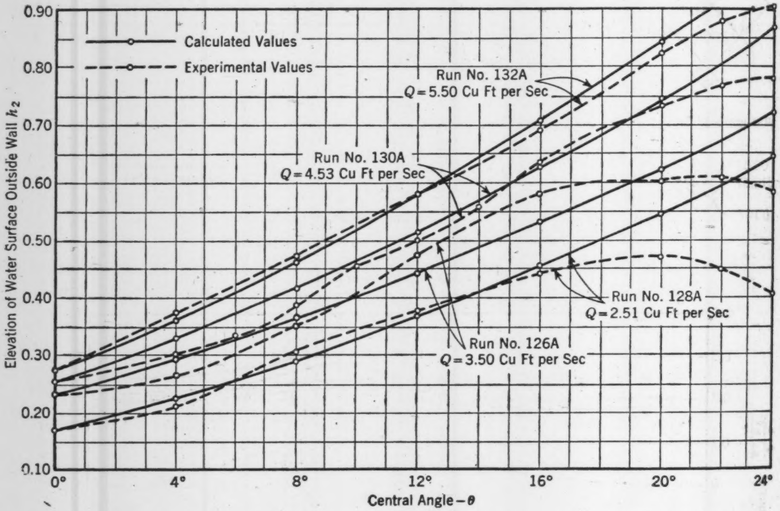
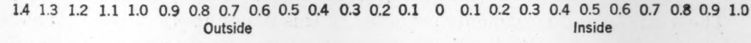


Fig. 32.—Comparison of Experimental and Calculated Values of Outer Wall Surface Profiles in a Trapezoidal Channel Curve of 25-Ft Radius

Trapezoidal Channels.—In discussing the methods of treatment, the characteristics of nonrectangular channels were referred to briefly, but no modifications of the analysis were proposed to permit their use. The trapezoidal channel is one of the most common forms of nonrectangular cross section. The approximate method of analyzing its behavior is rather obvious and can be outlined as follows: Since the depth is not constant, the wave velocity must vary throughout the cross section. However, approximate velocities can be computed by using the average depth. The effective channel width is influenced by the amount of superelevation. Again, a fair approximation of the wave pattern can be obtained by using the width of the water surface in the upstream channel. The rise along the outer wall can likewise be computed on the basis of the average channel depth, if the datum is determined by the average depth measured down from the water surface. Fig. 32 is a comparison between the calculated and measured elevations in a trapezoidal channel which had the same hydraulic characteristics as the rectangular channels just discussed. The slope, hydraulic radius, and mean radius of curvature were all identical. The central angle of the trapezoidal curve was 30° instead of 45° because of the space limitations of the laboratory. Fig. 33 shows the cross section within the curve for a discharge of about 80% of the design. This section illustrates one of the dangers of trapezoidal channels for high-velocity flow. As a first approximation, the water surface may be considered as an inclined plane. If the slope of the outer wall had been flatter, it is apparent that the superelevation would have been much greater. Presumably, if the water surface and the outer wall had been parallel, the superelevation would have been limited by only the velocity head available in the channel, assuming sufficient curve length to develop equilibrium.

A comparison of the disturbances in rectangular and trapezoidal channels indicates that the superelevations are always greater in the trapezoidal channel.



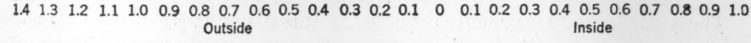


FIG. 33.—COMPOSITE CROSS SECTION SHOWING VARIATION WITH POSITION ALONG THE CURVE

This effect follows directly from the increased surface width. It was observed during the experiments that there seems to be a larger amount of energy stored in the "swing" of the trapezoidal disturbance pattern so that it appears to damp out less rapidly than in the rectangular section. Figs. 34(*a*) and 34(*b*) are views of the disturbance in and below the curve for two rates of flow. It will be noted that the maximum superelevation on the outer wall of the curve moves downstream toward the end of curvature as the rate of flow increases.

This downstream travel is accompanied by a disproportionate increase in the magnitude of the disturbance pattern in the downstream tangent. This result is to be expected from a consideration of the physical configuration involved since it was shown that the disturbance in the downstream tangent becomes greatest when the last maximum in the curve occurs on the outer wall at the end of curvature. The violent behavior of the flow in this particular example is due to the specific length of the curve and is not typical for all trapezoidal curves. For example, if the curve had been either appreciably shorter or longer with all other factors remaining the same, the disturbance pattern in the downstream tangent would have decreased as the flow increased instead of increasing as shown in Figs. 33 and 34.

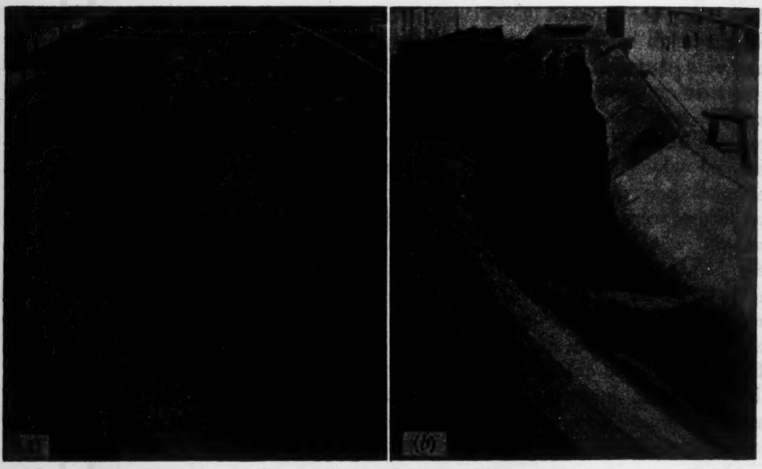


FIG. 34.—DISTURBANCE PATTERN IN AND BELOW THE CURVE IN A TRAPEZOIDAL CHANNEL: (a) HIGH FLOW AND (b) LOW FLOW

LIMITS OF APPLICATION

The analysis and treatments proposed in this paper apply only to channels with supercritical velocities. Furthermore, for channels operating with Froude numbers between 1 and 1.5, the results may be rather erratic because of the fundamental instability of the flow in this region. Minor disturbances cause disproportionate effects and may easily produce a jump which suddenly reduces the velocity to below that of the wave velocity. As the Froude number becomes greater than 1.5, the stability increases rapidly and, with it, the reliability of the calculations. One rather unexpected result of this situation is that these methods of calculation and treatment may be used with great confidence for very high velocities which would otherwise be most difficult to handle. The major limitations on the high-velocity end are air entrainment and cavitation. These limitations apply much less to compound curves and curves with banked bottoms than they do to simple curves and to the use of sills.

The scope of this paper has been limited to the conditions of flow in which the average velocity is constant—that is, when the friction loss is in equilibrium with the slope. The treatments are valid for the small accelerations or decelerations due to the constantly changing slopes usually encountered in the field. However, the high accelerations in spillways and similar structures are not covered. The physical principles underlying this treatment of open-channel flow for supercritical velocities are obviously applicable to nonequilibrium conditions as well. Examples of such treatments are found in other parts of this Symposium. There are undoubtedly many other cases not yet examined which can be solved by considering the unique characteristics of supercritical flow.

DESIGN OF CHANNEL CONTRACTIONS

BY ARTHUR T. IPPEN,[1] M. ASCE, AND JOHN H. DAWSON,[20]
JUN. ASCE

SYNOPSIS

Channel contractions for subcritical flow are designed for minimum energy losses by proper streamlining of the boundaries. Efficient and economic solutions are achieved with relatively little difficulty. If supercritical flow exists, the accent of design is shifted to the reduction or eventual elimination of the standing wave patterns which appear as a result of such flow, in accordance with the principles discussed in the first Symposium paper. So far these designs have envolved from experimental cut-and-try processes with models. This third Symposium paper endeavors to show, on the basis of experimental evidence, that the basic principles of supercritical flow can be applied in a satisfactory manner to the design of typical channel contractions and that solutions may be found with a minimum amount of surface disturbance patterns. Furthermore, the magnitude of the standing waves may be predicted adequately, as well as their location within the channel contraction. The method of eliminating waves in the downstream channel is discussed for a basic form of channel contraction.

GENERAL PROBLEMS OF DESIGN

The design of channel transitions has received the attention of hydraulic engineers in the past and has been formulated into a number of suggestions and procedures. The theoretical basis of these procedures is supplied by the principles of nonuniform flow applied, successively, to short channel sections. Variations of velocity and depth induced by the conveying boundaries are assumed to occur only along the channel axis. Basic surface curves identical for all longitudinal sections are derived in first approximation by assuming that the total head remains constant, and refined surface curves are obtained by considering the friction losses in the direction of flow. The velocities and depths at any station are assumed to be unaffected by curvature of the lateral boundaries and, hence, constant in transverse sections. The primary aim of economical design is a minimum of energy loss.

The design of channel transitions for supercritical velocities, on the other hand, must be attacked quite differently because of the occurrence of standing waves. The basic principles of the theory have been discussed in the first Symposium paper. The primary conclusions with respect to the problem in question are that, in economically feasible structures, standing waves cannot be avoided, and that their characteristics must therefore be explored carefully to insure successful design. Velocities in supercritical flow will vary in magnitude and direction in a systematic fashion in transverse sections, and surface elevations will not be constant. The effect of a transition is not confined to the immediate vicinity of the structure as in subcritical flow, but may affect the flow conditions downstream from the transition for very long distances. Design for a minimum of standing waves, therefore, is the particular goal for flow at supercritical velocities so that economical structures may result. The various phases of the flow through contractions were attacked by a number of investigators[21,22,23,24,25] at Lehigh University and at M.I.T. during the years from 1938 to 1947. Although considerable experimental work remains to be done, it is felt that a summary of the essential findings is desirable and may prove useful in similar work directly concerned with structures of this type.

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[20] Associate Prof., Civ. Eng. Dept., Oklahoma Inst. of Technology, Oklahoma Agri. and Mech. College, Stillwater, Okla.

CONTRACTIONS COMPOSED OF CIRCULAR ARCS

The first systematic attack on the problem was made in the Hydraulic Laboratory of Lehigh University with a channel contraction composed of two equal circular arcs along each wall as shown in the diagram of Fig. 35. This case was chosen as one that might be typical of a contraction designed from conventional knowledge of open-channel flow. The photographs of Fig. 36, taken in the downstream direction, show clearly the problem to be faced with supercritical flow in such contractions. The surface of the stream is traversed

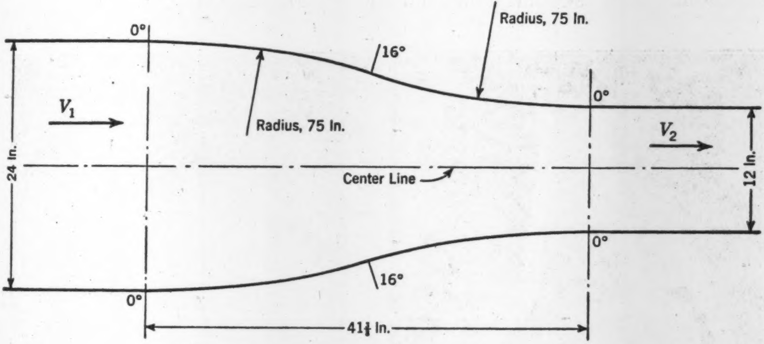

FIG. 35.—CONTRACTION COMPOSED OF CIRCULAR ARCS

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[21] "The Effects of Lateral Contractions on Supercritical Flow in Open Channels," by J. H. Dawson, thesis presented to Lehigh Univ. at Bethlehem, Pa., in 1943, in partial fulfilment of the requirements for the degree of Master of Science.

[22] "Design of a Sharp Angle Contraction in Supercritical Flow," by D. P. Rodriguez, thesis presented to Lehigh Univ. at Bethlehem, Pa., in 1943, in partial fulfilment of the requirements for the degree of Master of Science.

[23] "Experimental Relation Between Sudden Wall Angle Changes and Standing Waves in Supercritical Flow," by D. Coles and T. Shintaku, thesis presented to Lehigh Univ. at Bethlehem, Pa., in 1943, in partial fulfilment of the requirements for the degree of Bachelor of Science.

[24] "Theoretical Investigation of Standing Waves in Hydraulic Structures," by A. A. Stone, thesis presented to the Massachusetts Inst. of Technology at Cambridge, Mass., in 1946, in partial fulfilment of the requirements for the degree of Master of Science.

[25] "Standing Waves in Supercritical Flow of Water," by M. P. Barschdorf and H. G. Woodbury, thesis presented to the Massachusetts Inst. of Technology at Cambridge, Mass., in 1947, in partial fulfilment of the requirements for the degree of Master of Science.

by large standing waves which, in height, exceed considerably the initial depth of flow. The converging arcs in the upstream part and the reversed arcs in the downstream part of the contraction were chosen to be equal and with a 16° central angle each. The contraction was from a 2-ft width to a 1-ft width for the downstream channel. These two conditions, therefore, determined the radius of curvature of 75 in. and the length of the contraction. The channel contraction was set into a steel flume, which was 2 ft wide at its upper end and 1 ft wide in the longer reach of 30 ft below the contraction. For any flow the flume was adjusted in slope to give uniform flow conditions for the initial Froude number. Proper velocities and depths could be established for any Froude number between $\mathbf{F} = 2$ and $\mathbf{F} = 12$ by discharging the water into the flume through a rectangular nozzle, whose opening could be adjusted to give a certain desired depth. All water quantities were determined by a calibrated venturi meter in the supply line from a constant-head tank. The entire surface within the contraction, and for a sufficient distance downstream from the contraction, was mapped for the runs presented in Fig. 37. Normally, however, only the side-wall elevation and the center-line elevation were determined for all stations along the contraction and downstream tangent.

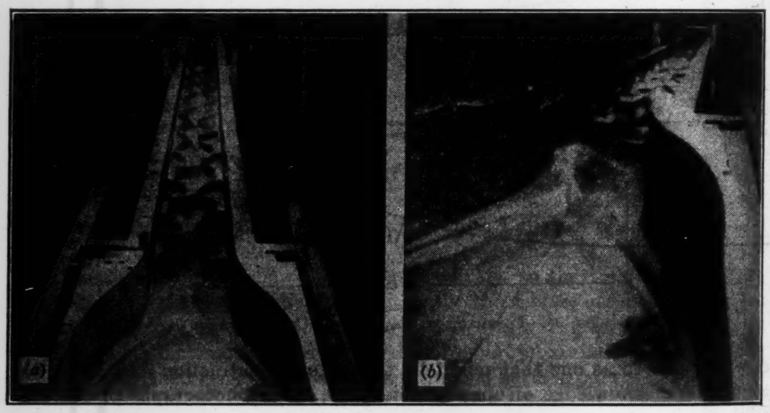
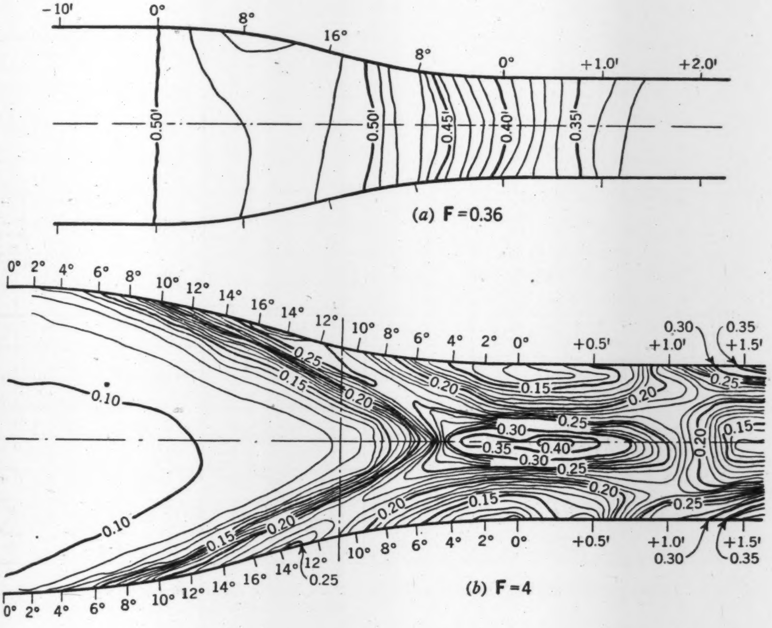


FIG. 36.—STANDING WAVE PATTERNS CAUSED BY CIRCULAR-ARC CONTRACTIONS

Surface Contours in Circular-Arc Contractions.—The differences in the surface elevations between subcritical flow and supercritical flow are strikingly apparent from the configurations of the constant-depth lines in Fig. 37. Fig. 37(*a*) shows distinctly that the usual assumption of essentially constant depths in transverse flow sections is basically fulfilled for this run at a Froude number of $\mathbf{F}_1 = 0.358$. Fig. 37(*b*), for a Froude number of $\mathbf{F}_1 = 4$, shows clearly the essentially different surface contours for cases of supercritical flow. The characteristic feature of flow remaining undisturbed within the center of the contraction is especially apparent. This can easily be explained on the basis of disturbance lines discussed in the first Symposium paper. The increases in depth caused by converging side walls can only be communicated along disturbance lines at a wave angle β_1 to the oncoming flow. Positive lines tending to converge originate along the first or concave part of the wall, and diverging or negative lines emanate from the convex part of the contraction. In Fig. 38 a theoretical solution showing the disturbance lines on the basis of the method of characteristics is given for comparison with the actually measured surface lines. The converging lines result in steep wave fronts which clearly have their counterpart in Fig. 37(*b*). The diverging or negative lines extend from Station 16° on downstream, resulting in a depth along the wall at the end of the contraction which reverts to the normal depth.



(*a*) **F** = 0.36

(*b*) **F** = 4

FIG. 37.—CONTOURS OF WATER SURFACE IN A CONTRACTION COMPOSED OF CIRCULAR ARCS

Wall Profiles.—The characteristic variations in depth along the side walls are revealed distinctly by the profiles in Fig. 39 for a range of Froude numbers and for the same contraction. The theoretical profiles are plotted for direct comparison with the measured surface lines. According to the principles discussed in connection with Fig. 3 in the first Symposium paper, all theoretical lines must show a sharp break at the beginning of the curved wall, since the curvature at that section increases from zero to $1/r$ and since the velocity at the wall is assumed equal to the normal velocity V_1. At Station 16° there must be a peak, because at this point the curvature is reversed suddenly in the example investigated. This point would correspond to the peak obtained along the outside of a channel curve where the first negative disturbance is reflected. The only difference is that, in the latter case, the negative impulse is created locally instead of being produced at the opposite wall and being transmitted along a negative disturbance line. Since the total angular turn of the reverse curve must also be 16°, the depth along the wall must revert to the initial depth h_1, and the Froude number **F** of the flow at the end of the contraction should again be the same as at the entrance, at least along the walls. The complication arising from this requirement is not difficult to see: The continuity condition obviously does not permit the same velocity and depth to exist in the narrower section as in the wider entrance. It follows that depth and velocity must vary across the end section in such a way as to satisfy the

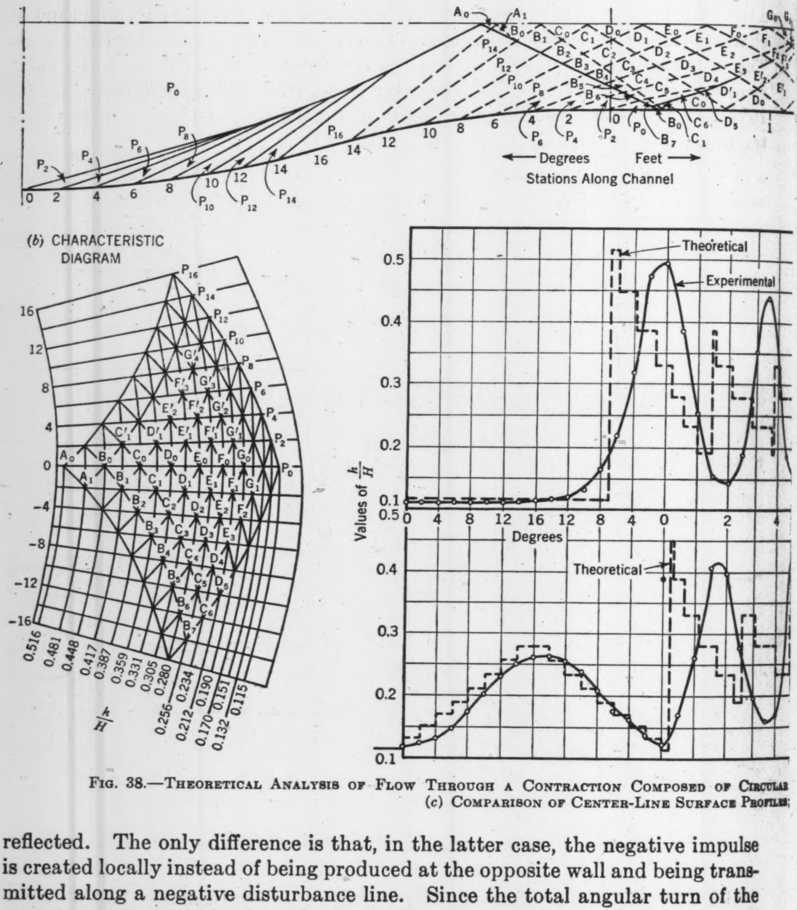
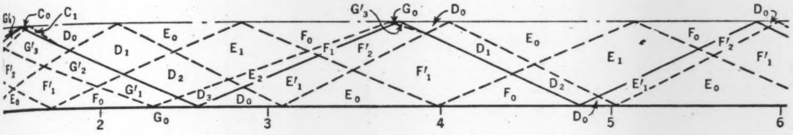


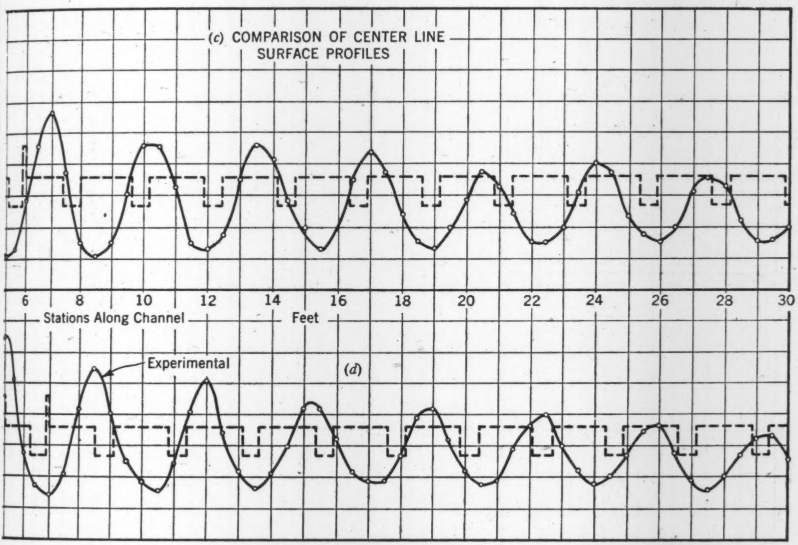
FIG. 38.—THEORETICAL ANALYSIS OF FLOW THROUGH A CONTRACTION COMPOSED OF CIRCULAR
(c) COMPARISON OF CENTER-LINE SURFACE PROFILES;

continuity condition. A fundamental weakness of this type of contraction is thus exposed.

The lowest initial Froude number that would not result in a hydraulic jump being formed in the contraction was $\mathbf{F} = 3$. The discrepancy between theoretical and measured profiles is explained by the fact that the Froude number of the wave is too near the critical, and therefore small changes in velocity result in large changes in depth. The counterpart of this case is found for a right hydraulic jump when the undulating stage is reached and unstable conditions are approached. In actual practice such conditions are to be avoided. The theory, it is to be remembered, too, neglects vertical accelerations, which assume relative importance in the present case. The agreement between theory and measurement is very close for $\mathbf{F} = 4$ to $\mathbf{F} = 8$ as far as height of the total disturbance is concerned. Only when $\mathbf{F} = 12$ is there a serious discrepancy apparent. This difference is explained by the fact that the maximum



(a) PLAN OF CONTRACTION AND DISTURBANCE LINES



Arcs, $\mathbf{F} = 4$: (a) Plan of Contraction and Disturbance Lines; (b) Characteristic Diagram;
and (d) Comparison of Side-Wall Surface Profiles

depth along the wall is now seven times the initial depth, a case in which vertical accelerations must become excessive. Such a ratio of h/h_1 would certainly not be permissible in practice and would clearly call for a longer transition with smaller central angles for the arcs composing the side walls.

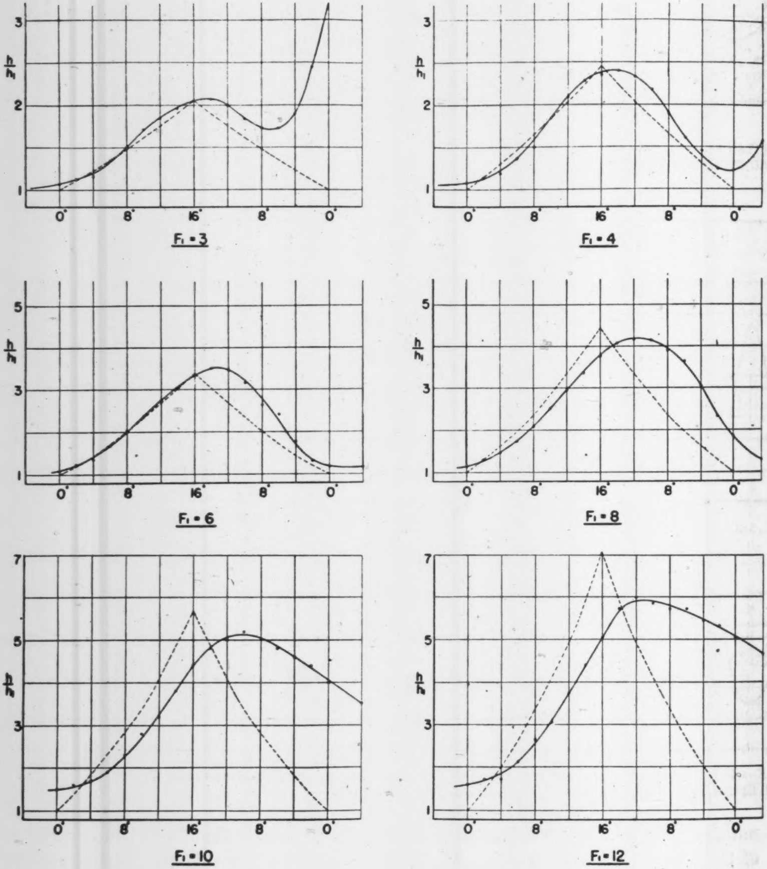


FIG. 39.—MEASURED AND THEORETICAL WALL PROFILES FOR CIRCULAR-WALL CONTRACTION

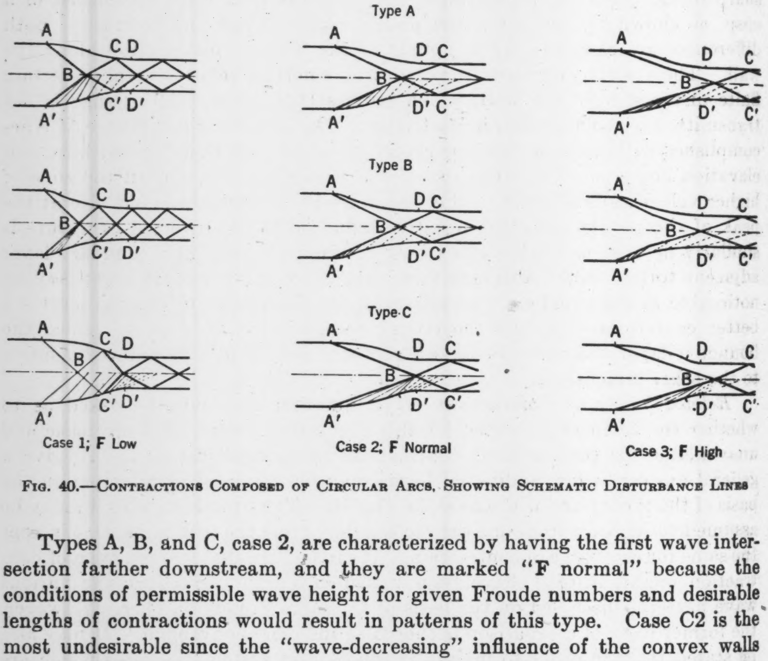
It is not the magnitude of **F** which causes the difficulty, but the fact that the central angle of the circular-wall sections is excessive for high values of **F**. A longer transition therefore is indicated, which might be designed for a curvature determined by a specified ratio of h/h_1. The permissible central angle can easily be read from the curve in Fig. 3 for such a ratio, once the starting point has been determined from the flow conditions at the entrance.

The theoretical profiles in the example chosen are symmetrical with the maximum point since the curvature of the two arcs composing the converging wall is the same. The maximum will not be changed, regardless of a change in curvature of both arcs or of one of the arcs, as long as the central angles remain the same. The wall profiles therefore may easily be made unsymmetrical if, for example, a larger radius is used for the downstream arc than for the upstream one. Nevertheless, a given depth change is always associated with a given change in direction, or in θ; and at the point of the same total angular change along the wall the same depth must occur for given initial conditions, always assuming, of course, that no disturbance line reaches the wall from the opposite side.

Boundary-Layer Effects.—In comparing the measured profiles in Fig. 39 with the theoretical profiles it is noted, first, that usually the measured profile shows a somewhat delayed and smooth rise as compared to the aforementioned sharp initial break at the entrance. Second, the peak does not consist of a cusp, as shown by the computed profile, but is broad and rounded. Both differences are explained by the effect of the velocity distribution along the wall. The velocity increases from zero to the normal velocity within a certain finite distance from the wall, which implies that the angular change is not transmitted to the main flow immediately. The initial angular change $\Delta\theta$ is accomplished within a zone of lower Froude numbers and, therefore, with smaller elevation changes. The steep rise of the profile is delayed until the zone of higher velocities is affected. This boundary-layer influence is felt also at the peak of the profile, since the negative influence of the reversed curvature is similarly delayed and takes effect only gradually through the boundary layer adjacent to the wall. Although this influence of the boundary layer is quite noticeable in the small-scale experiments, for conditions in large structures a better conformance to the theoretical profiles must be expected since the boundary-layer thickness for large structures is considerably reduced relative to the other dimensions.

Related Types of Contractions.—The question may now be asked as to whether the findings presented for this circular-arc contraction are basic and unavoidable for such channel contractions in supercritical flow. To give a general answer, a few additional forms may be discussed in principle on the basis of the theory and with the aid of Fig. 40. The Froude number $\mathbf{F}_1$ may be assumed to increase from one case to another while the wall alinement is kept the same for each type of contraction. It can be seen that a given type of contraction might, instead, have been varied in length to produce the various wave patterns discussed on the basis of the same Froude number. However, the former basis of comparison is chosen as the more convenient. It may also be stated that, in order to make the comparison systematic, cases which are obviously impractical are still included to illustrate certain points. The probable wave patterns have been sketched into three plan views (cases 1, 2, and 3 of the three types of contraction, A, B, and C). However, only the initial disturbance lines and the basic shock waves are indicated, so that the lines of reflected disturbances may not obscure the fundamental problem.

Types A, B, and C, case 1, have little practical significance. Since the wave angles are normally small, extremely long contractions would be required for the waves to intersect as far upstream as is shown. Cases A1 and B1 are better than case C1, since the positive waves reach the opposite wall within the convex range of wall curvature, whereas case C1 has some positive waves reflected within the concave range—thus causing still higher elevations along this wall. Case B1 will have the most pronounced shock wave because the converging positive disturbance lines are so crowded together; the rise in cases A1 and C1 will not be so steep, but of the same ultimate magnitude. Case A1 and, especially, case B1 are more advantageous than case C1 in that the positive disturbances are partly canceled, and somewhat lower diamond waves result in the downstream channel. However, these conditions may exist only for very long contractions or for large wave angles near critical depth, and, therefore, are not expected to occur in practice.



Case 1; F Low Case 2; F Normal Case 3; F High

FIG. 40.—CONTRACTIONS COMPOSED OF CIRCULAR ARCS, SHOWING SCHEMATIC DISTURBANCE LINES

Types A, B, and C, case 2, are characterized by having the first wave intersection farther downstream, and they are marked "F normal" because the conditions of permissible wave height for given Froude numbers and desirable lengths of contractions would result in patterns of this type. Case C2 is the most undesirable since the "wave-decreasing" influence of the convex walls is shifted downstream, and a maximum depression along the walls may lie in the same cross section as the intersection of the shock waves with its resulting maximum depth. The highest differentials of depths are obtained and, therefore, the maximum disturbances in the channel downstream. In solution A2 this tendency is somewhat relieved, and in B2 the converging lines are so far upstream that some of the negative disturbances may reach the shock fronts in advance of their point of intersection—thereby reducing their heights.

For types A, B, and C, case 3, all the shock-wave intersections are moved too far downstream, which results in large disturbances, if such conditions can be maintained at all. There are indications that for cases A3 and B3 the contraction is too short because the Froude numbers appear to be too high. Although peaks may be relatively somewhat lower for cases A3 and B3 than before (due to the effect of the negative impulses from the convex wall sections), the occurrence of maxima and minima near the same cross section indicates large downstream disturbances.

Summary.—Of all the contractions, B2 will probably perform best. The maximum depth in the center, however, will differ little from solution A2, and smaller downstream disturbances may be expected due to wave interference in this case. In designing such circular contractions, therefore, the characteristics of this type may be followed. In the following section, however, a more systematic approach is indicated toward basically better designs.

STRAIGHT-WALL CONTRACTIONS

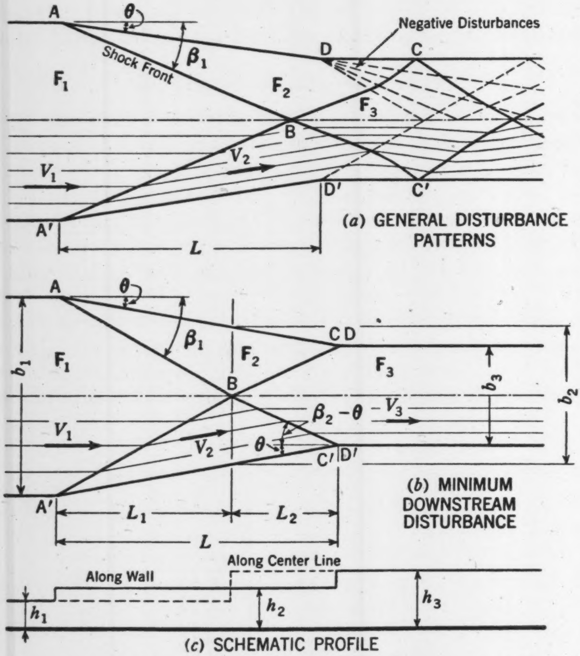
General Principles of Design.—The aim of rational design for supercritical flow must be oriented, first, toward lower standing waves and, second, toward reduction or possible removal of standing wave patterns in the channel section downstream from the contraction. As stated, the total deflection angle θ determines the wave height, regardless of the degree of curvature of the side walls. It is logical, therefore, to decrease this angle θ to a minimum. For the contractions discussed in the preceding section, decreasing values of θ are obtained by inserting longer straight-wall sections between the circular sections. Thus, for contractions of the same length, increasing curvatures result for the circular parts, but the deflection angles are decreased. The minimum angle θ is had by connecting the upstream and downstream tangent points by straight lines, possibly rounding the corners slightly for the sake of appearance.

TABLE 4.—REDUCTION IN MAXIMUM WAVE HEIGHT (SEE FIG. 8)

Froude No., F_1	$\theta = 8°$; STRAIGHT CONTRACTION			CENTRAL ANGLE = 16°; CIRCULAR-ARC CONTRACTION		
	h_2/h_1	h_3/h_2	h_3/h_1	h_2/h_1	h_3/h_2	h_3/h_1
2	1.35	1.35	1.82	(1.82)	....	Hydraulic jump
3	1.50	1.40	2.10	2.10	1.80	3.78
4	1.63	1.50	2.44	2.46	1.83	4.50
6	2.00	1.67	3.34	3.41	2.05	7.0
8	2.31	1.83	4.23	4.45	2.25	10.0
10	2.70	2.00	5.40	5.61	2.40	13.5

Table 4 serves to illustrate the reduction in maximum wave height for a contraction of given longitudinal extent as a function of the Froude number. The values of h_2/h_1 and h_3/h_1 are listed for a contraction composed of circular arcs of 16°, central angle, and for a straight-wall contraction of 8°, deflection angle. The ratio h_3/h_1 represents, in each case, the theoretical depth ratio for the zone immediately below the first shock-wave intersection and is assumed to be indicative of the highest possible disturbance. The ratios h_3/h_1 naturally become excessive for the higher Froude numbers from a practical point of view, but they are considered here only for the sake of comparison.

The design procedure for a straight contraction is simple: For an initial Froude number $\mathbf{F}_1$ and for a permissible depth increase h_2/h_1 the wall angle θ and $\mathbf{F}_2$ can be found from the diagram in Fig. 8 of the first Symposium paper. The first shock wave can be regarded as reflected at the center (see Fig. 41) by its image wave from the opposite wall. Since the depth and velocity beyond this point of reflection or intersection determine the height and location of the disturbances downstream; and, since the angle θ must be the same for the initial and reflected waves, the diagram is used again to determine h_3/h_2 and $\mathbf{F}_3$ with the values of θ and $\mathbf{F}_2$ found previously. The ratio h_3/h_1 can then be readily computed and should normally not be higher than three. Fig. 41(*c*) will show immediately whether the hydraulic quantities are safe in so far as they remain sufficiently supercritical. In practice, wall angles so small (and, therefore, contractions so long) that a second reflection along the wall takes place within the contraction are improbable. Theoretically, however, the story is not changed in this case; another set of values of h_4/h_3 and $\mathbf{F}_4$ are to be added to the computation with the same angle θ.



A
θ
Negative Disturbances
D
C
Shock Front
β₁
F₁
F₂
F₃
B
V₂
V₁
D′
C′
A′
L
(a) GENERAL DISTURBANCE PATTERNS
A
θ
β₁
C D
F₁
F₂
F₃
b₁
b₃
b₂
B
β₂ − θ
V₃
V₁
V₂
θ
C′
D′
A′
L₁
L₂
L
(b) MINIMUM DOWNSTREAM DISTURBANCE
Along Center Line
Along Wall
h₁
h₂
h₃
(c) SCHEMATIC PROFILE

FIG. 41.—DESIGNS OF STRAIGHT-WALL CONTRACTIONS

Wave Patterns in Channel Below Contraction.—In the preceding section it was shown that, on the basis of the theory, better hydraulic performance is always obtained for contractions with straight walls rather than with circular-wall alinements due to the smaller total angle of deflection. The additional aim of design is to reduce the standing waves in the channel downstream from the contraction. If the length of channel contraction in relation to initial flow conditions is ignored, the shock waves set up by the converging walls are subject to the negative disturbances originating at points D and D′ (see Fig. 41(*a*)) in a rather haphazard fashion. In the discussion of the various possible wave patterns sketched in Fig. 40, the location of the shock-wave intersection B with respect to points D and D′ is shown to have a definite influence on the wave height in the downstream channel. Maximum disturbances result with points B, D, and D′, lying in the same cross section. Minimum disturbances are to be expected if the reflected shock waves are made to meet the walls at points D and D′, since then the deflecting effects of shock waves and walls tend to cancel each other. This cannot easily be accomplished with **S**-shaped wall contractions unless straight sections are introduced between the concave and convex wall sections. However, it is possible to design straight-wall contractions as indicated in Fig. 41(*b*) to meet this requirement for low disturbances. The disturbance might even be reduced to zero in such a case, provided the basic assumptions of the theory could be satisfied.

To what extent these assumptions are violated remains for further discussion. At this point the geometric conditions in relation to hydraulic flow conditions may be established for such designs in accordance with the notations in Fig. 41(*b*). The deflection of the flow at points A and A′ at the channel entrance causes symmetrical shock waves which cross the channel at an angle β_1 and meet at point B at the center. They are reflected to the walls and on their way proceed through a new flow field characterized by the parameter $\mathbf{F}_2$. If they meet the wall (as assumed) at CD and C′D′, theoretically, there remain no disturbances because the flow has been directed parallel to the walls in the downstream channel by synchronizing the deflection effect of the waves BC and B′C′ with the deflection effect of the wall. Since the deflection angle θ is the same for both, no disturbance is obtained. It is difficult to correlate, mathematically, the geometric and hydraulic conditions for this special case although the setup appears simple enough. The following relations hold:

$$L = L_1 + L_2 = \frac{b_1}{2 \tan \beta_1} + \frac{b_3}{2 \tan (\beta_2 - \theta)} \qquad (40)$$

also,

$$L = \frac{b_1 - b_3}{2 \tan \theta} \qquad (41)$$

Continuity conditions yield $b_1 h_1 V_1 = b_3 h_3 V_3 = Q$; or

$$\frac{b_1}{b_3} = \frac{h_3 V_3}{h_1 V_1} = \left(\frac{h_3}{h_1}\right)^{3/2} \left(\frac{\mathbf{F}_3}{\mathbf{F}_1}\right) \qquad (42)$$

In agreement with Eqs. 19 to 22 of the first Symposium paper and Eqs. 40 to 42, all conditions are available to determine the shape of a contraction. It probably is impossible to eliminate from Eq. 40 the variables β_1 and β_2 and to replace them in terms of h_1/h_3, θ, and $\mathbf{F}_3/\mathbf{F}_1$. Therefore, the procedure must be one of trial and error as follows: For a given value of $\mathbf{F}_1$ a certain reduction in width is required from b_1 to b_3. The assumption of a desirable depth change h_3/h_1 is made, and $\mathbf{F}_3/\mathbf{F}_1$ is thus given by Eq. 42. Provided $\mathbf{F}_3$ is not too close to the critical value, Fig. 8 may be employed to determine the deflection angle θ by trial and error. Assuming a value of θ, the corresponding values of h_2/h_1 and $\mathbf{F}_2$ are read from that graph. A second determination using the same θ and replacing $\mathbf{F}_1$ by the $\mathbf{F}_2$ just obtained will yield a value of h_3/h_2. Multiplying h_2/h_1 by h_3/h_2, the trial value of h_3/h_1 is obtained. If this is not the desired value, the process has to be repeated with an adjusted value of θ until agreement is reached between the assumed h_3/h_1 and the one obtained by trial. The procedure using this diagram is extremely fast. The length L follows from Eqs. 40 and 41.

In general, long contractions will result for low values of h_3/h_1 and high values of $\mathbf{F}_1$. Ratios of $h_3/h_1 = 2$ and $h_3/h_1 = 3$ seem advisable in order to reduce the length of contractions, provided $\mathbf{F}_3$ stays well above the critical value.

If the contraction cannot be designed to the correct angle θ, disturbances must be expected to continue into the downstream channel, as indicated in Fig. 41(*a*). The maximum height of these disturbances may then be determined as in the foregoing procedure. The process is repeated twice with a given angle θ, since the maximum depth h_3/h_1 will occur at least within a narrow zone below the wave intersection B. It will then recur intermittently along side walls and center line, provided points C and C′ lie below points D and D′. The respective positions of points D and C tend to modify the height of the downstream waves, since points D and D′ are the origins of negative disturbance lines. It is felt, however, that the limitations imposed on the theory by the basic assumptions would render a possible theoretical analysis of surfaces downstream from the contraction of little practical significance.

The following summary may express the results of this theoretical investigation briefly:

1. Straight-wall contractions are always better than curved-wall contractions from the standpoint of maximum wave height and compared on the basis of equal length.
2. For given reductions in channel width, correct deflection angles θ may be found, which result in minimum disturbances in the downstream channel.

Vertical Accelerations and Boundary-Layer Influence.—The actual physical features of shock-wave fronts differ from the ones assumed in the basic theory to a considerable extent, as is indicated schematically in Fig. 42. First, vertical fronts, represented by single lines in plan and having negligible longitudinal dimensions, are arrived at in the elementary theory; and, second, the velocity is assumed constant over the depth.

The flow is assumed to expand under the front instantaneously, in accordance with the deflection imposed on the stream by the side wall. It has already been shown that the existence of a boundary layer along the channel sides will modify the rate of rise or depression of the water surface there. In the theory this assumption is one of convenience, and the actual appearance of the shock wave is considerably modified under the influence of the neglected factors. The flow will expand rather gradually in the case of low waves, whereas, for higher fronts, the surface slope approaches the vertical with eventual overturning or breaking of the wave. Surface rollers, familiar from the hydraulic jump, are formed. Steep fronts naturally cause high vertical accelerations and, therefore, considerable deviations from the assumed hydrostatic pressure distribution under the front. Thus, the wave fronts are characterized locally by depths which are higher than those calculated. However, within a short distance the depths revert closely to the theoretical depths as soon as the streamlines again are parallel to the bottom. The fact that wave fronts extend longitudinally and are distorted vertically in the case of high vertical accelerations must be considered in analyzing experimental results. The longtitudinal dimensions of the front should be small as compared to the dimensions of the structure to be tested. The phenomena of wave reflection and intersection are equally influenced and modified by the physical discrepancy between theoretical assumption and actuality. However, basic wave patterns must remain valid regardless of these local effects.

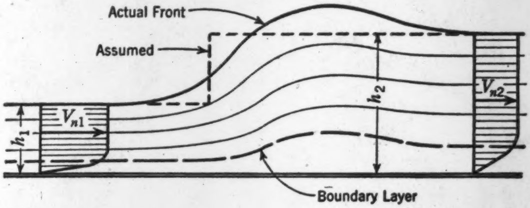


Fig. 42.—Schematic Comparison of Actual and Assumed Wave Front

In Fig. 42 the boundary layer along the bottom is indicated to an exaggerated extent by a dashed line. This boundary layer will increase disproportionately in thickness under the wave front since the low momentum of the fluid within this layer is not sufficient to overcome the adverse pressure gradient under the front. This fact will change the so-called displacement thickness of this layer and will be equivalent to a rise in elevation of the bottom, thus decreasing the specific head of the flow. A slight change in the height of the front and in its location may result. Although this discussion is helpful in the analysis of experimental trends, refinements of the theory do not seem warranted for the purpose of this paper.

Experimental Results with Straight-Wall Contractions

The two phases of the problem with straight contractions were the subject of several experimental studies at Lehigh University and at M. I. T. As shown previously the basic theory and the experimental results obtained for curved side walls agree satisfactorily (see Fig. 39), giving disturbances corresponding to the maximum deflection angles. The statement made previously that straight-wall contractions are more satisfactory due to smaller deflection angles has already been supported by theoretical evidence in Table 4; it remains, therefore, to present experimental confirmation of the basic equations. Two sets of results are available for this purpose.

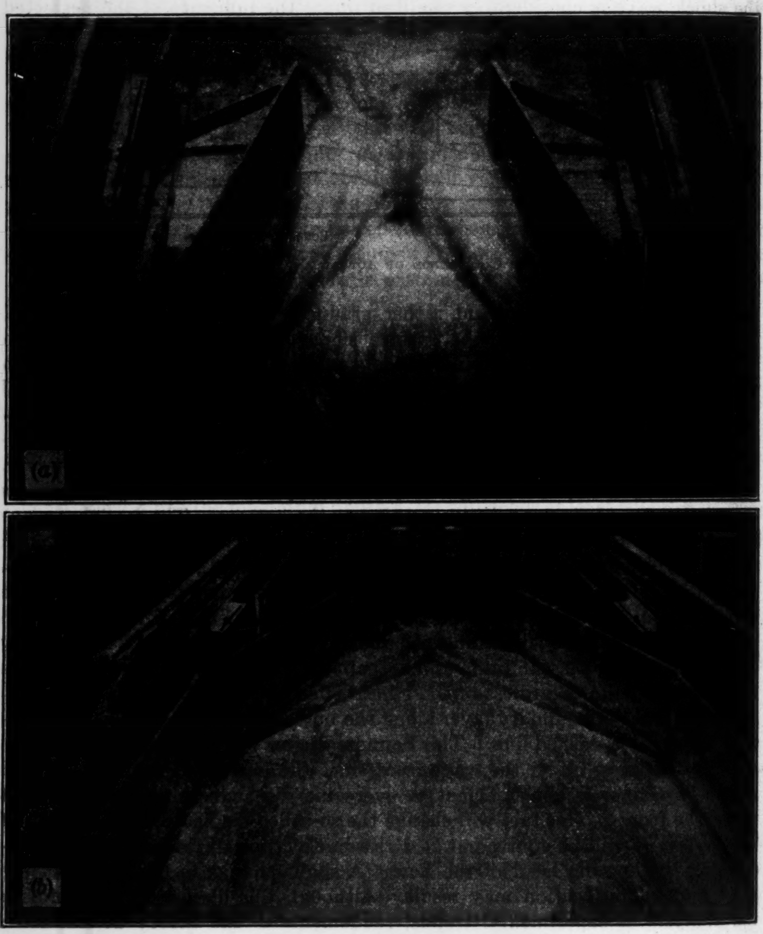


Fig. 43.—Views, Facing Downstream, of Contraction Composed of Straight Walls; $F = 3.86$ and $\theta = 15°$

M. P. Barschdorf and H. G. Woodbury,[25] in a flume constructed of aluminum plates for accurate work at the Hydraulic Laboratory at M. I. T., arranged for movable side walls, so that wall angles of θ from 3° to 30° could be established. For Froude numbers between 3 and 4 they conducted a series of experiments measuring the pertinent features of the standing waves produced for wall deflection angles from 3° to 30°. A sufficient number of transverse profiles were determined by point gage to obtain the wave heights and wave angles independently of the local effects discussed in the preceding section. The photographs of Fig. 43 are two views facing downstream into the contraction with a wall angle of $\theta = 15°$ and for a Froude number of $\mathbf{F} = 3.86$. The surface profiles for all runs were plotted in distorted scale in isometric form as shown for the preceding example in Fig. 44. The initial rise of the surface near the front above the equilibrium depth is apparent, whereas beyond this local disturbance the depth along the wall returns to a more or less constant value. Side-wall profiles are also plotted and show clearly the constancy of depth h_2 along the wall.

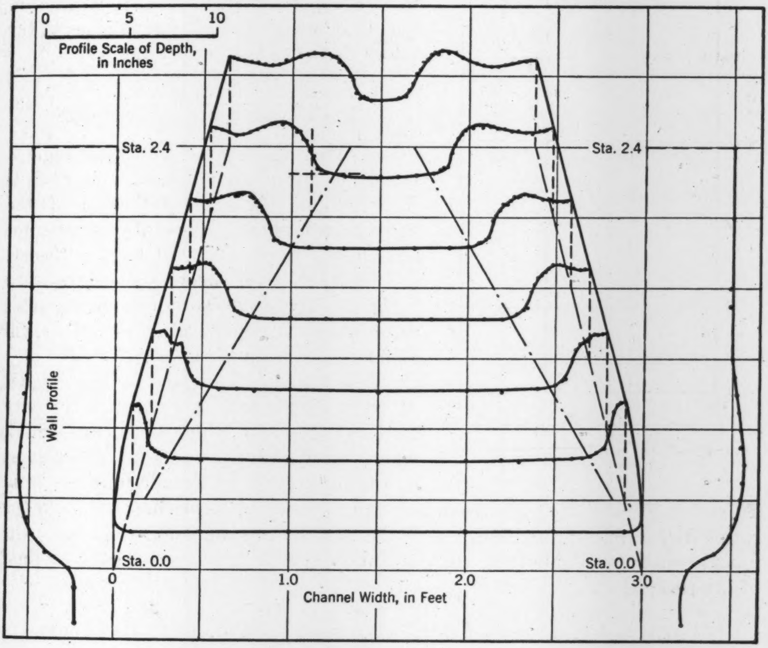


FIG. 44.—SURFACE PROFILES OF STANDING WAVES IN STRAIGHT-WALL CONTRACTION; $\mathbf{F} = 3.86$ AND $\theta = 15°$

In Table 5 a comparison is presented of measured and theoretical values of wave angle β_1 and of the depth ratio h_2/h_1. All theoretical values of β_1 and h_2/h_1 were determined from the known quantities of θ and $\mathbf{F}_1$ by graphs equivalent to Fig. 8. Wave angles β_1 were determined by averaging the wave fronts vertically at each station and by drawing straight lines through the wave front locations thus obtained. From Table 5 it is seen that excellent agreement was obtained for this Froude number $\mathbf{F}_1$ as far as ultimate values of h_2/h_1 are concerned. The values of β_1 varied from −12% to +5% about the theoretical value of β_1 for the different deflection angles from 3° to 24°. The general trend obtained consistently for only slightly different Froude numbers is best illustrated in Fig. 45 in which the ratio K of sines of the actual β_1 to the theoretical β_1—

$$K = \frac{\text{actual} \sin \beta_1}{\text{theoretical} \sin \beta_1} \ldots\ldots (43)$$

—is plotted against the deflection angle θ. Since the minimum possible value of β_1 is $\sin^{-1} 1/\mathbf{F}_1 = 15°0'$, the influence of the wave height on the value of β_1 is smaller for low values than indicated by the theory, and larger than the theoretical values when h_2/h_1 greatly exceeds values of 2. This result shows that vertical accelerations will affect the location of the wave front.

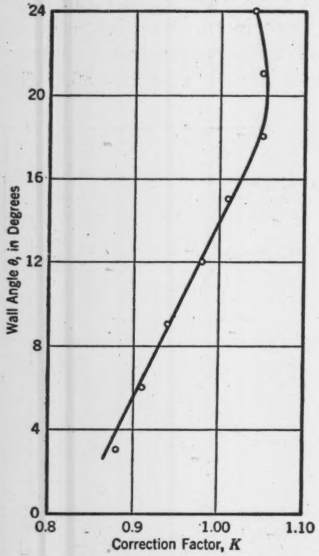


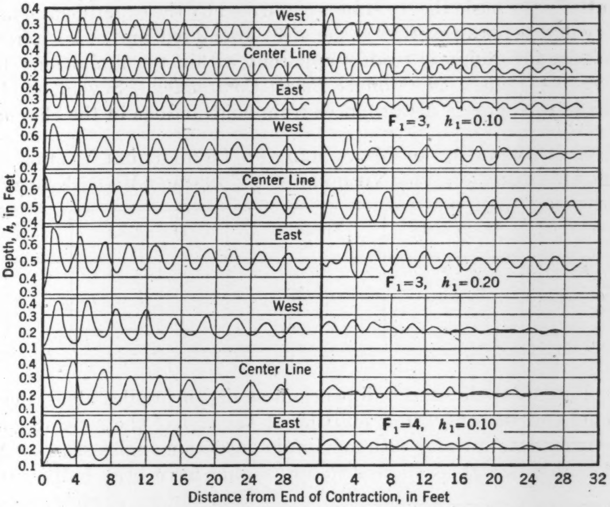
FIG. 45.—PLOT OF CORRECTION FACTOR K AGAINST WALL ANGLE θ

This tendency appeared also in a series of tests conducted at Lehigh University by D. Coles and T. Shintaku,[23] Jun. ASCE, when the wall angle was kept constant at $\theta = 6°$ for a range of Froude numbers from $\mathbf{F}_1 = 3$ to $\mathbf{F}_1 = 10$. Their results are summarized in Table 6. Although the ratios h_2/h_1 are obtained fairly easily from the measurements, the angle β_1 is not so readily available from the data and depends somewhat on the method of interpretation.

TABLE 5.—COMPARISON OF MEASURED AND THEORETICAL VALUES OF β_1 AND h_2/h_1 FOR A FROUDE NUMBER OF $\mathbf{F}_1 = 3.86$

Wall angle	Theoretical[a] β_1	Measured β_1	Theoretical h_2/h_1	Measured h_2/h_1	PERCENTAGE VARIATION FROM THEORY	
					β_1	h_2/h_1
3°	17° 35′	15° 30′	1.21	1.24	−11.8	2.3
6°	20° 15′	18° 15′	1.47	1.47	− 9.9	0
9°	23° 10′	22° 45′	1.70	1.73	− 1.2	1.7
12°	26° 30′	26° 7′	2.00	2.00	− 1.5	0
15°	29° 30′	29° 55′	2.22	2.22	+ 1.4	0
18°	32° 25′	34° 15′	2.49	2.47	+ 5.6	−0.70
21°	35° 45′	38° 00′	2.72	2.71	+ 6.3	−0.3
24°	39° 18′	41° 20′	2.99	3.01	+ 5.2	+0.6

[a] Minimum value is 15° 00′.



(*a*) CONTRACTION COMPOSED OF CIRCULAR ARCS OF 16° CENTRAL ANGLE

(*b*) STRAIGHT WALL CONTRACTION $\theta = 6°$

FIG. 46.—WALL AND CENTER-LINE PROFILES; $\mathbf{F} = 4$ AND $h_1 = 0.10$

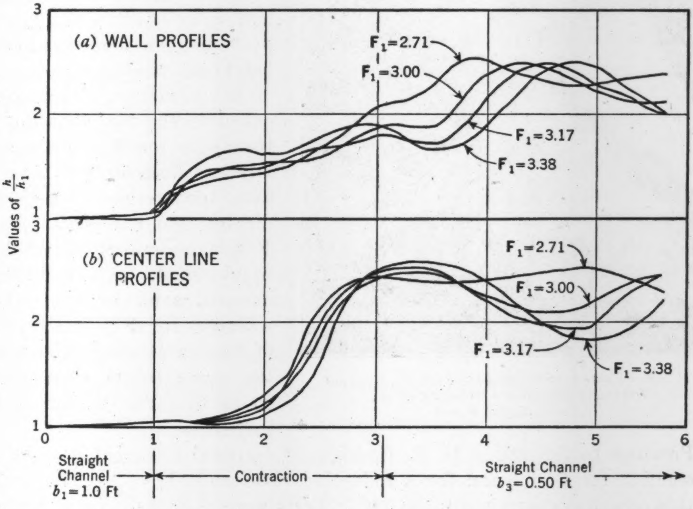


FIG. 47.—WALL AND CENTER-LINE PROFILES FOR STRAIGHT-WALL CONTRACTION; $\theta = 6.9°$

Essentially the basic theory stands confirmed by the experimental findings in laboratory flumes for a practical range of $\mathbf{F}_1$ from 3 to 8 and of h_2/h_1 from 1 to 3. Its usefulness for design is readily apparent within the limitations discussed.

Disturbance Pattern in Channel Below Contraction.—Several studies were finally made to compare disturbances from contractions in the channel downstream. Fig. 46(*a*) shows the dimensionless depth profiles along the walls of the channel and its center line, starting at the end of the contraction for the circular-arc contraction and with maximum deflection angles of 16°. Although the comparable straight-wall contraction should have been built with $\theta = 8°$ to obtain a contraction of the same length, a longer contraction with $\theta = 6°$ was built to extend its usefulness into the range of lower Froude numbers. These profiles are shown in Fig. 46(*b*) as before for the same hydraulic conditions. The improvements with respect to wave height are readily seen.

TABLE 6.—COMPARISON OF MEASURED AND THEORETICAL VALUES OF β_1 AND h_2/h_1 FOR VARYING FROUDE NUMBERS AND CONSTANT WALL ANGLE $\theta = 6°$

Froude No.	Theoretical β_1	Measured β_1	Theoretical h_2/h_1	Measured h_2/h_1
3.00	25° 10′	22° 00′	1.38	1.31
4.00	19° 30′	18° 40′	1.47	1.52
6.00	15° 00′	15° 10′	1.75	1.73
8.00	12° 20′	12° 45′	2.00	2.17
10.00	11° 10′	10° 30′	2.30	2.88

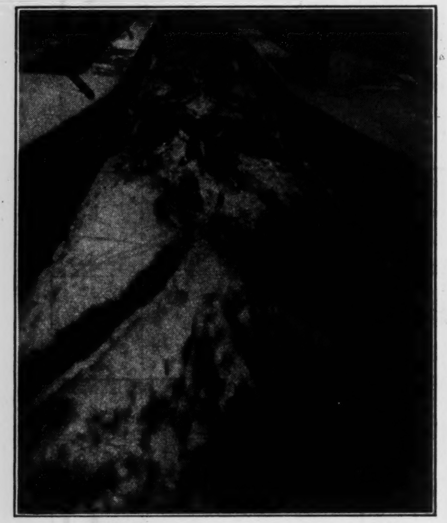
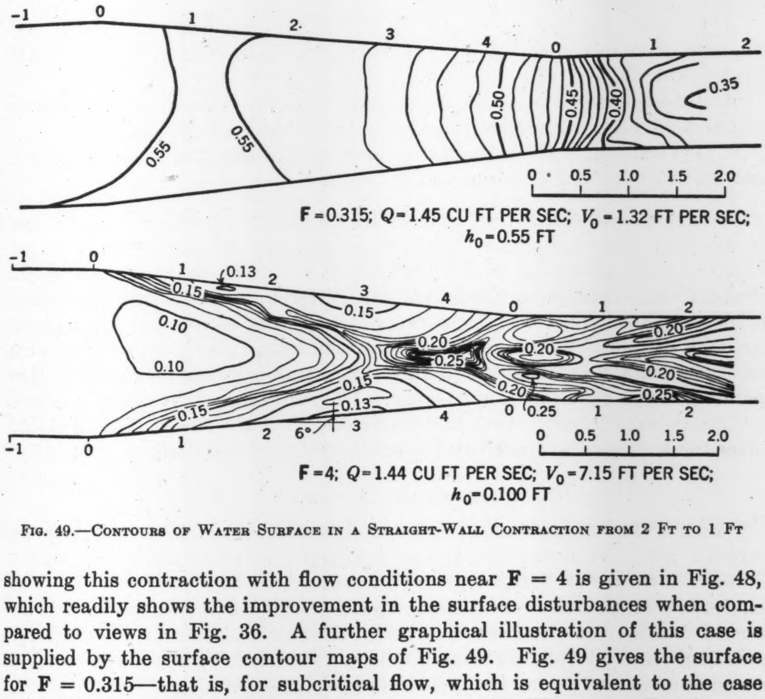


FIG. 48.—FACING DOWNSTREAM TOWARD SURFACE DISTURBANCES FOR A STRAIGHT-WALL CONTRACTION; $\mathbf{F} = 4$

In addition, attention is called to the fact that the disturbances for $\mathbf{F} = 4$ are actually less than for $\mathbf{F} = 3$ for the same contraction. This fact indicates that the contraction at $\mathbf{F} = 4$ more nearly satisfied the requirements of correct design as indicated in Fig. 41(*b*), whereas, for $\mathbf{F} = 3$, the points of intersection and reflection of the wave fronts were located so as to cause maximum disturbances.

Previous to this study D. P. Rodriguez[22] designed a correct straight-wall contraction for $\mathbf{F} = 4$ and for a reduction ratio of $b_1/b_3 = 2$. The angle θ for this case was computed to be 6.9°. Tests were then made on a one-sided contraction in the Lehigh flume. The best profiles were found not for $\mathbf{F} = 4$

but for **F**-values of about 3, as indicated in Fig. 47 showing the various wall and center-line depth profiles. This result, although not anticipated at that time, is due to the fact (established later) that the actual wave angle β_1 for this wall angle and Froude number is smaller than the computed value (see Table 5). Therefore, a longer contraction is indicated than is obtained from the theory. It is also for this reason that the wall angle of 6° was chosen in the study reported by Messrs. Coles and Shintaku.[23] With this smaller wall angle the improved profiles for **F** = 4 in Fig. 46(*b*) were obtained. A photograph showing this contraction with flow conditions near **F** = 4 is given in Fig. 48, which readily shows the improvement in the surface disturbances when compared to views in Fig. 36. A further graphical illustration of this case is supplied by the surface contour maps of Fig. 49. Fig. 49 gives the surface for **F** = 0.315—that is, for subcritical flow, which is equivalent to the case plotted in Fig. 37(*a*) and does not show any essential difference. For the case of supercritical flow with **F** = 4, Fig. 49 may be compared to Fig. 37(*b*), and the reduction in the depths produced along side walls and along center lines is seen to be considerable.



F = 0.315; Q = 1.45 CU FT PER SEC; V_0 = 1.32 FT PER SEC; h_0 = 0.55 FT

F = 4; Q = 1.44 CU FT PER SEC; V_0 = 7.15 FT PER SEC; h_0 = 0.100 FT

FIG. 49.—CONTOURS OF WATER SURFACE IN A STRAIGHT-WALL CONTRACTION FROM 2 FT TO 1 FT

CONCLUSION

The experimental evidence presented in the preceding section is not sufficient to establish corrections of general validity for the application of the elementary theory. However, by and large, it is adequate to show that the basic phenomena are predictable from the theory. Only minor corrections need be applied because of factors not included in the theory (such as the influence of gravity and friction forces) and because of the vertical accelerations under the shock-wave fronts. Further experimentation is to be directed toward a systematic coverage of these influences. The limited aims of the present paper may be summarized under the following points:

1. The behavior of a typical channel contraction composed of circular arcs was explored experimentally for a wide range of Froude numbers;
2. The flow conditions predictable from the theory were compared to experimental measurements;
3. On the basis of these results the physical requirements of channel contractions for supercritical flow were defined as distinct from those for subcritical flow;
4. A basic design form was developed for supercritical flow conditions to reduce wave heights and disturbances in the downstream channel; and
5. Typical contractions designed with straight converging walls were tested and found to conform essentially to the requirements.

It is thus established that channel contractions for supercritical flow can be designed specifically to avoid excessive standing wave heights by proper choice of both deflection angles and length for given reductions in width. Straight-wall contractions are normally superior to curved-wall contractions, as long as the channel bottom is level crosswise and of constant slope in the direction of flow. Warping of the bottom and large changes in longitudinal slope have not been included in this paper, although improvements in the standing wave patterns can be attained experimentally by such methods. Additional experimental work and an extension of the theory are needed to cover these phases of supercritical flow in channel contractions.

ACKNOWLEDGMENT

Much of the experimental work reported in this paper was done by graduate students in the hydraulics laboratories of Lehigh University and of the Massachusetts Institute of Technology. Their contributions are acknowledged by extensive reference to their theses submitted to the two institutions for advanced degrees. It remains to express grateful appreciation for the extensive assistance received from the respective laboratory staffs on the construction of the equipment and during the performance of the experimental work.

DESIGN OF CHANNEL EXPANSIONS

BY HUNTER ROUSE,[26] M. ASCE, B. V. BHOOTA,[27] ASSOC. M. ASCE, AND EN-YUN HSU[28]

SYNOPSIS

Following an introductory discussion of supercritical flow in divergent channels, the matter of channel design is discussed under three sequent headings: (1) Surface configuration at abrupt expansions; (2) efficient curvature of expanding boundaries; and (3) elimination of disturbances at the end of transitions. The extent of the agreement between elementary wave theory and experimental measurement is shown, and the results are presented in the form of generalized diagrams convenient for rapid exploration and preliminary design.

INTRODUCTION

In the design of hydraulic structures it is often necessary to provide for the lateral expansion of flow emerging at high velocity from a closed conduit, sluice gate, spillway, or steep chute. If such a transition section is made to diverge too rapidly, the major part of the flow will fail to follow the boundaries; if the divergence is too gradual, on the other hand, waste of structural material will result; and, finally, if local disturbances are produced by incorrect boundary form, either at these points or farther downstream the walls may fail to confine the flow. Any particular problem of this nature, to be sure, may be subjected to cut-and-try investigation through model tests, with results which are necessarily restricted to the specific model form. An exact and general analytic solution, unfortunately, is still a matter for the future, and may be approached only through application of sound physical principles as discussed in the first Symposium paper. However, a step in this direction has been made in the development of reasonably general relationships among the several major variables involved, by means of which an approximate solution may be obtained for a great range of boundary conditions.

As in other problems of steady flow in an open channel of nonuniform cross section, the variation in velocity and depth through a channel expansion will depend on the geometry of the channel boundaries, the rate of flow, and the fluid properties. Under boundary geometry must be considered the form of the channel walls, the slope and form of the floor, and the surface roughness of floor and walls. In a strict sense, under fluid properties one should consider the density, specific weight, viscosity, and surface tension; except in small models, however, or under conditions in which boundary shear is of particular moment, both surface tension and viscosity are of very minor importance, and the two remaining properties then reduce to their ratio $\gamma/\rho = g$, the gravitational acceleration.

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[26] Director, Iowa Inst. of Hydr. Research, State Univ. of Iowa, Iowa City, Iowa.

[27] Engr., Foreign Div., The Dorr Co., Bombay, India.

[28] Research Associate, Iowa Inst. of Hydr. Research, State Univ. of Iowa, Iowa City, Iowa.

If these different independent variables are combined by the Π-theorem[29] of dimensional analysis into a series of dimensionless ratios, as many length ratios will be obtained as are necessary to describe the relative geometrical proportions of the boundary, together with a flow parameter of the Froude type. The latter is generally written in the form:

$$\mathbf{F} = \frac{V}{\sqrt{g\,h}} \quad \text{......................} \quad (44)$$

in which V is the mean velocity and h is the mean depth of the approaching flow.

For given boundary conditions, the relative form of the free surface and the relative velocity distribution will depend solely upon the magnitude of the Froude number. As in all cases of open-channel flow, the critical magnitude $\mathbf{F} = 1$ marks the border between two wholly different types of surface configuration and velocity distribution. For Froude numbers less than unity, the depth then being greater than the critical, a gradual enlargement of the cross section will result in a gradual increase in mean surface elevation and a corresponding reduction in mean velocity. For Froude numbers greater than unity, the depth then being less than the critical, the same gradual enlargement of the cross section will result in a gradual reduction in mean surface elevation and a corresponding increase in mean velocity. However, only if the divergent boundaries are continuous planes (which is physically impossible if the transition is to begin and end with other than zero and infinite cross-sectional areas) will the depth of flow and the magnitude of the velocity be constant over any normal section. In other words, at the beginning and at the end of the transition the local curvature or angularity of the walls and floor will produce disturbances which make it impossible to handle such a problem satisfactorily on the elementary basis of mean velocity and mean depth. For Froude numbers less than unity (which are not the concern of the present paper), the boundary may be designed and the flow pattern may be evaluated in much the same manner as for the corresponding transition in a closed conduit. On the other hand, for Froude numbers greater than unity, the problem of design and evaluation becomes one of gravity-wave analysis, since each increment of the boundary deflection may be considered to generate an incremental surface wave which crosses the flow at an angle depending upon the Froude number and the boundary form; only through determination of the cumulative effect of all such waves may the depth and velocity at each and every point be predicted.

As has been described in the first Symposium paper, there is at hand a graphical method which permits the direct construction of streamlines, "isovels," and water-surface contours for any boundary form, provided that: (1) The channel walls are vertical and the floor is horizontal, (2) the energy loss due to boundary resistance is negligible, and (3) the pressure is hydrostatically distributed. These provisions may at first glance appear decidedly restrictive, but they become less so as they are considered individually. Vertical channel walls are common; indeed, sloping walls are generally to be avoided in nonuniform high-velocity flow because of their tendency to exaggerate surface disturbances. Channel floors are seldom horizontal, however, and the boundary resistance is never completely negligible; on the other hand, slight to moderate slopes of either the floor or the total-head line generally have an influence which is secondary to that of the wall expansion, and such effects are in fact compensative rather than additive. Moreover, only in the case of relatively abrupt curvature at the beginning or the end of the expansion is the existence of nonhydrostatic zones to be expected, and these may be effectively eliminated by proper easing of the transition curve. Only two factors, in actuality, tend to limit the graphical method in its use for the complete design of a well-proportioned expansion. First, its application depends upon prior knowledge or assumption of the boundary geometry, so that determination of the best form of transition involves the tedious process of trial and error. Second, if (as is usually the case) a hydraulic jump is to form at the end of the expansion, the method offers no clue as to the inherent stability (or, more likely, instability) of the phenomenon. As a matter of experience, the formation of a jump or standing wave by other than the boundary curvature (for instance, by backwater from a downstream control) may lead to an asymmetric pattern of flow within the transition which is still wholly unpredictable.

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[29] "Fluid Mechanics for Hydraulic Engineers," by Hunter Rouse, McGraw-Hill Book Co., Inc., New York, N. Y., 1938, pp. 13–18.

Since the purpose of this paper is the provision of general rather than specifically detailed information on the behavior of high-velocity flow in any channel expansion and on the preliminary design of particular expansion structures, primary attention (once the agreement between theory and experiment has been shown to be satisfactory) is focused upon the reduction of all experimental data to a few composite diagrams from which the basic details of design may be determined. Both the experiments and the generalization of the experimental results group themselves naturally into three subdivisions of the problem—first, the characteristics of a high-velocity jet expanding upon a level floor; second, the effects of boundary curvature in the zone of divergence; and, third, phenomena accompanying the return to uniform flow at the end of the transition.

All experiments described herein were conducted at the Iowa Institute of Hydraulic Research of the State University of Iowa under a project sponsored by The Engineering Foundation and the Committee of the Hydraulics Division, ASCE, on Hydraulic Research. The first part of the project, including the construction of equipment, was undertaken as a doctoral dissertation by Mr. Bhoota,[30] and the second part as a master's thesis by Mr. Hsu,[31] who then completed the investigation as a staff member of the Iowa Institute. Messrs. C. H.

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[30] "Characteristics of Supercritical Flow at an Abrupt Open-Channel Enlargement," by B. V. Bhoota, thesis presented to the State University of Iowa, at Iowa City, Iowa, in December, 1942, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

[31] "Characteristics of Supercritical Flow at a Gradual Open-Channel Enlargement," by En-Yun Hsu, thesis presented to the State University of Iowa, at Iowa City, Iowa, in February, 1946, in partial fulfilment of the requirements for the degree of Master of Science.

Hsia and M. M. Hassan, Assoc. M. ASCE, assisted in various phases of the analysis. The entire project was under the direction of Mr. Rouse.

Essentially the same equipment (see Fig. 50) was used for all experiments. Water was supplied from constant-level tanks, through 4-in. lines and 8-in. lines containing calibrated elbow meters, to a pressure tank 2.5 ft in diameter and 5 ft long. One end of the pressure tank was provided with three interchangeable nozzles yielding rectangular jets 0.4 ft by 0.4 ft, 0.3 ft by 0.6 ft, and 0.25 ft by 1.0 ft in cross section—that is, having width-depth ratios of 1, 2, and 4. Rates of discharge were such that flow at any Froude number from 1 to 8 could be established. Flush with the bottom of the nozzle outlet sections was a level table, normally horizontal but adjustable to a maximum slope of approximately 10° and provided with a suitably hooded waste trough. This table was originally 5 ft wide and 8 ft long, and was covered with oiled hardboard except for a plastic section with floor piezometers. It was later doubled in length and paved with finished concrete throughout. A gage carriage traveling on steel shafts above the table permitted three-directional movement of a point gage or pitot tube to any part of the test section. Through the earlier experiments one edge of the table was alined with the outer edge of each nozzle, a glass wall extending down the assumed line of symmetry of the transition for purposes of observation. That such elimination of one half of the flow pattern introduced negligible error was shown when the full transitions were later tested.

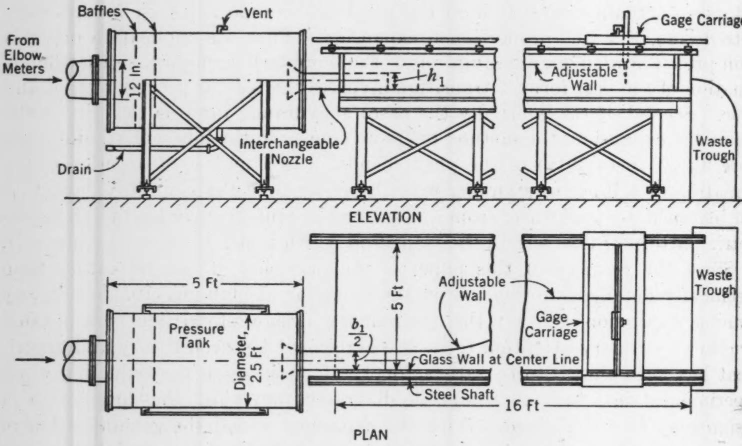
Baffles
Vent
From Elbow Meters
12 In.
h_1
Gage Carriage
Adjustable Wall
Interchangeable Nozzle
Drain
Waste Trough
ELEVATION
5 Ft
Pressure Tank
Diameter, 2.5 Ft
$\frac{b_1}{2}$
5 Ft
Adjustable Wall
Gage Carriage
Glass Wall at Center Line
Steel Shaft
Waste Trough
16 Ft
PLAN

FIG. 50.—SCHEMATIC REPRESENTATION OF EXPERIMENTAL APPARATUS

CHARACTERISTICS OF FLOW AT AN ABRUPT EXPANSION

The extreme case of a channel expansion is represented by the abrupt termination of the side walls, the channel floor continuing at the same slope.

If h_1 and V_1 represent the depth and mean velocity of the approaching flow, b_1 is the channel width, x and y are the longitudinal and lateral coordinates (measured from the outlet section and the center line, respectively) of a point of depth h, and if no other factors than the acceleration of gravity are assumed to influence the flow, these variables may be combined into the following dimensionless relationship:

$$\frac{h}{h_1} = f_1\left(\frac{x}{h_1}, \frac{y}{h_1}, \frac{b_1}{h_1}, \mathbf{F}_1\right) \quad \text{(45)}$$

Evidently, the relative depth at any point of the flow should depend upon the relative coordinate location, the relative width of the channel outlet, and the Froude number of the approaching flow. The form of this functional relationship, of course, cannot be predicted through dimensional considerations, but must depend upon either physical analysis or experimental measurement.

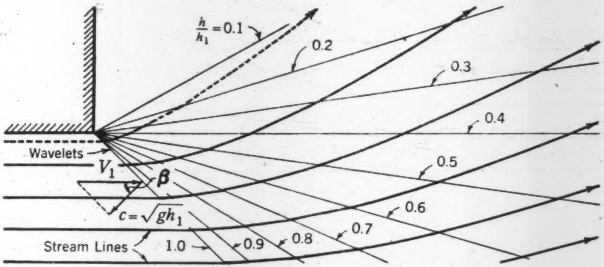

FIG. 51.—PATTERN OF FLOW NEAR THE ABRUPT END OF A CHANNEL WALL

The elementary wave theory indicates that the flow in the neighborhood of the end of either wall will begin to change in direction only as it passes the first negative wavelet (see Fig. 51), which lies at the angle $\beta = \sin^{-1} \sqrt{g\,h_1}/V_1 = \sin^{-1} 1/\mathbf{F}_1$ to the initial flow direction. From then on the streamlines may be considered to continue deviating through a series of infinitesimal steps, the angle of each succeeding wavelet depending upon the local magnitude of the continuously changing ratio of V to $\sqrt{g\,h}$—that is, the local Froude number. Each wavelet represents, in effect, a line of constant depth, so that proper selection among the infinite series of wavelets will yield the systematic series of surface contours shown in Fig. 51.

If now the zone is investigated in which the wavelets from the two opposite sides of the outlet begin to intersect, it will be seen (Fig. 52) that the resulting pattern of interference will yield a rather complex variation in depth and velocity of flow. The surface contours may again be determined by a rather laborious analysis of each element of the pattern in accordance with the elementary wave theory, but a far more rapid solution may be obtained by the graphical method of characteristics outlined in the first Symposium paper. Three such solutions, for different values of $\mathbf{F}_1$, are shown in Fig. 53.

The method of characteristics, in effect, reduces the functional relationship of Eq. 45 to the form:

$$\frac{h}{h_1} = f_2\left(\frac{x}{b_1}, \frac{y}{b_1}, \mathbf{F}_1\right) \qquad (46)$$

by combining the relative coordinate terms x/h_1 and y/h_1 with the initial width-depth ratio, b_1/h_1. This entails the inherent assumption of hydrostatic pressure distribution at all points—that is, the absence of appreciable vertical acceleration. As a matter of fact, at the abrupt end of either channel wall the pressure is far from hydrostatically distributed, as the water surface is practically vertical in such a zone. The extent to which this lack of fulfilment of the assumption causes the actual surface configuration to differ from the theoretical evidently depends upon the magnitude of the ratio b_1/h_1. In wide, shallow channels the zone of disagreement is of relatively small extent; in narrow, deep channels, on the other hand, the pressure distribution will be markedly nonhydrostatic from wall to wall.

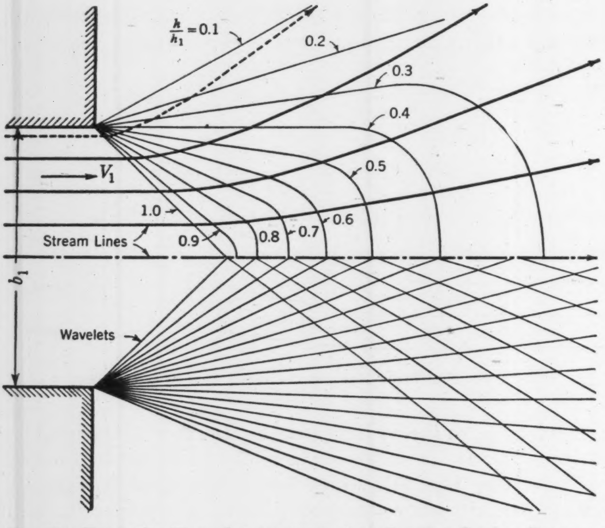

FIG. 52.—EFFECT OF WAVE INTERFERENCE FROM OPPOSITE SIDE OF CHANNEL

In illustrating the variation to be expected, experimentally measured surface contours for three different width-depth ratios are plotted in Fig. 54 for the same Froude numbers as those in Fig. 53. The deviations with b_1/h_1 are appreciable, but nevertheless secondary to the variation with the Froude number. In other words, using an average system of contours in preliminary design is quite in order. Even for the widest channel, however, it will be found that there is also a discrepancy between the measured contours and those

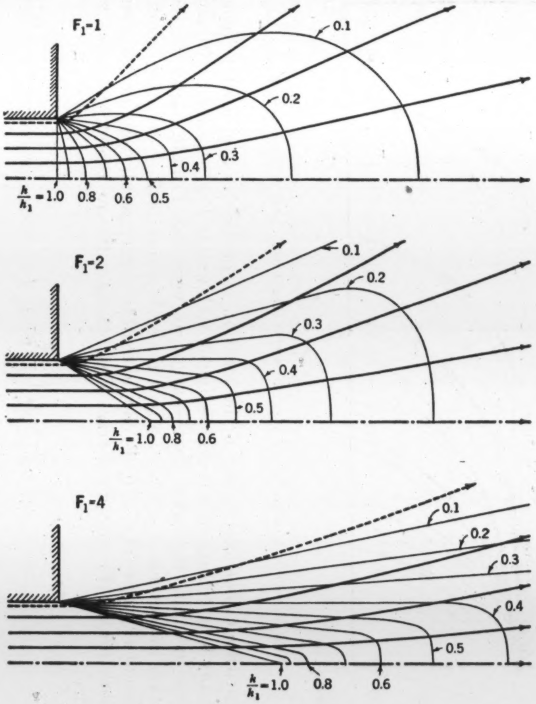
F₁=1
0.1
0.2
0.3
0.4
h/h₁=1.0
0.8
0.6
0.5
F₁=2
0.1
0.2
0.3
0.4
0.5
h/h₁=1.0
0.8
0.6
F₁=4
0.1
0.2
0.3
0.4
0.5
h/h₁=1.0
0.8
0.6

FIG. 53.—VARIATION OF THE FLOW PATTERN WITH THE FROUDE NUMBER

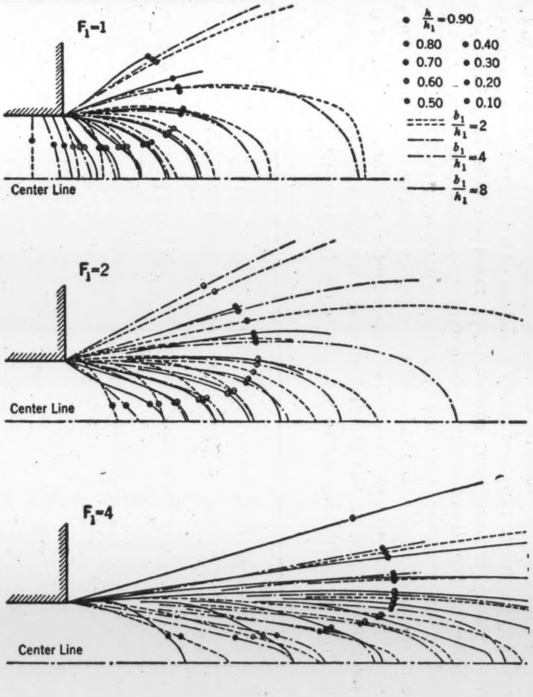

FIG. 54.—EXPERIMENTAL MEASUREMENTS OF SURFACE CONFIGURATION FOR VARIOUS FROUDE NUMBERS AND WIDTH-DEPTH RATIOS

obtained by the graphical method of analysis. This may be attributed to the failure of the analysis to provide for any variation of total head due to boundary resistance, since the analytical contours are invariably displaced upstream. Again, however, the discrepancy is secondary in comparison with either the measured or the predicted variation of the flow pattern with $\mathbf{F}_1$.

For purposes of rapid exploration of the various possible conditions of flow, it would be desirable to combine the Froude number with one or more of the other parameters, just as the method of characteristics effectively eliminates the ratio b_1/h_1 by combining the terms x/h_1 and y/h_1 (compare Eqs. 45 and 46).

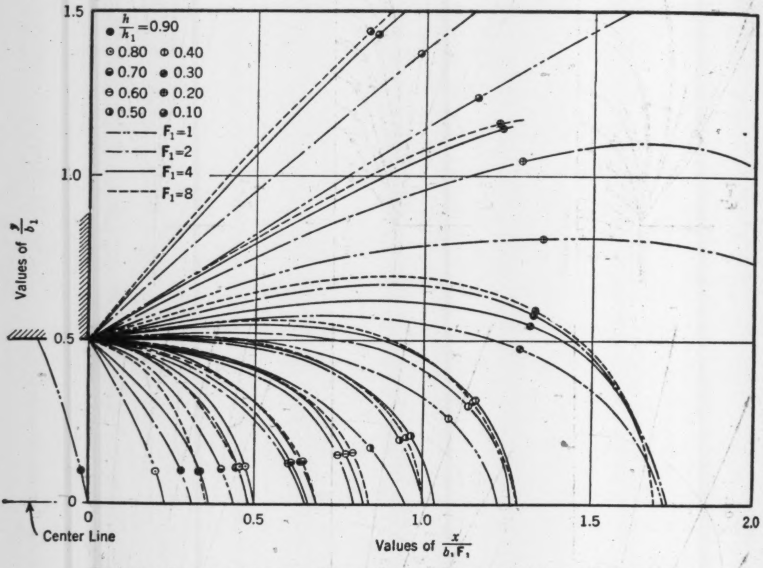

FIG. 55.—GENERALIZATION OF EXPERIMENTAL DATA FOR ABRUPT EXPANSIONS

A logarithmic plot against $\mathbf{F}_1$ of the location of particular contour intercepts along any longitudinal axis (say, the line of each wall in Fig. 54), made with this purpose in mind, indicated that—except at very low Froude numbers—the variation in location was essentially a linear function of $\mathbf{F}_1$. In other words, division of all values of x/b_1 by $\mathbf{F}_1$ should tend to superpose all equivalent surface contours for all Froude numbers. Eqs. 45 and 46 are thereby reduced to the form:

$$\frac{h}{h_1} = f_3\left(\frac{x}{b_1\,\mathbf{F}_1}, \frac{y}{b_1}\right) \qquad (47)$$

As may be shown analytically, this is certainly not a rigorous generalization. However, as will be seen from inspection of Fig. 55, when replotted in this manner the deviation of the mean contours taken from Fig. 54 is not great.

Although single average lines on this generalized diagram would evidently represent means of means, their departure from the contours of twenty-one runs at different values of $\mathbf{F}_1$ and b_1/h_1 is considered sufficiently small to permit the use of this diagram for the preliminary analysis of abrupt expansions at practically any value of either parameter.

EFFICIENT CURVATURE OF EXPANDING BOUNDARIES

There is a rule of thumb for the design of divergent boundaries in high-velocity flow which arbitrarily fixes the angle of divergence at $\theta = \tan^{-1} \frac{1}{6}$ to $\theta = \tan^{-1} = \frac{1}{9}$, regardless of the depth and the velocity of flow. As is apparent from the foregoing discussion of the expansion of flow without lateral constraint, the angle of divergence of any two neighboring streamlines is constant neither with the Froude number at a particular longitudinal distance nor with the longitudinal distance for a particular Froude number. At the abrupt beginning of such a uniformly divergent section the flow itself cannot abruptly change direction, and local separation as well as a concentration of negative wavelets will result; on the other hand, at a distance downstream which varies with the Froude number. the flow would naturally diverge more rapidly than the constant boundary angle will permit, thereby producing positive wavelets. Thus, as indicated by either of the contour maps of Fig. 56, obtained by the method of characteristics, such a divergent section is invariably inefficient at its beginning (and again before its end) unless the Froude number is so high that recovery from the initial zone is not accomplished before the transition ends.

It is obvious from such reasoning that an efficient boundary expansion should display a continuous change in curvature, and should have different proportions for every Froude number. The latter requirement suggests at once that, for purposes of preliminary design, the best form of boundary, as well as the resulting surface configuration, should be reducible to a generalized diagram such as that of Fig. 55. As a matter of fact, Fig. 55 was initially used in the arbitrary selection of a number of boundary curves for experimental and graphical investigation, the curves being formulated algebraically to approximate streamlines of the unconfined flow which enclosed about 90% of the total discharge.

The boundary equation eventually found to be most satisfactory was of the form:

$$\frac{y}{b_1} = \frac{1}{2}\left(\frac{x}{b_1\,\mathbf{F}_1}\right)^{\frac{3}{2}} + \frac{1}{2} \quad \text{. . . . .} \quad (48)$$

which is plotted in Fig. 57, together with surface contours for a mean value of b_1/h_1 and various values of $\mathbf{F}_1$.

As will be noted from this composite plot, the beginning of the transition is sufficiently gradual to reduce effects of nonhydrostatic pressure distribution to a minimum, so that the factor b_1/h_1 is no longer an essential variable. The gradual increase in boundary angle, moreover, is sufficient to prevent the formation of positive waves—yet not so great as to cause an undue change in depth across any normal section. In fact, using circular arcs to approximate normalcy to the streamlines at successive sections, it will be found that the variation in depth from wall to wall does not exceed 30% of the center-line value.

It is, of course, possible to reduce such depth variation between wall and center line by decreasing the rate of flare—that is, by decreasing the coefficient

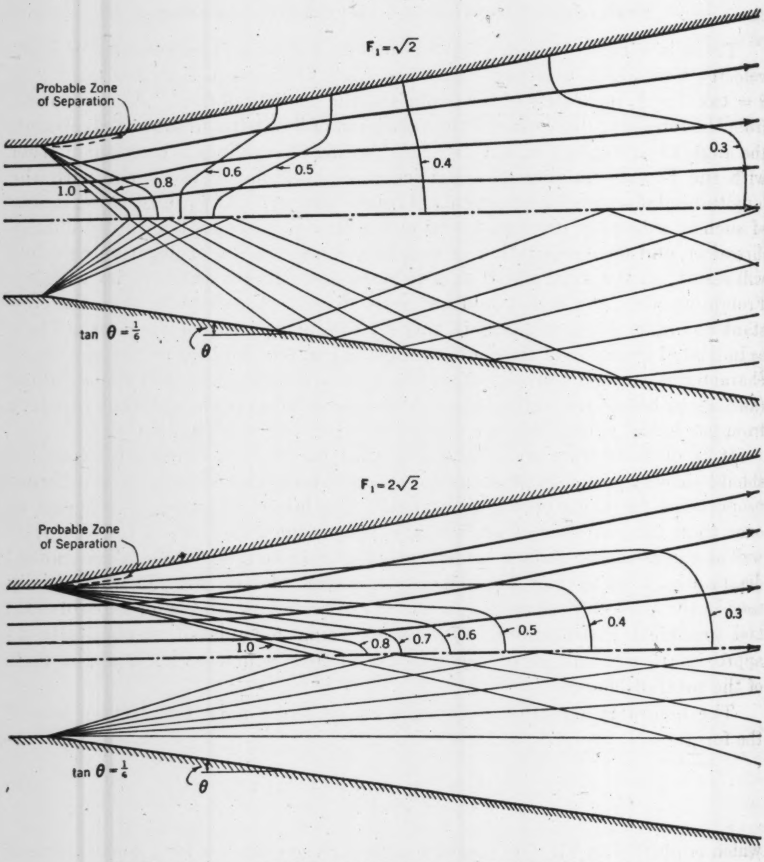
$F_1 = \sqrt{2}$
Probable Zone of Separation
0.3
0.4
0.5
0.6
0.8
1.0
$\tan\theta = \frac{1}{6}$
θ
$F_1 = 2\sqrt{2}$
Probable Zone of Separation
0.3
0.4
0.5
0.6
0.7
0.8
1.0
$\tan\theta = \frac{1}{6}$
θ

Fig. 56.—Patterns of Flow in a Uniformly Divergent Channel at Different Froude Numbers

of Eq. 48. This reduction, however, will result in a longer (and, hence, more costly) expansion for a given ratio of initial and final widths. Although the decision as to the greatest permissible depth variation is a matter either of judgment or of outlet requirements, it is believed that the curve shown in Fig. 57 will provide a satisfactory average basis for design.

The several curves reproduced in Fig. 57 typify the twenty or more which were determined experimentally for various Froude numbers and width-depth ratios and then checked by the graphical method of characteristics. Although a good general agreement was always obtained, the experimental results invariably yielded contours which were displaced downstream (from 0 to 35%, depending upon the ratio h/h_1). As this was attributed to the failure of the graphical method to take into account the gradual loss in total head due to boundary resistance, the same measurements were repeated on bed slopes varying from 4% to 10%. This, however, resulted in little displacement of the contours longitudinally, but in a considerable displacement laterally—that is, toward the center line, because the maximum bed slope was necessarily in the longitudinal direction rather than in the direction of each individual streamline.

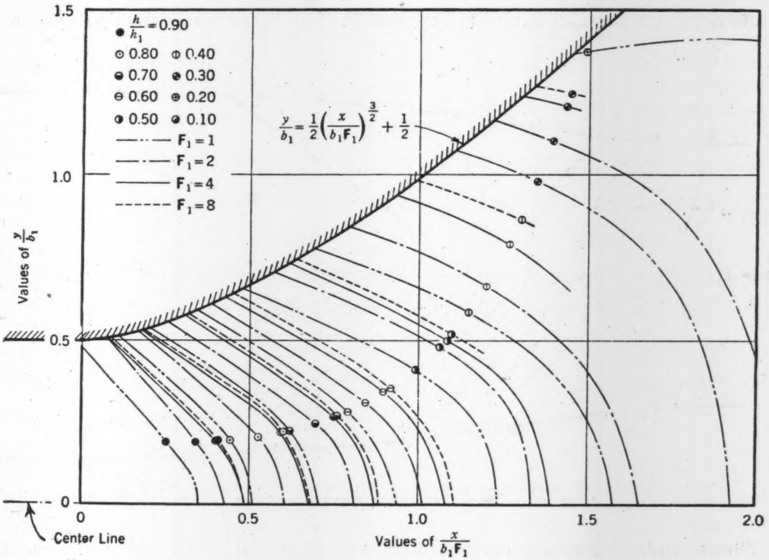

FIG. 57.—GENERALIZATION OF EXPERIMENTAL DATA FOR GRADUAL EXPANSIONS

From these considerations it would appear that the boundary form and surface contours for a level bed, shown in Fig. 57, may also be considered applicable to such moderate slopes as are normally encountered in open-channel design. Great slopes, on the other hand, would require a warped bed to prevent the major part of the flow from tending to follow the direction of maximum slope parallel to the center line. At present, the bottom surface can be warped satisfactorily only by trial and error at model scale. In the latter connection, however, it is to be noted that an effort was made to equalize both the surface elevation and the unit rate of flow across all normal sections by molding the bottom in conformity with the surface contours of Fig. 57. In other words, the initially horizontal bed was arbitrarily made higher at the center and lower at the walls in exact conformity to the indicated change in surface elevation across each normal section. Although full equalization of flow rate and surface elevation was not attained, conditions were improved perhaps 50%. Since application of this method of partial correction requires no further trial-and-error experimentation, its consideration is recommended where the added expense of bottom contouring is warranted.

ELIMINATION OF DISTURBANCES AT THE END OF AN EXPANSION

Just as the analysis of supercritical flow of water is closely related to that of supersonic flow of gases, an open-channel transition for such flow should satisfy essentially the same general requirements as the test section of a supersonic wind tunnel—a variation in cross-sectional area such that the velocity and depth (or pressure intensity) are evenly distributed across the final section.

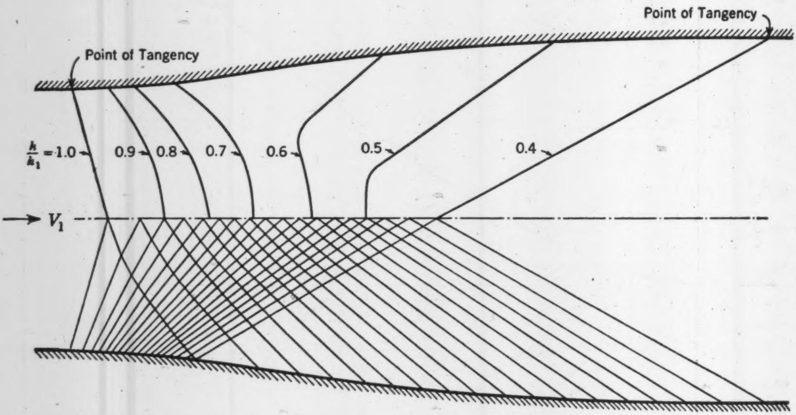

FIG. 58.—DESIGN OF AN EXPANSION WITH UNIFORM OUTFLOW FOR $F_1 = 2$, BY THE METHOD OF CHARACTERISTICS

The requirements of a supersonic wind tunnel are far more severe than those of an open-channel expansion, to be sure, even though experimental flow conditions are usually subject to arbitrary control; in fact, it is necessary to vary precisely the wall curvature of the tunnel test section from run to run in accordance with particular velocities of operation. Nevertheless, the same basic principles of boundary design could be applied quite generally to the case of open-channel expansions were it not for three practical limitations: First, under many circumstances the Froude number of the flow must be expected to vary over a considerable range; second, the length of transition required for either high Froude numbers or great expansion ratios will frequently be many times that permitted by structural economy; and, third, no method is provided thereby of stabilizing the hydraulic jump.

For the particular condition that an expansion represents merely a desirable increase in the width of a continuously paved channel, however, the same design procedure will yield at least the first approximation to an efficient design for a particular Froude number. As indicated in Fig. 58, the basis of the design technique is the control of wall curvature in such manner that the negative waves formed by successive elements of the outward curve just offset the positive waves formed by successive elements of the subsequent inward curve, so that the flow is restored to complete uniformity at the end of the transition. The procedure is, unfortunately, one of trial and error, and the resulting expansion ratio cannot be accurately foretold. A generalized series of boundary curves for successively greater expansion ratios is therefore presented in Fig. 59, as determined by interpolation from a series of solutions for various Froude numbers and expansion ratios by the method of characteristics. These curves must be regarded merely as guides in preliminary design, for the following reasons: (1) Since the primary purpose was the generalization of results, each curve represents the average form of several somewhat different curves for different Froude numbers; (2) since the initial outward curve was chosen to yield without change the greatest practicable expansion range, it is probable that a somewhat shorter expansion curve could be devised for a particular condition; and (3) since the length of any transition is far in excess of that for which the drop in total head could be ignored, the assumption of zero loss in applying the method of characteristics leads to a predicted outlet depth which is considerably smaller than that which will actually prevail. For example, experiments on several expansions constructed in the laboratory on the basis of Fig. 59 resulted in outlet depths as much as from 20% to 40% in excess of that indicated by the simple wave theory, even though the flow was essentially uniform at the exit.

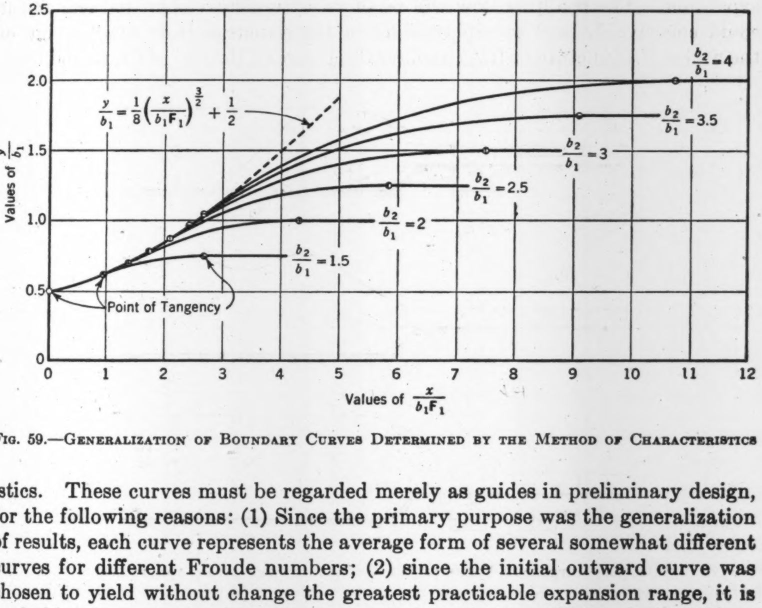

FIG. 59.—GENERALIZATION OF BOUNDARY CURVES DETERMINED BY THE METHOD OF CHARACTERISTICS

If, on the other hand, the channel expansion is intended to reduce the Froude number of the flow just prior to the formation of the hydraulic jump, it will be possible to hold the expansion to the shorter form of Fig. 57. If the divergent walls are followed by parallel walls with either an abrupt or a gradual transition, a positive wave will be formed at each wall junction and will extend diagonally across the flow at an angle varying with the local Froude number. Such waves would persist a considerable distance down the channel through repeated reflection if no jump were formed. If the toe of the jump lies at or near the end of the expansion, however, the diagonal waves will no longer form. On the other hand, should the jump as a whole advance even slightly into the expanding section, the smaller depth at each side will result in a progressively greater advancement of the jump along the walls, any slight asymmetry of the divergent flow finally giving rise to a deflection of the whole stream along one wall as the jump advances along the other almost to the upstream end of the expansion. The resulting flow will be of an extremely violent nature, and it could conceivably lead to rapid failure of the structure from overtopping of the wall at the juncture or from undercutting beyond the end of the paved floor.

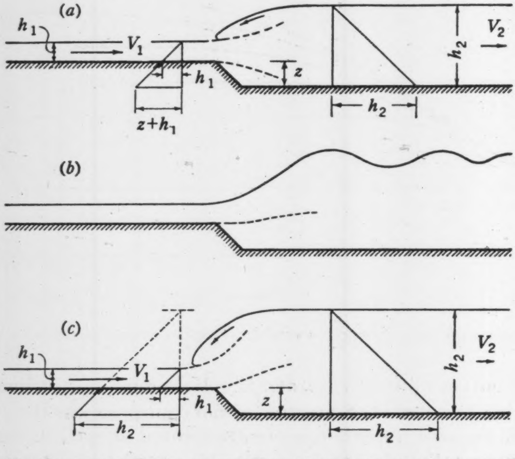

FIG. 60.—ALTERNATE FORMS OF THE HYDRAULIC JUMP AT AN ABRUPT DROP

In order to stabilize the jump at the end of such an expansion it appears necessary to provide a drop in floor level at the beginning of the parallel section. The relative magnitude of the change in elevation should depend primarily upon the Froude number of the flow as it leaves the expanding section. With reference to Fig. 60, it will be seen that the relationship between the Froude number, the relative change in depth, and the relative size of drop may be determined, like the equation of the jump itself, by the momentum and continuity relationships. There are, however, two different types of jump which may form, depending upon whether the downstream depth is below or above that which produces the standing wave indicated by Fig. 60(*b*). For the condition of Fig. 60(*a*), the pressure on the face of the drop will be determined by the upstream depth; for the condition of Fig. 60(*c*), the downstream depth will govern. The relationships for cases (*a*) and (*c*) are as follows:

$$\mathbf{F}^2_1 = \frac{1}{2}\frac{h_2/h_1}{1 - h_2/h_1}\left[\left(\frac{z}{h_1} + 1\right)^2 - \left(\frac{h_2}{h_1}\right)^2\right] \qquad (49a)$$

and

$$\mathbf{F}^2_1 = \frac{1}{2}\frac{h_2/h_1}{1 - h_2/h_1}\left[1 - \left(\frac{h_2}{h_1} - \frac{z}{h_1}\right)^2\right] \qquad (49b)$$

Curves for Eqs. 49 will be seen in Fig. 61, the right-hand (or lower) series corresponding to Eq. 49*a* and the left-hand series to Eq. 49*b*. The critical zone for the formation of the standing wave—case (*b*)—cannot be foretold therefrom, however, and recourse must be had to experimental measurement. Tests on both abrupt and sloping drops resulted in the points plotted in the figure, which not only verify the approximate analysis but indicate a systematic trend of the transition between the two regimes of flow.

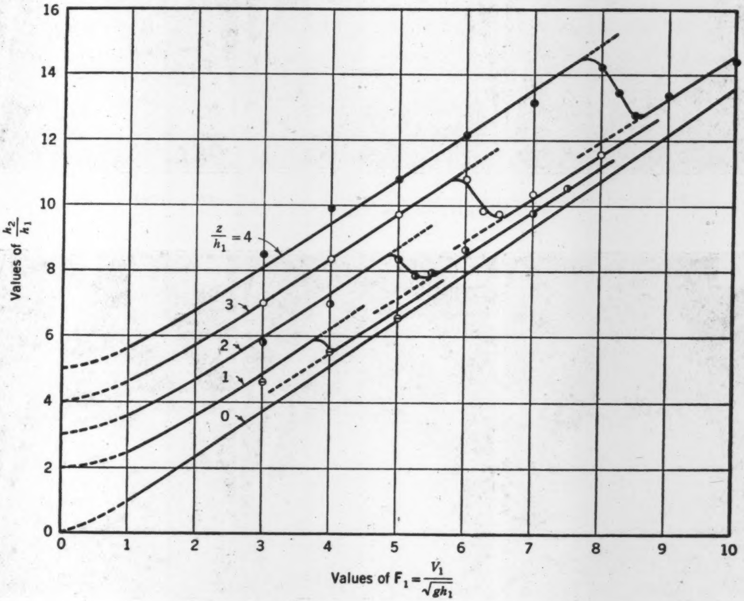

FIG. 61.—ANALYTICAL AND EXPERIMENTAL CHARACTERISTICS OF THE HYDRAULIC JUMP AT AN ABRUPT DROP

From this diagram the magnitude of the drop for a given tailwater depth, or vice versa, may be determined once the average depth and the Froude number of the flow at the end of an expansion have been established. For protection of the structure, the design should be such that at the maximum expected Froude number the tailwater depth will be the minimum required to produce a jump. Figs. 62(*a*) and 62(*b*) are photographs of conditions for expansion ratios of 8 and 4, respectively, the details of the expansions following the recommendations presented herewith.

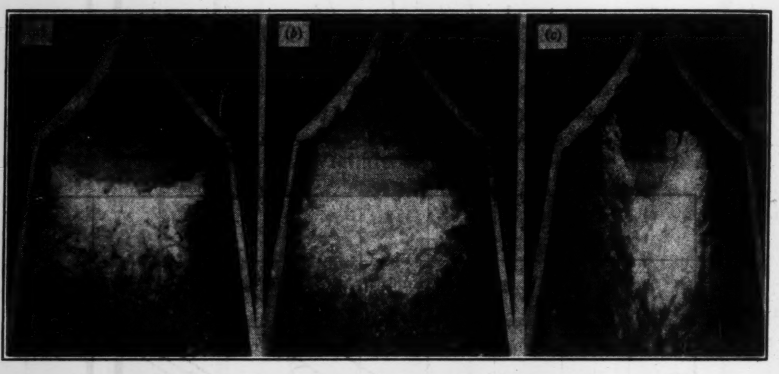

FIG. 62.—CHANGE IN JUMP CHARACTERISTICS AT A 1:8 EXPANSION DURING A 30% INCREASE IN THE TAILWATER DEPTH

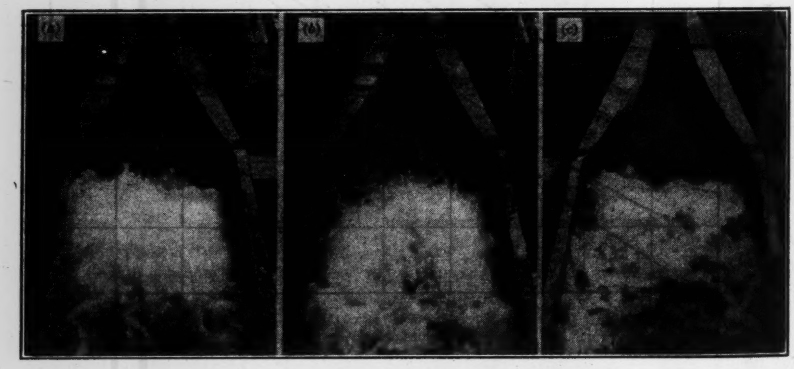

FIG. 63.—CHANGE IN JUMP CHARACTERISTICS AT A 1:4 EXPANSION DURING A 30% DECREASE IN THE RATE OF FLOW

If the tailwater depth is increased beyond the value for minimum jump requirements, the undular regime will first appear (see Fig. 62(*b*)), followed by the second form of the jump and then by an uneven penetration of the expansion (Fig. 62(*c*)). An asymmetric pattern may eventually result, but the presence of the drop makes this far less likely than would otherwise be the case. On the other hand, if the Froude number increases beyond the design value, or if the tailwater depth decreases, the jump will be carried downstream. This phenomenon again is rendered less sensitive by the drop; the condition shown in Fig. 62(*a*), for example, will prevail during a 10% change in the downstream depth, whereas the three different stages shown in Fig. 62 represent a 30% change. Finally, if the Froude number is decreased, essentially the same sequence will follow as for the increase in tailwater depth (see Figs. 63(*b*) and 63(*c*)). In this event, however, the decrease in discharge will correspond to an equivalent decrease in harmfulness of the flow, and—although asymmetry may eventually develop—the structure planned for higher flows should then be safe.

CONCLUSIONS

Application of the elementary wave theory to the analysis of high-velocity flow in open-channel expansions may be expected to yield results in essential agreement with experiment as long as the assumptions involved in the theory are approximately satisfied. For purposes of design, however, it is convenient to reduce all measured data for abrupt expansions to a single generalized plot of surface contours as a function of the initial Froude number and the relative coordinate location. A similar procedure for gradual expansions permits selection of an efficient wall form for any initial Froude number and width-depth ratio. To avoid dangerous asymmetry of the flow at the end of such an expansion, the hydraulic jump should be stabilized by a drop in the channel floor, the proper magnitude of which may be determined by the momentum equation.

DISCUSSION

PAUL BAUMANN,[32] M. ASCE.—In the years after 1929 dams have risen from maximum heights of from 300 ft to 700 ft and still the trend toward greater heights persists.

Since spillways are indispensable appurtenances of dams, it is obvious that the maximum velocity of water discharging from them has increased in direct relation to the height of the structure above the stream bed. Simultaneously flood channels below the spillway of debris barriers, located on the cones below canyons of the coastal mountain range of Southern California, have been designed and constructed for water velocities far above critical—that is, for velocities of the order of 70 ft per sec. The slopes of these channels often exceed 10%.

The Symposium on supercritical flow in open channels is most timely, therefore, particularly in view of the vagueness that existed until recent years in connection with the mechanics of this phenomenon. However, this Symposium is to be welcomed by the profession, not only because it is timely, but also (and indeed, particularly) because it is presented masterfully, vividly, and fascinatingly, in a manner that cannot but help stimulate the interest of the hydraulic experimenter and designer. The questions so freely admitted as being still unanswered by the authors will thereby be moved into the focus of broader attention, and the more complete, as well as exact, solution of the problem may well be much advanced.

The writer, being in charge of the general direction of hydraulic research of the Los Angeles County (Calif.) Flood Control District, naturally kept in close touch with the California Institute of Technology at Pasadena throughout the model tests and their interpretation so clearly presented in the papers by Messrs. Knapp and Ippen. The testing apparatus and plant were financed by the District and in part constructed by its forces. In the design of the model and the testing work itself the institute was assisted by district personnel. Among them Wilbert W. Baustian of the Hydraulic Division contributed many constructive ideas.

The validity of the approach to the problem of supercritical flow of water on the principle of supersonic flow of gases has no doubt been established as a result of the tests and their mathematical analysis. The abrupt and sharp boundary lines (formed by the waves due to a disturbance, such as a sudden change from a straight to a curved alinement of a channel) cannot help but suggest a striking similarity to the well-known V-shaped "wake" which forms at the point of a projectile, moving through air at a speed greater than sound. The oscillatory movement of the water is likewise apparent in all cases in which the radial water prism loses its balance. This happens when water enters a circular curve without suitable transition.

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[32] Asst. Chf. Engr., Los Angeles Flood Control Dist., Los Angeles, Calif.

Certain approximations had to be made to facilitate solution of the problem in the form of design equations. Of those, the one referred to by Mr. Knapp in his discussion of Eq. 2 is notable. However, it is not Eq. 2 itself that is an approximation, but rather that which follows as a result of the substitution, $V_n = V \sin \beta$, which leads to Eq. 30. It corresponds to Eq. 6.

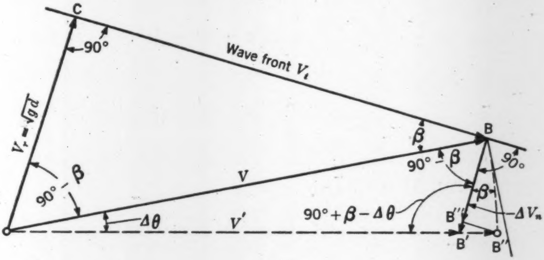
C
90°
Wave front V_t
$V_t = \sqrt{gd}$
$90° - \beta$
β
B
$90° - \beta$
$90°$
V
β
$-\Delta V_n$
$\Delta\theta$
V'
$90° + \beta - \Delta\theta$
B'''
B'
B''

FIG. 64

The apparent discrepancy could mathematically be justified for only very small wave angles β which is not necessarily the case. The approximation in question can be illustrated by Fig. 64. Whereas Eq. 6 is based on V changing to V', which is represented by AB', Fig. 64, and therefore $-\Delta V_n$ is represented by BB', Eq. 30 is based on V being constant. Therefore $-\Delta V_n$ is represented by BB''', and the tangential component represented by B''B''' is neglected. Since $\Delta\theta$ is small as compared to β, this neglect, although contrary to the basic premise, is well within the tolerance of error for design purposes.

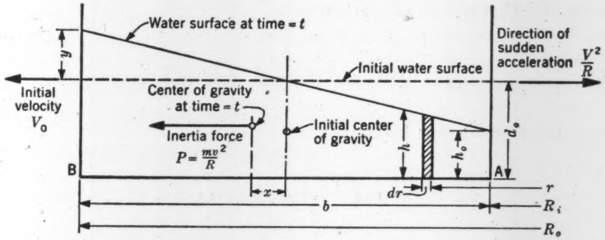
Water surface at time = t
y
Initial water surface
Direction of sudden acceleration $\frac{V^2}{R}$
Initial velocity V_0
Center of gravity at time = t
Inertia force $P = \frac{mv^2}{R}$
Initial center of gravity
h
h_o
d_o
B
A
x
dr
r
b
R_i
R_o

FIG. 65

For the sake of comparison with the results set forth in the papers, equations for the maximum superelevation and the distance from the point of maximum superelevation to the beginning of the curve are derived as follows: To represent the phenomena simply, the assumption is made that water in a tank of a rectangular section, such as a channel, is being moved at the constant velocity V_o (see Fig. 65). At a time $t = 0$ the direction of movement of the tank is suddenly reversed and the tank is accelerated in the opposite direction at a constant acceleration of V^2/R (symbol R denotes curve radius). The water will then (because of the inertia forces) flow in the opposite direction of motion and will bank up against the respective wall of the tank to a maximum superelevation, and then flow back. This motion is comparable with a harmonic oscillation under the influence of a constant force, P, acting either in the direction of flow or opposite to it. The differential equation describing this motion in the direction of P is

$$\frac{-m\, d^2x}{dt^2} - K\,x + P = 0 \dots\dots\dots\dots\dots\dots (50)$$

or

$$\frac{+\, d^2x}{dt^2} + \frac{K}{m}(x - a) = 0 \dots\dots\dots\dots\dots\dots (51)$$

in which $a = P/K$.

The solution of Eq. 51 is known, and is

$$x - a = A \sin \alpha\, t + B \cos \alpha\, t \dots\dots\dots\dots\dots\dots (52)$$

Determining the constants of integration A and B from the boundary conditions,

$$B = -\,a \dots\dots\dots\dots\dots\dots (53a)$$

and

$$A = \frac{V_o}{\alpha} \dots\dots\dots\dots\dots\dots (53b)$$

Hence, the equation of motion reads

$$x - a = \frac{V_o}{\alpha} \sin \alpha\, t - a \cos \alpha\, t \dots\dots\dots\dots\dots\dots (54)$$

The velocity at any instant, then, is

$$\frac{dx}{dt} = \alpha \left(\frac{V_o}{\alpha} \cos \alpha\, t + a \sin \alpha\, t \right) \dots\dots\dots\dots\dots\dots (55)$$

and the acceleration at any instant is

$$\frac{d^2x}{dt^2} = -\,\alpha^2 \left(\frac{V_o}{\alpha} \sin \alpha\, t - a \cos \alpha\, t \right) \dots\dots\dots\dots\dots\dots (56)$$

In Eq. 56 the term in the parenthesis may be seen to equal $x - a$, and the constant α, therefore, is equal to $\sqrt{K/m}$.

From conditions of equilibrium of the water in the tank (Fig. 65) it follows that the differential pressure P_x between inside and outside walls at any instant is equal to $2\, w\, y\, d_o$. Since x represents the horizontal displacement of the center of gravity of the oscillating prism, the superelevation y can be expressed as a function of x—namely,

$$y = 6\, d_o \frac{x}{b} \dots\dots\dots\dots\dots\dots (57)$$

Introducing

$$P_x = \frac{12\,w\,d^2_o}{b}\,x \tag{58a}$$

the constant K then equals $\frac{12\,w\,d^2_o}{b}$, and, since $a = P/K$, $P = m\,v^2/R$, and $m = \frac{w\,b\,d_o}{g}$:

$$a = \frac{v^2\,b^2}{12\,R\,g\,d_o} \tag{58b}$$

Introducing these values in Eq. 54,

$$x = \frac{v_o\,b}{\sqrt{12}\,\sqrt{g\,d_o}}\sin\alpha\,t + \frac{v^2\,b^2}{12\,R\,g\,d_o}(1 - \cos\alpha\,t) \tag{59}$$

The first member on the right side of Eq. 59 represents the influence of initial velocity whereas the second member represents the influence of constant acceleration during oscillation. Introducing x into Eq. 57,

$$y = d_o\left[\frac{\sqrt{3}\,V_o}{\sqrt{g\,d_o}}\sin\alpha\,t + \frac{V^2\,b}{2\,R\,g\,d_o}(1 - \cos\alpha\,t)\right] \tag{60}$$

In studying this result it must be kept in mind that Eq. 60 is based on the assumption that the entire mass is concentrated in the center of gravity. In other words, the water prism is treated as a mass point. This is not the case in reality, and instantaneous acceleration at the beginning of the curve is only partly taking place with water in a channel. The approximation is on the safe side, however. In the case of water confined in a tank, instantaneous acceleration is very nearly possible.

An equation for y similar to Eq. 60 could be derived for the motion in the direction opposite to force P. This would lead to smaller values of y and therefore would not be a governing consideration for design purposes. It is obvious that v_o may become important.

Checking back on the boundary conditions, it follows that the premises upon which the analysis is based are satisfied. It is also evident that $\frac{v}{\sqrt{g\,d_o}}$ (which is the cosecant of β, the wave angle) insists on appearing as a parameter. When $v_o = 0$ the maximum superelevation occurs for $\alpha\,t = \pi$. Hence,

$$x = \frac{v^2\,b^2}{6\,R\,g\,d_o} \tag{61a}$$

and

$$y_{\max} = \frac{v^2\,b}{R\,g} \tag{61b}$$

The period of oscillation T follows from

$$T = \frac{2\,\pi}{\alpha} = 2\,\pi\sqrt{\frac{m}{K}} = \frac{\pi\,b}{\sqrt{3}\,\sqrt{g\,d_o}} \tag{62}$$

The time required for the water to reach its first maximum superelevation from the neutral position would then be one half the values found by Eq. 62, and the distance from the beginning of the curve to that point would be:

$$D = \frac{v\,\pi\,b}{\sqrt{12}\,\sqrt{g\,d_o}} \qquad (63)$$

From this analysis it may be readily recognized why no oscillations develop in correctly banked channels. The driving force P is exactly balanced at all instants by the component of a gravitational force, due to a sloping channel bottom. Hence, in Eq. 59 the first member disappears because of no initial radial velocity, and the second member disappears because of a third member which is equal to it and opposite in sign. The displacement x (Fig. 65) of the center of gravity then remains zero throughout the movement of water around the curve.

It is interesting to note that Eq. 8 leads to Eq. 61*b* for long radius curves—that is, for the case in which the arc of the outside wall may be approximated by the tangent and θ is substituted for $d\theta$.

Eq. 31 is an approximation of Eq. 8. For long radius curves, and when θ is small in relation to β, Eq. 31 may also be written in finite terms (see Fig. 64):

$$h = \frac{v^2}{g}\sin\beta\cos\beta\,\theta = \frac{(V^2 - g\,d_o)\,b}{R\,g} = \frac{V^2_t\,b}{R\,g} \qquad (64)$$

Thus, instead of V its tangential component V_t appears in Eq. 64. Obviously, the discrepancy between the two formulas (Eqs. 31 and 8) grows with the wave angle β.

An interesting comparison of Eqs. 32*b* and 61*b* has been made by C. A. Hart, Assoc. M. ASCE, by plotting maximum superelevations observed during 80 model runs at the California Institute of Technology and 10 runs on the Puddingstone Flood Channel prototype. Values by Eq. 32*b* (minus the measured value) were plotted as abscissas against values by Eq. 61*b* (minus the measured values) as ordinates. By analyzing the scatter in quadrants, formed by 45° axes, of the chart it was observed that 45 of the 80 model observations came closer to checking Eq. 32*b* and that 6 of the 10 observations on the prototype came closer to checking Eq. 61*b*.

The comparison between Eq. 33 and Eq. 63 may best be shown by a numerical example. Let: $Q = 5{,}000$ cu ft per sec; $b = 20$ ft; $d_o = 5$ ft; $A = 100$ ft²; $R = 500$ ft; $v = Q/A = 50$ ft per sec; $\tan\beta_o = \frac{\sin\beta_o}{\cos\beta_o} = \frac{\sqrt{g\,d_o}}{\sqrt{v^2 - g\,d_o}} = 0.263$; and from Eq. 33, $\theta_o = \tan^{-1}\frac{20}{510 \times 0.263} = \tan^{-1} 0.15$. Hence $D = R\,\theta = 74.30$ ft; and from Eq. 63, $D = \frac{50 \times 3.14 \times 20}{\sqrt{12} \times 12.68} = 71.50$ ft. The difference of $74.30 - 71.50 = 2.80$ ft corresponds to a difference of 0.056 sec in time.

All the model tests at the California Institute of Technology were conducted on channels of uniform bottom width and uniform slopes for trapezoidal channels. For banked rectangular channels the bottom was straight in a radial direction. This is a satisfactory approximation for a width b which is small relative to the center-line radius of the channel. Actually, the water surface for balanced flow is always a curve, the equation of which reads (see Fig. 65):

$$h = \frac{v^2}{g} \log_e \frac{r}{R_i} + h_o \ldots\ldots\ldots\ldots\ldots\ldots\ldots (65)$$

Eq. 65 is the equation of a free vortex.

In a banked channel the bottom must be parallel to the water surface for balanced flow (that is, the depth must remain constant); therefore, Eq. 65 also holds true for the curvature of the bottom if h_o is replaced by a value that varies between 0 and $-\frac{v^2}{g} \log_e \frac{R_o}{R_i}$ and which causes the curve to pass through any point between A and B in Fig. 65. The wisdom as to which point should be chosen is governed by conditions in the field. Whenever practicable it is advantageous to use such a point between A and B for which the change in position of the center of gravity of the prism is a minimum.

In the fall of 1948, the Los Angeles Flood Control District completed a section of a major rectangular flood channel 50 ft wide. Spiral transitions were used in the design. As a result of this experience Mr. Knapp's apprehension regarding such transitions has not been found to be justified in regard to difficulties in design, layout, or construction. Hydraulically, a spiral transition at least one half of a design wave length in length is superior to compound curves, particularly for rates of flow considerably smaller than the design flow.

The writer has been a strong believer in, and proponent for, banked rectangular channels with spiral transitions for supercritical flow. Large-scale model tests conducted in the summer of 1947 by the United States Corps of Engineers, Los Angeles District, have shown remarkable confirmation of the small-scale tests at the California Institute of Technology and have strengthened the writer's confidence in the value of the latter.

The writer has often heard engineers express doubt as to the performance of banked channels for flows smaller than design flow. Fear was expressed that the flow would concentrate on the inside of the curve, forcing it to go over the top of the wall. A study of this problem will readily show that, because of the influence of the inertia force, $P = \frac{m\,v^2}{r}$, this cannot happen. This theory has been fully confirmed by model tests.

As to sills as described in Mr. Knapp's paper, there can be no doubt that, scientifically, they are an elegant suggestion. Practically, however, they have limited usefulness in flood channels mainly because of the almost unavoidable debris problem involved.

The disadvantages of trapezoidal channels can be greatly alleviated through the provision of vertical parapet walls at the top of the slopes. Of course, banking is not practicable.

In the foregoing the term "transition" was used to describe the change from a straight to a curved section of a channel without a change in width. In the papers on the design of channel expansions and contractions for supercritical flow the term "transition" refers to a sudden or gradual change in width.

These papers are closely related to the ones previously discussed, particularly to the one by Mr. Ippen. The simplicity of the mathematical solution by Messrs. Rouse, Bhoota, and Hsu of a problem of such complexity is remarkable. The paper by Messrs. Ippen and Dawson brings out the fact that for supercritical flow contractions must be designed with particular care. Fig. 40 is most helpful in demonstrating the significance of high Froude numbers relative to the most advantageous length and configuration of the transition. It shows clearly that gradual transitions commonly used for subcritical flow or low Froude numbers are not necessarily desirable. The formulas and graphs presented in these papers will be of considerable help to the designers. The Los Angeles Flood Control District has carefully avoided sudden changes in width of flood channels and, wherever possible, has confined gradual transitions to straight sections of channels. The length of these transitions has been determined by the wave angle theory. No trouble has ever developed under these conditions.

Concluding, the writer believes that the design of flood channels is in quite a different category from the design and installation of hydraulic machinery in connection with clear, well controlled water. Changes often occur in flood channels, such as in the discharge of storm drains into them, which may cause more serious disturbance than plain circular curves. Liberal freeboard is therefore advisable. In this case it is truly a matter of providing for a factor of ignorance.

N. N. Bhandari.[33]—The phenomenon discussed in the Symposium is particularly interesting in the light of the viewpoints from which the different authors have studied it. This problem is of great importance to the irrigation engineers in India as an increasing number of works is being constructed which involve fluming or constriction, and then expansion, of channels.

Experiments have been conducted by J. Nikuradse[34] on pressure flow in smooth channels with various angles between the walls. It was found that the symmetrical flow was stable when the channel was convergent or slightly divergent—even with a divergence of 4° there was no trace of instability. The velocity distributions for divergences of 5°, 6°, and 8° were not symmetrical about the middle plane. No region of back flow at either wall was evident with a divergence of 5°; separation of the forward flow in the boundary layer along one of the walls was just beginning to occur at 5°. Mr. Nikuradse found that the first occurrence happened between 4.8° and 5.1°, but could not be specified more exactly. At 6° the separation became apparent from the measurements taken; at which wall the separation occurred was a question settled by incidental changes in pressure, and once it occurred the pressure gradients were altered. For divergences of 5° and 6° there was no apparent tendency for separation to occur at the second wall, and the symmetrical flow once established seemed to be quite stable. A tendency appeared for the symmetrical distribution to switch over from wall to wall at 8°; each configuration was usually maintained for some time—long enough to allow measurements to be taken before a change occurred.

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[33] Asst. Secy., Central Board of Irrig., Simla, India.

[34] "Modern Developments in Fluid Dynamics," by S. Goldstein, Clarendon Press, Oxford, 1938, Vol. II, pp. 371–373.

To the writer the foregoing observations are of utmost importance. This kind of action of the flow downstream from the expansions has generally been observed in both experimental flumes and prototype channels.

The result of this action has also been observed in the asymmetrical shape of the scour hole created downstream from such structures. Attempts have been made to control the flow in such cases by "bowing" the downstream cross section of these structures by making the downstream beds curved across the width instead of leaving it flat. This idea of "bowing" the cross section was first introduced by A. M. R. Montagu[35] in 1934. The bowed bed concentrates the flow longitudinally in the center, makes the scour hole symmetrical, and reduces the side scours; but it increases the depth of scour hole in the center considerably, accentuating the dangers to the stability of the structure. To guard against this action additional precautionary measures in the form of staggered blocks,[36] baffles, etc., must be incorporated in the structure, further complicating an already complicated problem. No one seems to have proceeded on the lines indicated by Mr. Nikuradse to design the expansion. The expansion required may not actually be 5°. It would naturally depend, to a large extent, on the discharge required to be passed through the flumed structure. Unfortunately there is no hydraulic laboratory at the disposal of the writer in which he can complete an ideal simple design of a flumed structure on the foregoing principles. The flinging jets interfere with each other and create a number of wavelets, as indicated in the Symposium in some of the illustrations and photographs of flow in expansions. The writer would like to suggest further experimental work on the lines indicated herein, to be conducted by those who have the facilities to do so. This would be a great contribution to the knowledge of the profession.

T. Blench,[37] M. ASCE.—The Symposium is welcome as a logical extension of the methods developed by Julian Hinds,[38] M. ASCE. It relies on the Bernoulli equation and the momentum (hydraulic jump) equations under simplifying assumptions concerning the nature of the flow, and it uses graphical methods of solution; but a special assumption of no energy loss over a sudden rise of surface elevation—unnecessary in Mr. Hinds' work—seems to have potentialities for producing considerable error in certain cases. The value of graphical solutions based on simplifying assumptions has been well demonstrated in India, where Mr. Hinds' methods have been used since 1934 in the form of officially published graphs. The result has been a marked advance in the design of flumes and drops in erodible channels; the writer attributes this advance largely to the fact that every engineer was able to create a mental picture of ideal behavior as a basis for comparison with actual occurrences.

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[35] "Fluming," by A. M. R. Montagu, *Publication No. 6*, Central Board of Irrig., Simla, India, 1934.

[36] "Protection Against Scour of River and Canal Works," by N. N. Bhandari and H. L. Uppal, *Paper No. 214*, Punjab Eng. Congress, 1938.

[37] Cons. Engr. and Associate Prof., Civ. Eng., Univ. of Alberta, Edmonton, Alta., Canada.

[38] "The Hydraulic Jump and Critical Depth in the Design of Hydraulic Structures," by Julian Hinds, *Engineering News-Record*, Vol. LXXXV, July–December, 1920, p. 1034.

The authors' line of attack seems to have "paid dividends" already by indicating the first approximation to the production of interference patterns that will have a small maximum wave amplitude. This should put an end to radically incorrect designs and should reduce model experimentation to the improvement of basically correct models. Because of the basic value of the work, the writer is not perturbed by the rather poor numerical agreement that can occur between theory and experiment as in Fig. 38.

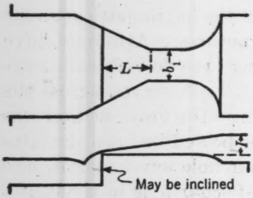

FIG. 66.—DIAGRAM OF SPREADING FLOOR DROP

The writer's principal practical interest is in Fig. 57 as a possible confirmation of the length, L, required for the spreading floor of a fairly new type of nonscouring fall.[39,40] Fig. 57 does not seem to agree well with model results obtained in India, and the question arises as to the sensitivity of the surface conformation of a hypercritical expansion to such factors as initial velocity distribution and friction, which the simple theory ignores. Fig. 66 illustrates the spreading floor design, in principle. Instead of starting a chute and the divergence simultaneously from the flumed throat, the crest level is continued a length, L, into the divergence until almost regime breadth of the downstream channel is reached, then the drop is provided. The spreading floor may slope upward in the downstream direction to insure a definite plunging of the water surface over the drop when the difference of level between the upstream

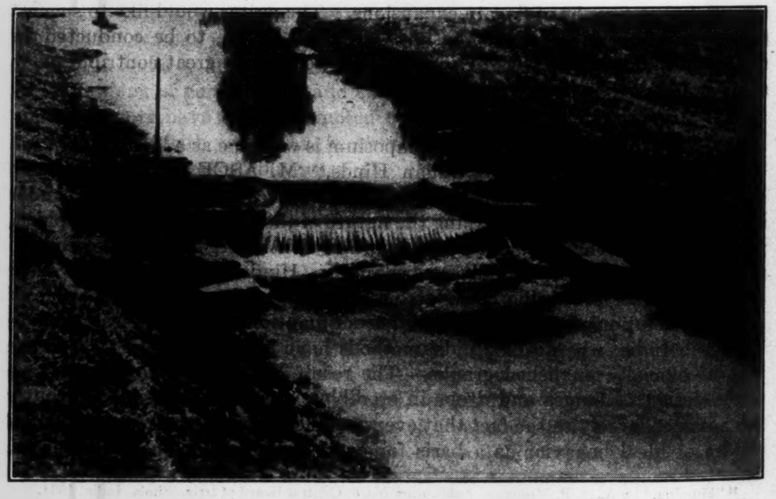

FIG. 67.—SPREADING FLOOR MODEL OF ABOUT 20 CU FT PER SEC

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[39] "Scour in Alluvium Below Falls," by T. Blench, Mushtaq Ahmad, and Nazir Ahmad, Rept. on the 22d Meeting of the International Assn. for Hydraulic Structures Research, Stockholm, 1948.

[40] "Design of Canal Falls," by T. Blench and Mushtaq Ahmad, *Research Publication*, Irrig. Research Inst., West Punjab, Pakistan, Vol. 11, No. 26.

and the downstream flow is small. Side-wall divergences of 1:2 are satisfactory. (The principal object of the design is to take advantage of the rapid divergence that can occur in hypercritical flow, because there is acceleration; the failure of subcritical divergences—from the viewpoint of scour—arises from the fact that they require deceleration and the friction that could produce it is too small to permit a divergence rate usually considered economical.) The length L was determined from a range of small model experiments to produce a practically uniform depth of flow just before the drop—not more than 5% variation. These experiments showed that L/b_1 was very dependent on H/b_1—in fact, 0.9, 1.0, and 1.4 correspond to 0.075, 2.4, and 2.9. In Fig. 57, taking F_1 as 1.0—which seems to correspond reasonably with the conditions of Fig. 66—and using $x/b_1 = 0.9$, the mean rate of side divergence up to that value of x/b_1 is 1:2.25 with a central depth of about 50% greater than the side depth; but the Indian experiments show that with the sharper divergence of 1:2 there would be practically uniform depth all the way across. Fig. 67 shows a model of about 20 cu ft per sec working; the spreading floor had a very slight upward slope.

The writer would appreciate any information on a counterpart of Fig. 57 for flow emerging from a parallel flume throat, with divergence walls arranged to insure the rapid assumption of uniform depth at right angles to the center line.

Clarence A. Hart,[41] Assoc. M. ASCE.—In the sentence immediately following Eq. 34 Mr. Knapp states, for a condition of equilibrium: "The depth along the outer wall should be $h_o + \frac{V^2 b}{2 r g}$." The next sentence is considered as meaning that the depth calculated by Eq. 32*b*, for conditions of nonequilibrium, amounts to nearly $h_o + 2\left(\frac{V^2 b}{2 r g}\right)$. Mr. Knapp then concludes: "The disturbance pattern that oscillates about the equilibrium depth, therefore, has a wave length of $2\,\theta_o$ and an amplitude of $\frac{V^2 b}{2 r g}$."

On the basis of the foregoing statements, the writer infers that the depth of water, at the position of first maximum depth, may correctly be taken as the sum of the normal depth, h_o, and an amount of superelevation equal to $\frac{V^2 b}{r g}$; and that the value so obtained will not be substantially different from that value calculated by the equation derived from wave mechanics. As a matter of fact, the validity of the expression, $\frac{V^2 b}{r g}$, as a measure of maximum superelevation for supercritical flow, is not essentially dependent upon proof by comparison. The value can be obtained by equating expressions for work and potential energy without requiring specific knowledge concerning wave fronts, pressure differences, and other details of the disturbance pattern.

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[41] Design Engr., Los Angeles County Flood Control Dist., Los Angeles, Calif.

In Fig. 68, which shows the cross section of a channel entering a circular curve without transition, the line AB represents the water surface of a unit length prism at the end of the tangent. Immediately after the prism passes the beginning of the curve, centrifugal force and wall reactions cause the water surface at the outer wall to rise above point B and the water surface at the inner wall to drop below point A. The center of mass of the prism moves from its original position, E, toward the outer wall. The work being done by the centrifugal force is being stored in the prism as kinetic energy and potential energy. When the cross slope of the water surface reaches the slope for static equilibrium, the kinetic energy is maximum and the movement continues. The maximum depth, $h_o + s$, will be reached when all kinetic energy has been converted to potential energy. At this time, the total work done by the centrifugal force in distorting the unit length prism to its final shape, A′B′CD, will equal the potential energy of the distorted prism with respect to its normal position of rest. Assuming that the mass of the prism can be considered as concentrated at its center, and since, by geometry, the horizontal and vertical distances between the original center, E, and the final center, F, are equal to $\frac{s\,b}{6\,h_o}$ and $\frac{s^2}{6\,h_o}$, respectively, it can be stated that

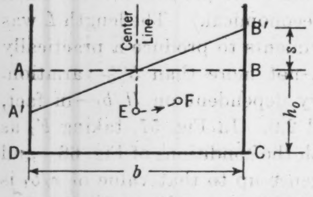


FIG. 68.—CHANNEL SECTION—SUPERELEVATION AT FIRST MAXIMUM DEPTH

$$\frac{m\,V_o^2}{r} \times \frac{s\,b}{6\,h_o} = m\,g \times \frac{s^2}{6\,h_o} \quad \text{.................(66)}$$

$$s = \frac{V_o^2\,b}{r\,g} \quad \text{.......................(67)}$$

and

$$h = h_o + \frac{V_o^2\,b}{r\,g} \quad \text{.......................(68)}$$

The foregoing derivation assumes that the mean velocity is constant throughout the reach considered and that the radius of the center of mass is that of the channel center line. It also assumes the line A′B′ as straight and as revolving about the intersection of AB and the channel center line. None of these assumptions is exactly correct but all of them are considered to be approximations within the range of permissible error.

One other assumption deserves mention. It is true, as Mr. Knapp states, that the individual filaments are not changed in direction until after reaching the wave front. However, these filaments in the central part of the channel will be deflected later and more sharply than those nearer the walls. Therefore, since the expression for work done is written for the entire reach within the indicated longitudinal limits, it appears reasonable to conclude that the mean value of the summation of all products of masses (m) and the accelerations will be numerically equal to $m\,V_o^2/r$. By analogy, the work performed in lifting a bag of buckshot through any vertical distance will be equal to the total work done in raising the bag, and then its contents, shot by shot, regardless of variability in the separately applied accelerative and decelerative forces.

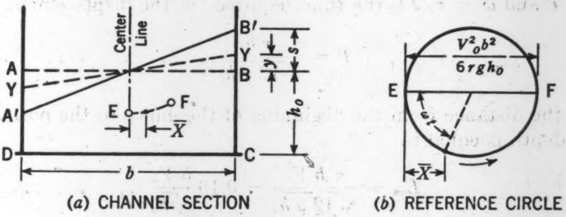
Eq. 68 is derived for positions of A′ between A and D, only (see Fig. 68), and should not be used where the flow prism becomes triangular. The writer wonders whether Eq. 32*b* does not become less applicable when the prism is triangular, inasmuch as the author states that "* * * the analysis assumes a true rectangular channel." However, if the general analysis used in the derivation of Eq. 68 be applied to conditions of flow in triangular prisms, it will be found that the maximum depth attains a value of

$$h = \frac{3}{4}\left[(h_o + s) + \sqrt{(h_o + s)^2 - \frac{32}{9}\,h_o\,s}\,\right] \ldots\ldots\ldots\ldots (69)$$

in which

$$s = \frac{V^2_o\,b}{r\,g} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (70)$$

In Mr. Knapp's paper, the position of the first maximum depth is determined by wave pattern geometry. The values obtained from Eq. 33 can be shown to check, quite closely, the values determined from an equation derived by a method involving simple harmonic motion. In Fig. 69(*a*), line AB represents the original water surface; line A′B′, the water surface at the position of first maximum depth; and line Y–Y, the water surface for any intermediate value of superelevation designated as *y*. At any instant of time, the resultant horizontal force acting upon the prism of unit length equals the centrifugal force minus the net reaction from the walls. For a value of superelevation equal to *y*, this reaction can be shown to equal $2\,w\,y\,h_o$, in which *w* is the unit weight of water. The value of the resultant horizontal force can therefore be stated as $w/g\;b\;h_o\;V^2_o/r - 2\,w\,y\,h_o$. This expression, when divided by the mass, gives the corresponding acceleration,



(*a*) CHANNEL SECTION (*b*) REFERENCE CIRCLE

FIG. 69.—POSITION OF FIRST MAXIMUM DEPTH

$$a = \frac{V^2_o}{r} - \frac{2\,g\,y}{b} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (71)$$

Eq. 71 can also be expressed in terms of the horizontal displacement of the center of mass from its original position, E. This displacement, designated as

$\bar{X}$, equals $\frac{y\,b}{6\,h_o}$. By substitution in Eq. 71,

$$a = \frac{V^2_o}{r} - \frac{12\,g\,h_o\,\bar{X}}{b^2} \dots\dots (72)$$

In either Eq. 71 or Eq. 72, the acceleration is zero when the superelevation equals $\frac{V^2_o\,b}{2\,r\,g}$. Either equation obeys the law of simple harmonic motion since the acceleration, on either side of the position of static equilibrium, is directly proportional to the displacement. Therefore, in accordance with the familiar relationship, the horizontal projection of the motion of the center of mass is the projection upon the horizontal diameter of a point moving at constant velocity on the circumference of a circle having a radius of one half the total horizontal displacement. As indicated in Fig. 69(*b*), the radius equals $\frac{V^2_o\,b^2}{12\,r\,g\,h_o}$. At any time, *t*,

$$\bar{X} = \frac{V^2_o\,b^2}{12\,r\,g\,h_o}\,(1 - \cos\omega\,t) \dots\dots (73)$$

and

$$a = \frac{d^2\bar{X}}{dt^2} = \frac{V^2_o\,b^2}{12\,r\,g\,h_o}\,\omega^2 \cos\omega\,t \dots\dots (74)$$

Equating Eq. 74 to Eq. 71, and imposing the conditions for the total displacement ($t = T$ and $\omega = \pi/T$), the time required for the displacement is

$$T = \frac{\pi\,b}{\sqrt{12\,g\,h_o}} \dots\dots (75)$$

Therefore, the distance from the beginning of the curve to the position of first maximum depth is equal to

$$L_m = \frac{\pi\,b\,V_o}{\sqrt{12\,g\,h_o}} = 0.16\,\frac{b\,V_o}{\sqrt{h_o}} \dots\dots (76)$$

This derivation has been made under the same general assumptions as described in the text following Eq. 68.

Eqs. 66 to 76 are for circular curves without approach transitions. The oscillations about the equilibrium cross slope are due to the abruptness with which the centripetal forces are applied to the stream filaments. One of the effective methods for the prevention of oscillations is that of using a spiral approach easement, in which the radius decreases at a gradual rate from infinity to that of the circular curve. If this varying radius be introduced in Eq. 72, with the length of spiral expressed as $V\,t$, the resulting differential equation involving $d^2\bar{X}/dt^2$ can be solved to show that the length of spiral required to produce the equilibrium depth at the beginning of the main curve will be twice the value of L_m in Eq. 76. This solution, of course, is subject to such qualifications as may be justified by the approximate nature of the previously described assumptions. Actually, the filaments in the central portion of the flow will not follow the channel curvature until the wave fronts are reached and so absolute equilibrium is not quite achieved.

Comparisons.—The first paragraph of this discussion quoted a statement by Mr. Knapp with regard to the amplitude and the wave length of the oscillation which occurs for supercritical flows on circular curves without approach easements. The writer has endeavored to present a rational proof of Mr. Knapp's evaluation of the amplitude and also, without recourse to wave mechanics, to determine the distance to the position of the first maximum depth, which occurs for a half wave length of the oscillation. Despite his statement relative to the amplitude, Mr. Knapp has presented Eq. 32*b*, and not $h_o + \frac{V^2_o\, b}{r\, g}$, as a measure of total depth. Therefore, it is interesting to compare the numerical results obtainable from the two equations. The solutions of Eq. 33, involved in the solutions of Eq. 32*b*, will also be compared with the values derived from Eq. 76.

TABLE 7.—COMPARISONS OF CALCULATED VALUES

β_o (1)	$\frac{b}{r}$ (2)	θ_o (3)	θ'_o (4)	$\frac{s}{s'}$ (5)	$\frac{s}{h_o}$ (6)	$\frac{s'}{h_o}$ (7)	$\frac{s - s'}{h_o}$ (8)	$\frac{h}{h'}$ (9)
12°	0.02	5° 00′	5° 19′	0.961	0.463	0.481	−0.018	0.988
	0.04	10° 00′	10° 27′	0.900	0.925	1.028	−0.103	0.949
	0.06	....	15° 19′	0.895[a]	1.448[a]	1.618	−0.170[a]	0.935[a]
20°	0.02	3° 23′	3° 07′	1.109	0.171	0.154	+0.017	1.014
	0.04	6° 05′	6° 09′	1.092	0.342	0.313	+0.029	1.022
	0.06	9° 07′	9° 05′	1.080	0.513	0.475	+0.038	1.026
28°	0.02	2° 13′	2° 08′	1.279	0.091	0.071	+0.020	1.018
	0.04	4° 26′	4° 13′	1.280	0.181	0.142	+0.039	1.035
	0.06	6° 38′	6° 15′	1.282	0.272	0.212	+0.060	1.049
36°	0.02	1° 46′	1° 34′	1.532	0.058	0.038	+0.020	1.019
	0.04	3° 32′	3° 05′	1.551	0.116	0.075	+0.041	1.038
	0.06	5° 18′	4° 35′	1.559	0.174	0.111	+0.063	1.056

[a] Computed by the formula for triangular prism flow (Eq. 69), since Eq. 68 cannot be applied when sin β is less than $\sqrt{b/r}$.

As shown in Table 7, the comparisons have been made for a wide range of variable conditions. With limits for β_o set at 12° and 36°, provision is made for velocities ranging between 4.8 and 1.7 times the critical velocities. The ratios of b/r cover the cases in which the radii vary between 50 *b* and 16.7 *b*. The equations have been compared with respect to θ_o and also for various ratios involving the superelevation, *s*, and the maximum depth, *h*. For convenience, the values of θ_o, *s*, and *h* as derived from Eq. 33 and Eq. 32*b* are represented as θ'_o, *s′*, and *h′*.

The values in Table 7 have been obtained from the equations in the following modified forms: Since $L_m = \frac{\pi\, b\, V_o}{\sqrt{12\, g\, h_o}} = \frac{\pi\, r\, \theta_o}{180}$,

$$\theta_o = \frac{51.96\, b/r}{\sin \beta_o} \qquad (77a)$$

and

$$\theta'_o = \tan^{-1} \frac{2\, b/r}{(2 + b/r) \tan \beta_o} \qquad (77b)$$

Since $\sin \beta_o = \sqrt{g\,h_o}/V_o$, Eqs. 67, 68, and 32*b* can be recast into such form as to obtain the following ratios:

$$\frac{s}{s'} = \frac{b/r}{\sin^2(\beta_o + \theta'_o/2) - \sin^2 \beta_o} \qquad (78a)$$

$$\frac{s}{h_o} = \frac{b/r}{\sin^2 \beta_o} \qquad (78b)$$

$$\frac{s'}{h_o} = \frac{\sin^2(\beta_o + \theta'_o/2) - \sin^2 \beta_o}{\sin^2 \beta_o} \qquad (78c)$$

and

$$\frac{h}{h'} = \frac{b/r + \sin^2 \beta_o}{\sin^2(\beta_o + \theta'_o/2)} \qquad (78d)$$

The comparative values for the angular distances to the positions of first maximum depths are shown in Cols. 3 and 4, Table 7. The best agreements are found in the high and middle range velocities. The greatest contrast is for sharpest curvature and slowest velocity where θ_o is 16% larger than θ'_o. The value of θ_o for $\beta_o = 12°$ and $b/r = 0.06$ is omitted since triangular prism flow is beyond the range for the application of Eq. 77*a*.

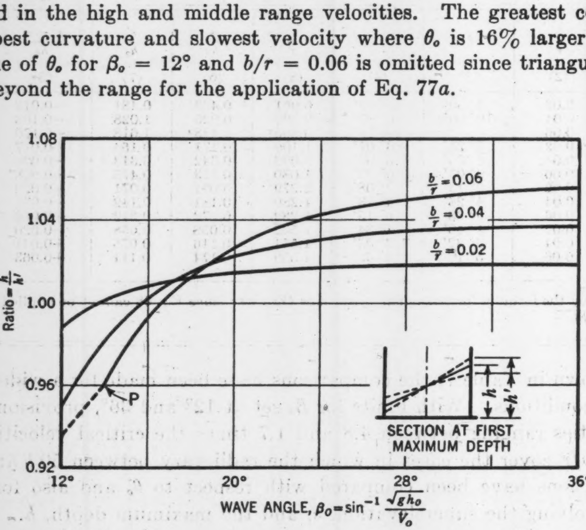


FIG. 70.—RATIO OF CALCULATED MAXIMUM DEPTHS

The ratios of superelevations are entered in Col. 5, Table 7. It will be noticed that the poorest agreements are for the slowest velocities. Although the comparative values for $\beta_o = 36°$ differ quite widely, the differences are not of much practical significance since the superelevations are small. This statement is borne out by the ratios given in Cols. 6, 7, 8, and 9, Table 7. In the entire range studied, values of maximum depth differ by a maximum of about 6%.

The depth ratios shown in Col. 9, Table 7, plus supplementary ratios between $\beta_o = 20°$ and $\beta_o = 12°$, have been plotted in Fig. 70. Equality of

Eq. 32*b* and Eq. 68, for the range studied, lies approximately between wave angles of 14° and 17°. The value of b/r apparently has very little effect upon the ratios when the wave angle equals 18°. The discontinuity at point P, Fig. 70, is due to the substitution of Eq. 69 for Eq. 68.

All the foregoing comparisons have been made with respect to values obtained by calculation. Table 2 furnishes experimental data which permit Eq. 32*b* and Eq. 69 to be compared with reference to observed values of first maximum depth. Table 8 makes this comparison. The values entered in Cols. 1 to 4 and Col. 7 are all obtained from Table 2 or derived from other furnished data. Since the four runs were made for a ratio of $b/r = 0.06$, the values of h/h' in Col. 5, Table 8, are obtained from Fig. 70. Col. 6 shows the values for maximum depth as of Eq. 69. In Cols. 7 and 8, ratios of measured maximum depths to calculated maximum depths are given for Eq. 32*b* and for Eq. 69, respectively.

TABLE 8.—COMPARISONS WITH OBSERVED VALUES

Run No. (Table 2) (1)	β_o (2)	Computed h' (Table 2) (3)	Measured h_m (Fig. 19) (4)	$\frac{h}{h'}$ (Fig. 70) (5)	$h = h' \times \frac{h}{h'}$ (6)	$\frac{h_m}{h'}$ (Table 2) (7)	$\frac{h_m}{h}$ (8)
117C	12° 39′	0.387	0.336	0.943	0.365	0.868	0.921
120A	12° 27′	0.496	0.450	0.940	0.466	0.905	0.968
119D	12° 29′	0.584	0.550	0.940	0.549	0.942	1.002
32A	11° 48′	0.693	0.700	0.932	0.646	1.010	1.084

In closing, the writer wishes to comment upon two of the statements made in the Symposium with respect to banking. Mr. Knapp states (sentence following Eq. 35): "* * * that, with the proper use of banking, all the force necessary to change the direction of the flow is supplied by the cross slope." If "cross slope" means the cross slope of the bottom, this statement is true provided that the slopes of the bottom and the water surface are parallel, and provided that the water prism possesses no transverse momentum due to any lack of equilibrium in upstream sections. On the other hand, if "cross slope" means the cross slope of the water surface, the statement would be true even if the banking be improper or if it be lacking entirely. The centripetal force which changes the direction of flow is the algebraic summation of wall reactions plus the horizontal component of the floor reaction. In the ideal case where the bottom slope is parallel to the water surface the wall reactions will be equal and opposite in direction, and the floor will be furnishing the entire turning force. In the general case (that is, where the banking is not in strict conformity with the ideal requirement), the walls must assist in the turning.

On the practical side, in some instances, the use of banking may have rather serious disadvantages. One disadvantage is that the low water flow is concentrated against the base of the inner wall. In case the flow carries sand and persists for considerable periods of time, erosion of concrete surfaces is likely to occur. Another disadvantage becomes apparent in those cases in which the channel grade is not steep enough to prevent the occurrence of an adverse grade along the inner wall in the downstream transition.

Fred W. Blaisdell,[42] Assoc. M. ASCE.—The papers of the Symposium are a fine addition to the small amount of literature available on high-velocity flow in open channels. They should serve to acquaint those who have never dealt with high-velocity flows with the entirely different methods that must be used handling flows at supercritical velocities as compared with the methods used when the velocity is in the more common and more familiar subcritical range. The presentation, in such a practical way, of information that can be used for the preliminary design and, in some cases, for the final design of hydraulic structures subjected to high-velocity flows is particularly welcome to the designer who has had to guess rather than to estimate what to expect both qualitatively and quantitatively and who has undoubtedly guessed wrong at times because of the lack of information such as is presented herein.

Comments on Theory.—The writer is favorably impressed with Mr. Ippen's efforts to make the descriptions of the mechanics of supercritical flow both clear and complete. Fig. 3, relating the deflection angle with the Froude number and the relative depth, should be particularly useful in preliminary investigations. It should be remembered that, even if Eqs. 10*a* and 10*b* can be used to give the depth, they must be used with a plan of the structure, and also with Eq. 5 to determine where these depths occur. The graphical method of characteristics, of course, gives both the depth and its location simultaneously and is to be preferred when determining the flow surface if it is safe to assume that there is no energy dissipation.

Mr. Ippen's statements (under the heading, "Mechanics of Wave Propagation in Supercritical Flow: Wave Interference") to the effect that a successful analysis of certain problems requires a fair prediction of the initial Froude number $\mathbf{F}_1$ is interesting. The writer interprets this to mean that a fairly good determination of $\mathbf{F}_1$ must be made. Theoretically it seems that the exact value of $\mathbf{F}_1$ would be required. However, since the simplifying assumptions used in developing the theoretical solution are at variance with what actually occurs, extreme efforts to secure an exact determination of $\mathbf{F}_1$ could hardly be justified. Nevertheless, some effort should be made to secure the best possible value of $\mathbf{F}_1$ for even the best may not be too good; the most probable value of $\mathbf{F}_1$ should be used, not the maximum or minimum expected values. The selection of the proper friction factor is only one obstacle to an exact determination of $\mathbf{F}_1$, and model tests are of little help here because the scaling up of many depths depends on the selection of the proper friction factor in both the model and its prototype. It should be realized that an error in determining $\mathbf{F}_1$ results in a shift in location of the disturbance lines and may cause the counterdisturbance source, which would theoretically cancel the original disturbance, to be improperly located. Although the counterdisturbance, unless grossly mislocated, should have some beneficial effect in reducing the size of the original disturbance, efforts should be made to locate the counterdisturbance as nearly as possible in its correct position and this requires the determination of the most likely value of $\mathbf{F}_1$.

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[42] Project Supervisor, Soil Conservation Service, U. S. Dept. of Agriculture, St. Anthony Falls Hydr. Laboratory, Minneapolis, Minn.

The graphical method of characteristics is certainly a valuable aid to the solution of problems involving disturbances in high-velocity flows. It is possible, as Mr. Ippen states, to solve problems quite easily once the fundamental procedures are understood. In fact, in less than a week the writer trained a subprofessional draftsman to solve problems of channel expansions by the method of characteristics. As a result of this experience the writer prepared detailed instructions for the use of the method of characteristics in solving problems involving expanding transitions.[43] However, these instructions were based on the paper by Ernst Preiswerk,[9] published in 1940. Although identical results can be obtained with both Mr. Ippen's method[10] and Mr. Preiswerk's method, the latter uses the critical depth $2\,H/3$ as his reference head instead of the specific head H used by Mr. Ippen. This results in Mr. Preiswerk's epicycloids being drawn between the limits $\sqrt{3}\ \bar{V} = 1$ and $3\ \bar{V} = \sqrt{3}$.

In Fig. 7 Mr. Ippen shows a scale of h/H which may be computed from Eq. 13. A scale of $\bar{V}$ can also be added to Fig. 7 although the writer has found little need for it. However, the writer has found a scale of $\mathbf{F}$, which can also be added to Fig. 7, to be most useful. In fact, the writer's instructions call for scales of h/H, $\bar{V}$, and $\mathbf{F}$ engraved on a transparent sheet. The equation for $\bar{V}$ in terms of $\mathbf{F}$ is derived as follows: By definition, $\bar{V} = V/\sqrt{2\,g\,H}$ and $\mathbf{F} = V/\sqrt{g\,h}$. Solving for V, and equating,

$$V = \bar{V}\,\sqrt{2\,g\,H} = \mathbf{F}\,\sqrt{g\,h}\ldots\ldots\ldots\ldots\ldots\ldots(79)$$

Solving for $\bar{V}$, and substituting for h/H its value as given in Eq. 13, it is found that

$$\bar{V} = \sqrt{\frac{\mathbf{F}^2}{2 + \mathbf{F}^2}}\ldots\ldots\ldots\ldots\ldots\ldots(80)$$

In practice the writer prefers to construct a family of epicycloids for each problem so that the ellipse will lie over, or equidistant from, the intersections of the epicycloids rather than near the intersections as Mr. Ippen states in the paragraph following Eq. 14. This greatly eases the graphical solution and increases its accuracy. The starting point for the family of epicycloids is found by measuring the distance $\bar{V}$ (or h/H or $\mathbf{F}$) from the origin of the diagram parallel to the initial flow direction. The first epicycloids are drawn in both directions through this point, the remaining epicycloids being drawn at equal angles apart. A typical solution is shown in Fig. 71. It seems likely that Mr. Ippen himself uses this method since in Fig. 38 of the third paper the characteristic diagram apparently was especially constructed instead of being a part of the diagram shown in Fig. 7.

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[43] "Graphical Construction of the Flow in Transitions by the Method of Characteristics," by Fred W. Blaisdell, *Manuscript Report MN-R-3-24*, SCS, U.S.D.A., in cooperation with the Minnesota Agri. Experiment Station and the St. Anthony Falls Hydr. Laboratory, Univ. of Minnesota, Minneapolis, Minn., May, 1944 (revised February, 1946).

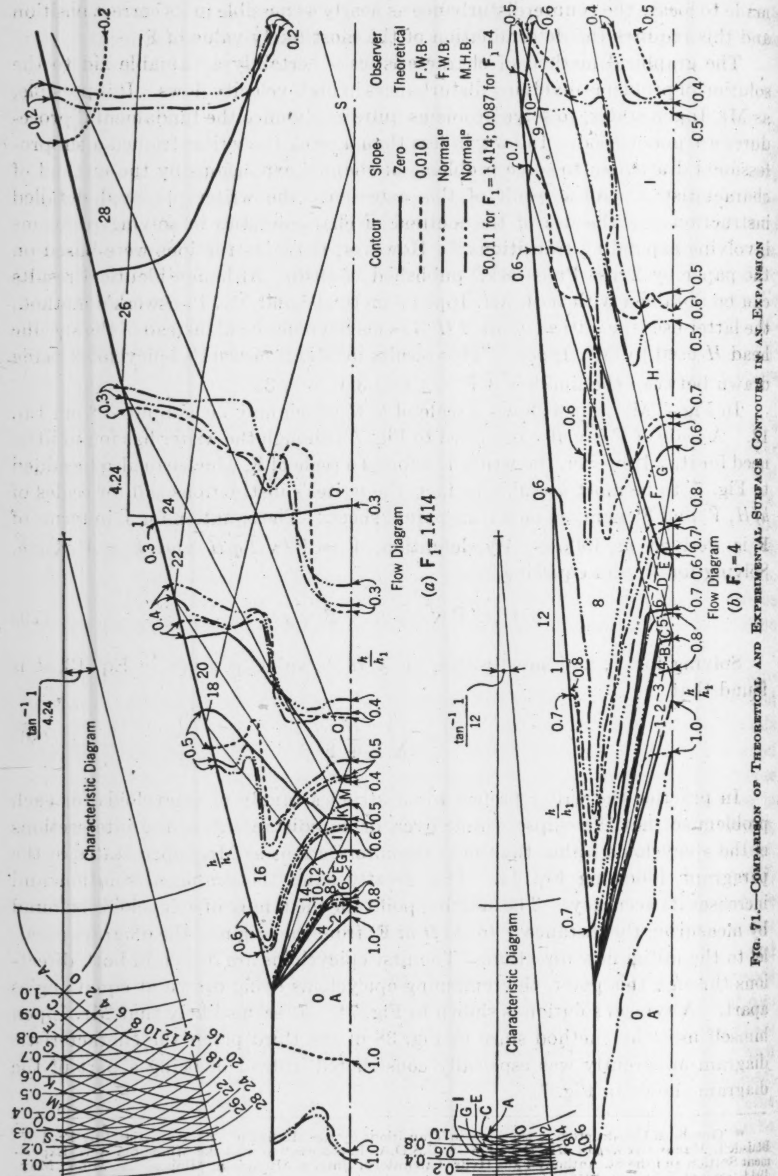

Characteristic Diagram
$\tan^{-1}\frac{1}{4.24}$
$\frac{h}{h_1}$
Flow Diagram
(a) $F_1 = 1.414$
Characteristic Diagram
$\tan^{-1}\frac{1}{12}$
Flow Diagram
(b) $F_1 = 4$
Contour | Slope | Observer
Zero | Theoretical
0.010 | F.W.B.
Normal* | F.W.B.
Normal* | M.H.B.
*0.011 for F_1 = 1.414; 0.087 for F_1 = 4
O A C E G I K M O Q S
H F G D E B C A
4.24
12

FIG. 71.—COMPARISON OF THEORETICAL AND EXPERIMENTAL SURFACE CONTOURS IN AN EXPANSION

It is interesting to note that each point on the coordinate system of Fig. 38(*a*) has a corresponding point on the coordinate system of Fig. 38(*b*). This permits an instant determination of the value of h/H at any point in Fig. 38(*a*) simply by referring to the correspondingly numbered point in Fig. 38(*b*). For example, it can be seen that in area F_o, $h/H = 0.234$ which has been plotted at Stations 1, 3, 5.5, etc., in Fig. 38(*c*). From Fig. 38(*b*) it can be seen that $h/H = 0.234$ also in areas C_6, D_4, E_2, etc. Incidentally, the theoretical surfaces in Figs. 38(*c*) and 38(*d*) could have been drawn as smooth lines instead of stepped lines; the steps are a result of the method of solution since θ was varied in finite steps—that is, the increments were in terms of $\Delta\theta$ rather than $d\theta$. However, the vertical lines representing the shock-wave fronts would necessarily be retained.

It would be helpful if Mr. Ippen would define the magnitude of the "small error" which, as he states, results when Eq. 19*b* is substituted for the exact form (Eq. 19*a*).

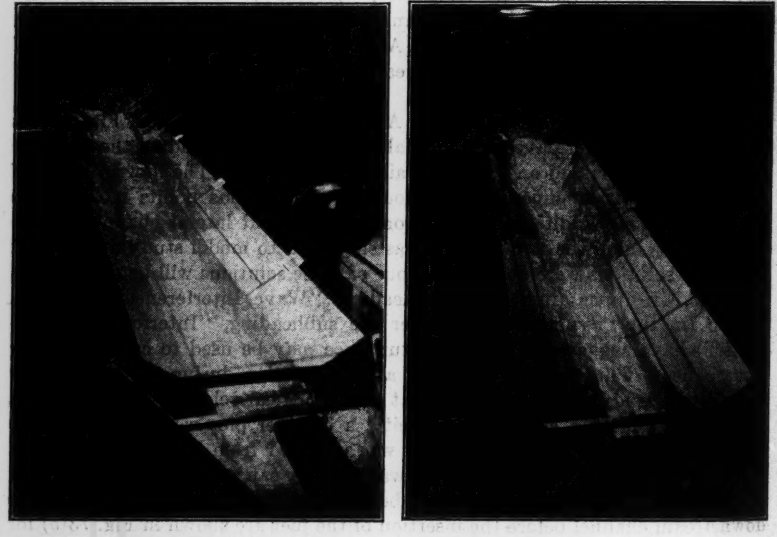
Channel Junctions.—It is too much to expect that this Symposium, composed of four papers treating only as many different phases of the problem, could be an "inclusive treatise" on the subject of high-velocity flow in open channels, as was hoped in the "Foreword." One major problem which the authors have not mentioned is the joining of two streams, one or both of which may be flowing at supercritical velocities. Several problems of this type were studied on models during late 1948 and 1949 by C. E. Bowers, Jun. ASCE, formerly research associate at the St. Anthony Falls Hydraulic Laboratory of the University of Minnesota, in Minneapolis, under the general supervision of the writer.

At Whiting Field Naval Auxiliary Air Station, near Pensacola, Fla., it was proposed to construct paved trapezoidal open channels to conduct the drainage water from the plateau on which the airfield is located. The flow in many of these channels is at supercritical velocities. At various points most of the channels pick up additional water from terraces and lateral channels. The problem of joining the two streams was subjected to model study and several successful solutions were obtained Four of these solutions will be described.

Both Mr. Ippen (under the subheading, "Wave Interference") and Mr. Knapp (in the "Synopsis" and under the subheading, "Interference Treatments") have suggested that one disturbance may be used to cancel another disturbance. This method was tried at one junction during the exploratory stage of the study; a channel flowing at a supercritical velocity was joined by a channel flowing at a subcritical velocity, a diverter pier was inserted in the main channel, and the diverted flow was reflected from the vertical wall as shown in Fig. 72. That this reflected wave largely canceled the wave caused by the flow entering from the lateral can be seen in Fig. 73(*a*). The waves in the downstream channel before the insertion of the pier are shown in Fig. 73(*b*) for purposes of comparison. Greatly improved flow conditions in the downstream channel were obtained for all flows as long as the flows from the two channels were approximately proportional to their design flows.

The design recommended for this particular junction, which originated with Mr. Bowers, is shown in Fig. 74(*a*) and the resulting flow conditions are shown

FIG. 72.—COUNTERDISTURBANCE PIER TYPE OF CHANNEL JUNCTION

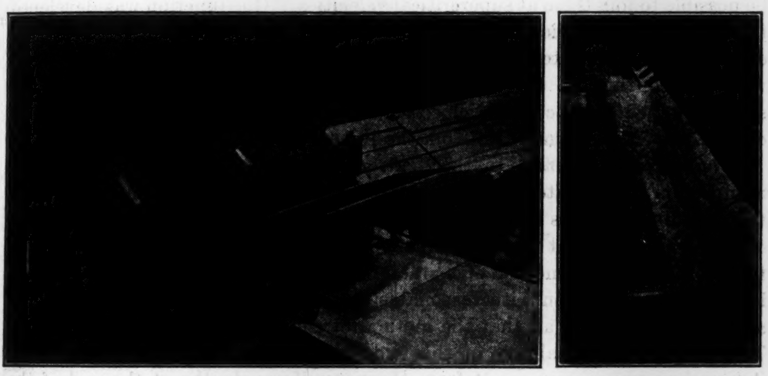


(a) Counterdisturbance Pier Design Almost Completely Cancels Out Cross Waves

(b) Serious Waves Are Present When a Simple Junction is Used

FIG. 73.—FLOW CONDITIONS IN CHANNEL DOWNSTREAM FROM JUNCTION; CONTOURS ARE AT 2-FT VERTICAL INTERVALS

in Fig. 74(*b*). The long low piers which are submerged at the design flow have much the same effect as do the multiple curved vanes mentioned by Mr. Knapp. They are remarkably effective in eliminating disturbances in the downstream channel at the design flow as well as at flows less than the design value, and for hydrographs which are so out of synchronism that there may be little or no flow from one channel while the other is flowing at its design capacity. For any particular combination of flows from the channels the piers could be much shorter than those shown in Fig. 74(*a*). However, for other combinations of flows the pier locations would have to be shifted along the channel to be of maximum effectiveness. The result is that long piers are required to cover the anticipated range of flow conditions.



(*a*) Piers and Vertical Wall Opposite Junction

(*b*) Flow Conditions in Channel Below Junction

FIG. 74.—SUBMERGED PIER TYPE OF JUNCTION DESIGN; CONTOURS ARE AT 2-FT VERTICAL INTERVALS; HEAVY LINE REPRESENTS TOP OF RECOMMENDED SIDE WALL

Another effective type of junction is known as the transverse weir type. In this design the lateral channel is carried over the main channel and the water from the lateral channel is dumped onto the water surface in the main channel. Several precautions are required, however: (1) The bottom of the "bridge" must be high enough above the water surface in the main channel so that waves and surges will never strike the transverse weir structure and cause a hydraulic jump to form that might overtop the channel sides; (2) the width of the nappe where it strikes the water surface in the main channel must be fully equal to the width of the water surface to prevent the formation of strong disturbances; (3) the ratio of the flow over the weir to the flow in the main channel should be quite small, probably; and (4) the Froude number in the main channel must be high enough to insure that the velocity will not decrease from supercritical to subcritical at the junction. Regarding precaution (3), ratios tested for different locations at Whiting Field had values between 0.16 and 0.06. From what little information is now available, it appears that this ratio should be less than 0.08. Regarding precaution (4), Froude numbers between 2.9 and 4.0 gave satisfactory results. A test with a Froude number of 1.7 and a discharge ratio of 0.06 yielded results that were barely acceptable. At two transverse weir junctions it was necessary to recommend a local steepening of the channel grade in order to increase the Froude number so that satisfactory flow conditions could be obtained. In addition to these precautions, the use of one of more of the submerged piers just described may be desirable to improve flow conditions further. The transverse weir type of junction originated with Arthur F. Moratz, head, District Operations Design and Construction, Soil Conservation Service, Milwaukee, Wis., who was in charge of the design of the structures used in the drainage system at Whiting Field.

In cases where the Froude numbers of the joining streams are low it may not be possible to join them at supercritical velocities. One junction was developed for use at Whiting Field where the Froude numbers were 2.8 and 3.9 in the entering main and lateral channels, respectively. For this junction the flows were passed through the hydraulic jump upstream from the junction, the streams joined at subcritical velocities, and the flow was accelerated to supercritical velocities downstream from the junction.

Channel Expansions.—In the "Introduction," the authors of the fourth Symposium paper state that the graphical method is limited in its application to expanding channels because it offers no clue as to the stability or instability of a hydraulic jump at the downstream end of the expansion. The writer feels that, if the graphical method of characteristics can be used to predict the surface levels in the transition with reasonable accuracy, then it is not limited in its application. This is all that the method of characteristics is supposed to do. The stability or instability of the hydraulic jump as defined in the fourth Symposium paper may result from the nonuniform depths at the end of the transition; but the performance of the jump is a separate, although related, problem that is beyond the scope of the method of characteristics.

In 1944 the writer and his associates compared selected cross sections of a jet expanding freely on a level floor with the cross sections determined by the method of characteristics. The results of this study throw some light on the effect that the assumptions used in developing the method of characteristics have on its practical application. The experimental data used in this study were obtained by Mr. Bhoota[30] at the State University of Iowa, in Iowa City. Presumably a part of this material is presented in Fig 54. The complete data cover streams having initial cross section width-depth ratios of 2, 4, and 8 and initial Froude numbers increasing in geometric progression by $\sqrt{3}$ from 1 to 27. Good agreement was obtained between the theoretical and experimental cross sections. The average difference at 78 cross sections between the data obtained by Mr. Bhoota and that computed by the method of characteristics was only 2% of the initial cross-sectional area. Although one difference was as high as 27%, there were only three differences greater than 9%, and all of these were for the highest Froude number where the greatest difficulty could be anticipated in obtaining accurate water-surface measurements due to the rough surfaces that undoubtedly existed. In general the areas obtained by Mr. Bhoota increased relative to the characteristic areas with distance along the jet. This tendency could possibly be ascribed to the effect of friction, which is neglected in the theoretical analysis. Nevertheless, it appears that the comparison of the theoretical and experimental results for this particular set of data is excellent.

The writer has been interested in expanding transitions since discovering some years ago that the use of expansions ahead of a stilling basin can produce significant economies in the cost of the complete structure.[44] The studies on transitions which followed this discovery covered the determination of the best side-wall flare for expansions having straight side walls and also the determination of surface contours and flow properties along transitions laid on a slope of 1% where the side-wall flare[44] was 1 in 3 **F**. The reason for selecting this side-wall flare is that the tests showed the most rapid side-wall flare to be 1 in 3 **F** to insure a reasonably good surface configuration and eliminate the higher waves. Although arrived at independently, this is also the flare recommended by Mr. Bhoota.[30] Subsequently, the writer conducted a few tests with the channel floor laid on the normal slope for the approaching stream and these tests were continued by Merlin H. Berg, Assoc. M. ASCE, in 1948 as a thesis study.[45]

To compare the experimental data with the theoretical predictions the writer has computed, graphically, the theoretical surface profiles for two values of **F**. The surface contours resulting from these computations are presented in Fig. 71. Also plotted in Fig. 71 are surface contours obtained by the writer on 1% and normal slopes and those obtained by Mr. Berg using normal slopes. The work of two observers is included to show approximately what precision can be expected when two supposedly identical test runs are made. Some of the contours have been omitted to avoid confusion. In studying Fig. 71 it should be remembered that the water-surface slopes are quite flat relative to the channel floor, especially for the lower relative depth contours, and that a considerable horizontal displacement in the position of any two contours denoting a given depth may indicate only a small difference between the depths at any specific point.

Surface contours for those transitions located on normal slopes would presumably agree better with the theoretical predictions than do those contours obtained when the floor is level. This is because (as noted by Mr. Ippen, under the subheading, "Basic Properties of Standing Waves," and by Mr. Knapp when discussing assumptions (*c*) and (*d*), under the heading, "Calculation of Liquid Surface Profiles Along the Walls") the normal slope should largely compensate for the friction and produce, in effect, flow on the frictionless surface used in developing the theory. It can be seen in Fig. 71(*b*) that the agreement between the theoretical and experimental contours is considerably improved if the transition is located on a normal slope. The difference between the normal and 1% slopes in Fig. 71(*a*) is not sufficient to cause a significant difference between the contours there. The writer is unable to understand why the authors of the fourth Symposium paper did not obtain similar results.

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[44] "Flow Through Diverging Open Channel Transitions at Supercritical Velocities," by Fred W. Blaisdell, *Technical Publication No. 76*, SCS, U.S.D.A., Washington, D. C., April, 1949.

[45] "Supercritical Flow in Straight-Wall Diverging Channels on Normal Slopes," by Merlin H. Berg, thesis presented to the University of Minnesota, at Minneapolis, Minn., in August, 1948, in partial fulfilment of the requirements for the degree of Master of Science.

Eq. 48 is presented as the most satisfactory form of the boundary curve; further discussion of the reasons for selecting this particular curve would be helpful. A question that arises in the mind of the writer is: Do the disturbance lines originating on one side of the transition strike the opposite wall, or is the wall curve formulated so as to eliminate the possibility of reflections?

As several of the authors have noted, much remains to be learned concerning flow at supercritical velocities. However, this should in no way detract from the fine beginning made by the Symposium authors.

J. H. DOUMA,[46] Assoc. M. ASCE.—Heretofore, hydraulic design of channel expansions, contractions, and bends has been based all too often on unproved empirical or rule-of-thumb criteria, not subjected to critical theoretical analysis. In some cases this results from a lack of research and development information; in other cases there seems to be lacking, also, an appreciation of the need for fundamental information on the part of personnel responsible for design.

In the third Symposium paper Messrs. Ippen and Dawson state that straight-wall contractions are the most satisfactory for supercritical flow in open channels. Past practice has conformed with this criterion in the case of contractions which are required when right-of-way restrictions or changes in invert grade of a concrete-lined, high-velocity channel make it economical to reduce the channel width. Here, invert grades are relatively mild, varying from about 0.5% to 3%, and contractions vary from about 10% to 25% of the channel widths. The practice has been to design straight-wall contractions of generous length to provide small angles of divergence, thereby causing the formation of small transverse waves that are confined within normal freeboard allowances. There is need for a simplified criterion to determine minimum desirable lengths of contractions for any combination of channel dimensions and flow elements, which it may be possible to develop from the data presented.

Another common type of channel contraction is encountered in the design of chute spillways, wherein it is usually economical to contract from a wide upstream section to a narrow downstream section. In this case, the width of control section or gate structure is determined by discharge requirements and economic considerations, and the width of downstream chute, by topography, character of foundation, and economical channel dimensions. The length of contraction, which is designed to provide efficient passage of flow from the control structure to the downstream chute, is dependent upon the percentage of contraction (which is sometimes as much as 300%, as in the case of transition from a gate structure 120 ft wide to a chute 40 ft wide) and upon the permissible height of transverse waves in the chute spillway.

When the channel downstream from a contraction is short, or when chute spillways discharge directly into side canyons, it is less costly to design the contraction with a large over-all angle of convergence and to provide greater wall heights to offer protection against overtopping by the resulting higher waves; but, when the downstream channel is long and overtopping by waves is likely to endanger the structure, it will be economical to provide a long contraction so that waves will be contained within the normal freeboard allowance. Infre-

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[46] Hydr. Engr., Office, Chf. of Engrs., Washington, D. C.

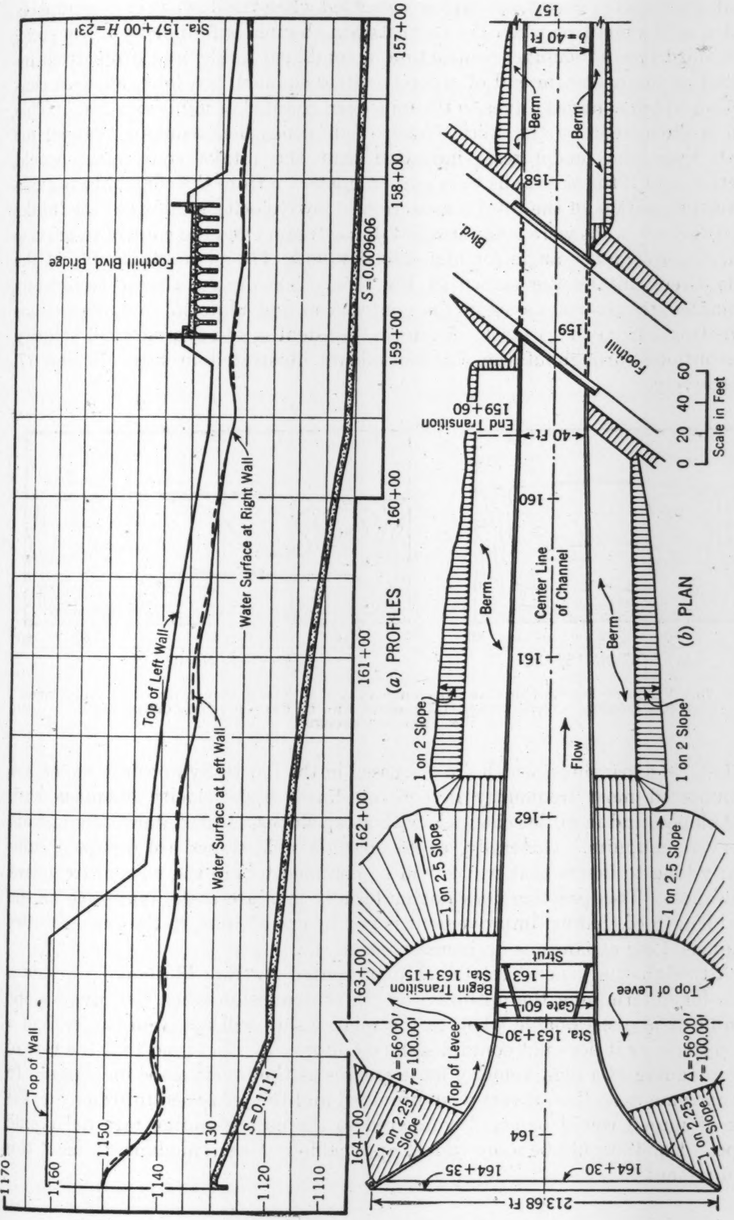
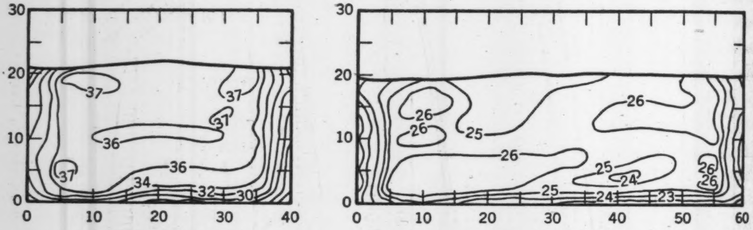

1170
1160
1150
1140
1130
1120
1110
Top of Wall
Sta. 157+00 H=23'
Foothill Blvd. Bridge
S=0.009606
Water Surface at Right Wall
Top of Left Wall
Water Surface at Left Wall
S=0.1111
157+00
158+00
159+00
160+00
161+00
162+00
163+00
164+00
(a) PROFILES
157
b=40 Ft
Berm
Berm
158
Blvd.
Foothill
159
End Transition
159+60
40 Ft
160
Center Line of Channel
Berm
Berm
161
1 on 2 Slope
1 on 2 Slope
Flow
162
1 on 2.5 Slope
1 on 2.5 Slope
Strut
163
Sta. 163+15
Begin Transition
Gate 60'
Sta. 163+30
Top of Levee
Top of Levee
Δ=56°00'
r=100.00'
Δ=56°00'
r=100.00'
1 on 2.25 Slope
1 on 2.25 Slope
164
164+35
164+30
213.68 Ft
0
20
40
60
Scale in Feet
(b) PLAN

Fig. 75.—Intake of Flood Control Channel, Lytle Creek, California

quent overtopping by waves may be permitted when the downstream channel is located in deep cut or when the downstream channel is founded on firm rock.

A third type of channel contraction, a combined intake and drop structure located at the upstream end of a flood control channel, is a long, efficient contraction which confines waves to the improved channel at minimum cost. This type is illustrated by the Lytle Creek (California) flood-control channel for which hydraulic model tests indicated that the intake contraction would function best if the side walls were curved upstream from the control to provide a large percentage of the total contraction in low-velocity sections of the intake and if the side walls were made straight downstream from the control to provide a small convergence angle for high-velocity flow. The plan and profile of the Lytle Creek intake are shown in Fig. 75. This design almost completely eliminated transverse waves in the contraction and uniform channel sections downstream from the control. Uniform flow depths and symmetrical velocity distributions throughout the contraction are illustrated in Figs. 76 and 77, respectively.



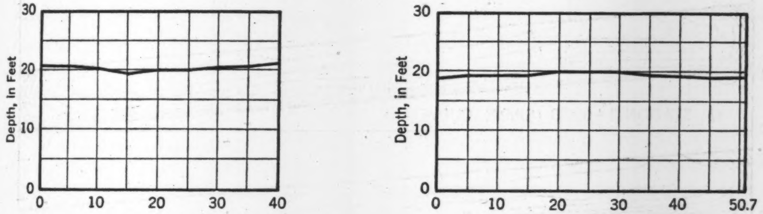
(*a*) STATION 159+60 (*b*) STATION 163+15

FIG. 76.—VELOCITY DISTRIBUTION AS MEASURED IN A 1:16 SCALE MODEL OF THE LYTLE CREEK (CALIFORNIA) CONTRACTION. (DISTANCES ARE IN FEET AND VELOCITIES ARE IN FEET PER SECOND)

Lateral expansions, which are discussed in the fourth Symposium paper, are encountered most frequently in concrete-lined, high-velocity channels with mild slopes, and in outlet channels with steep slopes, located downstream from reservoir conduits. Generally, expansions on mile slopes are designed sufficiently long to insure that waves will be confined within the normal freeboard allowance. This practice does not materially increase costs, inasmuch as the total length of channel improvement is not increased; but, in the case of outlet channels, long expansions do increase costs.

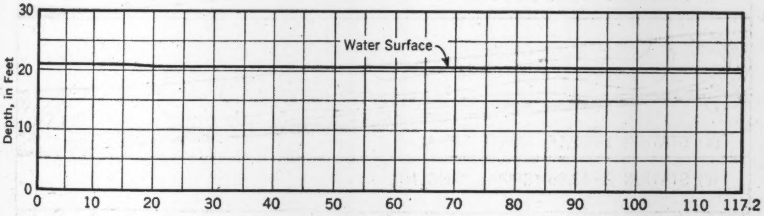
Although the general relationship presented by Eq. 48 should provide a basis for determining the optimum length of expansion when the invert slope is mild, it is questionable whether the relationship will apply in the case of a steep slope, as it does not contain a slope factor. It is believed that the rate of the spreading of a high-velocity jet decreases as the invert slope increases. In the extreme case, if no invert were provided and the jet issued into free air, the rate of spread would be small compared to the corresponding rate on a mild slope. Eq. 48 would be more widely applicable if it were modified to meet this requirement.

Mr. Knapp states, in the second Symposium paper, that compound and spiral transition curves prevent the formation of wave disturbances in channel bends and downstream tangents when flow is supercritical. However, as the experiments conducted at the California Institute of Technology did not include velocities lower than those corresponding to a Froude number of 1.5, it is not known whether the formulas presented for correction by compound curves are applicable when the flow is near that of the critical depth. Interference wave patterns produced by compound curves would be greatly complicated by undulating waves characteristic of flow at or near critical stage. Moreover, superelevation of the water surface is not eliminated, and attendant nonuniform depths of flow, caused by centrifugal force when the invert is not banked, require variable wall heights throughout the length of curve and unequal heights of the two walls at common stations, unless the inside wall is made higher than required by the design water surface to maintain equal heights.



(*a*) STATION 158+00

(*b*) STATION 161+50



(*c*) STATION 164+00

FIG. 77.—TRANSVERSE WATER-SURFACE PROFILES AS MEASURED IN A 1:16 SCALE MODEL OF THE LYTLE CREEK (CALIFORNIA) CONTRACTION

Nonuniformity of wall heights may be eliminated by banking the invert so that it is constantly parallel to the transverse slope of the design water surface in accordance with Eq. 35.

In order to produce gradual changes in flow, a transition, consisting of spiral curved walls and an invert whose transverse slope varies linearly from zero at the beginning of the spiral to the transverse slope required for equilibrium along the circular curve at the end of the spiral, is required upstream and downstream from each curve. On the basis of theoretical analysis by the Los Angeles County Flood Control District, the minimum length of spiral, L_s, required in precluding the formation of transverse waves in a rectangular curved reach may be expressed by

$$L_s = 1.82 \frac{b\,V}{(g\,d)^{\frac{1}{2}}} \quad \text{(81)}$$

in which d is the design depth of flow.

Although, theoretically, banking the invert for the design discharge will produce the desired correction without the use of spiral transition curves, for lower and higher discharges the transverse invert slope is incorrect, which results in flow disturbances. Furthermore, on the basis of large-scale model tests conducted by the Los Angeles District, Corps of Engineers, unless spiral transitions are used in a curved reach of short radius, wave disturbance occurs for the design discharge. For projects in which the channel alinement contains a number of curves in series and flow is near that of the critical depth (as is the case of many channels in the Los Angeles area), it is considered important to eliminate wave disturbances for all flows, whenever practicable, because of the potential synchronizing effect of wave disturbances in successive curves.

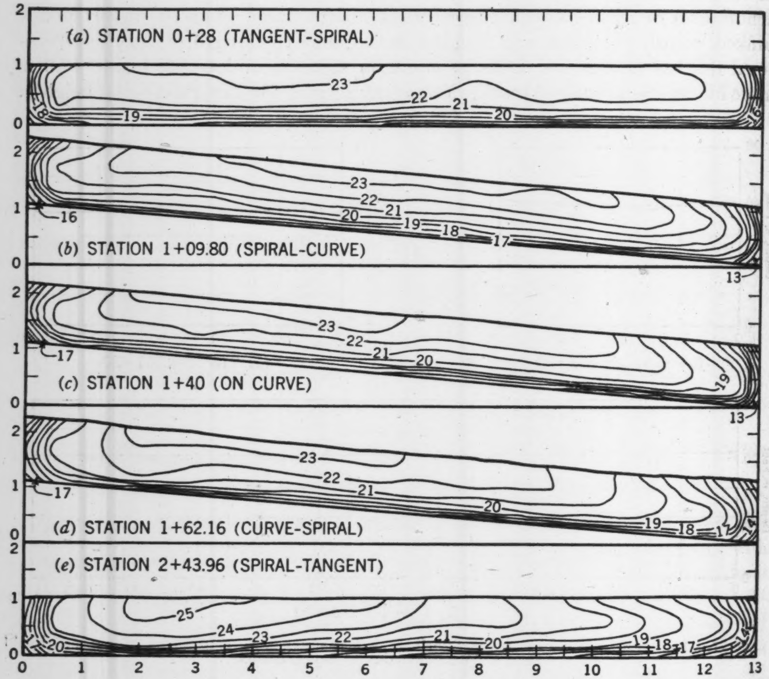


FIG. 78.—VELOCITY DISTRIBUTION FOR SUPERELEVATED FLOW IN A CURVED CHANNEL (RADIUS, 150 FT; WIDTH, 13.0 FT; VELOCITY, 20.5 FT PER SEC; DISCHARGE, 280 CU FT PER SEC; DISTANCES ARE IN FEET; AND VELOCITIES ARE IN FEET PER SECOND)

The results of this series of large-scale model tests (which were conducted to determine the adequacy of design criteria for superelevated curves of rectangular channels in which the flow velocity is greater than critical) indicate that the design criteria are satisfactory for all channels in which the ratio of velocity head to the center-line radius of the curve does not exceed 0.04. Examination of Fig. 78 shows that: (1) The depth of flow is nearly uniform across the channel throughout the curved reach; (2) in all sections, the water surface is approximately parallel to the banked invert; and (3) transverse surface waves do not form in the curved reach. As a result of the model tests, the design criteria were considered adequate and curved reaches of rectangular concrete flood control channels are now being designed with spiral transitions and banked inverts.

On the basis of detailed, comparative estimates of cost made by the Los Angeles District, Corps of Engineers, for wide (from 60 ft to 130 ft) rectangular concrete channels with high walls (from 15 ft to 20 ft), relatively small radii of curvature (from 500 ft to 1,500 ft), and high velocities (from 30 ft per sec to 40 ft per sec), banked inverts and spiral transition curves result in a saving of 10% in construction costs. The estimates show that the additional cost of constructing banked inverts and spiral transitions, rather than horizontal inverts and simple or compound curves, are small compared to the saving effected by lower walls of uniform height.

Application of the analyses and procedures suggested, in the Symposium, to the design of high-velocity channels, will result in improved design criteria, although (as in the case of any rapidly expanding pioneer field) some of the basic assumptions and formulas may have to be revised later to conform with new developments.

Arthur T. Ippen,[47] M. ASCE.—The discussions submitted in response to the Symposium on "High-Velocity Flow in Open Channels" show a gratifying amount of additional confirmation of the theories and ideas given in that Symposium. Particularly valuable are the comments from practical engineers concerning past and present design procedures.

Mr. Douma mentions the practice of designing straight-wall contractions of generous length to provide small angles of convergence. Extra length is not necessarily the answer to small disturbances since, in accordance with Fig. 9(*a*), the saving is minimized with additional length due to reflections within the transition which in turn will cause continuously increasing depth, although the wave height may remain small. This increase of depth, of course, is often prevented by providing additional bottom gradient and thus continuously accelerated flow.

Simplified criteria for dimensions of channel contractions are not easily derived since, in contrast to gradual channel expansions, the wave angle for large disturbances depends on wave height and Froude number and since energy dissipation may have to be considered. However, rapid approximations can be made if, in accordance with notations in Fig. 41(*b*), the front ABC′D′ is assumed straight and if the angle of β is initially approximated from sin $\beta_1 = 1/F_1$. Then $\tan \beta_1 = \frac{b_1 - (b_1 - b_2)}{L}$ or

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[47] Prof. of Hydraulics, Dept. of Civ. and San. Eng., Massachusetts Inst. of Technology, Cambridge, Mass.

$$L = \frac{b_1 + b_2}{2 \tan \beta_1} \qquad (82)$$

With this value of L, an approximate value of θ is obtained from

$$\tan \theta = \frac{b_1 - b_2}{2\,L} \qquad (83)$$

Starting from this initial value of θ, the correct length L can then be evolved by means of Fig. 8 or the corresponding equations.

It is to be noted that a maximum angle of deflection is associated with every Froude number and that two deflections take place in each contraction. If flow conditions below fronts AB and A'B in Fig. 41(*b*) cannot sustain a second deflection through angle θ, a normal hydraulic jump must form at the intersection of the wave fronts in B which will then move upstream beyond line AA'. Supercritical flow through the contraction is then impossible. In other words, in accordance with Fig. 8, the value of θ required would exceed the maximum permissible deflection under the fronts BCD and BC'D'.

Ample confirmation of the theory expressed in summary by Fig. 8 is now available from extensive experimental work at the Hydrodynamics Laboratory of the M.I.T. (Cambridge, Mass.) for a large range of Froude numbers and deflection angles.[48]

The application of any formula presented to conditions near critical depth is open to serious question since, by nature of this flow, any approximations made in the theory are no longer permissible and since the smallest disturbance will cause extremely large undulations. For this reason, any transferability of model results to prototypes must also be in doubt.

Eq. 81 concerning the length of the spiral approach is not the result of any new theory but is an approximation for the half-wave length required by the theory given and modified on the basis of experiments. Banking must always be accompanied by transition sections for the confining walls. Whether these confining walls in a transition are of the spiral, circular, or any other type, the essential criterion for the counterdisturbance is: (1) That the length of the transition be one half the wave length; and (2) that the total deflection at the end of the transition be one half of the central angle of a corresponding length of curve with normal radius.

It is only necessary to produce a counterdisturbance of the proper magnitude which, in accordance with Fig. 3, does not depend on the curvature of the channel, but only on the total deflection produced within the transition.

There is complete agreement with Mr. Baumann and Mr. Douma that banking should be the normal solution for important projects. There is no reason, however, why compound curves as proposed in the Symposium should not have extensive application whenever banking is not feasible on account of structural or economic conditions.

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[48] "Studies on the Validity of the Hydraulic Analogy to Supersonic Flow," by Arthur T. Ippen and Donald R. F. Harleman, *Technical Report No. 5985*, U. S. Air Force, October, 1949, Pts I and II.

That banking does not eliminate all disturbances is well brought out by Mr. Douma in Fig. 78 where, for some reason, the velocity distribution upon entering the curve, Station 0 + 28, is uneven. Considerable deceleration is indicated at Station 1 + 09.80 when the maximum velocity shifts away from the outside. This tendency would have been more pronounced if the velocity distribution at Station 0 + 28 had been symmetrical about the axis. Considerable acceleration takes place between Station 1 + 62.16 and Station 2 + 43.96 along the outside, resulting in considerable differences in velocity for the downstream tangent of the curve between outside and inside walls. This effect of uneven kinetic energy distribution may cause serious difficulties with respect to producing a uniform hydraulic jump if a stilling pool follows downstream.

The theories dealing with transverse oscillations of flow in curved channels presented by both Mr. Baumann and Mr. Hart must be recognized as applicable only to large-radius curves. They impose a serious limitation by their assumption of a uniform transverse surface gradient which is not found in practice. The treatment is strictly applicable only to supercritical flow in curves and therefore does not contribute to the basic understanding of supercritical flow in general, which should be acquired by all those responsible for design work in this field.

The introduction of curvature $1/r$ into any of the problems is not very helpful, and Eq. 68 fails, for example, if $1/r$ approaches infinity. Eq. 10*a*, which is identical with Eq. 28 and which is graphically represented in Fig. 3, has no such shortcoming. Thus the importance of recognizing the primary role of wall deflection angles and the nature of propagation of transverse disturbances is emphasized again. Also the expressions given hold for curved wall sections of any length whereas Eq. 68 applies only for lengths greater than one half of the wave length. The concept of centrifugal force in connection with any of the supercritical flow problems is not a fruitful one since fluid particles are not rigidly tied to each other and, therefore, methods dealing with the bulk flow are incorrect.

Mr. Blench refers to the relatively poor agreement between theory and experiment in the example given in Fig. 38. The surface curves agree excellently within the contraction itself, and there exists no other method of determining such surface profiles than the one given. In the downstream tangent, the agreement becomes rapidly worse due to the limiting factors discussed before—primarily, the effect of successive intersection of steep waves and the cumulative side-wall shear. The theory given for sudden changes in surface elevation summarized in Fig. 8 takes full account of energy losses.

Mr. Bhandari suggests a comparison between free surface transitions for supercritical flow and those in closed conduits. The physical behavior of the two types of flow is entirely different. In closed conduits and in subcritical flow contracting channel walls without change in bottom gradient cause accelerated flow and expansions produce deceleration. For supercritical flow, the reverse is true, contracting flow causes deceleration, and expanding flows cause acceleration as clearly shown in papers 3 and 4. There exists no geometric similarity for flows in the same transitions for different Froude numbers due to widely varying standing wave patterns.

Mr. Blaisdell's discussion contains excellent confirmation of the theory presented and suggests only minor variations in technical procedure. Special thanks are due him for his clear presentation of additional supercritical flow variants. The writer does not agree with respect to the increased accuracy claimed in applying the method of characteristics. It seems that a densely drawn network of epicycloids should make interpolation easy when the values of $\overline{V}$ do not correspond to those given along an epicycloid. To draw special diagrams for each particular problem seems unnecessary and would not result in increased accuracy beyond that warranted by the basic assumptions of the theory.

The example given concerning the use of submerged piers to produce interference waves is very interesting. Naturally, any design of a supercritical flow structure can be correct only for a certain value of the Froude number. Model results for higher Froude numbers are readily transferable to prototype conditions if Froude number values are the same. Since in most practical cases these values are somewhat in doubt, model tests must be expanded to include possible variations. The Reynolds numbers of effects in almost all studies have been found to be negligible except when long channels with successive wave intersections and reflections are encountered.

A very effective contribution to the Symposium has been made by discussing various types of channel junctions, and the methods presented for eliminating disturbances in these cases are ingenious as well as helpful for future designs.

Toward the development of more rapidly expanding flows than are possible by side-wall flare and uniform bottom gradient, the "humping" of the bottom or the use of transverse slopes from the center toward the sides may be mentioned. Such a problem was solved successfully by the writer through model tests at Lehigh University (Bethlehem, Pa.) on the Clark Creek Spillway which was constructed in 1940, and a uniform jump below the expansion was obtained.

In conclusion, the writer would like to express his appreciation for the notable contributions made by the discussers and hopes that a helpful beginning has been made in developing rational design procedures which will be applied by fellow engineers engaged in the design of hydraulic structures for supercritical flow.

ROBERT T. KNAPP,[49] M. ASCE.—Several assumptions are made in connection with the derivation referred to in the paper and the deviations from actual conditions involved in some of them are quite serious. These deviations have been re-emphasized by Mr. Baumann. It was the writer's intention to state the nature of the assumptions clearly so that the user might be in a position to evaluate, for each application, whether or not the assumptions are valid within the required accuracy. It is believed that in most cases the other uncertainties in the design data are so large that refinements in the assumptions made in the paper will not be warranted.

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[49] Director, Hydrodynamics Laboratory, California Inst. of Technology, Pasadena, Calif.

Mr. Baumann presents an ingenious method for deriving the maximum height and the position of the first maximum in the curve. However, the validity of the physical analogy upon which the method is based is not completely clear. The agreement of most of the laboratory results with Eq. *32b* and the agreement of most of the field results with Eq. *61b* is probably more fortuitous than real because of the deviation in both cases of the real conditions from the assumptions of the flow characteristics.

Mr. Baumann's discussion of the cost of building the spiral transitions is very informative. It appears that there is little to choose between spiral and compound transitions from the point of view of cost. Therefore, since both accomplish the same end result hydrodynamically, it would appear that the choice between them is a matter of individual taste. Mr. Baumann's comments regarding banked channels is very interesting. As stated in the paper, they are an elegant solution, particularly for channels operating mainly near design capacity. Mr. Baumann also comments upon the inadvisability of using sills in channels carrying debris. The writer would agree that they are less desirable in debris-carrying water, but it should be remembered that the recommended installation leaves free spaces between both ends of the sill and the channel walls. Since there is a vigorous circulation along the upstream face of each sill, it is doubtful if debris can collect there during a major flow. It is believed that the deposits which will undoubtedly be found above each sill at the end of a storm period are formed during the receding stages of the flood when the need for the sill action is no longer present. However, it should be remembered that sills are recommended primarily as remedial measures, not for the design of a new channel.

It is not surprising that Mr. Hart finds other methods of deriving some of the equations than those employed by the authors of the Symposium. There are usually many physically correct approaches to the analysis of a physical problem, and all correct approaches will result in the same answer within the limits of the simplifying assumptions made in each. The writer is disturbed, however, at Mr. Hart's reluctance to accept and utilize the concepts of wave motion to simplify, not only the calculations, but also the physical concepts of the mechanics of the flow. The foundations of wave mechanics are identical with those of other phases of mechanics. Wave motion is a classical example of simple harmonic motion.

Mr. Hart and a few others seem to share an erroneous impression concerning the validity of the use of wave mechanics to treat the problem of high-velocity flow in open channels. They imply that the justification for the use of the methods of wave mechanics is only that the high-velocity flow bears a resemblance to some wave problems. This falls far short of the truth. The fact is that high-velocity flow in open channels is a wave problem and that the equations derived for supersonic flow in gases are also the equations that express the physical realities of the open-channel flow. Hydraulic engineers should be thankful that so much work has been done in theoretical wave mechanics in other fields of science and engineering so that the prior findings can be applied with confidence to hydraulic problems without retracing all the laborious steps that have brought them to their present useful state.

One further point raised by Mr. Hart is the writer's use of the word "cross slope" with reference to banking of the channel bottom. He is correct in assuming that the word refers to the cross slope of the bottom. In his discussion of the action of the cross slope of the water surface as compared to the cross slope of the bottom, however, it appears that there is some confusion between cause and effect.

Mr. Douma calls attention to the difficulty of working with flows whose Froude numbers are in the vicinity of 1. This is a "cross" that must be borne. It corresponds directly to the dreaded "sonic barrier" which plagues the aerodynamicists. Gross approximations are usually not adequate for work in this region so that much more knowledge is needed in this area of open-channel flow before such problems can be handled with confidence. One of the chief difficulties is that the flow energies for the subcritical and supercritical regimes in this region differ so little that the flow is inherently unstable and may swing from one regime to the other with the slightest disturbance.

In the discussion of the use of banking of the channel bottom, Mr. Douma states:

> "Although, theoretically, banking the invert for the design discharge will produce the desired correction without the use of spiral transition curves, * * *"

The writer cannot agree with this statement if it implies that a correctly designed wall transition curve is not required as a part of the design of a banked curve. As was pointed out in the discussion of banking, adjacent to Eq. 35, the radius of curvature of the walls should decrease as the bottom cross slope increases and should always conform to the curvature of the flow produced by the bottom slope at each point on the transition. If this procedure is not followed, the walls will do some of the turning of the flow with the development of the usual pattern of cross waves. Of course, it would be possible to design deliberately to do part of the turning of the flow by the bottom cross slope and part by the walls of the transition, but this appears to be a needless complication in normal cases.

The writer wishes to thank all the discussers for their interest and helpful suggestions as well as for their kind comments regarding the value of the Symposium.

Hunter Rouse,[50] M. ASCE, and B. V. Bhoota[51] and En-Yun Hsu,[52] Assoc. Members, ASCE.—As is often true of comments on such relatively new concepts as were presented in this Symposium, the discussions ranged from a keen appreciation of the principles involved to a total disregard for the pertinent differences between supercritical and subcritical flow. It is therefore particularly heartening that the majority of the discussers, by actual use of the concepts, had already been convinced of their fundamental importance in hydraulic design.

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[50] Director, Iowa Inst. of Hydr. Research, State Univ. of Iowa, Iowa City, Iowa.

[51] Engr., Foreign Div., The Dorr Co., Bombay, India.

[52] California Inst. of Technology, Pasadena, Calif.

Mr. Baumann, who was among the first in the United States to realize the necessity of designing for high-velocity flow by new methods rather than standard methods, comments on the simplicity of the writers' correlation of expansion characteristics according to the Froude number. Mr. Bhandari, on the other hand, presumes—the theory of supercritical flow to the contrary—that the same angle of permissible divergence which applies to closed-conduit expansions and to open-channel expansions for subcritical flow should also be applicable to open-channel expansions for supercritical flow. The writers have shown both analytically and experimentally—and their findings are substantiated by Mr. Blaisdell's discussion—that such is by no means the case; in fact, this distinction between the two flow regimes is the very crux of the Symposium. (In view of these circumstances, it is almost impertinent to note in addition that even in the case of closed conduits or of open channels for subcritical flow there is no fixed angle below which separation will not occur, since the limiting angle, despite Mr. Nikuradse's earlier indications, has been shown by more recent boundary-layer analysis to decrease with increasing length of transition.)

Mr. Blench makes a noteworthy point in regard to separation tendencies when he points out that supercritical flow in an expansion is inherently accelerative whereas subcritical flow in an expansion is decelerative—and it is in zones of deceleration that boundary-layer separation is to be expected. However, what is called separation in supercritical flow is not primarily a boundary-layer effect (although the latter may be involved to some degree), but rather the result of wall deflection at a greater rate than the finite celerity of wave propagation will permit the flow at once to follow. Such local phenomena of wave propagation are definitely three-dimensional, although each paper of the Symposium simplifies them to two dimensions for ease in analysis. The work of Mr. Hinds, to which Mr. Blench refers, further simplifies the treatment to a single dimension. Whereas the latter simplification is very useful for subcritical conditions, it can yield significant results for supercritical conditions only if the flow actually possesses the comparatively rare one-dimensional characteristics—that is, continuous convergence or divergence at a constant angle without beginning or end effects. One-dimensional analysis of this nature can obviously give no clue as to the local effects of wave formation, interference, and reflection which formed the basis of the papers under consideration. On the other hand, the two-dimensional wave analysis permits just as ready inclusion of shock losses if they are actually involved. In reply to Mr. Blench's question as to the possibility of designing an expansion with constant depth across each normal section, this must be answered categorically in the negative, if only because of the local variations essential to wave propagation. It is possible, to be sure, to design an expansion which will produce a constant depth across the outlet, as discussed in the paper. At intermediate sections, however, the desired constancy can only be approached as a limit by making the expansion uneconomically long and gradual. As far as conditions in the vicinity of the critical (chosen by Mr. Blench for comparison purposes) are concerned, these are difficult to fit very satisfactorily into as simple a generalization as the writers proposed, if only because of the appreciable curvilinear (that is, nonhydrostatic, or three-dimensional) effects which come into play as this limit is approached.

Mr. Blaisdell's verification, not only of the writers' experimental results but also of the agreement between the experiments and the simplified theory, is almost too good to expect; in any event it should be most encouraging to those faced with the necessity of using the methods proposed. The writers therefore again call attention to the fact that incorporation of such factors as bed resistance and slope in the analysis represents at best a very rough approximation, since such flow is highly nonuniform and any gross correction of this nature can be based only on average conditions. The situation becomes more inexact as the surface contours for various widely different flow conditions are superposed as in the writers' further simplification, for the effects of bed resistance and slope then change not only from point to point and section to section but also from case to case. Rather similar circumstances may be cited in answer to Mr. Blaisdell's query as to the form of the recommended wall curve. For a given case, the continuous curve which the disturbance lines from the opposite wall just touch (or just miss) probably represents an optimum theoretical solution for the given Froude number. Such a curve is not algebraically expressible, however, and superposition of results in the writers' manner for various Froude numbers would yield as many different curves. The single power function which the writers used in their investigations is merely a convenient approximation which was selected, for the various reasons cited, upon a purely experimental basis.

Mr. Douma is quite correct in presuming that relationships determined for a very small slope will be of little use if the slope is great. As he surmises, and as indicated by the writers in their brief discussion of this factor, the optimum rate of expansion for a given Froude number will necessarily decrease more and more rapidly as the slope increases. Since the method of including slope as a primary variable in the wave theory is still undeveloped, the only solution which the writers can suggest is the preparation, based on experimental measurement, of a series of interpolation diagrams similar to Fig. 57 for successively greater slopes. Although it is questionable whether even approximate superposition of curves could then be obtained, the procedure would at least yield preliminary design information and should also provide a quantitative basis for eventual further analysis.

The writers join with the authors of the other Symposium papers in expressing to the discussers their appreciation for the interest which they have shown and the additional contributions which they have made. Since new engineering concepts are invariably slow to be accepted, capable discussions of this nature often prove as useful to the profession as the papers themselves.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852

TRANSACTIONS

Paper No. 2435

CAPILLARY PHENOMENA IN COHESIONLESS SOILS

By T. William Lambe,[1] Jun. ASCE

With Discussion by Messrs. M. R. Lewis, Peng Swe-tze, D. P. Krynine, and T. William Lambe.

Synopsis

This paper explains the fundamentals of capillarity in cohesionless soils, by first describing analogies in capillary tubes and then presenting data from soil tests to substantiate the conclusions drawn from the analogies. A number of theoretical methods, in common use, for computing the rate of flow of water under the influence of capillarity are investigated and evaluated. Improvements in several of these theories are suggested.

The limiting values of capillary head which a cohesionless soil may have are presented and discussed. The role of the different capillary heads in the various types of capillary flow are also explained. Names are suggested for these capillary heads.

Introduction

In the design of roads and airport runways, one of the most important considerations is the proper control of surface and subsurface water. Likewise, the design of earth dams, earth slopes, and walls that retain soil backfills may be controlled by the water that passes through and remains within the soil in question. In addition to its influence on man's structures, water governs the growth of nature's plants since they rely on the ground water their roots supply to them. The movement and retention of this important soil water can be largely dependent on the phenomenon of surface tension as exhibited in capillaries.

Capillarity, as used in soil mechanics, is that property which enables the soil to draw and hold water above the elevation at which atmospheric pressure exists in the water. The importance of capillarity in soils has been realized for many years by agricultural scientists as well as by engineers who design highways, airports, dams, retaining walls, and foundations. As a result, a large amount of research on soil capillarity has been reported, but the purpose of

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Note.—Published in February, 1950, as *Proceedings-Separate No. 4*. Positions and titles given are those in effect when the paper or the discussion was received for publication.

[1] Asst. Prof. of Soil Mechanics, Massachusetts Inst. of Technology, Cambridge, Mass.

nearly all this study has been to obtain numerical measures of capillarity on many different soils.

Some incorrect concepts appear in the literature because of the incomplete fundamental understanding of the role of capillarity in the movement and retention of soil water. Therefore, it is the purpose of this paper to help explain the fundamentals of capillarity in cohesionless soils and to evaluate the assumptions involved in some of the current theories of capillary water in soils. In addition, improvements in several of the theories will be suggested.

All experimental work reported in this paper was done on one soil under test conditions that were made as similar as possible. The soil used was a fine, uniform, natural sand which had a 60% size of 0.19 mm and a 10% size of 0.08 mm. Since only one soil and one set of test conditions were used, the numerical results of the laboratory tests have meaning only for this case.

CAPILLARY FLOW INTO COHESIONLESS SOIL

Description of Horizontal Capillary Flow.—To explain the horizontal movement of water in soils under the pull of capillarity, a study of flow in capillary tubes will first be made. Fig. 1 shows a horizontal capillary system comprised of tubes 1 and 2, which have no interconnection. The tubes have diameters, *D*, and capillary heads, *h*, as shown. (All pressures considered in this paper are based on atmospheric pressure taken as zero pressure.)

By Poiseuille's law, the rate of flow, *q*, in a tube is given[2] by

$$q = \frac{\gamma\, r^2}{8\,\mu}\, i\, A \ldots . (1)$$

in which γ is the unit weight of water; r is the radius of the tube; μ is the viscosity of the flowing water; i is the gradient; and A is the cross section of the area.

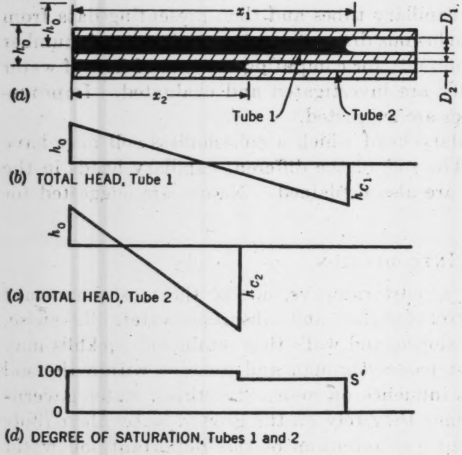

FIG. 1.—FLOW IN INDEPENDENT HORIZONTAL CAPILLARY TUBES

Since for each tube the expression $\frac{\gamma\, r^2}{8\,\mu}$ is constant, it may be taken equal to k', which corresponds to the Darcy coefficient of permeability, k. Eq. 1 can be written

$$q = k'\, i\, A \ldots\ldots\ldots\ldots\ldots\ldots\ldots . (2)$$

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[2] "Fundamentals of Soil Mechanics," by D. W. Taylor, John Wiley & Sons, Inc., New York, N. Y., 1948.

When the meniscus has moved a distance of x from the left end,

$$i = \frac{h_o + h_c}{x} \tag{3}$$

Substitution of Eq. 3 in Eq. 2 gives

$$q = k' \frac{h_o + h_c}{x} A \tag{4a}$$

Since the flow is continuous, the rate must also be equal to velocity of movement of the wetted surface, $v = \frac{dx}{dt}$, multiplied by the area, or

$$q = \frac{dx}{dt} A \tag{4b}$$

Equating Eqs. 4a and 4b gives

$$q = k' \frac{h_o + h_c}{x} A = \frac{dx}{dt} A \tag{4c}$$

Solution of Eq. 4c gives

$$x = \sqrt{2\,k'\,t\,(h_o + h_c)} \tag{5}$$

By using Eq. 5, one can locate the wetted surface at any time in either tube of the system in Fig. 1. For example, let $D_1 = 2\,D_2$; and $h_o = h_{c1}$. For this specific example, since h_c varies[3] as $\frac{1}{D}$, solution of Eq. 5 gives

$$x_1 = \sqrt{\frac{8}{3}}\,x_2 = 1.63\,x_2 \tag{6}$$

Thus, in this example the air-water interface in tube 1 is always 1.63 times as far from the left end as that in tube 2.

Although the foregoing analogy illustrates some useful concepts applicable to soils, there is a most important difference between system 1 in Fig. 1 and the capillary systems of soils—in soil there exists an infinite number of interconnections between the various effective capillary tubes.

To simulate the action in soils more closely, the system of Fig. 1 will be modified by interconnecting the two tubes at many points along their length. No difference in total head can now exist between the tubes because the interconnections would permit cross flow to equalize any tendencies toward such a difference. Therefore, at any distance less than the x of the lagging meniscus, the total head is the same in the two tubes. Fig. 2 shows the modified system in which the movement of water has progressed to x_1 in the large tube and to x_2 in the small tube. Fig. 2(*b*) illustrates that the heads in the two tubes are alike at any distance for 0 to x_1; therefore, it follows that the gradient in this range is the same in both tubes.

An important difference between the systems of Figs. 1 and 2 is that in the latter the meniscus in the small tube precedes that in the large one by a distance

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[3] "Colloid and Capillary Chemistry," by Herbert Freundlich, Methuen and Co., Ltd., London, 1926.

$x_2 - x_1$. The meniscus in the large tube cannot precede that in the small tube because flow would occur from the large tube to the small tube, as a result of the higher negative pressure in the small tube, until the meniscus of the large tube was lagging. In addition, the value of x_2 can never be such that the gradient over this distance is constant because this would mean that, although the tubes, in effect, had no influence on each other, the value of x_2 would be greater than x_1. The system in Fig. 1 shows this latter condition to be untrue. Therefore, x_2 must be greater than x_1 but can never be large enough for a constant gradient throughout its length. (The foregoing qualitative reasoning is sufficient for the purposes of this analogy. A more exact study of the problem shows that at x_1 there is a small difference of head between the small and the large tubes, as at point M, Fig. 2(*b*). There is flow from the large into the small tube at point M due to this head difference. This cross flow decreases the gradient in the large tube at x_1, thus reducing the velocity of the meniscus in the large tube. A solution to the problem indicates that the distance between the menisci, $x_2 - x_1$, increases with an increase of x_1 and decreases with an increase in cross flow.)

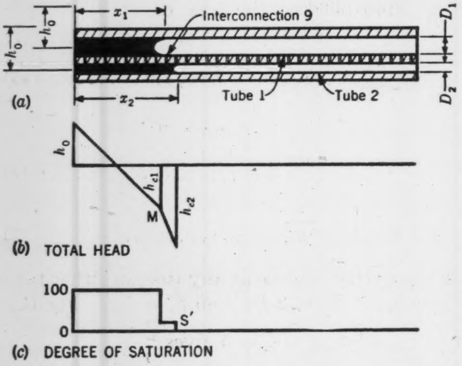

Fig. 2.—Flow in Interconnected Horizontal Capillary Tubes

The flow in the system of Fig. 2 has a number of characteristics which have counterparts in capillary flow through soils, the most important of which are:

1. The meniscus in the smaller tube precedes that in the larger tube;
2. The only tube diameter that is effective at any instant and at any point in developing capillary head is that diameter where a meniscus is formed;
3. The gradient in both tubes back of the air-water interface in the large tube is essentially constant, because lateral flow between the tubes equalizes heads;
4. The average gradient for the system depends more on the larger tube, since a large part of the greater capillary head of the smaller tube is lost in the distance between the two menisci; and
5. In this zone between the two menisci, k' (which corresponds to permeability in soils) and the degree of saturation for the system are lower than those back of the lagging meniscus.

Some of the more important inherent differences between soils and a system of capillary tubes, which make their flow processes unlike, are:

(*a*) In the system in Fig. 2, the sole function of the interconnecting tubes was to permit sufficient flow to equalize pressure in the two tubes. Since one water surface always preceded the other, the connections could always fill themselves by capillarity (see interconnection 9, Fig. 2(*a*)). This ideal situation does not exist in soils, since there may be two pores that have the same effective diameter and thus become filled simultaneously; therefore, a small connecting tube between these pores would not be saturated. In such a connecting tube, flow would proceed from one or both ends until the air pressure balanced the capillary head and stopped the flow. As a result of this and other air entrapping phenomena in soils, the maximum degree of saturation attained by capillary action is less than 100%.

(*b*) In soils, the equivalent capillary tubes change diameters from point to point. (Microscopic study by the writer showed capillary flow to be very jerky. This sudden change of velocity is due to sudden changes in void sizes along the path of flow.)

(*c*) There is an almost infinite number of effective tube sizes in soils rather than the two considered in the previous analogies.

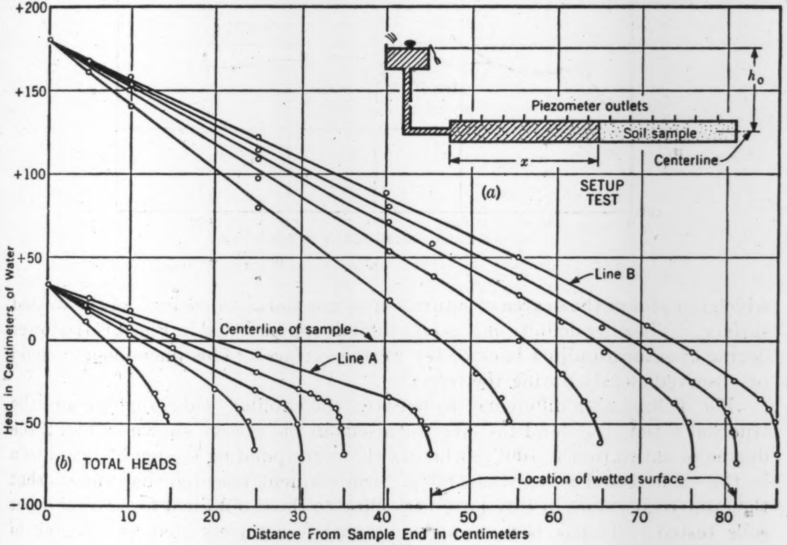

FIG. 3.—HORIZONTAL CAPILLARITY TEST IIIE

With the foregoing three differences in mind, the extension of description of flow processes from tubes to soil can better be understood.

Horizontal Capillarity Test.—Fig. 3(*a*) shows the setup for test IIIE which was a special laboratory soil test in which pore water pressures were measured. This test corresponded somewhat to the flow system for the tubes shown in Fig. 2. Water was admitted at the left end of the horizontal sample of soil under a static pressure of magnitude h_o. From the beginning of the given test until $x = 49.8$ cm, h_o was 35.0 cm; at $x = 49.8$, h_o was increased to

181.7 cm. Water pressures were measured at the piezometer outlets spaced every 5 cm along the length of the soil tube. Water pressures and tensions were read to an accuracy of 0.3 cm of water.[4] Fig. 3(*b*) is a plot of these water pressure readings taken at different values of x. For example, when the wetted surface has traveled to a distance of 45.5 cm, line A is the line connecting the total water heads shown on the vertical scale opposite the points at which they exist. Thus the piezometer 25 cm from the end of the soil read −8 cm and is so plotted on line A. The slope of line A is the gradient and corresponds to the plots of total head versus distance shown in Figs. 1 and 2.

To determine the varying degree of saturation along the length of a sample two similar tests were run in a segmented "lucite" tube. At any instant, this segmented tube could be sliced up to obtain the degree of saturation of the soil in each segment. The results of slicing up two samples are shown in Fig. 4 which is a plot of the degree of saturation versus the distance back of the wetted surface. The only definite characteristic that can be noted in Fig. 4 is the lower degree of saturation just back of the wetted surface. This phenomenon could be observed clearly during the tests.

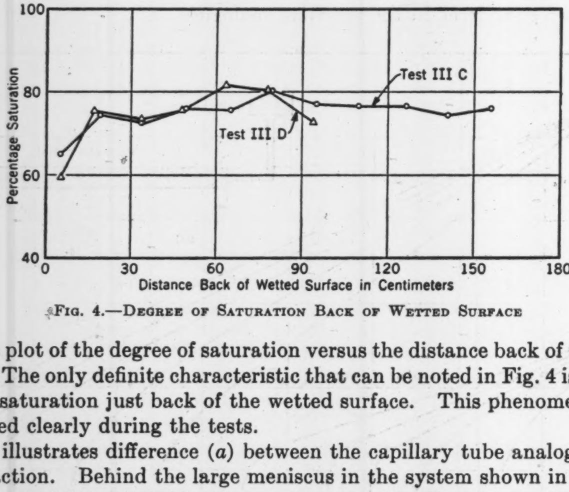

FIG. 4.—DEGREE OF SATURATION BACK OF WETTED SURFACE

Fig. 4 illustrates difference (*a*) between the capillary tube analogy and the true soil action. Behind the large meniscus in the system shown in Fig. 2 the degree of saturation is 100%, whereas the corresponding degree of saturation in the sand tested is less than 80%. (Subsequent research[5] has shown that the numerical values in Fig. 4 are very close to those obtained for several other soils tested.) It has been thought by many engineers that the degree of saturation obtained by capillarity was very close to 100%, and in previous work in soil mechanics 100% saturation frequently has been assumed. The observed low value of "capillary saturation" points to inaccuracies in several currently used formulas and laboratory techniques of soil mechanics.

Theoretical Methods of Analysis of Horizontal Capillary Flow.—The currently used theory considers flow into soils to be similar to flow in one capillary tube as shown in Fig. 5. Also shown in Fig. 5 are the assumed distributions of head, degree of saturation, and permeability.

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[4] "The Measurement of Pore Water Pressures in Cohesionless Soils," by T. William Lambe, *Proceedings*, 2d International Conference on Soil Mechanics and Foundation Eng., Rotterdam, 1948, paper No. 3.18.

[5] "Experimental Investigation of the Degree of Saturation in Sands," by H. R. Parfitt and N. E. Pehrson, thesis presented to the Massachusetts Inst. of Technology, at Cambridge, in 1948, in partial fulfilment of the requirements for the degree of Master of Science.

For water flowing in soil as shown in Fig. 5(*a*), the rate of flow,[6] *q*, by Darcy's law, is equal to

$$q = k\,i\,A = k\,\frac{h_o + h_c}{x}\,A \ldots\ldots\ldots\ldots\ldots\ldots (7a)$$

in which k is the Darcy coefficient of permeability; i is a gradient; A is the cross-sectional area; and h_o, h_c, and x are as shown in Fig. 5. The flow q can also be expressed as

$$q = \frac{dx}{dt}\,n\,A \ldots\ldots\ldots\ldots\ldots\ldots (7b)$$

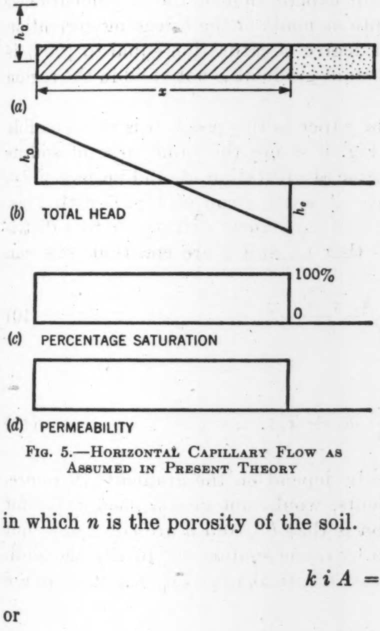

FIG. 5.—HORIZONTAL CAPILLARY FLOW AS ASSUMED IN PRESENT THEORY

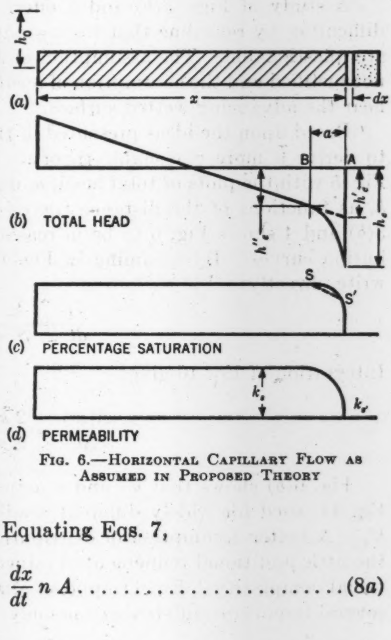

FIG. 6.—HORIZONTAL CAPILLARY FLOW AS ASSUMED IN PROPOSED THEORY

in which n is the porosity of the soil. Equating Eqs. 7,

$$k\,i\,A = \frac{dx}{dt}\,n\,A \ldots\ldots\ldots\ldots\ldots\ldots (8a)$$

or

$$k\,\frac{h_o + h_c}{x} = \frac{dx}{dt}\,n \ldots\ldots\ldots\ldots\ldots\ldots (8b)$$

The solution of Eq. 8*b* gives

$$\frac{\Delta(x^2)}{\Delta t} = \frac{2\,k}{n}\,(h_o + h_c) \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

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[6] "Capillary Phenomenon in Cohesionless Soils," by T. William Lambe, thesis presented to the Massachusetts Inst. of Technology, at Cambridge, in 1948, in partial fulfilment of the requirements for the degree of Doctor of Science.

Eq. 9 has been used as a basic equation for a laboratory test, called the horizontal capillarity test, to determine k and h_c. The slope of any (x^2 versus t)-curve has been taken as equal to $\frac{2\,k}{n}(h_o + h_c)$. By running the test at two different values of h_o, two equations of the form of Eq. 9 can be obtained; the simultaneous solution of the two equations gives values of the unknown soil properties k and h_c. In past years many tests run on several different soils in the Soil Mechanics Laboratory at the Massachusetts Institute of Technology (M.I.T.), in Cambridge, have shown that the slope of the (x^2 versus t)-curve is very nearly constant after the early stages of the test; but these tests have given values for k that could not be checked by other types of permeability tests. Also the head h_c obtained from such horizontal capillarity tests was lower than was thought reasonable.

A study of Figs. 3(*b*) and 4 offers an explanation of the aforementioned difficulties by revealing that the assumptions made in the foregoing derivation are questionable. The assumption of a degree of saturation equal to 100% is not justified and the assumption of a constant gradient is not valid in the region near the advancing wetted surface.

Based upon the ideas presented in the paper to this point, it is now possible to derive a more reasonable theory. Fig. 6 shows the same tube of soil as Fig. 5 with the plots of total head, h, degree of saturation, S, and permeability, k, as functions of the distance traveled. A comparison of Fig. 6 with Figs. 3(*b*) and 4 shows Fig. 6 to be in reasonable agreement with the actual distribution curves. By assuming in Fig. 6 that h'_c and a are constant, one can write, directly:

$$\frac{dx}{dt} = \frac{k_s}{S\,n}\,\frac{h_o + h'_c}{x} \quad \text{.......................(10)}$$

Integration of Eq. 10 gives

$$\frac{\Delta(x^2)}{\Delta t} = \frac{2\,k_s}{S\,n}(h_o + h'_c) \quad \text{...................(11)}$$

Fig. 6(*b*) shows that h'_c and a actually depend on the gradient; therefore, Eq. 11, used for widely different gradients, would not give a good value for h'_c. A better assumption of distribution is that h''_c and a are constants; but the little additional refinement obtained by doing so does not justify the additional complexity. Eq. 11 appears almost identical with Eq. 9, but there are several important differences—namely:

1. Eq. 11 includes a term for the degree of saturation;
2. The permeability (Eq. 11) is that which occurs at a given value, S, the degree of saturation, and not the saturated value; and
3. The term h'_c is not a true capillary head, but is actually the value of water tension at the wetted surface required to give a straight-line gradient for the entire length of x.

To illustrate the use of Eq. 11, an application of it will be made to test IIIE. Fig. 7 is a plot of the squares of various x-values against the total elapsed times, t, at which they were observed. The slopes of the (x^2 versus t)-curve are measured in square centimeters per minute and are, for $h_o = 35.0$ cm—

$$\frac{\Delta(x^2)}{\Delta t} = 11.2$$

and, for $h_o = 181.7$ cm—

$$\frac{\Delta(x^2)}{\Delta t} = 39.3$$

The flow in test IIIE was stopped at a value of $x = 93.4$ cm and the quantity of water, V_w, that had flowed into the soil at that point was 392 cu cm. These data and the porosity of 0.375 can be used to compute the average degree of saturation as: $S = \frac{V_w}{V_v} = \frac{V_w}{L\,n\,A} = 0.745$, in which V_w is the volume of water; V_v is the volume of voids; and L is the length of wetted soil. (The degree of saturation can also be determined by weighing the permeameters of soil in both the dry and wet states. The difference of these weights is the weight of pore water. Knowing the void ratio of the soil, one can easily compute the degree of saturation.) Since the degree of saturation is essentially constant for the distance from the left end of the soil to $x - a$, the value of $S = 0.745$ will be taken as constant. Substituting the foregoing data in Eq. 11 to obtain two simultaneous equations, and solving these equations, the result is a value of permeability which, reduced from the test temperature to 20° C, is $k\left({}^{s=0.75}_{T=20^\circ\mathrm{C}}\right) = 0.0254$ cm per min, in which T is the temperature.

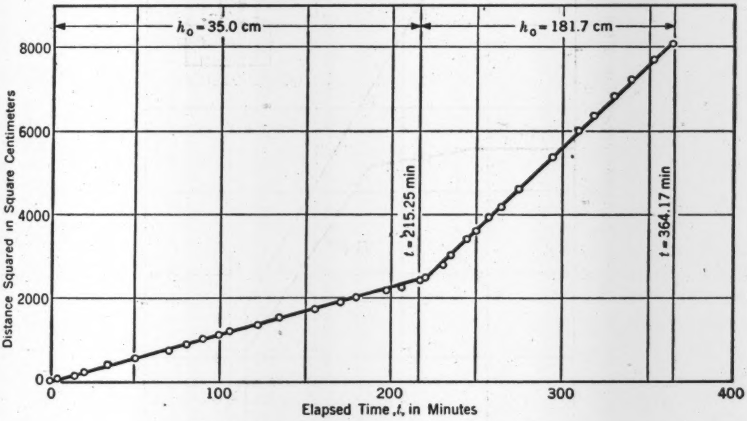

FIG. 7.—APPLICATION OF EQ. 11 TO TEST IIIE

The results of permeability tests run on the tube of soil from test IIIE give, for $k\left({}^{s=0.745}_{T=20^\circ\mathrm{C}}\right)$:

(*a*) Constant head test; $k = 0.0266$ cm per min
(*b*) Falling head test; $k = 0.0235$ and $k = 0.0264$ cm per min

The differences in the foregoing values, obtained from the standard permeability tests, are due to different initial pressures in the entrapped air and represent the extremes that were obtained with the apparatus used. To compare, properly, the results of a horizontal capillarity test and a falling head test, one must consider in detail the pressure in the water and entrapped air initially and during each test. Nevertheless, one can see that the permeability from the horizontal capillarity test is in good agreement with the values determined from the other tests.

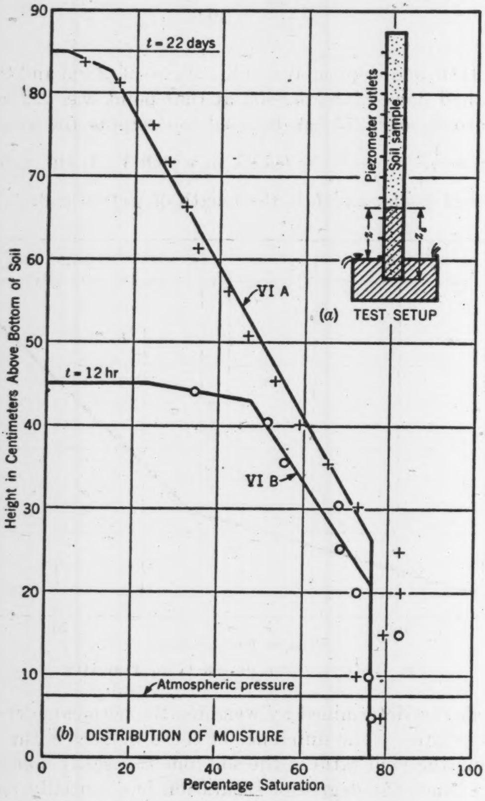

Fig. 8.—Capillary Rise Test

In summary it may be stated that the use of Eq. 11 with the data from a horizontal capillary test gives a good measure of the permeability of a soil that has been wet under the influence of capillarity. The value of capillary head obtained is an effective head and bears little relation to that determined by other laboratory capillary tests.

Capillary Rise Test.—The ideas developed for horizontal capillary flow are applicable in general to vertical flow, except that in vertical flow elevation head is more important. In capillary rise, the effect of elevation head is to decrease total head, thus reducing the rate of flow, and to make the range of low degree of saturation much larger and more important than in horizontal flow. No capillary tube analogy will be given for vertical flow because the important concept that flow in the small tubes precedes that in the larger ones, with the resulting larger gradients in the region just back of the wetted surface, can be carried forward from the previous analogy.

In order to obtain curves of the degree of saturation and the total water head versus the distance, a series of tests were run using the setup shown in Fig. 8(*a*). In these tests water from the constant head reservoir is permitted to rise by capillarity into the soil sample. During the test, the distance above the bottom of the soil, z_s, the water pressures at the piezometric outlets, and the elapsed time are recorded. Fig. 8(*b*) is a plot of the degree of saturation versus the distance above the bottom of the soil for two stages in the rise—that is, test VIB is after the wetted surface has risen 45 cm in an elapsed time of 12 hours and test VIA is after the wetted surface has risen 85 cm in an elapsed time of 22 days. The shape of these distribution curves seems to be characteristic. The abrupt end to the curve at the top is caused by stopping the capillary rise before a static condition has been reached. Long-time tests by another experimenter[7] indicate that the wetted surface would continue to rise at an infinitely small velocity for years and eventually rise to more than 100 cm for the type of soil used.

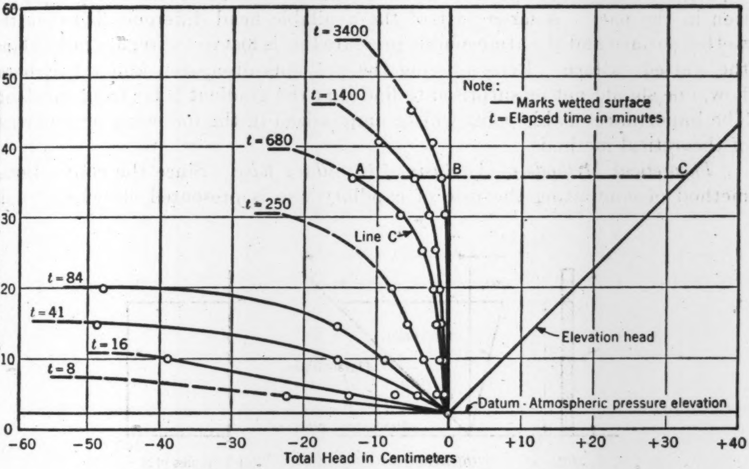

FIG. 9.—DEGREE OF SATURATION VERSUS DISTANCE ABOVE BOTTOM OF SOIL

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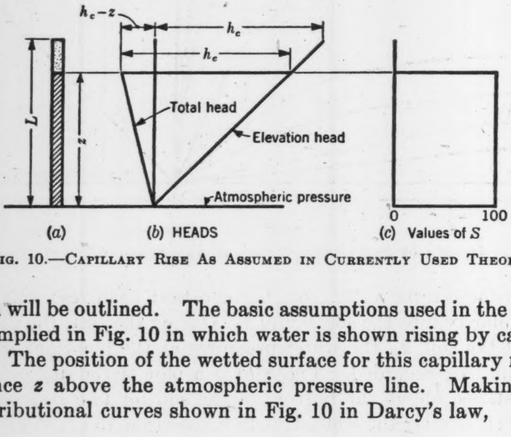
[7] "Capillarity Tests by Capillarimeter and by Soil Filled Tubes," by K. S. Lane and D. E. Washburn, *Proceedings*, Highway Research Board, National Research Council, Washington, D. C., Vol. 26, 1946, p. 460.

Fig. 9 is a plot of the measured pore water pressure versus the height above the bottom of the soil for various positions of wetted surface in its process of rising. (The broken line part of each curve was obtained by extrapolating back to intersect the wetted surface elevation line.) For example, at an elapsed time of 680 min when the wetted surface has risen 39.5 cm from the bottom of the soil, line C is the locus of total head in the soil sample. Thus, at a point (point A, Fig. 9) 35.5 cm above the bottom of the soil, the total head (AB) is -13.2; the elevation head (BC) is $+33.0$; and the pressure head (AC) is -46.2 cm of water. The slope of line C, or the head lost in a given height, is the gradient in the sample at time, $t = 680$ min; line C in Fig. 9 corresponds to line A in Fig. 3(*b*).

Fig. 9 shows a very important point which has not received proper appreciation in the past. A large part of the available head difference between the wetted surface and the atmospheric pressure line is lost in the region just behind the wetted surface. Remembering the previous discussion under horizontal flow, one should not be surprised to find that the gradient is far from constant. The importance of this point will be emphasized in the following presentation of theoretical methods.

Theoretical Methods of Analysis of Capillary Rise.—Since the conventional method of computing the rate of capillary rise is presented elsewhere,[8] only its derivation will be outlined. The basic assumptions used in the conventional method are implied in Fig. 10 in which water is shown rising by capillarity in a tube of soil. The position of the wetted surface for this capillary rise is located by its distance z above the atmospheric pressure line. Making use of the assumed distributional curves shown in Fig. 10 in Darcy's law,



(*a*) (*b*) HEADS (*c*) Values of S

FIG. 10.—CAPILLARY RISE AS ASSUMED IN CURRENTLY USED THEORY

$$q = k\, i\, A = k\, \frac{h_c - z}{z}\, A \ldots\ldots\ldots\ldots\ldots\ldots (12a)$$

in which h_c is defined as the maximum height to which capillary water will rise. Also,

$$q = v\, A = \frac{dz}{dt}\, n\, A \ldots\ldots\ldots\ldots\ldots\ldots (12b)$$

in which v is the velocity of movement of the wetted surface. Equating Eqs.

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[8] "Theoretical Soil Mechanics," by Karl Terzaghi, John Wiley & Sons, Inc., New York, N. Y., 1944.

12 and integrating,

$$t = \frac{n\,h_c}{k}\left(\log_e \frac{h_c}{h_c - z} - \frac{z}{h_c}\right) \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Eq. 13 is used for computing the time of capillary rise, in which the following assumptions have been made or implied:

(1) The degree of saturation of all wetted soil is 100%;
(2) The permeability is the saturated permeability; and
(3) The gradient is uniform.

The test data in Fig. 8(*b*) prove assumption (1) to be incorrect; and, since the permeability is a function of the degree of saturation, assumption (2) is also incorrect. In view of the previous observation, Eq. 12*a* is more accurately interpreted as applicable to only the bottom of the sample. Since, for small heights, the degree of saturation seems to be essentially constant after the first few minutes, the permeability remains almost constant at the bottom. The permeability to be used in Eq. 13 is the value for a degree of saturation of about 80% (see Fig. 8(*b*)). (For the soil used in these tests, the permeability at 80% saturation is about a third of the value when the saturation is 100%.) Assumption (3) is even less valid than assumptions (1) or (2), as Fig. 9 clearly shows.

The writer has developed[6] a theory which is founded on more reasonable assumptions than is Eq. 13. This theory, based on the use of an effective capillary head, is not presented here for several reasons: First, Fig. 9 shows that about half of the capillary rise occurred while the gradient at the bottom of the soil was less than a few hundredths. This small gradient not only is approximately equal to the experimental accuracy of the laboratory apparatus used to measure it, but also is so small that the results of an actual capillary test would be needed to indicate how it varies; and, second, if a more correct expression is used for the gradient, the resulting formula for the rate of rise becomes so complicated as to be of little value.

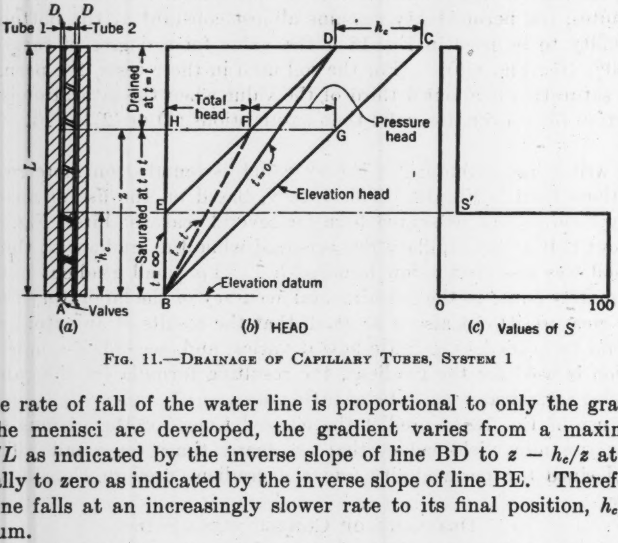
The use of Eq. 13 to predict the elapsed times for the various stages of capillary rise may yield values that are many times too small. This is as expected since the permeability and the gradient used in Eq. 13 are both too large.

DRAINAGE OF COHESIONLESS SOIL

Description of Drainage.—As in the case of capillary flow into soils, the description of the drainage process of cohesionless soils can be aided by first studying the drainage of simple systems of capillary tubes. In Fig. 11 are shown two capillary tubes of diameter D; the tubes are connected to each other at many points. Initially, the tubes are completely filled for their entire length, L, flow being prevented by closed valves at elevation A.

Drainage of the two tubes is started by opening the valves at point A and maintaining a free water surface there. Upon opening the valves the pressure head at all points disappears, thereby giving a total head distribution equal to the elevation head distribution. Water begins to drain because of the difference of total head between adjacent points. When the total head becomes equal to the elevation head the gradient is L/L or 1. After a small amount of flow occurs, menisci are developed at the receding water-air interface. The menisci apply a capillary head or tension to the water, giving a total head distribution shown by line DB in Fig. 11. The elapsed time required to obtain the head distribution BD is negligible and is taken as zero.

The plot of heads versus depth in Fig. 11 is a complete picture of all heads at any stage of drainage. Thus, at time, t, the water level has fallen to elevation, z. A horizontal line was drawn from elevation z in the tube to intersect line DE and give point F. Line DE was obtained by drawing a line parallel to BD but offset a horizontal distance of h_c to the left of the elevation head line. The elevation head at time t is $\overline{HG}$; the pressure head is $-\overline{FG}$; and the total head is $\overline{HF}$. A line from points F to B represents the distribution of total head in the water in the tubes at time t, and the reciprocal slope of FB is the gradient. Since every other factor in the Poiseuille law is constant in this case, the rate of fall of the water line is proportional to only the gradient i. After the menisci are developed, the gradient varies from a maximum of $L - h_c/L$ as indicated by the inverse slope of line BD to $z - h_c/z$ at time t, and finally to zero as indicated by the inverse slope of line BE. Therefore, the water line falls at an increasingly slower rate to its final position, h_c, above the datum.



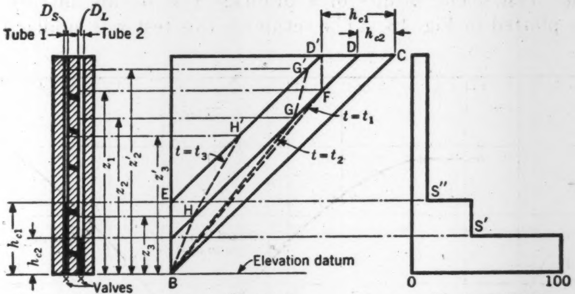
(*a*) (*b*) HEAD (*c*) Values of *S*

FIG. 11.—DRAINAGE OF CAPILLARY TUBES, SYSTEM 1

The final distribution of water in the two-tube system is shown in Fig. 11 by a plot of the degree of saturation, S, versus the height. To a height, h_c, above elevation A, the tubes are completely filled with water, giving a degree of saturation of 100%. From an elevation of h_c to an elevation of L, the only water present is the water entrapped in the lateral connecting system, giving a degree of saturation equal to S'. Since this entrapped water is not continuous, S' is not a function of the elevation. In this system there is no intereffect of the flow in the two tubes because the tubes are the same size.

Although the system in Fig. 11 is like an actual draining mass of soil, it is different in several important respects. One of these differences is that in soils the tubes are not of the same size, but vary over a large range. System 2, Fig. 12, goes one step further than that of Fig. 11, in having tubes of diameters D_S and D_L; in every other respect the tube setup in Fig. 12 is like that in Fig. 11.

During the drainage of system 1, the water levels in the tubes were always at the same elevation. In system 2, after the beginning of flow, the two levels will never be together since the tubes are of different size. Until menisci are developed, the draining processes in the two systems are similar. Because it is larger in diameter, tube 2 will develop its capillary head fully before tube 1 does. Thus, at time, $t = 0+$, the total head line $\overline{\mathrm{BD}}$ is determined by tube 2. The water level in tube 2 will fall to elevation z_1 before the full capillary head at the top of tube 1 is mobilized. Until h_{c1} is mobilized, there is no movement of the water level in tube 1. The condition of incipient movement of the level in tube 1 is represented in Fig. 12 as that occurring at $t = t_1$. Line BF represents the total head distribution to the elevation z_1; from z_1 to L the total head is constant, as shown by FD′.



(a) (b) HEAD (c) Values of S

FIG. 12.—DRAINAGE OF CAPILLARY TUBES, SYSTEM 2

At time $t = t_2$ the level in tube 2 is at z_2, and in tube 1, at z'_2. Line BG represents the total head distribution to elevation z_2 and GG′ from z_2 to z'_2. The gradient from z_2 to z'_2 is less than that below z_2.

The rate of movement, q, of water in either tube is proportional to $D^2 i$ (Poiseuille's law) since all other factors are constant in this setup. Thus, the rate of fall in tube 2 continues to be faster than that in tube 1 until i_2 is equal to the gradient above the line of saturation, multiplied by $\frac{D^2{}_S}{D^2{}_L}$. Position 3, as indicated by z_3 and z'_3, represents a further stage of drainage in which the gradient above the lower level is greater than below. It can thus be seen that gradients above and below the lowest water level would be equal to each other only by coincidence.

Before the analysis is carried from the tubes into the soils, a number of conclusions, based on the preceding discussion of the three analogies, will be stated:

1. Drainage proceeds at an increasingly slower rate due to the gradient, which decreases with time;

2. In a system of two tubes with different diameters, as in system 2, the larger tube has the greater influence on the rate and the quantity of drainage; and

3. The water held in the tubes above the elevation of the maximum capillary head is not continuous, but is water entrapped in the cross connections and, therefore, the quantity retained is not a function of the elevation.

In addition to the differences between tubes and soils, listed in the paragraph following Eq. 11, there is another difference that can be important in drainage—natural soils can be partly saturated from the beginning. Entrapped air causes time lags for changes in water pressure to occur; if the entrapped air is not distributed uniformly the gradient is affected.

Drainage Test.—The results of a drainage test on an initially saturated sample are plotted in Fig. 13. The setup for this test was similar to that for the capillary rise shown in Fig. 8(*a*); the elevation of the piezometer outlets is shown on the left-hand scale in Fig. 13. For example, at an elapsed time of 225 min, the total head at piezometer No. 7 was 29 cm (shown by point A, Fig. 13). The pressure head at piezometer No. 7 is the total head minus the elevation head or 29 − 101 = − 72 cm. Since point A is located where the water became discontinuous at piezometer No. 7, the value of −72 cm is the maximum water tension measured, or the maximum capillary head measured, at piezometer No. 7. At an elapsed time of 225 min, the curve marked "visual line of saturation" shows the line of saturation to be approximately at the

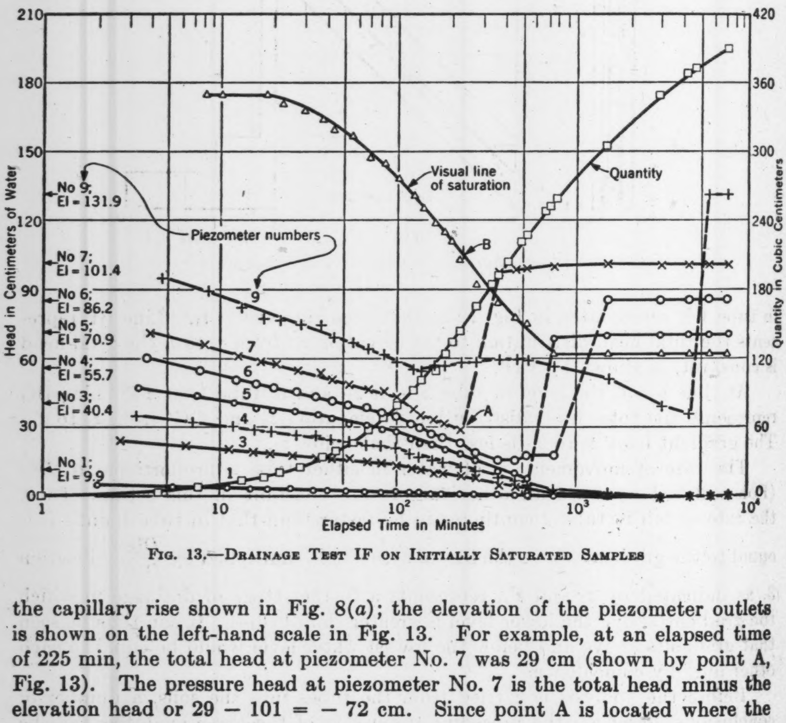


FIG. 13.—DRAINAGE TEST IF ON INITIALLY SATURATED SAMPLES

elevation of piezometer No. 7 (point B, Fig. 13). The "quantity" curve is a plot of the cumulative quantity of flow at any time.

In Fig. 14 the pressure data from Fig. 13 are plotted in a manner similar to the water pressure in the tubes shown in Figs. 10 and 11. The inverse slope of the lines connecting the plotted pressure data (Fig. 13) is the gradient. Point A, Fig. 13, can be seen in Fig. 14 with its total head of aA, elevation head of ab, and pressure head of −bA.

After test IF was run until practically all drainable water seeped out, the sample was sliced up and the degree of saturation of the soil segments was

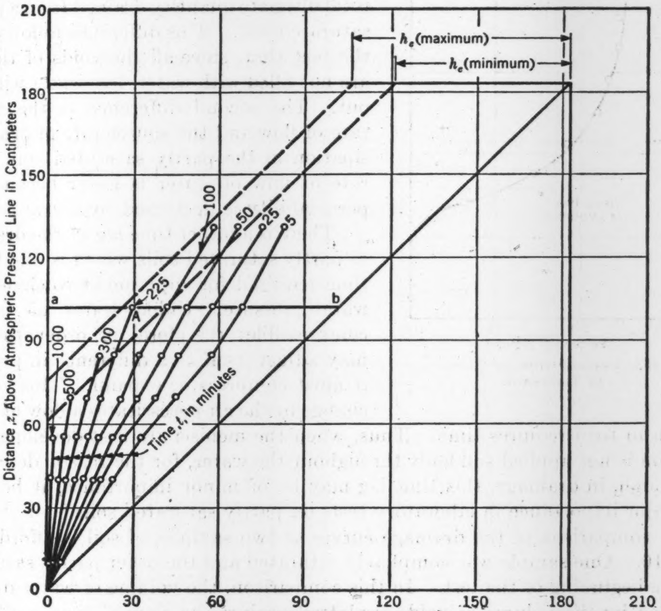


FIG. 14.—DISTANCES ABOVE ATMOSPHERIC PRESSURE LINE, TEST IF

obtained. Fig. 15 is a plot of the degree of saturation versus the distance above the atmospheric pressure.

A study of Figs. 13, 14, and 15 (which present actual test data) shows that the true drainage process can be well explained by the simple, capillary tube analogies shown in Figs. 11 and 12. The differences between the action of the analogy and the actual process are attributed to the previously listed differences in their physical make-ups.

Two interesting conclusions can now be drawn:

(1) A considerable amount of drainage occurs after the visual line of saturation appears to have reached its ultimate position (Fig. 13); and

(2) The visual line of saturation is not the true line of saturation (Fig. 15).

The drainage of a partly saturated soil is of much practical interest, since as shown in connection with Fig. 4, a soil "saturated" by capillary action may have a degree of saturation considerably less than 100%. A test (test IG) was run on the soil sample from test IIIE which was a horizontal capillary flow test, and thus the sample had an average degree of saturation of 79%. This test reveals two main differences between the drainage of a saturated and partly saturated soil mass. The first difference is the lower total ultimate quantity drained from a partly saturated soil. This difference follows from the fact that, since all the voids of the soil are not filled with water, less water will flow out. The second difference is the slower rate of flow and the slower rate of head dissipation in the partly saturated soil. The rate of flow of water is lower because the permeability is decreased by entrapped air.

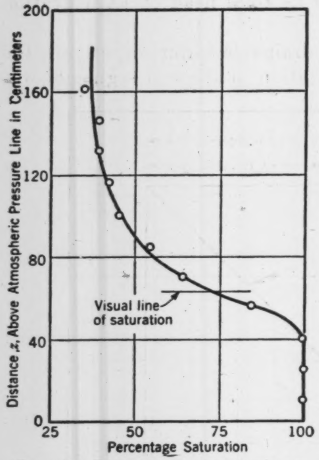


FIG. 15.—DISTRIBUTION OF MOISTURE IN DRAINAGE

There is another time lag in the drainage of partly saturated soils which is due to the time required for adjustment to changes of water pressure. Unlike water, air is very compressible; therefore, in order that air may adjust itself to a difference in pressure it must compress or expand. This volume change in the air necessitates a flow of water which in turn requires time. Thus, when the menisci are first developed, the tension is not applied suddenly throughout the water, for the reason described. Although, in drainage, this time lag may be of minor importance, it becomes of major importance in laboratory tests on partly saturated soils.[4]

A comparison of the drainage curves of two samples of soil is afforded by Fig. 16. One sample was completely saturated and the other partly saturated at the beginning of the test. In this comparison, the volume of water drained divided by the volume of voids is plotted against the elapsed time multiplied by the permeability of the soil. The agreement of the two curves in the early stage signifies that, in this early stage, the rate of drainage is proportional to the permeability.

Theoretical Methods of Computing Drainage.—A number of methods of computing drainage has been analyzed and evaluated; and solutions from the various theoretical methods were compared with the drainage curve of test IF. The theoretical evaluation and comparison with test results[9] showed all theoretical methods studied to be approximate. A major problem in the use of the theoretical methods is the proper selection of soil properties.

CAPILLARY HEADS

General.—Much of the literature implies that a given soil under given conditions possesses a single, definite capillary head, as does a capillary tube. Recently, to this concept was added the one which considered a soil to possess two capillary characteristics called the "active capillary rise" and the "passive capillary rise." To accompany this concept, a number of laboratory tests have been devised to measure the "active rise" and "passive rise," with little agreement being reached between the values obtained in the several types of tests. Much of the confusion that now exists is due to the oversimplification introduced by comparing the movement of capillary water in soils to that of water in a bundle of nonconnected capillary tubes. This paper shows that the capillary characteristics of a given soil cannot be completely represented by one or two head values; several capillary heads are required for adequate representation.

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[9] Discussion by T. William Lambe of "Investigation of Drainage Rates Affecting Stability of Earth Dams," by F. H. Kellogg, *Transactions*, ASCE, Vol. 113, 1948, p. 1294.

The Range of Capillary Heads.—Fig .17 is a plot of the degree of saturation versus the height above the atmospheric pressure line for test IF, a drainage test on a sample that was initially saturated, and test VIA, a capillary rise test on a sample that was initially dry. It would seem logical that point A (Fig. 17) (which is approximately the point where the curve ceases to be vertical) is the highest elevation at which there exists any continuous channel of water from the free water surface below. Therefore, the distance from point A to the free water surface is taken as the maximum capillary head, h_{cx}. Another critical point on the degree-of-saturation curve for a drainage test is the highest elevation at which complete saturation exists (point B, Fig. 17). The distance from the free water surface to this point is called the saturation capillary head, h_{cs}.

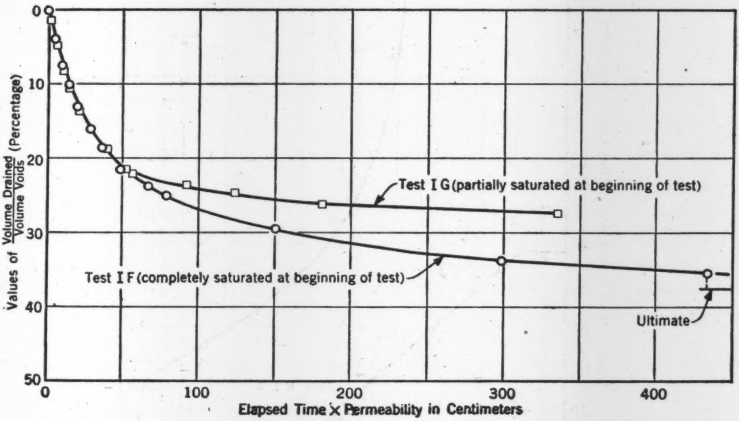

FIG. 16.—DRAINAGE OF PARTLY AND COMPLETELY SATURATED SOILS

On the distributional curve from the capillary rise test, there are two critical points. The distance from the free water surface to the highest elevation to which capillary water rose (point C, Fig. 17) is called the capillary rise, h_{cr}. The distance from the free water surface to the highest elevation at which the maximum capillary degree of saturation exists (point D, Fig. 17) is named the minimum capillary head, h_{cn}.

The four capillary heads are limits of the possible range of capillary heads that a soil can have. Any capillary head associated with drainage must lie between h_{cx} and h_{cs}—similarly, any associated with capillary rise must lie between h_{cr} and h_{cn}. Since it is the size of void at the air-water interface which determines the capillary head, it is possible for a void to support water that filled larger voids below its surface, yet not raise the water past these larger voids. Therefore, h_{cx} is greater than h_{cr}, and h_{cs} is greater than h_{cn}, as might be expected.

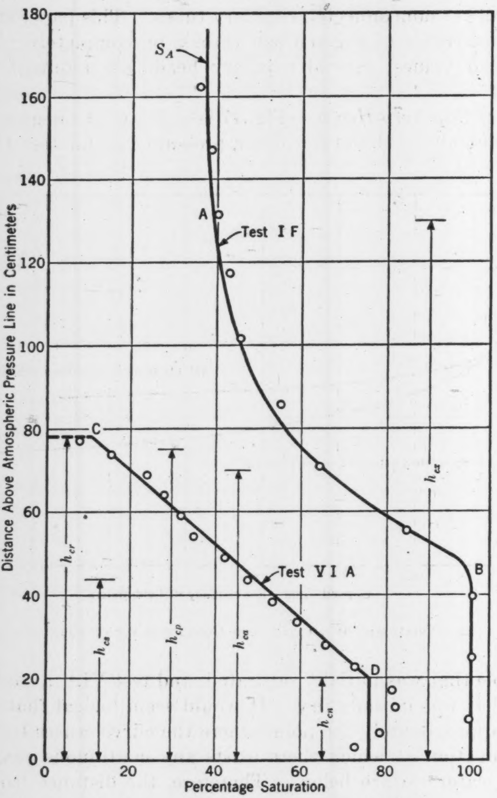

FIG. 17.—RANGE OF CAPILLARY HEADS

In addition to the previously described limiting capillary heads, two others are shown in Fig. 17. One of these is a weighted, average capillary head, h_{ca}, selected to make $(L - h_{ca})\, n\, A\, (100 - S_A)$ equal to the quantity of water drained. The other capillary head was obtained by cutting off the water supply in a horizontal capillarity test (test IIIE) and measuring the water tension that developed. By cutting off the water supply, the movement of water was retarded but not prevented since the smallest voids stole water from the larger voids. It is to be noted that the maximum water tension measured (75 cm, or h_{cp}) is a conservative measure of the capillary rise of 78 cm. Since flow was greatly slowed down an approximate value of the maximum capillary pull could be obtained; but h_{cp} would not be equal to h_{cr} unless the flow were absolutley prevented.

The Effective Capillary Head.—Between the two extremes, h_{cx} and h_{cn}, there exists an infinite number of capillary heads. In any soil problem involving capillarity the value of capillary head that is effective is a value within the range of the limiting values. The effective value of capillary head to be used in any problem depends on the particular problem. The many problems involving capillarity in soils can be divided into three classes as follows:

(*a*) The determination of the height above free water surface to which capillary water will rise or be retained;
(*b*) The determination of the rate of movement of capillary water; and
(*c*) The determination of the quantity of water that rises or is retained at any height.

The selection of an effective capillary head for problems of class (*a*) is the simplest. The height at which continuous water will be held by capillarity is h_{cx}; the height to which capillary water will rise is h_{cr}. (The value of h_{cr} shown in Fig. 17 would be larger, probably about 100 cm or 110 cm had test VIA been of several years' duration instead of 22 days.[9]) Recent inspection[10] of many airfields has led to the conclusion that it is nearly always h_{cx} rather than h_{cr}, that is the maximum height to which continuous water exists. The explanation of this fact is that the rain water, or other runoff, percolates down from above.

The selection of an effective capillary head for problems involving rate of flow is much more difficult than the selection for problems of class (*a*). In draining a soil, the tension back of the retreating menisci in the large voids is approximately h_{cs}, whereas that in the small voids approaches h_{cx}. The selection of an effective capilllary head in a drainage formula depends upon the nature of the particular formula.

In computing the rate at which water flows into a soil by capillarity, the selection of an effective capillary head again depends on the conditions inherent in the formula used. Since all the methods discussed in this paper assume a uniform gradient, the effective capillary head to be used is a value that gives a straight-line gradient which causes the same rate of flow as the true variable gradient. One type of effective capillary head can be obtained from the results of a horizontal capillarity test in which two magnitudes of applied head are used; from test IIIE the effective capillary head was found to be about 24 cm. This capillary head is fictitious and thus cannot be shown with meaning in Fig. 17.

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[10] "Report on Frost Investigation," New England Div., Corps of Engrs., War Dept., Boston, Mass., April, 1947.

The effective capillary head used for computing the corresponding rise is also a fictitious value which gives an equivalent uniform gradient. As there is no applied pressure in the capillary rise process to keep the flow in the large voids near that in the small voids, the gradient is variable and there is no range of height in which it approaches a constant except that below point D (Fig. 17). Because of this fact the development of a formula for the rate of capillary rise is a most difficult problem. As seen in Fig. 9, for the early part of the rise, the effective capillary head was about 40 cm. Above $z = 40$ cm, the effective capillary head, based on an assumed uniform gradient, becomes larger. Again the effective capillary head of 40 cm cannot be shown, with meaning, in Fig. 17.

The selection of an effective capillary head for problems of class (*c*), or the determination of the quantity of water that will rise or be retained at any given height, can be easily made from curves such as are shown in Fig. 17. Since problems of class (*c*) are distributional problems, they can be solved better when curves of the degree of saturation versus the elevation, rather that any particular values of capillary head, are available. For example, if, in any given problem, it is known that a degree of saturation of 60% or more is not desirable at the surface of the soil, a depth of 75 cm or more of soil must exist above the free water surface. Likewise, if the requirements of any other problem are known, the effective capillary head can be easily taken from a plot of the degree of saturation versus the elevation which was the procedure by which the 75-cm value was obtained in the foregoing example.

The long time required to develop h_{cr} and h_{cs} fully must be kept in mind. It was noted that the capillary head determined by the large voids in drainage, or the capillary head effective in the initial gradient, was greater than h_{cs}. This is explained by the considerable time allowed for the tortuous flow necessary to develop h_{cs}. Also all indications point to the probability that the maximum capillary head during drainage is less than h_{cx} because of the low velocity of flow required to develop h_{cx}. The previous statements as to when large and small voids are effective depend on the rate of flow.

Laboratory Tests to Determine Capillary Head.—Most laboratory tests determine only two values of capillary head, h_{cr} and h_{cs}, known as "active capillary rise" and "passive capillary rise," respectively. For a time, some engineers thought the two should be equal. A research project[11] reported in 1944 was aimed at finding the relation between the two "rises."

The current terminology of capillary heads seems inappropriate. The use of the terms "active" and "passive" is reasonable, but to add the word "rise" to each is not reasonable. Certainly there is nothing about h_{cs} which justifies it being called a rise, in fact, "passive" and "rise" imply opposite meanings. In view of the field observations[10] which indicate that soils obtain their capillary water by infiltration from above, one may question the value of obtaining the magnitudes of h_{cr} or h_{cn} except for comparative studies. Certainly the full magnitude of h_{cr} would have little meaning, since it is hard to conceive a soil deposit not being subject to water from above for as long a period as is required to develop h_{cr}. In general, it can be concluded that h_{cr} is one of the least used and one of the most difficult to obtain of the capillary heads.

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[11] "Capillarity in Sands," by Raul Valle-Rodas, *Proceedings*, Highway Research Board, National Research Council, Washington, D. C., Vol. 24, 1944, p. 389.

The h_{cs}-value is the one determined by Gunnar Beskow[12] and is one of the less difficult values to find. Also the practical use of h_{cs} to determine the height to which a soil mass will retain its maximum degree of saturation makes h_{cs} more important than h_{cr}.

The maximum capillary head has importance, but for a soil such as the one used in this project having a degree-of-saturation versus an elevation curve which does not break sharply at the end of the continuous water, the exact numerical value of h_{cx} is not too important. For example, it does not make much difference if h_{cx} is taken as 100 cm instead of 130 cm because there is very little continuous water in the range from 100 to 130.

The most meaningful and useful capillary head is h_{ca}. This value of capillary head reflects h_{cx} and h_{cs} plus the distribution between them and is thus a more indicative property of the soil than any of the other values of capillary head. If a single value of capillary head were to be adopted as a soil property and called "the capillary head of the soil," it probably should be h_{ca}. A test similar to test IF would give h_{cx}, h_{cs}, h_{ca}, and the distribution of retained water. Although a drainage test in a tube that could be sliced up is not as easy to run as a capillarity test in a device similar to that devised by Mr. Beskow, it is certainly not as time consuming as the capillary rise test.

CONCLUSION

It is suggested that the term "capillary head" be used in place of "capillary rise"; and that at least five meaningful capillary heads be recognized (Fig. 17) as: h_{cx}, h_{cn}, h_{ca}, h_{cr}, and h_{cs}. Also, the development of standard capillary tests should be considered with serious thought given to the adoption of some kind of drainage testing apparatus as standard laboratory equipment. The development of a test to measure h_{cp} as an indication of h_{cr} should be studied.

Again it is pointed out that all the research reported herein was performed on one soil under one set of test conditions. There is a great need for much more research on the subject of capillarity. Many of the trends suggested in this paper should be studied in detail on several soils under different conditions. There may be some definite relationship between various capillary heads; if such a relationship exists, it may be possible to predict several of the capillary heads from one.

ACKNOWLEDGMENT

The writer gratefully acknowledges the work of Frederick F. Flanders; his mechanical skill made possible the construction of special apparatus without which the research work reported in the paper would have been seriously retarded. M. J. Holley, Jr., Assoc. M. ASCE, and Victor F. B. De Mello, Jun. ASCE, of M.I.T. deserve thanks for reading a draft of this paper and making a number of helpful suggestions. The writer expresses his appreciation to D. W. Taylor, Assoc. M. ASCE, associate professor of soil mechanics at M.I.T., for his aid in conducting the reported research and in preparing this paper Mr. Taylor's suggestions and criticisms led to solutions that may not have otherwise been reached.

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[12] "Soil Freezing and Frost Heaving with Special Application to Roads and Railroads," by Gunnar Beskow, Technological Inst., Northwestern Univ., Evanston, Ill., 1935.

DISCUSSION

M. R. LEWIS,[13] M. ASCE.—Agricultural workers and particularly those concerned with irrigation have been interested in the movement of water under the influence of capillary forces. The writer[14] conducted experiments with different soil types in which water moved by capillary action both vertically upward and downward at different rates and he determined the moisture distribution after uniform flow had been secured. These tests indicated that flow due to surface tension took place only from zones of higher moisture content to zones of lower moisture content. The apparent exception in the case of downward flow at high moisture contents and low rates of flow is only apparent because under these conditions the flow is caused by the gravitational force. In the same way it is probable that the flow through distance x-a in Fig. 6(a) is due to the head h_o and that the straight part of the total head curve in Fig. 6(b) should meet the line of atmospheric pressure at point B, rather than pass through the point at distance h''_c below point B.

Moisture content versus capillary tension curves of soils determined by the use of porous plates, by both tension and pressure, offer very accurate and complete data on the water retaining capacities of soils. These plates can also be used to give helpful information on the drainage characteristics of soil.

PENG SWE-TZE.[15]—Some incorrect concepts of capillarity in soils have been clarified in this paper. The three classes of problems involving capillarity in soils, described by the author, show the desirability of adopting some drainage testing apparatus as standard laboratory equipment. In this discussion, the writer will emphasize the significance of the moisture distribution curve of capillary water in connection with unsteady seepage problems. The writer approves of the idea of making a test determining the moisture distribution of capillary water standard laboratory procedure.

Capillary water may be regarded principally as a zone of partial saturation. Except in very uniform soils, the so-called zone of capillary saturation is generally much smaller than the zone of partial saturation.[16] From drainage observations on a rather uniform dune sand, the writer found that a moisture distribution curve yields a zone of complete saturation, h_{cs}, equal to 14 cm and a zone of partial saturation, $h_{cx} - h_{cs}$, equal to 66 cm. The three-zone concept proposed by Karl Terzaghi,[8] Hon. M. ASCE, has the disadvantage that it offers no idea of the relative value of these two zones. Sometimes too much emphasis has been given to the zone of complete saturation and relatively little emphasis has been given to the zone of partial saturation.

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[13] Chf., Div. of Irrig. Operations, Bureau of Reclamation, U. S. Dept. of the Interior, Washington, D. C.

[14] "Rate of Flow of Capillary Moisture," by M. R. Lewis, *Technical Bulletin No. 579*, U.S.D.A., Washington, D. C., 1937.

[15] Civ. Engr., Soil Mechanics Laboratory, Delft, Holland.

[16] "Soil Physics," by L. D. Baver, John Wiley & Sons, Inc., New York, N. Y., 1940, Chapter VI, pp. 200–211.

In unsteady state, the zone of partial capillary saturation may become greatly extended, as shown by the capillary tube analogy for the drainage process used by the author.

By applying Poiseuille's law to the case shown in Fig. 12,

$$-A_L\,dz = c'\,i_2\,r^2{}_L\,A_L\,dt - c'\,(i_1 - i_2)\,r^2{}_s\,A_s\,dt \tag{14}$$

in which A_L and A_s are the cross-sectional areas of the large and the small tube, respectively; r_L and r_s are the radii of the large and the small tube, respectively; i_1 is the gradient above the line of saturation; and

$$c' = \frac{\gamma}{8\,\mu} \tag{15}$$

Expressions for the gradients are

$$i_1 = \frac{z' - z + h_{c2} - h_{c1}}{z' - z} \tag{16a}$$

and

$$i_2 = \frac{z - h_{c2}}{z} \tag{16b}$$

Therefore,

$$-\frac{dz}{dt} = c'\,r^2{}_L - c'\,\frac{h_{c2}}{z}\left(r^2{}_L + r^2{}_s\,\frac{A_s}{A_L}\right) + c'\left(\frac{h_{c1} - h_{c2}}{z' - z}\right)r^2{}_s\,\frac{A_s}{A_L} \tag{17}$$

Since

$$d(z' - z) = -\,dz - i_1\,c'\,r^2{}_s\,dt \tag{18}$$

it follows that

$$\frac{d(z' - z)}{dt} = -\,\frac{dz}{dt} - i_1\,c'\,r^2{}_s \tag{19}$$

from which

$$\begin{aligned}\frac{d(z' - z)}{dz} = -\Bigg[\,&(r^2{}_L - r^2{}_s)\,z\,(z' - z) + (h_{c1} - h_{c2})\left(\frac{A_s}{A_L} + 1\right)r^2{}_s\,z \\ &+ \left(-\,h_{c2}\,r^2{}_s\,\frac{A_s}{A_L} - h_{c2}\,r^2{}_L\right)(z' - z)\Bigg] \\ \div \Bigg[\,&r^2{}_L\,z\,(z' - z) + (h_{c1} - h_{c2})\,\frac{A_s}{A_L}\,r^2{}_s\,z + \left(-\,h_{c2}\,r^2{}_s\,\frac{A_s}{A_L} - h_{c2}\,r^2{}_L\right)(z' - z)\Bigg]\end{aligned} \tag{20}$$

in which $z' - z$ represents the height of the zone of partial capillary saturation.

One simple example will demonstrate the expansion of the zone of partial saturation during drainage. If the pressure at the bottom of the tubes is kept at $-h_{c2}$, $i_2 = \dfrac{z}{z - h_{c1} + h_{c2}} = 1$. Then

$$\frac{d(z' - z)}{dz} = -\frac{\left(1 - \dfrac{r^2{}_s}{r^2{}_L}\right)(z' - z) + (h_{c1} - h_{c2})\left(\dfrac{A_s}{A_L} + 1\right)\dfrac{r^2{}_s}{r^2{}_L}}{z' - z + (h_{c1} - h_{c2})\dfrac{r^2{}_s}{r^2{}_L}\dfrac{A_s}{A_L}} \quad \ldots (21)$$

By integration and using the initial conditions ($t = 0$, $z = H - (h_{c1} - h_{c2})$ and $z' - z = h_{c1} - h_{c2}$),

$$\frac{z' - z}{e\,H} + \frac{1}{H}(h_{c1} - h_{c2})\left(1 - \frac{1}{e}\right) - \frac{b - e\,f}{e^2\,H}\log\frac{\dfrac{e(z' - z)}{H} + \dfrac{b}{H}}{\dfrac{h_{c1} - h_{c2}}{H} + \dfrac{f}{H}} = \frac{H - z}{H} \quad \ldots (22)$$

In Eq. 22,

$$e = 1 - \frac{r^2{}_s}{r^2{}_L} \quad \ldots\ldots\ldots\ldots (23a)$$

$$b = (h_{c1} - h_{c2})\left(\frac{A_s}{A_L} + 1\right)\frac{r^2{}_s}{r^2{}_L} \quad \ldots\ldots\ldots\ldots (23b)$$

and

$$f = (h_{c1} - h_{c2})\frac{r^2{}_s}{r^2{}_L}\frac{A_s}{A_L} \quad \ldots\ldots\ldots\ldots (23c)$$

As an example, let $\dfrac{r_s}{r_L} = \tfrac{1}{2}$ and $h_{c1} - h_{c2} = \dfrac{H}{100}$. Then $e = \tfrac{3}{4}$; $\dfrac{A_s}{A_L} = \tfrac{1}{4}$; $b = \dfrac{5\,H}{1{,}600}$; $f = \dfrac{H}{1{,}600}$; from which Eq. 22 yields

$$\frac{4}{3}\frac{z' - z}{H} - \frac{1}{300} - \frac{17}{3{,}600}\log\frac{1{,}200\,\dfrac{z' - z}{H} + 5}{17} = \frac{H - z}{H} \quad \ldots\ldots (24)$$

When $z = 0$, $\dfrac{z' - z}{H} = 0.77$.

The zone of semicontinuous water spreads until it occupies 77% of the height of the tubes during drainage, whereas in a static state it occupied only one hundredth of the height of the tubes. Therefore, its influence may be small under static conditions but considerable under unsteady conditions.

Soils may be visualized as a system of tubes of many sizes. If the moisture distribution can be regarded as representing the distribution of the tube sizes, the moisture distribution curve will have great influence on the flowing condition in an unsteady state. The writer has considered this factor in studying some unsteady seepage problems, treating the rate of the quantity of flow and the rate of pressure variation separately.

In fact, as a result of rainfall or ground-water movement, the moisture in soils is often in an unsteady state. To explain the effect of capillarity more accurately in such cases, the moisture distribution curve seems indispensable. Therefore, the adoption of a test determining the moisture distribution curve, especially the curve representing conditions after drainage, is recommended.

The capillary potential theory also shows the importance of the moisture distribution curve. A mositure distribution curve with significant turning points will define the capillary water better than the three-zone concept.

The shape of the moisture distribution curve will vary with time, but at a decreasingly slow rate. It seems reasonable to specify some standard periods of drainage for different soils, in accordance with the permeability of the soils.

D. P. Krynine,[17] M. ASCE.—The highway or airport engineer usually builds pavements (or, more accurately, "riding surfaces") on artificially compacted bases and subgrades. Compaction is supposed to be done at the optimum moisture content and maximum density of the earth material (or at a certain high percentage of that maximum density, such as 90%). Measurements of the moisture content in that densified earth mass a certain time after the construction of the pavement, however, show that in a general case the actual moisture content differs from that used during construction, either on the positive side or the negative side. Moreover, measurements of the moisture content in the subgrade have disclosed fluctuations from month to month, sometimes with an occasional 100% saturation.[18] Failures of pavements resulting from the accumulation of moisture under the riding surfaces have been reported; and this circumstance makes the study of the movement of moisture under the riding surfaces of vital importance.

Moisture can reach the space under a riding surface (*a*) through the joints, from meteoric water; (*b*) from the water table in the form of capillary rise of water or water vapor, or both; and (*c*) from the edges of the pavement, in a horizontal motion (capillary movement or motion resulting from a head) of meteoric or other moisture accumulated there. Previous works on capillarity by civil engineers (as the author of this paper rightly states) were mostly concerned with obtaining statistical data on the capillary motion of water in different soils. In addition the investigators were dealing mostly with vertical capillary motion and they neglected the horizontal. In practice, however, horizontal motion of water from the edges of the pavement (item (*c*)) is perhaps more important than the vertical capillary rise from the water table (item (*b*)). In contrast, Mr. Lambe deals extensively with horizontal capillary motion and intends to explain the fundamentals of capillarity, in a theoretical manner that is decidedly to its advantage. An especially interesting and important feature of this work is the measurement of pore pressures.

If a brief characterization of this paper were required the writer would describe it as an excellent treatise on the geometric aspect of capillarity. On the other hand, some additional clarification of the physical aspect of the phenomenon would be welcomed by designers of highway and airfield pavements. Without any doubt, water moving above the elevation at which pressure in water is just atmospheric is in tension. Mr. Lambe assumes that this tension is caused by the negative head h'_c uniformly distributed in a cross section of the tube, as can be concluded from his comments[19] in 1948. Some further treatment of the forces causing this head would be of interest. Mr. Lambe assumes the existence of motion of an idealized fluid within a mass, the physical properties of which are not considered. This indeed was difficult since only one soil was tested, and apparently always at the same porosity. At first glance, the horizontal capillary test seems to be exceedingly simple. Technically, however, it is really quite difficult and some experimental details would be very interesting.

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[17] Cons. Engr., San Francisco, Calif.

[18] "Observation of Moisture Contents and Densities of Soil Type Bases and Their Subgrades," by L. D. Hicks, *Proceedings*, Highway Research Board, National Research Council, Washington, D. C., Vol. 28, 1948, pp. 422–432.

Degree of Saturation.—Engineers who experiment in the field of capillarity have never thought that the capillary moisture fills all the pores of a given soil—that is, that the average degree of capillary saturation is 100%. They believed (and this is perfectly correct) that, if a vertical tube filled with soil (prevented from falling out) just touches a free water level, the degree of saturation at that plane is 100% and decreases toward the top of the tube. It was always understood by experimenters that the average degree of saturation in an experimental tube simply cannot be 100%. Fig. 4 shows that the degree of saturation in the given series of tests was less than 80%, with a value of 0.745 for the case of Fig. 7. In the summer of 1949 the writer performed numerous experiments on the horizontal capillarity of Ottawa sand and two kinds of California sands.[20] One of the California sands passed entirely through sieve No. 200 (Kings River sand) whereas the other contained about 76% of medium sand (minimum size 0.1 mm) 18% of fine sand (0.05 mm), and 6% of silt size particles (Antioch sand). The tubes used by the writer were glass tubes $1\frac{1}{4}$ in. in diameter, 30 cm and 40

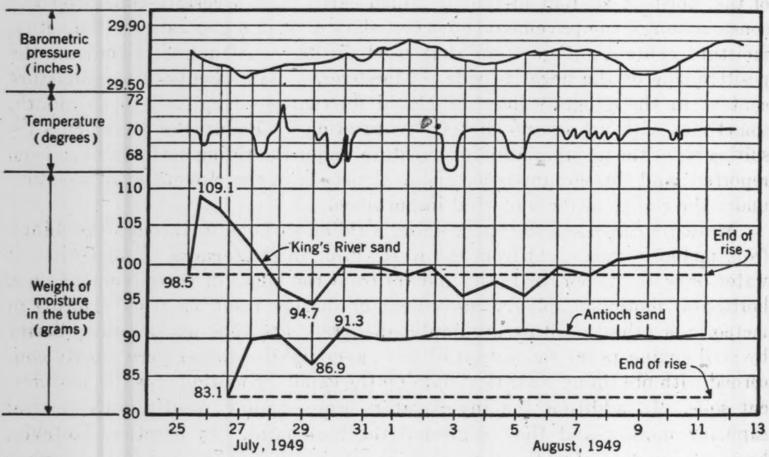

FIG. 18.—BOUNDARY EFFECT IN CAPILLARITY

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[19] "The Measurement of Pore Water Pressures in Cohesionless Soils," by T. William Lambe, *Proceedings*, 2d International Conference on Soil Mechanics and Foundation Eng., Rotterdam, 1948, Vol. VII, p. 40.

[20] "Some Capillary Phenomena in Sandy Materials," by D. P. Krynine, *Proceedings*, Highway Research Board, National Research Council, Washington, D. C., Vol. 29, 1949.

cm long. In no case, in that series of tests, was the average degree of saturation 100%, but it was very close to that value in horizontal capillarity tests on the Ottawa sand. As to the two California sands the average degree of saturation in both vertical and horizontal capillary tests on Kings River sand was close to 83%. This value held also for the horizontal test on Antioch sand. The average degree of saturation for vertical tests on Antioch sands was about 77%.

Fig. 18 shows the "boundary effect" in vertical capillarity. In two special vertical capillarity tests by the writer, the degree of saturation was determined as soon as the wetted line reached the top of the soil column, and afterward was under observation for 17 days, the lower end of the tube being kept in contact with a free water level. After the wetted line reached the top of the soil column, the volume of moisture within the tube increased to a maximum on the same day; then it dropped down, and afterward fluctuated about a certain average. The values of the degree of saturation (percentages) as observed in these tests were:

Description	Kings River sand	Antioch sand
At the end of rise	86	71
At the maximum saturation	95	81
Final average	86	76

Condensation Phenomena in Capillary Tests.—Let the end of the experimental tube, where the water enters, be called "origin of flow." The opposite end will be conventionally termed "exit end."

Since the degree of saturation of the soil in a test tube is less than 100%, there is air in the interstices. Soil scientists state that the relative humidity of the soil air is 100%—that is, that it is fully saturated with water vapor. Hence, a drop of temperature of the ambient air should cause condensation that really occurs in such cases on the inner side of the tube wall. In none of the tests by the writer was condensation observed next to the origin of flow (Fig. 19). The writer's explanation is that there is no air in the tube next to the origin of flow; in other words, the degree of saturation in an experimental tube may be 100% in the corresponding sections of the tube. Referring to Fig. 19, the absence of condensation at the exit end of the tube is explained by the easy escape of excess moisture into the ambient air.

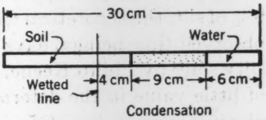

FIG. 19.—EXAMPLE OF CONDENSATION IN AN EXPERIMENTAL TUBE

Air, Entrapped and Free.—As water enters the test tube it gradually displaces the air from the pores. Since, however, the degree of saturation of the material in the tube is always less than 100%, it follows that the capillary water just cannot displace all air. The writer's point of view is that sand grains are able to develop and retain on their surfaces films of a certain thickness only, leaving some air between the grains (Fig. 20). Such a situation is also known as "funicular saturation."[21] Since the motion of the capillary moisture is slow, it may be assumed that, at the air-moisture interfaces, pressures in the air and the moisture are the same. In the parts of the test tube close to the origin of flow where the degree of saturation is high, the air is entrapped in the form of bubbles. As the capillary moisture moves, pressure at each cross section of the tube gradually increases (Fig. 3); hence, the volume of the bubbles of entrapped air decreases, with the corresponding gradual increase of the degree of saturation.

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[21] "Die Kapillarität des Bodens," by J. Verluys, *Internationale Mittelungen Bodenkunde*, Vol. 7, 1917, pp. 117–140.

Again, a simple examination of the curves in Fig. 3 shows that pressure in water decreases from the origin of flow toward the exit end of the test tube; hence, the degree of saturation in a test tube also decreases in the same direction and such a thing as uniform degree of saturation does not exist. Incidentally, the curves in Fig. 3 explain, in a perfectly clear way, why the velocity of capillary movement gradually slows down as the capillary process progresses. They also suggest a certain similarity between the capillary flow in a tube and the radial (or two-dimensional) flow of ground water to a sump or a ditch.[22,23] In both these cases a large part of the head is lost when the stream approaches free air.

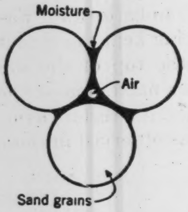

FIG. 20.—PARTIAL SATURATION

The physical condition of the air at the head of the moving capillary stream is not very clear. Obviously, this air is not entrapped there. At the air-moisture interfaces, however, air should be in tension if the head in a cross section is uniform. This is difficult to visualize, and a question arises as to whether the hypothesis of uniformity of pressure head in a cross section really holds.

Speed of Capillary Motion.—As stated by the author, Eq. 13, developed to determine the elapsed times t for various stages of capillary rise, may yield values that are many times too small. A detailed study of this formula has been made by the U. S. Army Engineers.[7] For a sandy material with about 9% of silt, the theoretical time of rise to a height of 60 cm is 0.22 day, the actually observed time being 6 days (ratio 27 times). Observations of capillary rise in a silty soil[24] by Philip Keene, Assoc. M. ASCE, have shown that Eq. 13 "* * * is of little value in the determination of capillary rise in soils." This formula is given, however, in a few authoritative textbooks on soil mechanics and other fundamental publications[25] The inadequacy of Eq. 13 cannot be too strongly emphasized.

Interfacial Curvature-Saturation Function.—Capillary rise and drainage of soils have also attracted the attention of agricultural soil scientists and petroleum engineers and scientists. Among the latter, considerable contributions in this field have been made by M. C. Leverett who has presented the curves for drainage and "imbibition" or vertical capillary movement (capillary rise). Mr. Leverett's experimental data[26] on two clayey sands are very close to the

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[22] "The Lines of Flow of Water in Saturated Soil," by L. F. Richardson, *Scientific Proceedings*, Dublin Royal Soc., Vol. 11, 1908, pp. 295–316.

[23] "Fundamentals of Soil Mechanics," by D. W. Taylor, John Wiley & Sons, Inc., New York, N. Y., 1948, p. 193.

[24] "Underdrain Practice in the Connecticut Highway Department," by Philip Keene, *ibid.*, Vol. 24, 1944, p. 384.

[25] "Physics of the Earth," Vol. IX, McGraw-Hill Book Co., Inc., New York, N. Y., 1942, p. 344.

[26] "Capillary Behavior in Porous Solids," by M. C. Leverett, *Transactions*, Am. Inst. of Mining and Metallurgical Engrs., Vol. 142, 1941, p. 161, Figs. 4 and 5.

curves in Fig. 17, whereas for four clean sands he found very little saturation at the upper part of the tube. Both Mr. Lambe and Mr. Leverett plot the degree of saturation horizontally; whereas, in the vertical plot instead of the height h (Fig. 17), Mr. Leverett uses a dimensionless value proportional to h—namely, the mean interfacial curvature of the given sand multiplied by a factor depending on the permeability and porosity of that sand. The analytical approach to the determination of the curvature-saturation function has also been presented by W. O. Smith.[27,28,29,30]

Capillary Heads and Capillary Rise.—Very ably, Mr. Lambe proves the existence of various characteristic points on the curves in Fig. 17 controlling the lengths of some vertical distances h (with various subscripts), which he terms "heads." Although the writer recognizes the five heads given in the "Conclusion," he believes that their introduction into engineering practice may possibly create even more confusion than before. Another objection against all these heads, as shown in Fig. 17, is that they refer to vertical capillary rise and may or may not be of the same value for horizontal or oblique motion.

What is really needed in practice is a test which will give even approximate answers to the questions on capillarity that a highway or airport engineer faces all the time; for instance—

(*a*) A subgrade has been compacted at a certain moisture content, say, 10%. What will be its moisture content 3, 4, or 5 years hence?

(*b*) An embankment of a certain height is to be built at a place where the water table is very shallow—that is, close to the earth's surface. Is it dangerous or not, for a given embankment material? Perhaps some other earth material for the embankment would be better?

(*c*) There is a possibility of lowering the water table by a system of ditches. How deep should these ditches be? Does this depth depend on the type and material of the structure, and, if so, how?

(*d*) There is danger of considerable accumulation of moisture at the edges of the pavement. What kind of material should be used for the base and subbase —pervious or impervious material?

The writer believes that, before devising a standard capillary test or tests, all such questions should be given due attention. So far there is no good capillary test, and there is a great need for one.

The writer is decidedly unfavorable to the author's suggested use of the term "capillary head" instead of "capillary rise." The term "head" in hydraulics and allied fields means a column of water of a certain height; and it is measured in units of length. The term "rise" generally means the action or process of rising, as, for instance, sunrise. These terms "head" and "rise" belong to two different classes of words. Apparently, what the author means by "capillary rise" is the "height of capillary rise" as used in the literature.[31] In turn,

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[27] "Capillary Rise in Sands of Uniform Spherical Grains," by W. O. Smith, Paul D. Foote, and P. F. Busang, *Physics*, July, 1931, pp. 18–26.

[28] "Capillary Flow Through an Ideal Uniform Soil," by W. O. Smith, *ibid.*, September, 1932, pp. 139–146.

[29] "Minimum Capillary Rise in an Ideal Uniform Soil," by W. O. Smith, *ibid.*, May, 1933, pp. 184–193.

[30] "The Final Distribution of Retained Liquid in an Ideal Uniform Soil," by W. O. Smith, *ibid.*, December, 1933, pp. 425–438.

[31] "Soil Mechanics in Engineering Practice," by K. Terzaghi and R. B. Peck, John Wiley & Sons, Inc., New York, N. Y., 1948, p. XII.

the term "capillary rise" should be used only to mean the vertical capillary movement. The latter term ("capillary movement") is more general than "capillary rise."

Conclusion.—This discussion is not meant to detract from the high value of Mr. Lambe's work. If curves can speak, the curves in Fig. 3 are eloquent in the sense that they offer much material for thought to the engineer who is willing to question them.

T. WILLIAM LAMBE,[32] JUN. ASCE.—A suggestion is made by Mr. Lewis that the pressure to the left of point B in Fig. 6(*b*) should be greater than atmospheric pressure. The comparison used to arrive at this suggestion does not appear to be a valid one. At any rate, test data prove the presence of pressures less than atmospheric on the straight line part of the total head curve—that is, line A in Fig. 3.

The writer appreciates Mr. Swe-tze's emphasizing that the zone of partial saturation is larger in unsteady flow than in a static condition. This point is illustrated in Fig. 12 where one can see that $z' - z$ increases from an initial value of zero to a maximum and finally returns to $h_{c1} - h_{c2}$. Mr. Swe-tze is also in favor of a test that determines the moisture distribution curve after drainage.

The writer is grateful to Mr. Krynine for giving experimental data and theoretical explanations concerning capillary flow. Mr. Krynine is correct in putting more emphasis on the inadequacy of Eq. 13.

Although a single laboratory test that would answer all the questions of the highway or airport engineer is certainly desirable, its development appears unlikely. A good standard test would be of value in comparing various soils, and it would give the engineer some help in his particular problem.

The writer does not consider that Mr. Krynine's second objection to Fig. 17 is sound. On the contrary, he feels that the heads shown in Fig. 17 are fundamental values dependent on the size of the meniscus at the air-water interface. The direction of motion is important only in that it influences the selection of the effective meniscus. The heads in Fig. 17 were determined from drainage (h_{cx}, h_{cs}, and h_{ca}), capillary rise (h_{cr} and h_{cn}), and horizontal flow (h_{cp}).

In conclusion the writer thanks the discussers for increasing the value of his paper.

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[32] Asst. Prof. of Soil Mechanics, Massachusetts Inst. of Technology, Cambridge, Mass.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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Paper No. 2436

MOVEMENTS IN THE DESICCATED ALKALINE SOILS OF BURMA

BY F. L. D. WOOLTORTON[1]

WITH DISCUSSION BY MESSRS. HANS F. WINTERKORN, J. E. JENNINGS, AND F. L. D. WOOLTORTON.

SYNOPSIS

Vertical movements in the tropical, desiccated, alkaline soils of the Mandalay District, Burma, are not considerable. An extreme maximum free soil surface movement of 3 in., based on volume changes over a calculated depth of 26.5 ft, is probably less than occurs in Texas. Horizontal shrinkage may be 8% at the surface over soil cracks 3 ft apart. Movements imparted to brick structures, when the dry soil (moisture content approaching or below the shrinkage limit) at foundation depths takes up moisture over the residual shrinkage range (about 5%), are sufficient to lead, by accumulative differential effects, to distortion and cracking when such structures transmit loads less than the effective soil swelling pressure. The deflection curves for such buildings are dome shaped. A maximum building deflection of 2.88 in. was recorded.

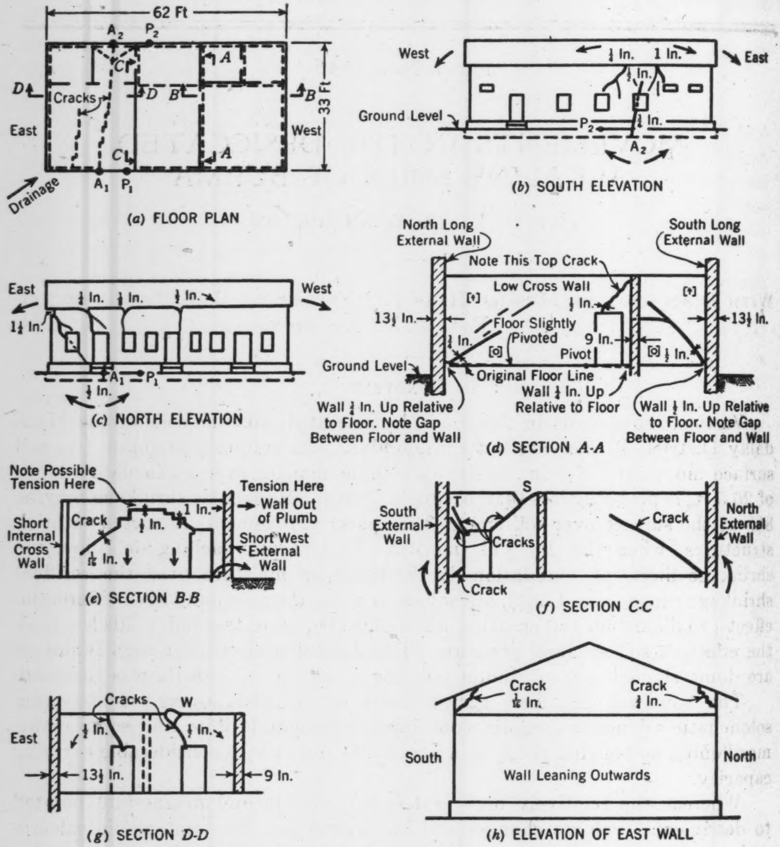
The soil was classified agriculturally as a calcium-magnesium-sodium solonchatic-solonetzic complex containing considerable illite and some montmorillonite, and engineeringly as a nonplastic A-6 clay of considerable carrying capacity.

Whereas the relatively high replaceable sodium and magnesium pointed to detrimental volume changes and impermeability, the replaceable calcium and magnesium suggested permeability. Although indicated as impermeable by the permeability test, mechanical, chemical, and moisture-content analysis showed the soil to be permeable. Moisture-content and volume changes thus occur rapidly. Badly cracked buildings were not over soil with higher clay content or shrinkage coefficient but with higher permeability. Within the moisture range, the swelling pressure of the clay is cyclically mobilized producing periodic horizontal and vertical loadings on foundations leading to accumulative movements.

NOTE.—Published in January, 1950, *Proceedings*. Positions and titles given are those in effect when the paper or the discussion was received for publication.

[1] Constr. Engr., Road Constr. Div., Public Works Dept., Nakuru, Kenya Colony, East Africa.

Building damage occurs at periods leading up to maximum moisture content (seldom exceeding 19%) and arises from the high reverse deflections whose effects are accentuated by sinusoidal modifications. Buildings transmitting loadings appreciably greater than the swelling pressure were not damaged and their deflection curves were dish shaped.


62 Ft
A_2
P_2
C
A
D
Cracks
D
B
B
33 Ft
East
West
C
A
Drainage
A_1
P_1
West
¼ In.
1 In.
East
½ In.
Ground Level
P_2
¼ In.
A_2
East
¼ In.
¼ In.
½ In.
West
1½ In.
A_1
P_1
½ In.
North Long External Wall
South Long External Wall
Note This Top Crack
Low Cross Wall
[+]
[+]
½ In.
13½ In.
13½ In.
Floor Slightly Pivoted
¼ In.
[o]
9 In.
Pivot
[o]
½ In.
Ground Level
Original Floor Line
Wall ¼ In. Up Relative to Floor
Wall ¼ In. Up Relative to Floor. Note Gap Between Floor and Wall
Wall ½ In. Up Relative to Floor. Note Gap Between Floor and Wall
Note Possible Tension Here
Tension Here
1 In.
Wall Out of Plumb
Short Internal Cross Wall
Crack
½ In.
$\frac{1}{16}$ In.
Short West External Wall
S
T
South External Wall
Cracks
Crack
Crack
North External Wall
Cracks
W
East
½ In.
½ In.
13½ In.
9 In.
Crack $\frac{1}{16}$ In.
Crack ¼ In.
South
North
Wall Leaning Outwards

(a) FLOOR PLAN

(b) SOUTH ELEVATION

(c) NORTH ELEVATION

(d) SECTION A-A

(e) SECTION B-B

(f) SECTION C-C

(g) SECTION D-D

(h) ELEVATION OF EAST WALL

Fig. 1.—Office Building, Mandalay Race Club, Mandalay, Burma

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1. Introduction

The study of soil movements in the desiccated soils of the dry zone of Central Burma arose out of a specific investigation into why, soon after construction, buildings in this region developed a most elaborate, and often symmetrical, pattern of cracks. In five known examples the cracking was so persistent and so severe that structures had to be demolished and rebuilt.

The main reason given for the cause of cracking was that, as the soil was considered to be soluble and hence plastic, the foundation loadings were too high. This view had resulted in designs to transmit, in the main, very low load concentrations.

The object of the investigation was principally to discover the phenomena that caused buildings to crack, and, also, why such cracking continued, in many instances, year after year without abatement. An explanation for the phenomena could not be found by pure soil mechanics; the investigation soon led to a pedological study of the soil and later to a study of the building deflection curves.

An example of the basic problem can be demonstrated by Fig. 1, a simple, rectangular, single-story, brick building, in Mandalay. The exterior walls were 13.5 in. thick and 16 ft high, the interior walls were 9 in. thick and 8 ft high, capped by a roof supported on steel trusses. The external walls rested on a cement concrete strip footing which in turn rested on a 12-in. cushion of sand. The underside of the foundation (3.75 ft below the ground level) was in the transition layer, between the surface "black cotton" soil and the lower "kyatti" soil. (Soil types are described, subsequently, in Sections 2 and 3.)

The building had cracked so badly as to be beyond repair. The cracks were as wide as 2 in. and the short end walls on the east were 2.5 in. out of plumb, the tops leaning outward. (In Fig. 1, the east end wall was out of plumb 1 in. at the ends and 2.5 in. at the center. The west end wall was out of plumb ¾ in. at the ends and 1.5 in. at the center. The cracks indicated in Fig. 1 illustrate both upward and downward movements of the center. The dimensions refer to the width of cracks, and the arrows show the direction of movement.) Although the walls were deflected, there was no sudden deflection due to uneven settlement and consequent shear. The bearing pressure was only 0.25 ton per sq ft. Levels along the plinth (water table) on the north and the south gave definite curves indicating that the building might be considered to have pivoted about some central axis, P_1P_2 (Figs. 1 and 2), at right angles to the long walls. The maximum variation in levels over the entire foundation was 2.5 in., other observed maxima being:

Wall	Movement (in.)
North—	
Laterally	0.5
Vertically (assuming water table originally level)	2.4
South—	
Laterally	0.75
Vertically	1.68

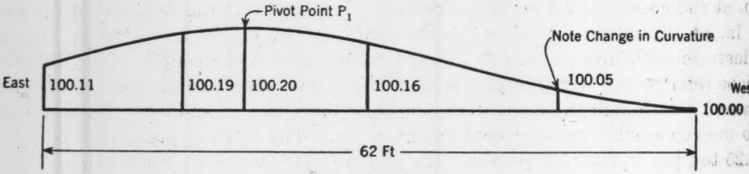
The variation in level of the long foundation walls toward the west ends was not uniform but indicated a second inflection point. The lowest part was the west end, not the east end which received the brunt of the drainage.

An examination of the cracks revealed three main forms—(1) bending tension, (2) tension by reverse bending, and (3) diagonal tension.

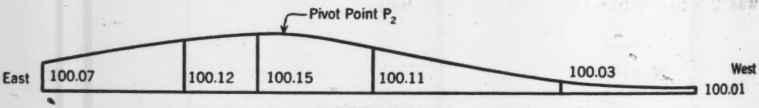
(1) In the long walls there were tension cracks in the tops extending about one third the way down—cracks which were in accordance with the variation in foundation levels and the deduction that the building had pivoted about a central axis, $P_1 P_2$, perpendicular to the long walls, as shown in Figs. 1(*a*) and 1(*b*).

(2) One bending tension crack, near the pivot points in each of the long external north and south walls, was in the bottom of the wall (see points A_2 and A_1, Figs. 1(*b*) and 1(*c*)), suggesting that bending in the reverse direction to that then evident had occurred at some period, although to a lesser degree. This was also indicated by floor cracks, running between points A_1 and A_2 (Fig. 1(*a*)) close to, and approximately parallel with, the line $P_1 P_2$, although the width of these cracks might have been due in part to longitudinal movement. Furthermore, some of the cracks in the top of the long internal low walls suggested compression failure—namely, point W in section D-D (Fig. 1(*g*)).

(3) The points of failure in the 9-in., low, short interior walls were diagonal tension cracks caused by shear. From the point where the cross walls joined the exterior walls a diagonal crack extended three fourths of the way up the cross walls (see Fig. 1(*f*)). Around the entire exterior wall (except for the area near the north wall pivot P_1), and along the cross walls (except in the center of the building), there was a relative raising of the wall to the floor (see Fig. 1(*d*)).



(*a*) NORTH WALL



(*b*) SOUTH WALL

FIG. 2.—SETTLEMENT OF THE FOUNDATION

An examination of about eighty buildings showed that the types of cracks described in the foregoing example were representative of most of the cracks found.

2. DESCRIPTION OF LOCAL SOILS

Two soils were involved known locally as kyatti and black cotton. Neither term is a pedological one, although the latter conveys a little more information to an outsider than the former.

Kyatti Soil.—The term kyatti merely conveys the idea of stickiness pertaining to certain layers in most of the local soil profiles. J. Charlton[2] has remarked that:

> "In actual practice it has been found that soils called 'Kyatti' (sticky) by Burmese cultivators are very liable to give bunds in which holes or caverns appear and, or, which ravine badly under rainfall."

Kyatti is the local name for the sticky soil below, or until comparatively recently below, the surface layer which might have been a local black cotton soil in areas where the relief would permit such soils to be formed. It represents a zone of accumulation of fine material and salts.

In general, kyatti soil has the following properties:

a. It is strongly alkaline, having a pH-value greater than 8.4 and sometimes in excess of 9.75.[3]

b. The presence of soluble salts and replaceable sodium causes the (disturbed) soil to be soluble in running water when the soluble constituents are removed in solution or colloidal suspension.

c. It is highly calcareous (nodules).

d. It lacks organic matter and hence is deficient in protective colloids although rich in dispersible colloids.

e. It is said to be impermeable.

The foregoing properties appear to apply more specifically to well-drained, uncultivated areas of kyatti surface soil.

Black Cotton Soil.—The surface layer (generally from 3 ft deep to 4 ft deep) overlying a kyatti, or accumulation, layer below, is designated black cotton soil. It has a pH-range of from 6.5 to 8.4. The presence of organic matter and the leaching of some of the deleterious salts appear sufficient for kyatti dispersion properties to be absent, but a number of dispersion tests suggested that the locally so-called black cotton soil might have kyatti characteristics in an extreme form.

The origin of the black cotton soils of Mandalay is very different from those of the central provinces and Hyderabad, India. Whereas the former are alluvial, the latter are said to be weathered products of the decomposition of Deccan trap or basalt.[4]

The Indian black cotton soil, or "regur," is claimed to be a highly argillaceous, calcareous, fine-grained soil containing a high percentage of calcium and magnesium carbonates and up to 10% of organic matter. (B. V. Nath[5] throws doubt on this claim and states: "The black cotton soils, or, the 'regurs' do not contain much organic matter.")

Mandalay black cotton soil, as subsequently found, is a siliceous, calcareous, fine-grained soil containing relatively high replaceable sodium and magnesium and only a little organic matter.

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[2] "A Note on Soils Regarding Their Suitability for Making Irrigation Works Exposed to Water," by J. Charlton, in "Agriculture and Live Stock in India," Vol. 1, 1931.

[3] "A Note on Soils Regarding Their Suitability for Construction of Irrigation Works, Exposed to the Action of Water," by J. Charlton, *Proceedings*, Assn. of Engrs. in Burma, Vol. 111, 1931.

[4] "Theory and Design of Foundations in Black Cotton Soil," by Y. D. Kumar, *Journal*, Inst. of Engrs., India, Vol. X, 1931.

[5] "Black and Red Soils of India," by B. V. Nath, *Bulletin No. 2*, Indian Soc. of Soil Science, 1939.

Soil Structure.—Throughout the profiles of the soils examined, the microstructure appears single grained. In a strict sense, the soil has no macrostructure; but, nevertheless, cracks form during the dry weather, causing the formation of very large and approximately hexagonal columns. Toward the end of the hot dry weather these cracks are from 2 in. to 4 in. wide, extending to a depth of 7 ft or more. The corners of the hexagons sometimes become detached and sink, leaving an irregular hole which, in time, often becomes a permanent hole or depression.

In disturbed kyatti soil the structure (during the hot dry weather) tends to be loose, friable, and "nutty" at the surface to cloddy at the depths beneath the surface (undisturbed). It is in such soils that roundish vertical piping (about 6 in. to 18 in. in diameter) is found, and in which ravines and caverns form so rapidly.

3. Agricultural Classification

The absence of pedological classification necessitated an attempt to classify the Mandalay soils examined. A monsoon rainfall of 34 in. occurs between May 15 and October 15; the average temperature is 28° C; and there is an apparent mixed tropical podsolic and lateritic weathering system. Under these conditions the soils are believed to be transitional yellow earths occurring north of the tropical laterites of Lower Burma and south of the believed brown forest soils of Upper Burma. In particular, they appear as "intrazonal-secondary-solonchatic-solonetzic" complexes, although drainage is generally free and no obvious permanent water table exists for a considerable depth. It appears, however, that a temporary water table exists within a depth of 30 ft during the rains, as evidenced by shallow wells (30 ft deep), and within a depth of 12 ft toward the end of the hot dry weather, as evidenced by moisture-content data, despite the fact that air is always present.

The profile is reasonably consistent in the areas studied, and consists of a 3.5-ft-deep, dense hard layer of black cotton soil, overlying a dense hard accumulation zone (4 ft deep) of black to yellow, or mottled black-yellow kyatti soil. A number of "sticky" horizons may be found in any deep profile. At deeper strata lies the dense yellow parent alluvium extending below a depth of 100 ft where it overlies sand.

The soil is mildly siliceous (silica-sesquioxide ratio about 2.6) indicating cohesion but little plasticity. The predominant replaceable base varies and may be calcium, magnesium, or sodium. The stickiness of kyatti soil is due to the presence of sodium salts and replaceable sodium in the clay complex. The relatively high molecular ratio of replaceable sodium to replaceable calcium justifies the clay being generally classified as one that has the properties of a sodium clay. For such a weathering system, the high replaceable magnesium probably also plays an important role in the properties of the soil. It indicates both expansive properties and aggregation, or perhaps permeability, and the aggregation would be accentuated by the presence of the free and replaceable calcium.

The clay minerals for samples taken from a depth of 21 in., and examined by X-ray methods by George L. Clark, were a small amount of montmorillonite and, apparently, considerable illite.

Such a clay would be expected to exhibit considerable and rapidly attained swelling and shrinkage, high density, high cohesion, and low plasticity. Hence, it would be a clay of considerable carrying capacity. It would be expected to differ from a pure sodium clay, however, in its diffusive properties and moisture-holding capacity, in place, because of the flocculating effect of calcium and magnesium. This flocculation might be sufficient to prevent impermeability which is associated with sodium clays.

The untreated and pretreated mechanical analyses examined, in conjunction with the chemical analyses, had also indicated aggregation. They showed that the dispersed clay content increased from about 25% in the surface (loam) layer to a maximum of 61% at the 3.5-ft (clay) layer, and thereafter decreased to 48% at the 12-ft (clay) depth, with an accumulation layer between 3 ft and 7 ft marking the limits of upward and downward weathering. Clay-loam lenses were occasionally recorded between the depths of 8 ft and 12 ft but their thicknesses appear to be limited to about 1 ft and, from the many borings taken, their plan dimensions are believed to be small.

To the depth of 12 ft examined, and especially that in the accumulation band, the soil might have been assumed to be impermeable (as generally believed); but from the analyses it was found to be aggregated, with a maximum aggregation at the depth of the maximum clay content—that is, 3.5 ft. This aggregation was not visibly obvious. By these analyses, the depth of 7 ft (not that of 3.5 ft) was indicated to be the least permeable. This finding was supported by the observation that the maximum apparent density of 2.4 and the maximum pH of 9.1 occurred at a depth of 7 ft.

4. SOIL MOISTURE

As it was considered that structural damage was correlated with moisture changes, arrangements were made to determine the monthly variation in moisture content for the depths, 3.5 ft to 4.5 ft, 6 ft to 7 ft, and 11 ft to 12 ft, at four residential areas to determine the maximum and minimum values and to observe whether a zone of constant moisture content existed within these depths (Fig. 3). In addition, the percentage,

$$S = \frac{\text{moisture}}{\text{saturation}} \times 100 \dots\dots\dots\dots\dots\dots\dots (1)$$

was determined. In this percentage value, the denominator was determined for a dried pulverized sample. Mean values of S are plotted in Fig. 4 for three representative residential areas. The average rainfall was 34.36 in.; the mean annual temperature, 82° F; and the total rainfall for the twelve months tested, 34.87 in. Within this period a load test (see Fig. 4) of 2 tons per sq ft was applied to kyatti sodium clay at a depth of 5.38 ft. The intention of the second measurement for determining S was to allow for any variation in soil texture which might occur in the monthly samples extracted from any particular depth.

The second task was to estimate the effect of seepage water, chemically comparable with rain, when the soil was in a condition similar to the thin layer immediately below local shallow foundations—that is, slightly puddled.

5. Summary of Soil Moisture Observations

The results of soil moisture observations can be itemized in three groups under "direct results," "indirect results," and "the effect of water on disturbed Mandalay soils."

Direct Results for the Residential Area.—

1. Monthly readings to a depth of 12 ft showed there was no zone of constant moisture content throughout the year and, hence, that some potential movement must be experienced by any structure whose foundations are laid within this depth. Nevertheless, the observations showed the undisturbed soil to be saturated for much of the year although this condition (as will be shown) is accompanied by an entrapment of air.

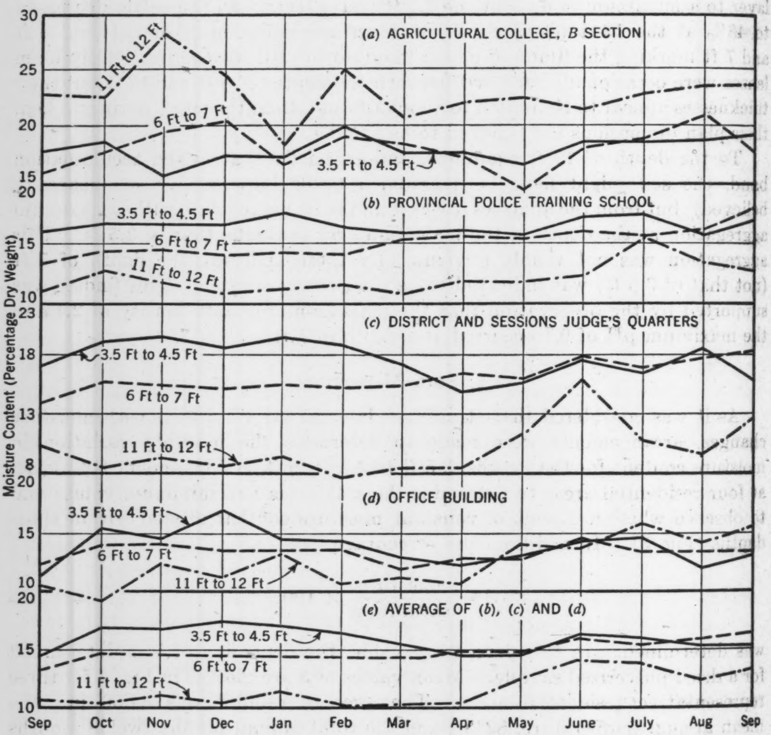

Fig. 3.—Yearly Variation in Moisture Content, September, 1935, to September, 1936

2. A "good" site, as judged by absence of cracking, is indicated by minimum moisture variations and minimum rates of change of moisture. In other words the value of the site decreases as its permeability increases.

3. The variation in, and the rate of change of, moisture content tended to be at a minimum at the depth of 7 ft (maximum pH-values), showing this level to be, in general, the least permeable and, by virtue of lower and slower moisture changes, the most suitable for foundations.

4. During the period from March to April, before the rains broke, the moisture content began to increase between 12 ft and 6 ft, attaining maximum values in May, which was still before the rains broke. This anomaly (see, subsequently, in Section 18) can be explained by the "distillation effect"[6,7] which, for arid soils, led to the first horizon of ground water, in summer (dry weather). In other words, during dry weather, a temporary perched water table was formed—by condensation, and not by rain.

5. The rapid changes in moisture content during the rains suggest that the soil may be permeable to rain water.

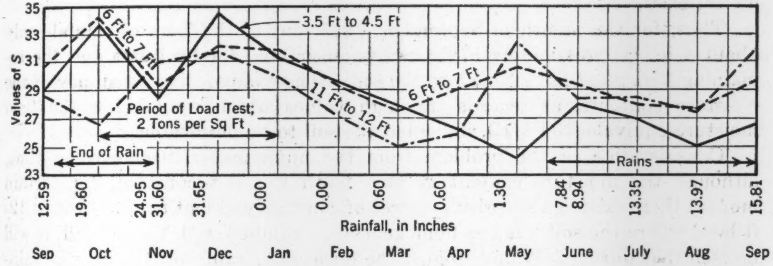

FIG. 4.—VARIATION IN MOISTURE SATURATION CAPACITY, SEPTEMBER, 1935, TO SEPTEMBER, 1936

Indirect Results.—

6. The indirect results of the moisture data were not immediately apparent, principally because the local belief that the soil was a sodium clay and, therefore, impermeable (supported by the high dispersion factor and high apparent density) had not been doubted. When the permeability of the undisturbed soil was at last suspected, the moisture values became of great interest; the key was found in the light rainfall for July and August preceding the first moisture determination in September and in the question of what happened to the heavy rainfall (7 in.) occurring in September. If the undisturbed soil were permeable, this percolating water would flow through the profile under the influence of certain laws and restrictions, but if it were impermeable the moisture would be retained in one of the layers. The heavy rainfall in September continued during October (5.21 in.) and November (7.87 in.), affecting the results for November and December.

The Irrigation Department of Burma could not give the runoff for local uncultivated areas but stated that it could not exceed 30%. Leonard C.

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[6] "The Movement of Ground Water and Soil Water," by A. F. Lebedev, *Proceedings and Papers*, 1st International Cong. of Soil Science, Vol. I, 1927.

[7] "Relation of Water to Soil," by A. F. Lebedev, 2d International Cong. of Soil Science, Vol. VI, Comm. VI, Moscow, 1930.

Urquhart,[8] M. ASCE, quotes the late E. Kuichling, M. ASCE, as giving a relative imperviousness of 0.05 to 0.25 for lawns and meadows,[9] the value used depending on the surface slope and character of the subsoil. Therefore, a value of 20% would be accurate enough for this investigation. For evaporation losses, an interpolation was made from the values found by A. D. Hall[10] as quoted by G. W. Robinson.[11] This suggested a maximum loss of 2.7 in., or 39%, at the prevailing average temperature during the rains. This value is a maximum because Mr. Hall's data were determined for a tilled surface, whereas in Mandalay the surface is covered by a slight mulch which, according to some authorities, tends to prevent evaporation. As a check, the formula of Walther Leather (as given by Harry R. Kempe[12]), involving temperature, humidity, and wind velocity, gave an evaporation of about 1.5 in., or 21% of the rainfall. The plant transpiration factor for uncultivated land is normally small and negligible in Mandalay where, for the areas examined, plants virtually do not exist.

Thus, for the month of September, 1935, out of the 7 in. of rainfall only about 4 in. (approximately 57%) can be accounted for by losses and the remaining 3 in. must have gone into the soil. The comparative constancy of the moisture contents, as regards the high values which were below the field moisture equivalent (FME), indicated the soil to be permeable.

Consideration of the problem from the numerical values revealed that, although the moisture content at any depth was not constant, for certain months there existed a surprising degree of constancy except for the 11-ft to 12-ft level where the soil was less homogeneous. Studied with the rainfall, it will be seen that during July and August there was little rain; but the heavier rainfall of 7 in. in September was reflected in an increase of moisture up to approximately the "field capacity" for the October values—in two instances down to 7 ft (the limit of downward weathering), and in the third down to 12 ft. The average increase for the three areas, allowing for the decrease at the 12-ft level in two areas, was 0.95%, 0.41%, and 3.29%, or an over-all average increase of 1.55%. This increase corresponds to $1.55/100 \times 12 \times 1 \times 100 = 18.6$ lb, or 3.5 in. of rain, against the value of 3 in. computed previously as having entered the soil. This comparison suggests that, on the average, all the available rain had percolated the profile and was just sufficient to reach a depth of 12 ft in one month. These values give an average movement of rain water, in a downward direction, of 5 in. per day.

For October, November, and December, when the rainfall was 5.21 in., 7.87 in., and 0 in., respectively, the moisture contents, after their initial increase in October (which, in two cases, was delayed for the 12-ft depth), remained constant (with one exception) at 3.5 ft to 4.5 ft and 6 ft to 7 ft and nearly constant at 11 ft to 12 ft. Such constant values show that, for such a soil, the moisture content during these months represented the Veihmeyer or "agricultural field" capacity (the volume of water held in the soil after excess gravitational water has drained away) and that, to a depth of 12 ft, the soil was saturated under field conditions. Otherwise the moisture would have continued to increase. During January there was no rain and the results for February and March indicated a general decrease.

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[8] "Civil Engineering Handbook," by Leonard C. Urquhart, McGraw-Hill Book Co., Inc., New York, N. Y., 2d Ed., 1940, p. 793.

[9] *Transactions*, ASCE, Vol. LXV, December, 1909, p. 400.

[10] "The Book of the Rothamsted Experiments," by A. D. Hall, Rothamsted Experimental Station, Harpenden, Hertsford, England, 1905.

[11] "Soils, Their Origin, Constitution and Classification," by G. W. Robinson, Thomas Murby & Co., London, 1936.

[12] "Engineer's Year-book," by Harry R. Kempe, Morgan, London, 1939.

The writer considers that the foregoing observations prove that the undisturbed soil is permeable and that, for much of the year, it is in an undisturbed state of saturation. This was not visibly obvious because of the high density of the soil and the percentage of entrapped air. The continuance of fairly constant saturation can only be explained by the fact that the field capacity is nearly equal to the limit of "osmotic imbibition" (defined as the maximum moisture content that clay particles will adsorb under osmotic type of forces which, for a highly expansive clay, approximates the FME). This argument is supported by the rapid and considerable changes in S that occur during the rains, as shown by Fig. 4. In other words, except when water is actually percolating the profile, there is generally little capillary water. This argument was corroborated by consolidation data.

7. Furthermore, it follows from the definition of "field capacity," when applied to a permeable soil, that it must be approximately equal to the "shrinkage limit" of undisturbed soil.

The Effect of Water on Disturbed Mandalay Soils.—

8. A site adaption of the slaking test, with remolded samples, showed that in this condition the deeper the soil (to 12 ft) the more slowly it dried and absorbed water. This finding was supported by the wetting times of the previous experiment on pulverized soil for the determination of the saturation capacity. The inference is that, when disturbed, the soil's resistance to changes in moisture content by fairly pure water increases with depth to 12 ft and that its permeability decreases with depth. This is in fair agreement with the discussion on the chemical analysis (see Section 3) but, as in that discussion, it leads to results contrary to what might have been expected from a study of the mechanical analyses alone.

9. It may appear also that these results are inconsistent with items 6 and 7, but it must be remembered that conditions are not the same in the two experiments. In items 6 and 7 variations in moisture are due to soil, or saline water, and are caused by natural changes in the moisture content of a soil with an undisturbed structure. In this test on disturbed soils the results are due to relatively pure water, such as might occur during rain when water percolates between the building and the soil. In the former instance the flocculated condition is preserved; in the present instance the water apparently deflocculates the remolded and structureless clay, below 7 ft, causing impermeability.

10. If the soil at foundation depths is superficially puddled, it will thus resist the seepage water entering between the soil and the foundation wall; and, for the short periods of local rainfall, the water is likely to have little effect. Once through the puddled and relatively impermeable layer, the water

will be led away by the increase in permeability. Thus, little building damage may be expected as a direct result of percolation.

6. Relation Between Climate and the Cracking of Buildings

From observations and records of the cracking of buildings it seemed that some cracks began to appear during August. In October and November they became obvious, and had been known to widen in November and December, but the worst month for cracks was undoubtedly May. Records further showed that cracking had been more pronounced during certain years. Therefore, it was decided to examine these records with meteorological records and the moisture-content data.

The results clearly showed that there were two main periods during which the cracking appeared—at the end of the rains, and at the end of the dry weather. The former period, occurring during or after a rapid series of maximum and minimum moisture contents, corresponded to a period of maximum moisture content, maximum humidity, and minimum mean temperatures. The latter period occurred when the dry weather minimum moisture content was reached at higher levels, accompanied by a rapid moisture increase between 12 ft and 7 ft. At the same time the mean temperatures were a maximum and the humidity was a minimum.

Cracking, therefore, appears to be caused by volume changes and to occur at periods of minimum and maximum soil volumes. It is augmented by the rapid changes following these conditions. Data showed that the cracking of buildings was most noticeable at the end of a long dry period following a long wet period. The periods investigated were each of two-year duration. Cracking was also found to be associated with periods of exceptionally heavy periodic rain. The moisture conditions of the foundation soil at the time of construction probably have a considerable influence over the future history of the structure.

7. Loadings Transmitted by Existing Structures and Site Loading Test

When it was found that the foundations of the condemned Mandalay Race Club office (see Fig. 1) were only transmitting 0.25 ton per sq ft, it was decided to investigate other loadings to correlate building cracks with loading and depth of foundations and to ascertain if there were any truth in the belief that the soil was being overloaded. It was also arranged to make a site loading test to determine the maximum load the soil could safely withstand, to estimate the accompanying sinkage, and to verify the belief that the plastic yield of the undisturbed soil was small.

An analysis of the loadings transmitted by sixty-six buildings showed:

a. A variation in loading transmitted to the soil of between 0.25 ton per sq ft and 2.1 tons per sq ft;

b. No direct correlation between cracking and depth of foundations; and

c. A direct connection between loading and cracking; few cracks were found in buildings transmitting more than 1 ton per sq ft; and for a loading less than 1 ton per sq ft, and foundation depths between 3 ft and 7 ft, all the buildings had cracked and thirty two out of fifty five were described as badly cracked. This refuted the belief that cracks were due to overloading the soil.

A confined site loading test was made on an area of 1 sq ft, in a hole about 15 in. square, at a depth of 5.38 ft and with a soil moisture content of about 15%. It was thus determined that (see Table 1), under its maximum normal moisture content, the soil could safely carry 2 tons per sq ft with little settlement (0.36 in.) and no plastic yield. For conditions corresponding to percolation between building and soil, and for flooding sufficient to create a 5-ft head, there was some immediate increase in the settlement (0.09 in. and 0.16 in., respectively) but no continuous and accumulative plastic yield. This proved the soil to be definitely nonplastic and refuted the belief that excessive sinkage was caused by rain and percolation.

TABLE 1.—SETTLEMENT OBSERVATIONS ON UNDISTURBED KYATTI SOIL, UNDER A LOAD OF 2 TONS PER SQ FT

Period (1)	Condition of pit (2)	Settlement[a] (in.) (3)	Weather (4)	Period (1)	Condition of pit (2)	Settlement[a] (in.) (3)	Weather (4)
October, 1935				November, 1935:			
8 to 9	Dry	0.36[b]	Fine	3	....	0.44	Rain
9 to 12		0.36	Fine	4	....[c]	0.45	Rain
12 to 26		0.36	7-in. rain	5	Flooded to depth of 5 ft	0.53	Rain
26 to 31		0.36	Fine	6		0.61	Rain
31	0.5 in. of water added daily	0.36	Fine	6 to 12		0.61	Fine
November, 1935:				12	No[d]	0.61	Fine
1		0.44	Fine	12 to 20	Muddy	0.61[e]	Fine
2		0.44	Fine	20 to February 4, 1936	Dry	0[f, g]	Fine
3 to 7	....	....	5-in. rain				

[a] Cumulative, total settlements. [b] As load increased from 0 to 2 tons per sq ft the settlement increased from 0 to 0.36 in. [c] 1 in. of water added. [d] Water pumped out. [e] Load 1 ton per sq ft. [f] No settlement, but some evidence of cyclic movements. [g] Load 0.75 ton per sq ft.

The possibility of cyclic changes in the levels of the settlement indicator pegs and bench mark on a near-by post footing was suggested and the phenomenon was examined in detail at a later date (see Sections 18 and 19).

8. SOIL CONSTANTS AND CLASSIFICATION

Mechanical Analyses.—A compilation of all data to show the variation in the clay fraction and the soil constants with depth is given in Tables 2 and 3, respectively.

Apart from the shrinkage data, the most interesting values are those for the plastic limit (PL) and the FME, when compared with the moisture contents of undisturbed soils. As the FME (which is held to represent the maximum quantity of water that the disturbed soil can freely hold and to be a value greater than the undisturbed soil may hold) is less or equal to the PL (which is greater than the maximum moisture content of undisturbed soil), the soil will remain stable during wet weather, there will be no plastic yield, and the bearing value will be unimpaired. This observation was confirmed by the load test reported previously in Section 7.

General considerations of the data and comparison with the old correlation curves issued by the United States Public Roads Administration place the soil in the A-6 group.[13] From correlation curves of the coefficient of friction and the PL reported in 1938,[14] the angle of friction would be 16°. It is probable that, with its low moisture content, the soil retains its granular nature in shear and that this angle is always effective for undisturbed soils.

TABLE 2.—PROFILE CLAY FRACTIONS, WITH AND WITHOUT HYDROGEN PEROXIDE PRETREATMENT

DEPTH (FT)		CLAY (%)		COLLOIDAL CLAY (%)		Soil type
From:	To:	Not treated	Pretreated	Not treated	Pretreated	
0	0.5	20.0[a]	....	10.0[a]	....	Black cotton soil
2	3	39.5	58.4[b]	16.5	49.0[b]	Black cotton soil
3	4	....	61.0	....	....	Kyatti soil
3.5	4.5	47.2[c]	....	44.7[c]	....	Kyatti soil
6	7	44.4[c]	....	39.0[c]	....	Kyatti soil
7	8	....	54.2[b]	....	44.2	Parent soil
11	12	38.7[b]	....	34.0[b]	....	Parent soil
12	13	....	48.4	....	....	Parent soil

[a] Mean of fifty-seven samples. [b] Mean of two samples. [c] Mean of three samples.

TABLE 3.—SOIL CONSTANTS

Structure (as identified in Fig. 3)	Lower liquid limit	Lower plastic limit	Plasticity index	SHRINKAGE:			MOISTURE EQUIVALENT		Specific gravity[b]	Volumetric change	Linear shrinkage[c]
				Limit (remolded)	Ratio[a]	Residual	Centrifuge	Field			
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
Office Building:											
2 ft to 3 ft deep......	67.4	19.0	48.4	9.1	2.10	3.5	42.3	18.6	2.68	20.0	6.0
7 ft to 8 ft deep......	70.0	20.1	49.9	11.0	2.06	3.5	45.4	20.2	....	18.9	5.8
Quarters of District Judge:											
2 ft to 3 ft deep......	81.1	24.3	56.8	....	....	....	51.4	20.5	....	....	....
12 ft to 13 ft deep....	56.2	18.3	37.9	....	....	....	....	....	2.70	....	....

[a] Approximate apparent density. [b] Specific gravity of soil particles. [c] On wet length.

An examination of considerable data pertaining to many soils has indicated that trouble may be expected, in both building and road construction, when the FME materially exceeds the PL—that is, when there is considerable osmotic swelling. Many such soils had constants identical with those for Mandalay except in respect to the FME and the apparent density. The former were actively plastic; the latter were not plastic despite their high plasticity indexes.

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[13] "Classification of Soils and Control Procedures Used in Construction of Embankments," U. S. Public Roads Administration, Vol. 22, No. 12, February, 1942.

[14] *Paper No. 88*, 8th International Roads Cong., The Hague, 1938.

Possibly the difference in the plasticity and in the FME is reflected in the fact that the FME also is a measure of the activity of the soil colloid after drying, when low values indicate a colloid undamaged by drying and high values indicate an irreversible colloid; or perhaps low values are a reflection of the structure and the air-entraining nature of the clay. The FME would certainly appear to represent a measure of the total energy available for absorbing moisture under the colloidal and structural properties of the soil at the time of testing.

9. THE SHRINKAGE CURVES

The phenomena of swelling and shrinkage are of more importance to the soil engineer than is realized. The reconstruction of the site shrinkage curves (that is, curves for undisturbed material) is demonstrated by reference to Fig. 5. Consider the depth 3.38 ft at the office compound first mentioned in Fig. 3. The apparent density ρ for an undisturbed sample for the depth was 1.61 and the specific gravity G for the soil particles was 2.68. The field capacity is 14.71% dry weight, in a specimen of 100 g. Therefore, the volume of water at this field capacity, in cubic centimeters, is $V_0 = 14.71$; $V_1 = 62.20$; $V_2 = 37.30$; and $V_3 = 24.90$. In the foregoing, V_3 is the undisturbed site shrinkage limit for no entrapped air. As a check on V_3, when the density ρ equals 1.61 and the void ratio e equals 0.665, the shrinkage limit is the volume of the voids, or $0.665 \times 37.3 = 24.8$ cu cm. The shrinkage limit for undisturbed soil (line SU, Fig. 5) is approximately the field capacity for a permeable soil.

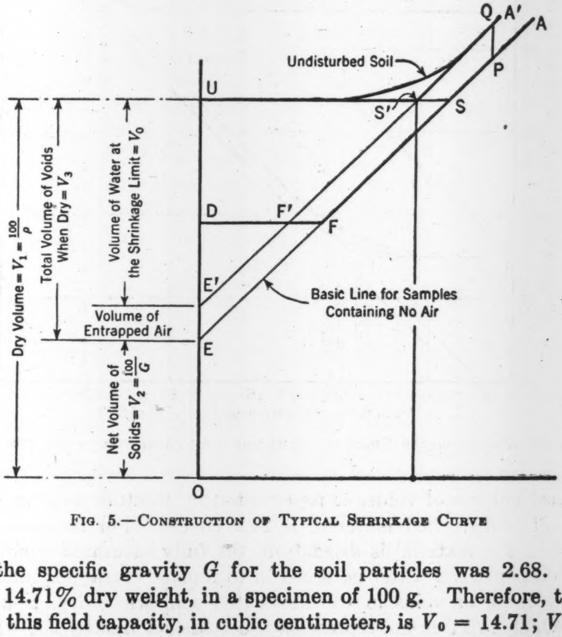
Q
A′
A
Undisturbed Soil
P
U
S′
S
D
F′
F
E′
E
O
Basic Line for Samples Containing No Air
Dry Volume = $V_1 = \frac{100}{\rho}$
Total Volume of Voids When Dry = V_3
Volume of Water at the Shrinkage Limit = V_0
Volume of Entrapped Air
Net Volume of Solids = $V_2 = \frac{100}{G}$

FIG. 5.—CONSTRUCTION OF TYPICAL SHRINKAGE CURVE

The volume of entrapped air at a moisture content of 14.71% (representing the site shrinkage limit when air is present) is $(V_3 - V_0) = 10.19$ cu cm. If the soil be such that some capillary water still exists at the estimated field capacity, the desiccated shrinkage limit for the undisturbed soil may be slightly less than 14.71. From the foregoing data the reconstructed site shrinkage curves may be drawn, as shown in Fig. 6, in which point E indicates the remolded shrinkage limit, point E_1 is the corrected shrinkage limit, point E_2 is the undisturbed shrinkage limit for the condition in which there is no entrapped air, and point E_3 denotes the actual undisturbed shrinkage limit.

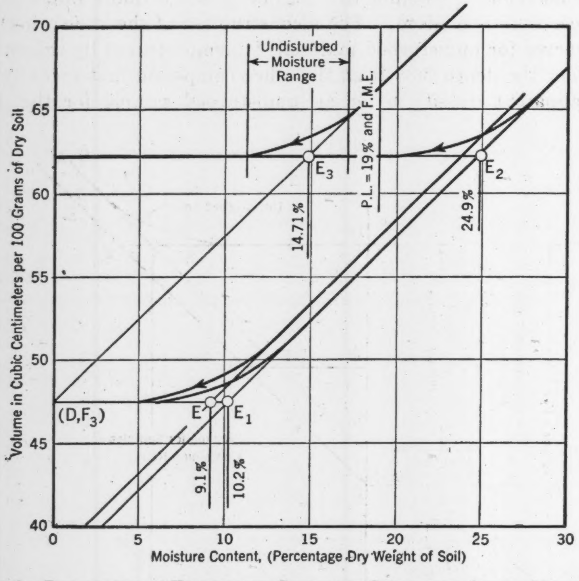

FIG. 6.—RECONSTRUCTED UNDISTURBED SHRINKAGE CURVES FOR OFFICE COMPOUND

The total volume of voids, as represented by the fully saturated shrinkage limit (w_s) of undisturbed material, is 24.9 cu cm. For instance, this is the volume when the material is dried from the fully saturated condition, which can only happen when water is forced in to replace entrapped air. Similarly, the total volume of voids in the desiccated soil, when dry, is likewise 24.9 cu cm of which 14.71 cu cm represents the voids set free by the water held at this shrinkage limit, and 10.19 cu cm represents "entrapped air" or structure.

A point on the site desiccated curve cannot move to its similar position on the fully saturated curve unless the entrapped air is released. If it were not for its structure the soil would be plastic and this condition represents a possible important difference between some soils in the A-6 and A-7 classifications. Diagrammatically the structure (which has gradually been built up and possibly made more permanent by a form of cementation) could be represented by many cycles of drying (below the shrinkage limit) and wetting between the time of the first fully saturated shrinkage curve and the present curve. This structure is believed to be a very important property of the soils in the classification A-6 examined:

Percentage of entrapped air = 16.2, dry volume
Percentage of pore space = 40.0, dry volume

The undisturbed moisture content range lies within the residual shrinkage range in Fig. 6.

Shrinkage from Undisturbed and Disturbed Curves.—As the site moisture content apparently varies over the residual shrinkage range (see Fig. 6) any changes in moisture content will result in volume changes as determined by these ranges. If the fluctuation of 8.12% (which is found at the 11-ft to 12-ft depth at the quarters of the district and sessions judge) occurred above the residual shrinkage range, it would account for a variation of 8.12 cu cm per 58.5 cu cm (approximately) of dry soil, or a volume change of 13.8%—that is, it would provide for an approximate linear variation of 4.5% on the wet length, whether the soil were remolded or undisturbed. However, as the variation occurs in the residual shrinkage range, the formula for volumetric change gives values that are too high, and the corrected values would be about 11.1% (volumetric) and 3.7% (linear), respectively.

A comparison of percentage changes, volumetric and linear, for undisturbed soil, as determined from various curves, is as follows:

Curve	Volumetric	Linear
Shrinkage	7.0	2.3
Compression (wetting from 1.6 tons per sq ft to 0.0 ton per sq ft)	7.3	2.4
Swelling pressure	9.0	3.0

The agriculturally determined linear shrinkage for the same depth was 3.0% for a remolded sample drying from the sticky point.

It should be noted from the consolidation (rewetting) curve that, although the volume changes are relatively small, the corresponding internal pressure changes over this moisture range are considerable.

10. SUMMARY OF ENGINEERING STUDIES

Undisturbed soil, classified as group A-6 flocculated cohesive clay of high carrying capacity—nonplastic, expansive, dense, easily consolidated, and capable of absorbing water only when manipulated—is unimpaired by natural increases in moisture content. At foundation depths the soil has a voids ratio corresponding to that required to carry a load of 6 tons per sq ft when fully saturated, although the presence of air, free to escape, may mean a slightly smaller value. Comparative shrinkage studies gave the results listed in Table 4.

As moisture changes occur within the residual shrinkage range, high capillary suction may be expected in the field when the soil is in contact with a free water face. This is supported by the low shrinkage limits, high densities, and the consolidation swelling curve.

High density values show a high degree of consolidation. The effect of structure is to reduce the density by some 25% and to increase the saturated disturbed shrinkage limit by an average of about 100%.

TABLE 4.—SHRINKAGE CHARACTERISTICS AT VARIOUS DEPTHS

Description	3.5 ft to 4.5 ft	11 ft to 12 ft
Percentages:		
Entrapped air (on dry volume)	16.2	21.5
Undisturbed shrinkage limit (air present)	14.7	8.95
Undisturbed shrinkage limit (air absent)	24.9	21.5
Disturbed shrinkage limit (air absent)	10.2	12.5[a]
Apparent Densities:		
Undisturbed soils	1.61	1.71
Disturbed soils	2.10	2.02[a]

[a] Approximate values.

Undisturbed linear shrinkages vary from about 1½% at depths of from 3.5 ft to 4.5 ft to about 3½% at depths of from 11 ft to 12 ft. These values, limited by moisture changes, are not high; but, occurring in the residual shrinkage range for the wetting curve, they are accompanied by high internal pressure changes. Naturally, agricultural shrinkage values were somewhat higher and gave lower values for a "bad" site than for a "good" site.

Experiments to determine the ratio between horizontal and vertical shrinkage were inconclusive but nevertheless indicated values greater than unity.[15]

From the consolidation curve, for depths from 12 ft to 13 ft, it was deduced that the maximum possible undisturbed natural moisture content was 26.1% (or 13.6% allowing for entrapped air) against the recorded maximum of 16.05%, thus confirming the limited capillary water and the presence of gravitional water at the end of the dry weather.

The permeability (by test) of 3.5×10^{-8} cm per min at a load of 2 tons per sq ft appears low for a liquid limit of 56.2, but it is considered that this test does not give true values for undisturbed soil. An analysis of rain-water movements, based on the moisture curves and the observed change in direction of swelling as evidenced by building cracks, gave the following velocities through undisturbed soils (in feet per month):

Direction	Velocity
Downward	12
Upward	5
Horizontal	2

These values show that the moisture movements caused by the early heavy rains cannot reach the central core under a structure until about the beginning of the hot weather.

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[15] "Vertical and Horizontal Shrinkage of Black Cotton Soils at Mandalay, Burma," by F. L. D. Woolterton and A. T. Sen, *Current Science*, Vol. XI, 1942.

11. CHEMICOCOLLOIDAL SWELLING PRESSURE

From the moisture variations of Fig. 3 it is clear that the voids ratio must change during the year, even to depths of 12 ft. Any change in voids ratio implies (as in the consolidation curve) a change in externally applied pressure, or (as in the shrinkage curve) a change in the internal forces. The externally applied forces on undisturbed material remain constant, but the internal forces vary with change in moisture content—their effect being equal to a change in external forces. Swelling is equivalent to a reduction, and shrinkage to an increase, in external compression.[16] During swelling any applied loading tends to suppress the swelling pressure; during contraction the applied load aids the volume decrease. This research deals with the equivalent reduction in applied pressure arising out of swelling, or the pressure necessary to stop swelling when the soil is in contact with water. Unless it is distributed evenly under a building, swelling has the effect of an applied load on the structure.

If the change in voids ratio is uniform over a site, any movement or deflection will be uniform; but, if the site is partly covered by a building, any change in voids, resulting in the moisture changes, cannot be uniform over the covered area because of the time lag in moisture movement. It is clear, therefore, that, not only will any movement in the structure be of a differential nature, but also the swelling can be prevented only if the soil is loaded to a value greater than the swelling pressure. With more usual loads of, say, from 2 tons per sq ft to 3 tons per sq ft (which are greater than the swelling pressure of this soil, or for a soil that is always fully saturated), the swelling pressure effect is entirely suppressed and movements due to shrinkage (which for the capillary range of Fig. 5, correspond to low pressure changes) are "ironed out" by capillary flow under external pressure, and thus occur more uniformly. It is only for the residual shrinkage range that moisture changes under low external loading become important.

An estimation of the maximum value of the swelling pressure has been found possible by four independent methods:

Swelling Pressure Determined from the Reconstructed Undisturbed Shrinkage Curve Data.—When the moisture content is equal to the engineering shrinkage limit, marking the minimum value of the voids ratio for samples whose line US′ (Fig. 5) is horizontal, the voids ratio is not less than $\frac{21.5}{27.0} = 0.581$. On the consolidation curve for fully saturated soil this value corresponds to a load not greater than 2.2 tons per sq ft—that is, when (wetting under no external load from the shrinkage limit) an applied load not greater than 2.2 tons per sq ft is necessary to prevent swelling in a free surface state. At the test depth of 12 ft the approximate loading required would be about 1.7 tons per sq ft. If an allowance is made for the entrapped air, in the undisturbed soil, the pressure required may be somewhat less.

Swelling Pressure Determined from a Failure Record.—The failure of a cement concrete tie beam, 24 in. by 15 in. by 8 in., connecting two pile tops, had been recorded during the construction of a building at the Agricultural College in Mandalay. This occurred after a night of rain and was obviously caused by the expansion of the soil under the beam which had only been reinforced in the bottom. Computations show that the swelling force necessary to crack the beam must have been greater than 1.07 tons per sq ft.

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[16] "Principles of Soil Mechanics: Part I. Phenomena of Cohesion of Clay," by Charles Terzaghi, *Engineering News-Record*, November 5, 1925, p. 742.

Swelling Pressure Determined by Special Test.—The Building Research Station in Watford, England, devised a test to measure, directly, the swelling pressure of a saturated undisturbed sample. Results indicate a value of between 1.5 tons per sq ft and 2 tons per sq ft for samples both parallel and perpendicular to the plane of bedding.

Swelling Determined from Building Deflection Data.—Subsequently, in Section 16, a value of 1.6 tons per sq ft is reported in this field.

12. SUMMARY OF OBSERVATIONS ON SWELLING PRESSURE

Estimates of the value of the swelling pressure show it to be of the order of 1.75 tons per sq ft. It is the pressure that must be exerted to prevent a free surface of the soil from swelling when in contact with water. As a building prevents uniformity of moisture content under the building, differential vertical movement must occur if the loading does not exceed about 1.75 tons per sq ft.

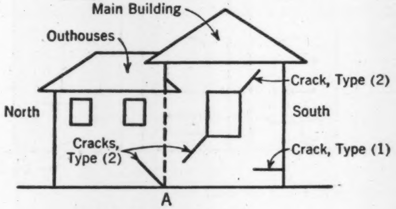
This fact, allowing for supercharge, offers an explanation for the results reported in Section 7, in which it was shown that for loadings in excess of 1 ton per sq ft there were few, if any, cracks in the superstructure. It also explains the results reported in Section 16 showing deflection curves to change from dome shape to dish shape for loadings in excess of 1.75 tons per sq ft.

13. CLASSIFICATION OF BUILDING CRACKS AND MOVEMENTS

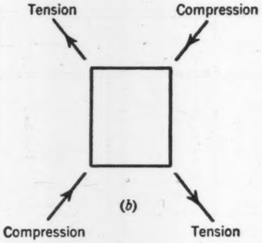
After eliminating the possibility of building cracks being due to earthquake shocks, the cracks in brick buildings were divided into six main types: (1) Horizontal cracks in the exterior corners of main buildings; (2) diagonal cracks from the foundation; (3) diagonal cracks in the exterior arched walls, supported by pillars; (4) vertical and diagonal tension cracks and shear tension cracks; (5) vertical tension or diagonal shear tension cracks in the bottoms of long and short walls; and (6) vertical cracks in the wall bottoms due to lateral movement.

Type 1.—Horizontal cracks in the exterior corners of main buildings usually occurred about 2 ft above ground level, suggesting that the corners were suspended in midair or that the foundations had sunk relative to the wall. In a block of thirteen single-story cottages constructed at the Agricultural College, these cracks were found in the east and west short exterior walls and occurred in nineteen of the fifty-two corners. In three instances (all in east walls) the cracks extended for the full length of the wall.

Type 2.—Diagonal cracks from the foundation, being often associated with type 1, suggested a local upward movement at the crack focus, resulting (as a secondary effect) in cracks of type 1. Of the twenty-six exterior east and west short walls of the foregoing thirteen buildings these cracks appeared in nineteen. They consisted of a diagonal crack that began at the northeast and northwest corners of the main building, at the junction of the main building and the outhouses, and radiated diagonally outward at 45° sometimes in one direction and sometimes in both directions. In some cases the crack in the main building wall stopped at the window corner (see Fig. 7), and in others it extended above the opposite corner of the window. In only five walls did the crack reach the top of the wall. It is significant that at point A, Fig. 7, there is a downtake water spout and drain (generally defective) in every wall. Furthermore, similar diagonal cracks were found in many of the return walls.



(*a*) WEST END ELEVATION



(*b*)

Fig. 7

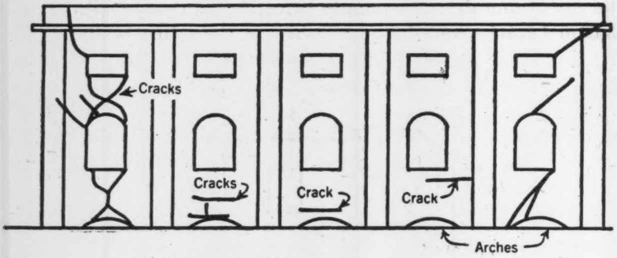
Type 3.—In the exterior arched walls, supported by pillars, diagonal cracks appeared as the result of relative vertical and horizontal movement. The light arches over the exterior end and exterior intermediate pillars show frequent diagonal cracks. They suggest relative upward and (in an end pier) horizontal movement. The reverse type of diagonal crack is also found suggesting downward movement and, in the end piers, horizontal movement.

These two types usually extend only from 2 ft to 3 ft and do not necessarily extend to the top of a low wall.

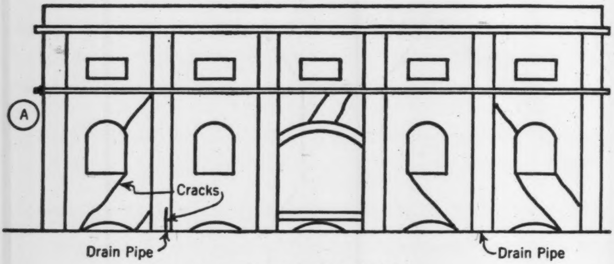
Sometimes both types occur close together, or superimposed, as illustrated in Fig. 8. (In Fig. 8(*c*), note the diagonal cracks in opposite directions.) Repeated distortion (but not settlement) could account for this combination.

Type 4.—Vertical and diagonal tension cracks (in relatively low walls) and shear tension cracks (in relatively high walls) in the tops of long walls occur as the result of relative upward movement of the center of the wall. These are the most common cracks in long walls. They are accompanied by hogbacking, which throws the short end walls out of plumb (see Figs. 1 and 9). A variation of this characteristic consists of a crack that begins as a top tension crack extending for a few feet down the wall. The crack is then extended and enlarged by horizontal movement.

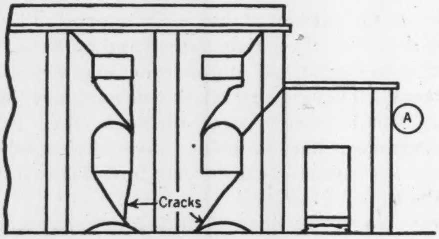
Type 5.—Vertical tension or diagonal shear tension cracks in the bottoms of long and short walls are due to forces the reverse of which cause the cracks of



(*a*) ELEVATION OF EAST WALL



(*b*) ELEVATION OF WEST WALL



(*c*) PART ELEVATION OF NORTH WALL
WITH WEST END PROJECTION

FIG. 8.—OFFICE OF DEPUTY DIRECTOR OF AGRICULTURAL COLLEGE, MANDALAY, BURMA

type 4. They appear in both long external walls and internal cross walls (see Fig. 1). In some instances both types 4 and 5 were found, as shown in Fig. 9. In short walls diagonal cracks caused by shear are found at the junction of the wall with the end walls running at right angles (see Figs. 1(*d*) to 1(*g*)).

Type 6.—Vertical cracks in wall bottoms from lateral movement, occurring close to the end of the wall, appear to be caused by horizontal swelling of the soil enclosed by the exterior foundations.

14. Further Examples of Building Cracks

Cracks in Brickwork Over Arched Foundations.—The best example of cracks in brickwork over arched foundations was found in the walls of the condemned office of the deputy director of agriculture. The walls were riven with cracks and the building appeared to have opened out both at the top and the bottom, as shown in Fig. 8.

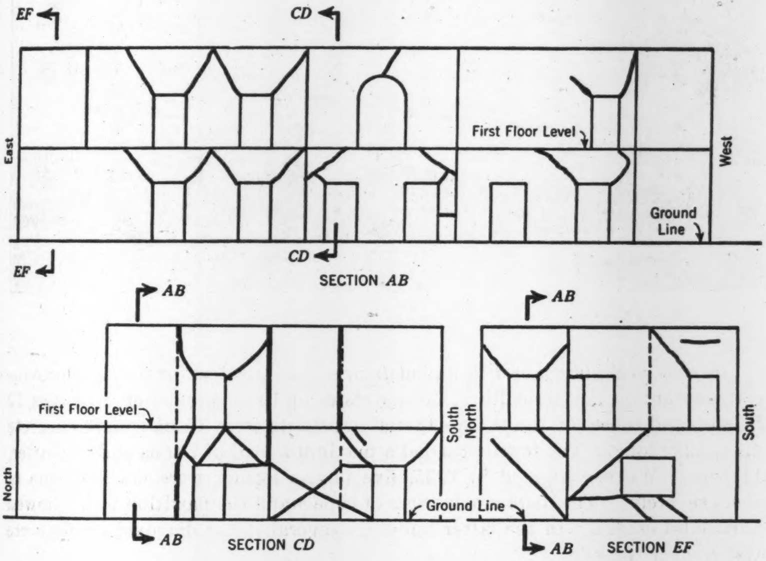


Fig. 9.—Manager's Quarters, Mandalay, Burma

Vertical Cracks in Brick Water Tank Walls.—A number of small brick water tanks constructed below ground level was examined. These had invariably failed by the side walls being pushed inward, resulting in a wide, vertical central crack in each wall, and in the flooring being pushed upward.

Cracks in, and Damage to, Bricknogging Buildings.—These buildings show few cracks and those all of one type. They are mainly confined to the lowest panels and consist of a diagonal crack, at 45°, extending upward from where the post meets its foundation. The floors are uneven, and levels on post footings indicate a wave formation.

Damage to Timber Building on Stilts.—In the "Introduction" reference has been made to the wave deflection of buildings. It occurs in brick buildings but less pronouncedly (see Fig. 2). Because of their elasticity and isolated footings, timber buildings furnish extreme examples of wave deflection and give a new conception of the difficulties met with in the design of local buildings, especially those of brick. The elevation of barracks (about 180 ft long) shows a distinct and consistent waved formation as illustrated by Fig. 10, facing the south and east walls. The lowest post is not the end one, but a post about 20 ft from either end. The most pronounced dip is still at the west end of the south wall. The highest posts of the south line are the end posts and those about 50 ft from each end. In the intervening 80 ft the wave is not generally quite so pronounced.

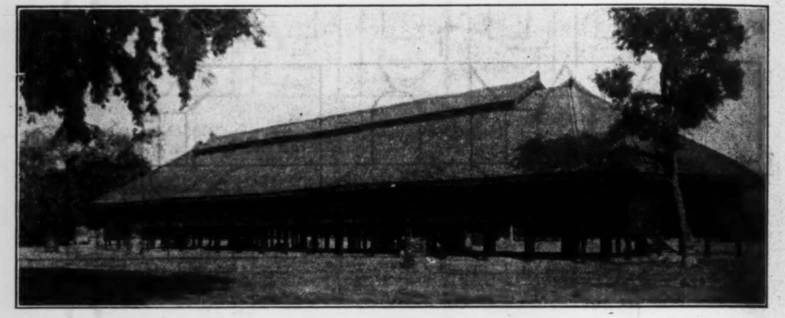


Fig. 10.—Wave Form of Settlement, Timber Building on Stilts, 180 Ft Long

Damage to Buildings on Pile Foundations.—The quarters for the eight provincial assistants at the Agricultural College rested on 10-in.-square concrete piles, 12 ft long, and those for the three imperial assistants were 15-in.-square concrete piles, 12 ft long. The former carried a maximum load of 7 tons and the latter, 11 tons. When examined in 1935, five of the former were cracked, one of them severely. The main cracks were of type 4 and the partition walls showed horizontal cracks. In the latter buildings several minor diagonal cracks were visible over the piles.

Damage to Buildings on Raft Foundations.—Only one attempt at a raft-type design has been made in Mandalay and that was for the seed laboratory at the Agricultural College. When examined in 1935, many cracks of type 4 were found. Although the failure of this design scarcely merits the condemnation of a good raft design, it furnishes the evidence that, instead of transmitting load to the soil, the soil was loading the slab and, in doing so, the slab had to span lengthwise instead of crosswise, with the result that the raft cracked in a direction perpendicular to its major axis.

Damage to Buildings on Arch Foundations.—Light culvert arches very soon develop cracks and, if not repaired, soon become derelict, as may be seen in almost any part of the dry zone of Burma. Whereas foundations transmitting loadings as great as 1 ton per sq ft crack badly, arches transmitting more than 2 tons per sq ft show little distress.

15. SUMMARY OF OBSERVATIONS ON BUILDING DAMAGE

In general, the foregoing study of building damage shows it is caused by a combination of vertical and horizontal movements and strongly suggests that the cracks arise out of the peculiar deflection curves assumed by the structures. For such cracks to occur the deflections must be relatively great; they must occur in a time interval insufficient to mobilize the full strength of the structure; or the induced stresses, increased over the theoretical by waving, become too severe for the design. As a further alternative, the foregoing possibilities may be compounded and, in addition, the wave may possibly change during the year.

16. DEFLECTION CURVES FOR BUILDING ON MANDALAY SOILS

Basic Structural Deflection Curves.—In the preceding sections, the more visible evidence of structural cracks was examined and the cracks were classified as explainable by vertical and horizontal movement, or by a combination of the two. In this section the basic deflection curves, with their secondary modifications, are discussed in relation to observations on soils in Mandalay.

The normal basic settlement curve for a flexible rectangular structure, transmitting a uniform pressure to a horizontally homogeneous cohesive soil, is dish shaped. This curve may not be uniform because of the lack of horizontal homogeneity of the soil; but, unless any change in the horizontal profile is very marked, the normal basic line remains dominant.

In Mandalay, buildings—mostly on strip or isolated footings—are flexible and the soil is a siliceous, cohesive clay of reasonably uniform homogeneity. Therefore, the basic curve should be dish shaped; yet, the normal dish curve for buildings transmitting less than 1.5 tons per sq ft was not found. Although the soil is cohesive, the curves for some seventy buildings resembled those for a flexible structure placed on the surface of a shallow stratum of cohesionless sand[17]—that is, they were fundamentally dome shaped. It follows, therefore, that most of the basic deflection curves cannot result from the Boussinesq distribution of stress.

Actually the basic dome curve, although dominant, is never uniform, especially at the ends. It shows secondary interference which is probably not connected with any change in the horizontal homogeneity of the soil material.

Estimation of Maximum Settlement.—Although no particular significance can be attached to theoretically computed settlements, when conditions do not lend themselves to theoretical treatment, estimates based on the site loading test, the "perimeter-shear" theory advanced by W. S. Housel, M. ASCE, and the Boussinesq and consolidation theories were made for the maximum settlement that might have occurred under the office of the severely damaged Mandalay Race Club.

From the computations the maximum settlement should have occurred under the center of the building, and should have been of the order of 1.32 in. maximum, whereas the maximum relative deflection occurred at one end and was in excess of 2.4 in.

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[17] "The Actual Factor of Safety in Foundations," by Charles Terzaghi, *Structural Engineer*, March, 1935.

Sinusoidal Curves of External Wall Deflections.—A study of the form of the deflection curves for short buildings is of interest when compared with the curves for long rectangular timber buildings. Let AG, Fig. 11, represent the curve of a building 180 ft long, as illustrated by Fig. 10, taking the form ABCDEFG, common to all such buildings, where the section CE is similar to, but not so pronounced as, AC or EG. The bricknogging office is 120 ft long, loaded 0.5 ton per sq ft, and its curve (see Fig. 12) can be represented by AL (112½ ft, in Fig. 11) in which peak C is dominant as compared with peak D. As indicated in Fig. 12(*a*), levels were plotted with the northwest corner post footing at an assumed datum elevation of 100.000. There is no justification for this assumption, but it enables a comparison to be made of the several curve shapes. Actually, it is quite clear that in this building there is no area of constant elevation. Figs. 12(*a*) and 12(*b*) show that, with the datum used, the maximum movement in the exterior lines occurs at the center, whereas, for the interior lines, it occurs at the third points. The maximum movements between the curves in Figs. 12(*a*) and 12(*b*) are 1.5 in., occurring, relatively, at the center of the veranda line (see Fig. 12(*c*)). The maximum deflection recorded was $d = 2.88$ in. The length of the chemical laboratory is 100 ft and its cracks and outline are fairly well represented by BE (112½ ft in Fig. 11) in which peak C is dominant as compared with peak D. The race club office was 62 ft long and its curve (see Fig. 2) can be represented by BD, Fig. 11, which is 67.5 ft. The electrical substation is 48 ft long and its curve can be represented by AC (Fig. 11), 45 ft long.

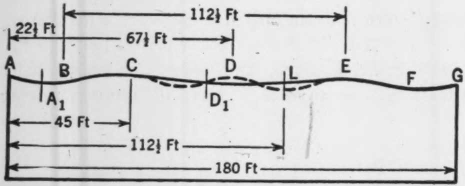


Fig. 11

The foregoing examples of simple, light, rectangular buildings suggested a common fundamental sinusoidal deflection curve of a constant wave length for all buildings and supported the belief in soil buckling. There is no suggestion, however, of a common wave length in the curves for more irregular or heavier buildings. The phenomenon appears to apply only to rectangular buildings transmitting loadings of less than 1.5 tons per sq ft.

Connection Between Basic Deflection Curve, Deflection, and Loading for All-Brick Buildings.—Deflection curves, for the limited number of brick buildings for which curves could be obtained, show variations in maximum relative vertical movement from +4.68 in. (dome) to −1.92 in. (dish) and in maximum deflection from +2.88 in. (dome) to −1.44 in. (dish), for a loading range varying between 0.25 ton per sq ft and 2.1 tons per sq ft. In most curves there is secondary interference especially at the ends. The deflection results are plotted in Fig. 13 and (with one exception to be analyzed, subsequently) show a straight-line relationship between the loading P and the ratio of deflection, d, to the length of building, L.

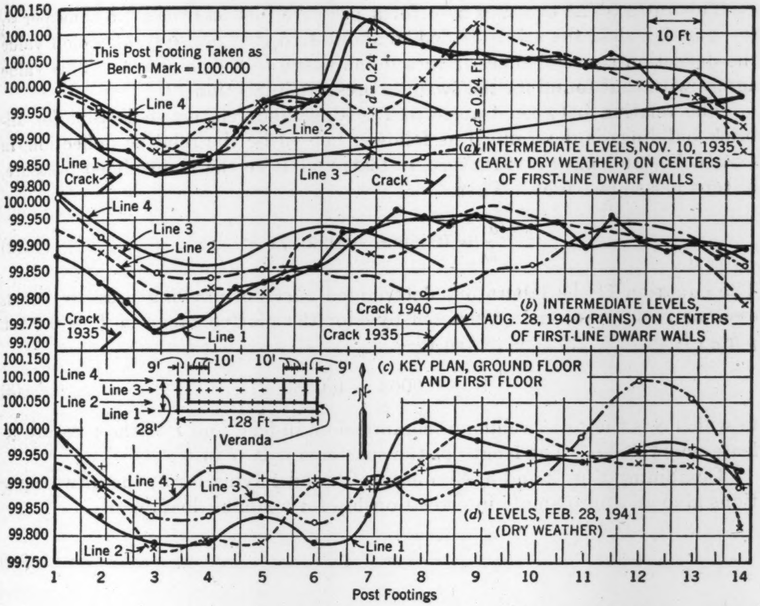


FIG. 12.—FOUNDATION LEVELS ON OFFICE BUILDING, MANDALAY, BURMA

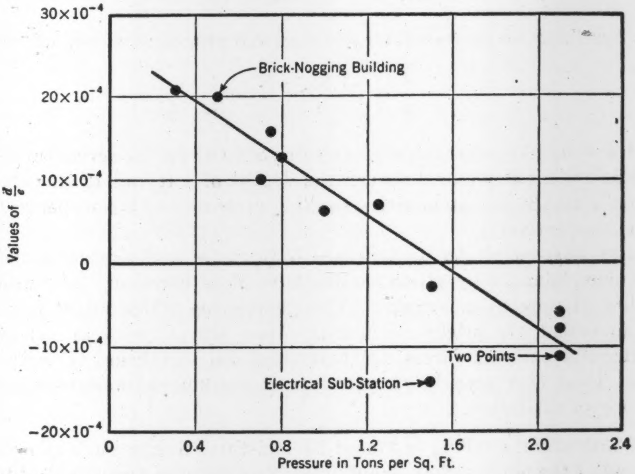


FIG. 13.—RELATION BETWEEN LOADING AND THE RATIO d/L FOR ALL-BRICK BUILDINGS

This ratio (d/L) becomes zero for a pressure just less than 1.5 tons per sq ft—that is, when the soil is loaded to about 1.75 tons per sq ft, at which value the deflection curve changes from a dome shape to a dish shape. This value agrees with that found for the swelling pressure in Section 12.

The relation between transmitted loading and relative deflection for all-brick buildings is determined as follows: Let—d be the relative deflection, in feet; L be the length of structure, in feet; and P be the transmitted pressure, in tons per square foot. Then

$$\frac{d}{L} = 0.00266 - 0.00171\,P \quad \text{(2)}$$

The equation of the laboratory determined curve of swelling pressure versus swelling : compression is expressed by—for the plotted range of $P = 0.5$ ton per sq ft to $P = 2.0$ tons per sq ft—

$$S = 0.0063 - 0.0039\,P \quad \text{(3)}$$

in which S is the test swelling or compression, in feet, and P is the pressure release on swelling. The length of the specimen is 0.87 in.

Combining Eqs. 2 and 3, eliminating P, and solving for S, these two equations give an approximate interrelationship for a similar loading of

$$S = 2.3\,\frac{d}{L} \quad \text{(4a)}$$

and the (test) osmotic range strain is expressed as

$$\epsilon = 32\,\frac{d}{L} \quad \text{(4b)}$$

Eq. 2 can thus be expressed, for any load and proportional osmotic strain, as

$$\frac{d}{L} = \epsilon\,\frac{1.55 - P}{1.55} \times \frac{1}{32} \quad \text{(5)}$$

in which $\epsilon = 0.085$ against the average value of 0.087 for the agricultural shrinkage coefficient for five sites between the depths of 4 ft and 12 ft. Curves of both Eqs. 2 and 3 gave an effective swelling pressure of 1.6 tons per sq ft (1.55 and 1.65, respectively).

Factors Responsible for Deflection.—With dome and sinusoidal deflection curves unexplainable by stress distribution, it is necessary to consider the possibility of upward movement. The phenomena which might lead to upward movement are briefly discussed. They are (1) vertical soil swelling, (2) horizontal external forces, (3) horizontal soil shrinkage, (4) soil circulation, (5) local clay areas with pronounced sodium characteristics, and (6) thixotropy or dilatation.

(1) Vertical soil swelling is caused by moisture changes such as may occur at the end of the hot weather, or by differential changes arising out of the time lag in the flow of rain moisture. These changes are accompanied by changes

in pressure between the foundations and the soil and would be expected to lead to movement if the external loading were less than the swelling pressure of 1.75 tons per sq ft.

(2) Horizontal external forces on the foundation walls due to external soil swelling may produce a dome-shaped curve and force the end walls out of plumb.

(3) Through adhesion between the soil and the foundation, horizontal soil shrinkage under a foundation may cause central upward movement in a light building. Approximate calculations show that a vertical loading of about 1 ton per sq ft is necessary to overcome this friction force.

(4) During the dry weather, when soil cracks appear, soil and other matter fall into crevices and the process is aided by various external agencies. When the rains start, water-borne material is also deposited. If a building is founded on an artificial sand cushion, some of the sand must enter any soil cracks that may occur. This circulating material may cause a local settlement but its presence when the soil begins to swell must tend, by conjugate pressure, toward an upward or wave movement of the soil. On the whole, this would be expected to occur, particularly at the ends of buildings. Proof of this soil circulation was noted during borings at the Agricultural College (dry area) when lumps of mottled black and yellow clays were occasionally extracted from as low as 16 ft.

(5) Local clay areas with pronounced sodium characteristics may lead to buckling if, for such soils, the suspected difference between horizontal and vertical shrinkage is an actuality.

(6) Thixotropy (the property or phenomenon, exhibited by some gels of becoming fluid when shaken) or dilatation, or a combination of each, may be responsible for some local buckling in the same way as soil circulation.

Evidence of Upward Movement.—Through rather a revolutionary conception, difficult to detect, upward movement follows as a corollary to the pressure release of 1.75 tons per sq ft during swelling over the residual shrinkage range unless the loading exceeds this value. The direct evidence in support of its actuality is as follows:

(*a*) The southwest corners of foundations, although most exposed to alternate drying and wetting, are often higher than the foundations to the east, or even those inside the building.

(*b*) Stone, or concrete, floors appear to settle considerably, especially in the southern half of the buildings. This settlement (if it is a settlement) may be presumed to occur uniformly until a condition of equilibrium be reached. At the end of the rainy season it has been noticed that footings in the flooring around the exterior walls—especially south walls protected by a veranda—indicate relative upward movement accompanied by occasional cracks of type 2 (Section 13). This movement occurs particularly under southerly exterior posts, subject to full weather effects, and, hence, to maximum changes in voids ratio. Therefore, it is considered to be direct evidence of upward movement of the footings (Fig. 14).

(*c*) Dwarf veranda walls between post footings often develop a dome curve with top, central cracking. This type of settlement is considered, principally to be due to the upward movement between the posts caused by soil swelling against no appreciable resistance, although it may be due in part to some downward movement of the posts. An example of a reinforced-concrete tie beam, connecting two pile tops falling under the same forces, is described in

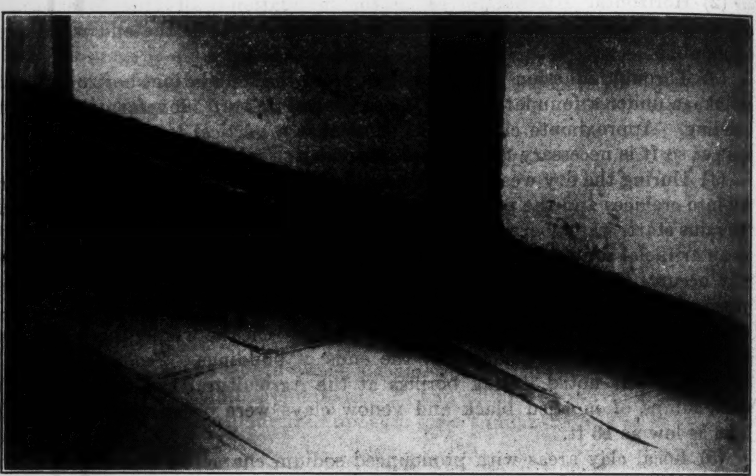


FIG. 14.—STONE OR CONCRETE FLOORS APPEAR TO SETTLE CONSIDERABLY

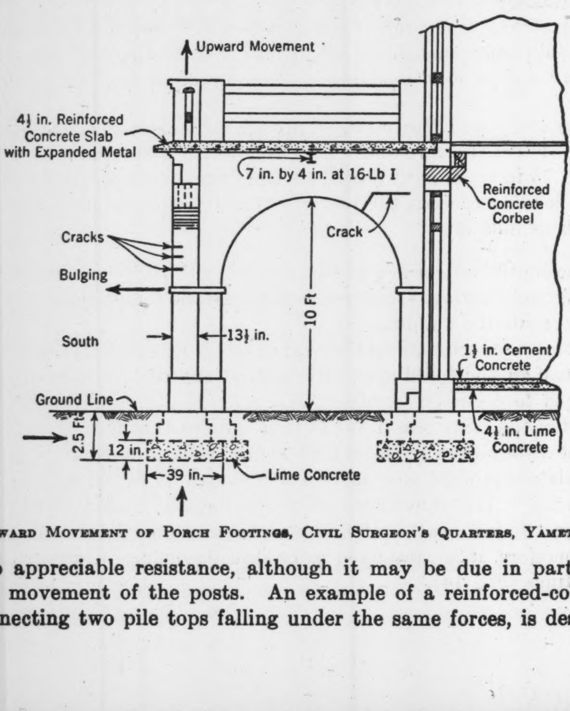

Upward Movement
4½ in. Reinforced Concrete Slab with Expanded Metal
7 in. by 4 in. at 16-Lb I
Reinforced Concrete Corbel
Crack
Cracks
Bulging
10 Ft
South
13½ in.
1½ in. Cement Concrete
Ground Line
2.5 Ft
12 in.
39 in.
Lime Concrete
4½ in. Lime Concrete

FIG. 15.—UPWARD MOVEMENT OF PORCH FOOTINGS, CIVIL SURGEON'S QUARTERS, YAMETHIN, BURMA

Section 11. The deflection curves for veranda posts in Fig. 12 supply an additional example.

(*d*) Cracks, particularly of types 2 and 4 are indicative of upward movement.

(*e*) A test, especially devised to prove the actual occurrence of upward movement and to measure its magnitude, is described in Section 19. The test confirmed the movement. The effect of upward movement is illustrated by Fig. 15 which shows the cracking and movement of a lightly loaded porch way of a house in Yamethin, Burma. To a depth of 4 ft the soil was a stratum of sandy clay loam, overlying a stratum of colloidal black cotton clay to a total depth of 7 ft.

17. Summary of Factors Responsible for Deflection Curves

It is clear that, as the factors productive of normal settlement are relatively insignificant, the secondary phenomena responsible for movement in an expansive soil, whose normal moisture content is confined to the residual shrinkage range of undisturbed soil, must be taken into consideration. Most of the phenomena discussed have been proved to exist and the forces induced by them have been shown to be of appreciable magnitude. The most important of these, applicable to any site, appears to be that of soil swelling involving the conception of upward movement.

18. Cyclical Movements

Movements Detected During Load Test.—A study of the levels taken on the tank which was used for the loading test (Section 7) and on the four indicator pegs, with one of the office post footings as datum, showed (from the results over the period from October 14, 1935, to April 2, 1936—that is, from the most saturated period to the most dry period) cyclical variations in level. The level of the tank relative to the bench mark on the post footing, founded on black cotton soil at a depth of 4 ft, showed similar variations in level. In particular, they indicated that toward the end of March, just before the lower horizons begin to take up nonrain moisture (see Fig. 4), the relative difference in level between the loaded test area and the post footings changed by 0.48 in., suggesting that the post footing had settled by 0.48 in. Within twenty days this difference had recovered to 0.36 in., which is explainable by the upward arrival of moisture under the test load, resulting in some upward movement of the test load relative to the post footing.

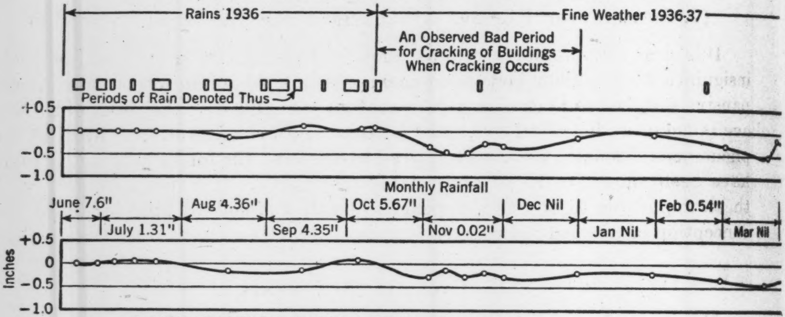
The apparatus was not considered suitable to permit drawing any definite conclusions and the test described in Section 19 was devised to check this phenomenon.

19. Cyclical Movement Demonstrated by Pile Test

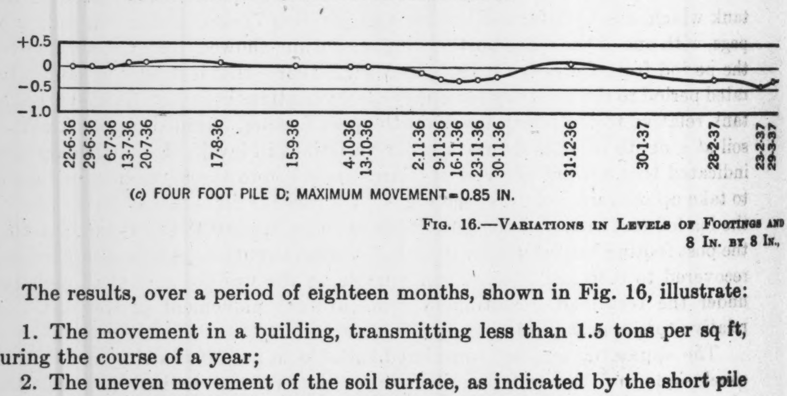
To verify the upward movement of buildings, use was made of the seasoned "pyinkado" (hard timber), 8 in. by 8 in., pile driven to refusal at a depth of 21.5 ft. Two short piles, each 4 ft long, were driven in line with the 21.5-ft pile and spaced at 11-ft and 5.5-ft centers. Levels were taken periodically on each pile top, flush with the ground, and on one corner of the foundation.

The possibility of the long pile varying in length with possible moisture changes in the surrounding soil was considered, and it was found, for the various conditions examined, that any changes in pile length were unappreciable compared with movements actually measured. The variations in the levels of the short piles and in the godown footing, relative to the long pile as datum, are shown in Fig. 16. They are of a cyclic nature. Comparison of these curves with those for variation in moisture: saturation capacity, in Fig. 4, shows close agreement although the two periods observed are not entirely identical.

(*a*) COLUMN FOOTING A, FOUNDATION LEVEL, 5.5 FT BELOW GROUND LEVEL; APPROXIMATE LOADING 0.85 TONS PER SQ FT; MAXIMUM MOVEMENT = 0.75 IN



(*b*) FOUR FOOT PILE C; MAXIMUM MOVEMENT = 0.62 IN.



(*c*) FOUR FOOT PILE D; MAXIMUM MOVEMENT = 0.85 IN.

FIG. 16.—VARIATIONS IN LEVELS OF FOOTINGS AND 8 IN. BY 8 IN.,

The results, over a period of eighteen months, shown in Fig. 16, illustrate:

1. The movement in a building, transmitting less than 1.5 tons per sq ft, during the course of a year;

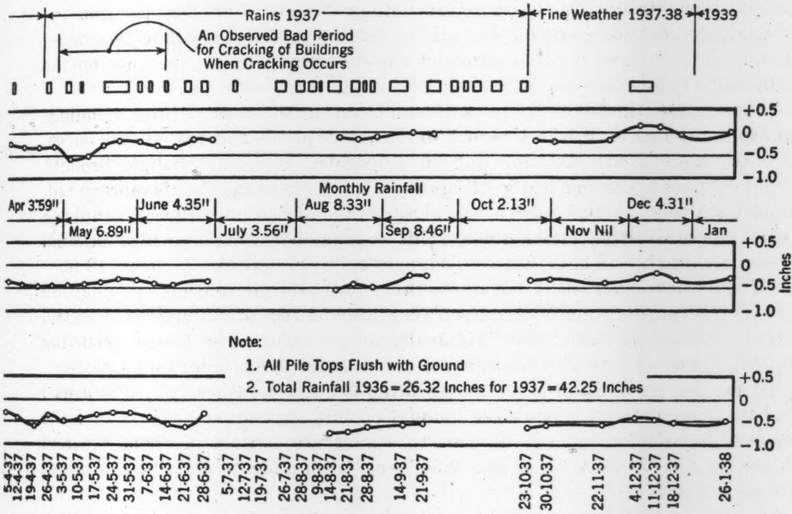
2. The uneven movement of the soil surface, as indicated by the short pile curves, simulating buckling and suggesting a moving wave; and

3. The accumulative effect of item 2, illustrated by the difference in levels of piles C and D at the end of the test period.

20. THE PHENOMENA

Old Theory Reviewed.—The main feature of the old theory of the cause of cracking of buildings in Mandalay was that the soil was of low carrying capacity, especially during the rains. The fact that kyatti soils turned rapidly into a soapy fluid in the presence of water was accepted as supporting evidence. This observation is correct only when applied to disturbed soils and dynamic water. In Mandalay the high dispersion factor, caused by the replaceable sodium, certainly accentuates this condition, but clays under foundations are neither fully disturbed nor subject to the eroding action of free dynamic water.

It has been shown in Section 7 that the soil will readily carry a pressure of 2 tons per sq ft under very severe moisture conditions and that even under



of Short Piles Referred to a Timber Pile,
and 21 Ft Long

these severe conditions there was no continuous plastic yield. The actual yield was small and instantaneous. The consolidation test supported the site test and showed that the soil is normally in a condition corresponding to a carrying capacity, when saturated, of nearly 6 tons per sq ft. (This pressure was reached during a loading test at Yamethin when the apparatus—not the soil—failed.) Observation supports the high carrying capacity of the soil in test observations and in the comparative absence of cracks in the heavier buildings.

The presence of building cracks close to a leaky drain has been quoted as evidence in favor of judging the carrying capacity to be low. Examination shows that these cracks result from a local upward movement over the seepage area and not from downward movement due to soil yield. These facts dispose of the theory accrediting the occurrence of cracks in buildings to a low bearing value.

The Cause of Dome-Shaped Basic Deflection Curves for Pressures Less than 1.5 Tons per Sq Ft.—It has also been shown that a load of at least 2 tons per sq ft can be carried safely with little settlement and that for loadings greater than 1.5 tons per sq ft little cracking appeared in buildings. The structural, basic deflection curve has been shown, generally, to change from the usual dome to the expected dish shape for loading in excess of that corresponding to the swelling pressure of 1.75 tons per sq ft. This suggests that the accepted conception of a dish-shaped deflection curve applying to structures sited on a horizontally homogeneous cohesive soil requires amplification to yield a more general law introducing the moisture-content factor. For the clay soils examined, the form depends on the ratio of loading to active swelling pressure—that is, it is apparently a function of the moisture fluctuation range on the undisturbed shrinkage curve.

The explanation offered for the dome curve in the case of brick buildings transmitting less than 1.75 tons per sq ft, in lieu of the theoretical dish curve, is based not on consolidation but on differential moisture changes brought about by natural drying and wetting over a moisture range that is represented, almost entirely, by the residual shrinkage range of the undisturbed shrinkage curve where during wetting small voids changes correspond to high internal pressure changes. These moisture changes occur to depths of at least 12 ft.

The process involved in the dome curve formation is considered to depend on: (1) The original and final moisture content of a core of soil under the central part of the structure; and (2) the high alternating volumetric changes occurring outside the central core—that is, against and under the exterior foundations.

Assuming excavation to be done during the dry weather (which is usual) and that, as so often happens, the foundations are open for some time, then the exposed foundation soil will dry out to a moisture content of about 6% and shrink to a voids ratio of, about, 0.567—corresponding to a loading of about 6 tons per sq ft.

After the first heavy rain the soil outside the building and immediately under the exterior foundations will swell to a voids ratio of about 0.705, where unloaded and exposed, and to about 0.63, when the loadings are 1 ton per sq ft.

In addition to a general tendency toward upward movement, this swelling must cause an upward movement of the building ends relative to the building center because of the forces on the end walls, the flexibility of the building in the direction of its major axis, and the time lag in moisture flow. The condition is similar to a beam held at the center and loaded upward at its ends; and it tends to produce a dish curve, but actual movements must depend upon the moisture content of the central core at the time of construction. However, because of rapid drying, this initial upward movement is of short duration.

After a series of such movements, the central core will (in time) absorb moisture to some maximum value which will probably occur during the dry weather; and, by preventing evaporation, this maximum value will be retained more or less permanently. This action will result in an ultimate central support through the cross walls and the inertia of the structure. The building then acts as a centrally supported, double cantilever and the deflection curve is in the the process of being changed from a dish shape to a dome shape. During the same period there is a general tendency for the exterior walls to settle to their original positions. This settlement will be most pronounced at the building ends because of the weight of the end walls, the flexibility in the direction of the major axis, and the support of the central core. The result is the deflection along the major axis and the formation of a dome-shaped curve.

During this change-over period, any late rains (as may occur in November or December) may cause some incremental additional end settlement through the local breaking down of the soil structure, which results in very small accumulative amounts of plastic yield.

As evaporation and temperature changes are a minimum for the central core, it is considered that its moisture intake will be retained, in general, and that any later fluctuations will be small compared with those under the external foundations. Consequently, the central core is left at a higher level than when constructed during the dry weather, and thus it acts as a final support through the inertia of the building.

The upward movement of the central core does not cause relative movement in the internal walls sufficient to cause severe cracking. If it did, the cracks illustrated in Fig. 1 would have been in the opposite direction. Since there must be a tendency for this cycle to be repeated yearly, it would appear that, through possible accumulative incremental end sinkage, the dome curve becomes somewhat more pronounced with time.

Alternatives and Exceptions.—In the event of the central core being saturated during construction, variations in the aforementioned cycle are possible. Each will depend upon the relative moisture changes and such secondary causes as drainage. Thus it is conceivable that the curve will pass straight into the dome shape, or, even, that the final curve will be dish shaped. Furthermore, it seems possible that, when construction is followed by a number of wet years and then by a number of dry years, an originally developed dish curve may change into a dome-shaped curve after several cycles, although in such an example the final central level will probably be lower than when constructed. Should the soil be fully saturated continuously, the swelling pressure would be negligible and a dish curve appears inevitable.

Reference has been made in Section 16 to an exception to the straight-line relationship between the loading and relative deflection. This is the Agricultural College electrical substation, and its pronounced dish curve is a result of the unusual local conditions under which one of the foregoing alternatives has developed. This building is situated on low land, waterlogged for much of the year, and within 50 ft of a main irrigation distributary. Under such conditions a pronounced and natural dish curve may be expected to develop.

21. THE EFFECT OF FLEXIBILITY ON THE BASIC CURVE FOR LOADINGS OF LESS THAN 1.5 TONS PER SQ FT

Bricknogging Buildings and Timber Buildings on Isolated Footings, But Not on Stilts.—From the discussion on the formation of the basic curve for all-brick buildings it might be expected that nogging buildings would exhibit the same deflection curve although not in such a pronounced form because of the discontinuity between the interior and exterior foundations. The lines of interior footings should also follow the general form in a lesser degree, with a suitable lag. Levels are in accord with this expectation.

Timber Buildings on Stilts.—For such buildings, at which moisture fluctuations may occur over the entire area, the basic curve should be the least evident and deflection might be considered to arise principally from tertiary effects. However, as the moisture fluctuations of the central core must be less than those under the exterior footings the theory suggests that the major axis should be higher than the exterior foundation lines. Barracks show sinusoidal deflection curves but no basic dish or dome curve. The end elevations of the barracks, shown in Fig. 10 and applicable to all barracks, illustrate that this does occur in the field.

The theory evolved by the findings reported in this paper explains the less pronounced basic curve for buildings on isolated footings and the almost complete absence of any basic curve for buildings on stilts. In general, it supplies a reason for the tertiary effects increasing with the flexibility of the building as it changes from an all-brick building to a timber building on stilts. By the proposed theory, rafts and piles (unless fixed deep in a constant volume horizon, or kept at a constant volume by the superimposed loading) should move in the same way as a brick building on strip footings. This again has been found to be so.

22. THE CAUSE OF THE SINUSOIDAL CURVES OF DEFLECTION

In Section 16 reference was made to the fact that the basic dome curve, for loadings less than 1.5 tons per sq ft, is subject to tertiary effects leading to a sinusoidal curve. In Section 21, it was demonstrated that the degree of interference depended not only on the loading but also on the flexibility and the type of design.

Since the possibility of soil buckling arising out of unequal horizontal and vertical shrinkage should be excluded (for the present) as a generally applicable major cause of wave action, the only explanation appears to lie in the accumulative effects of the factors producing upward and downward movement (see Section 16), and in the inertia of the structure. Of these possibilities the only ones that are generally pertinent are differential movement due to moisture changes and soil circulation. For a long brick building, both of these factors would be expected to be most apparent at the ends where the interference is most noticeable.

23. THE STABILITY OF THE DEFLECTION CURVES

Movement is not solely confined to one direction to form one of the basic curves, but may alternate as demonstrated by the coexistence of cracks from both upward and downward movement. As explained in Section 20, the aforementioned dish curve is considered to form for low loads, if it ever forms, during or soon after construction and may in time change to a dome curve. Whether further changes in the basic curves may occur is not known but it is significant that, when the Agricultural College was examined in 1935, it was recorded that the building levels showed slight hogbacking, whereas when it was leveled on April 1, 1940, it was dish shaped. Further releveling on February 28, 1941, indicated a distinct change in the curve, suggesting that any change-over from dish curve to dome curve, or vice versa, may apply particularly when the loading is in the region of the swelling pressure, as is true of this building.

Whether the positions of the crests of the sinusoidal curves are subject to lateral movement has not been proved. Perhaps the inertia of the building prevents this while permitting accumulative differential movement in other respects. It is clear, however, that cyclic vertical and horizontal movements occur, and, from Section 19, that some minor lateral movement in the crests should occur. The releveling of the office building in Fig. 12, after five years, showed the level lines to be similar regarding positions of crests and hollows, although there were considerable displacements between the level lines and differences in the relative heights of the crests. Further relevelings at this building and at the Provinical Police Training School showed similar relative movements occurring within the shorter periods of one month and six months.

Hence, once a condition of complete, or partial, equilibrium of moisture content of the central core has been established, it appears that the main features of the deflection curve are maintained; but the curves, nevertheless are subject to cyclical variations.

24. Acknowledgment

The material for the preparation of this paper is from a report submitted by the writer to the Government of Burma in 1945, entitled "The Phenomena Involved in the Failure and Partial Failure of Buildings Constructed on the Desiccated Alkaline Soils of the Dry Zone of Burma." For constructive assistance, the writer is indebted to: The director and staff of the Agricultural College, Mandalay; Mr. Robinson, University of North Wales, Bangor; Mr. Clark, University of Illinois, Urbana; R. E. Stradling, director, Building Research Station, Watford; R. Kenworthy-Schofield, Rothamsted Experimental Station, Harpenden, Hertford, England; Hans F. Winterkorn, Assoc. M. ASCE, Princeton University, Princeton, N. J.; L. F. Cooling, Building Research Station, Watford; and G. R. Clarke, School of Rural Economy, Oxford University, Oxford, England.

DISCUSSION

Hans F. Winterkorn,[18] Assoc. M. ASCE.—The phenomena observed and reported in this paper, as well as the underlying soil-physical causes, have been of considerable interest to the highway engineer ever since low-cost road construction and soil stabilization began to be transformed from an art into an applied science. Thus far, however, these phases have not been considered to any great extent by the foundation engineer grounded in soil mechanics alone and not in the wider aspects of soil physics.

One of the reasons for this lack of attention on the part of the latter may be that the type of light building that suffers damage as a result of the swelling or shrinking of the underlying soil represents too small an investment to require detailed foundation studies and protective measures. This reason no longer holds, however, if such trouble occurs over a considerable area and if either the government or a private organization is financially involved in a large number of light structures in such an area. For instance, in the drought years of the 1930's a considerable number of light buildings founded on the Putnam soil series in Missouri showed distress fundamentally similar to that described by Mr. Wooltorton. Oil companies owning a large number of service stations in this area suffered appreciable financial loss and were definitely worried about the situation.

The Burma soil under consideration was a calcium-magnesium-sodium solonchatic solonetzig complex. Mr. Wooltorton points out that the high sodium and magnesium content suggested detrimental volume changes and impermeability but that the replaceable calcium and magnesium suggested permeability, and that, although the soils were quite impermeable in laboratory tests, actual moisture movement in the soils on the sites proved them to be rather pervious.

Although the permeability of disturbed soils is definitely a function of the exchangeable ions, it also depends on the way the cations have been introduced into the exchange complex. This is illustrated in Table 5 which gives permeability coefficients as a function of the exchange ions for three different soils, prepared by leaching with chloride solutions of the respective cations (Cols. 2, 3, and 4). For one of these soils (Putnam), Col. 5, Table 5, also shows the effect of neutralization of the H-soil with the respective hydroxides. The Cecil clay is kaolinitic, whereas the Hagerstown and Putnam clays are predominantly illitic, as are the Burma clays reported by Mr. Wooltorton. The ionic influence is less for the clays prepared by leaching than for those prepared by neutralization. A further influence is the salt content of natural soil water to which soils are more pervious than they are to the distilled water normally used for laboratory tests.

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[18] Industrial and Eng. Consultant, Princeton, N. J.

However, even considering the greater permeability to natural soil water containing electrolytes, it would seem that the observed moisture movements required the action of agencies other than the normally considered ones. An important point in this connection is the increase to a maximum moisture content between 12 ft and 6 ft in May before the rains broke. Mr. Wooltorton (see Section 5) considers, as such an agency, the distillation effect described by A. F. Lebedev.[6,7] Results of studies at Princeton University (Princeton, N. J.) make it very probable, if not certain, that this "distillation" effect is actually a thermo-osmotic film flow, the crucial proof for the latter concept being the electrokinetic phenomena observed as a result of thermo-osmotic flow.

TABLE 5.—COEFFICIENTS OF PERMEABILITY, COMPUTED FROM CONSOLIDATION EXPERIMENTS ON HOMO-IONIC SOILS (DATA, MODIFIED BY 10^8, ARE IN CENTIMETERS PER SECOND; LOAD RANGE, 1 TO 2 KG PER SQ CM OR 14 TO 18 LB PER SQ IN.)

Ions (1)	SOILS LEACHED WITH SALTS[a]			Neutralization of H-soil[b] (Putnam) (5)
	Cecil (2)	Hagerstown (3)	Putnam (4)	
Natural	4.3	4.4	1.0	2.29
H	5.4	3.8	3.4	0.75
Na	4.8	0.9	1.1	0.056
K	5.0	1.5	1.3	2.58
Mg	5.5	4.8	3.4	0.60
Ca	4.8	2.3	1.5	1.50
Ba	4.8	1.8	4.0	....
Al	3.8	4.0	13.0	....
Fe	5.1	3.6	5.7	....

[a] "Surface Chemical Factors Influencing the Engineering Properties of Soil," by Hans F. Winterkorn, *Proceedings*, Highway Research Board, National Research Council, Vol. 16, 1936, pp. 293–308. [b] "A Study of Changes in Physical Properties of Putnam Soil Induced by Ionic Substitution," by Hans F. Winterkorn, *ibid.*, Vol. 21, 1941, pp. 415–434.

The Princeton experiments permitted the computation of thermo-osmotic permeability coefficients in centimeters per second, per degree centigrade per centimeter, in a manner similar to that of the electro-osmotic permeability coefficients. Typical data reported[19] by the United States Navy in 1949 are given in Table 6 for three pure H-clays of montmorillonitic, illitic, and kaolinitic character, as well as for the natural (illitic) New Jersey Hagerstown soil and six of its homo-ionic modifications. The data show that the observed thermo-osmotic permeability coefficients are of the order of magnitude of 10^{-6} as compared with 10^{-8} to 10^{-12} for hydrostatic coefficients, which means that a 1° C temperature difference is at least equivalent to a pressure of about 100 cm, hydrostatic head. It appears that thermo-osmosis is likely to play a very considerable role under the conditions described by Mr. Wooltorton.

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[19] "Report on Beach Sand Stabilization," by K. R. S. Gowda and Hans F. Winterkorn, Bureau of Yards and Docks, U. S. Navy, Washington, D. C., November 15, 1949, Part V.

The heat released in wetting of a dry clay which is a representative function of its water affinity normally decreases about 3% for a 10° C increase in temperature. On the other hand, the thermo-osmotic water conductivity of a soil is a very sensitive function of its moisture content having a maximum in the vicinity of the plastic limit. For this reason it is not surprising that the thermo-osmotic coefficients given in Table 6, and the respective hydrostatic equivalents of temperature differentials, are considerably lower than those computed from the well known temperature function of the water affinity.

TABLE 6.—THERMO-OSMOTIC AND ELECTRO-OSMOTIC COEFFICIENTS OF PERMEABILITY

Soils	THERMO-OSMOTIC EXPERIMENTS		ELECTRO-OSMOTIC EXPERIMENTS		Volts per centimeter = 1° C per centimeter (Col. 3 divided by Col. 5)
	Average moisture content (%)	Permeability, in centimeters per second per degree centigrade per centimeter	Moisture content (%)	Permeability, in centimeters per second per volt per centimeter[a]	
(1)	(2)	(3)	(4)	(5)	(6)
(a) PURE H-CLAYS (MONTMORILLONITIC, ILLITIC, AND KAOLINITIC)					
H-bentonite . .	61.82	1.91 × 10^{-6}	....	....	....
H-grundite . . .	26.50	1.61 × 10^{-6}	36.45	2.86 × 10^{-6}	0.563
H-kaolinite . . .	30.62	3.39 × 10^{-6}	30.84	12.8 × 10^{-6}	0.265
(b) NEW JERSEY HAGERSTOWN SOIL					
H-soil	22.53	1.92 × 10^{-6}	22.60	10.30 × 10^{-6}	0.186
Al	23.05	1.49 × 10^{-6}	23.08	7.49 × 10^{-6}	0.199
Mg	22.70	2.24 × 10^{-6}	22.57	5.10 × 10^{-6}	0.439
K	21.85	2.5 × 10^{-6}	21.83	3.75 × 10^{-6}	0.667
N	21.46	0.885 × 10^{-6}	22.29	1.29 × 10^{-6}	0.686
Ca	22.66	1.63 × 10^{-6}	24.39	1.06 × 10^{-6}	1.54
Natural	23.12	3.69 × 10^{-6}	23.42	2.32 × 10^{-6}	1.59

Furthermore, it should be realized that nature very seldom operates in accordance with one single mechanism; rather, it uses all available paths (although to different extents) to achieve its purpose. For this reason reference is made[20,21,22] to several papers that contain data and concepts pertinent to the problem of water movement under the soil and climatic conditions with which Mr. Wooltorton's paper is concerned.

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[20] "Mechanism of Water Attack on Dry Cohesive Soil Systems," by Hans F. Winterkorn, *Soil Science*, Vol. 54, 1942, pp. 259–273.

[21] "Theoretical Aspects of Water Accumulation in Cohesive Subgrade Soils," by Hans F. Winterkorn and Henry Eyring, *Proceedings*, Highway Research Board, National Research Council, Vol. 25, 1945, pp. 422–434.

[22] "Importance of Volume Relationships in Soil Stabilization," by Hans F. Winterkorn and Dutta Choudhury, *ibid.*, Vol. 29, 1949 (in press).

J. E. Jennings.[23]—At a very appropriate time, Mr. Wooltorton has called attention to the behavior of foundations on desiccated clays, since many workers in various parts of the world are facing the same problem. It appears that similar conditions are encountered in the greater part of southern Africa, parts of Australia, Palestine, and also in the high-lying areas of some states adjoining the lower Mississippi River in the United States. The Texans were probably the first to make serious attempts to solve the troubles experienced with buildings on these soils, but it is also very probable that the exact mechanism underlying the movements which these buildings experience has only recently been given detailed consideration.

It is unquestionable that, in South Africa, the movements experienced under such conditions may be broadly described as dome-shaped heaving, with the greatest rise at the centers and the smallest rises at the corners of the buildings. The cracking pattern described by Mr. Wooltorton is quite typical, and has been classified in South Africa as apparent "corners down"—the same pattern that would be expected where settlement is primarily caused by shear distortion, except that the movement is upward instead of downward. It is important to appreciate that the cracking does not result from settlement of the corners, but from the fact that the corners have risen less than the other parts of the building.

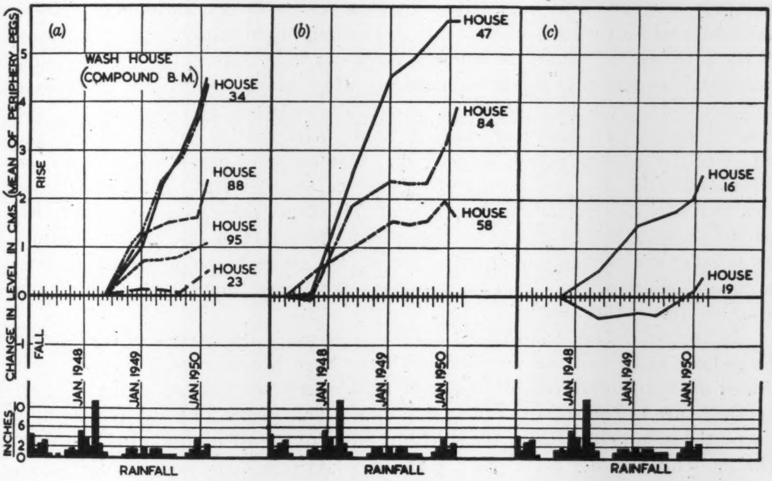

Fig. 17.—Heaving of Buildings, St. Helena Gold Mine, Orange Free State

The general cracking pattern described is frequently masked by local irregularities in the soil, and buildings immediately adjacent to each other demonstrate entirely different heaving patterns, as can be noted from Fig. 17, which shows level observations on a group of ten single-story concrete houses, most of which are within a radius of one-half mile. The pattern is further distorted by local conditions of drainage; and broken drain pipes, or other poor drainage conditions, immediately adjacent to the buildings, have frequently resulted in a dish-shaped rather than a dome-shaped heaving.

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[23] Director, National Bldg. Research Inst., Pretoria, South Africa.

Fig. 17 shows actual level observations referred to bench marks established in open fields at depths of approximately 10 ft. Each bench mark consists of a rod, set in concrete at the bottom of a pit. This rod is sleeved with a pipe set clear of the concrete pad, and the hole is then filled back. Check levels between such bench marks have indicated that small movements (± 0.2 cm) are taking place, but these are considered to be of second order of magnitude as compared to the movements of the buildings. In future efforts to obtain absolute movements, bench marks in South Africa will be placed in boreholes at least 50 ft to 100 ft deep and provision will also be made for temperature compensation of the rod.

Accepting these bench mark variations as being of second magnitude, it has been observed that pegs driven into the surface of open ground experience upward and downward movement, depending upon rainy or dry weather. With buildings, however, the conditions appear to be different, since they show a general tendency to settle immediately after construction, and then heave progressively for a long time. A correlation between this heaving and the rainfall seems problematical; and, in South Africa, it is considered that the effect of the seasons is only secondary. One series of observations, begun in 1941, indicates that an asymptotic value of the heaving will eventually be reached; and it is probable that, only at this stage, the cyclic movements due to the seasons will have a more important effect. It is also probable that a seasonal cyclic variation is superimposed on the general heaving curve and very accurate, short-period, level observations are being undertaken to ascertain whether this is the case.

In view of the apparent independence of season, other causes for the heaving have been examined and two theories have been propounded. One theory is based on the thermo-osmotic transfer of moisture toward the region of lower temperature in the soil under the building. This region of lower temperature is created in hot climates by the shielding effect of the structure. In this theory the moisture transfer takes place in the capillary fringe and consequently it is a necessary condition that the soils are partly saturated. The other theory relates to the relief of capillary stress, which results from the interference with the surface evaporation following erection of the building. Both theories have been discussed in other papers.[24]

Because of critical variations in the soil and the need to allow for a changing distortion pattern, it is apparent that a solution covering both dish-shaped settlement and dome-shaped heaving must be sought. Three solutions have been proposed in South Africa. The first adopts the principles of the Texas underreamed piles which are used to anchor the building in the zone of constant moisture content. This zone is believed to lie below the free water line in the soil; and, in soils of this type, the free water line is not very easy to recognize, generally requiring a deep excavation that must be examined by a trained observer. The piles are anchored at this level and the building is kept clear of the surface. The second solution is to reinforce the walls of the structure in such a manner that the building can tolerate the movements without cracking. This process is in the development stage and the washhouse in Fig. 18 is a successful example of such practice. The third method makes use of a flexible structure to allow for movements without deleterious or dangerous cracking. Dry-block construction is an example of such a procedure.

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[24] "The Theory of Heaving of Buildings on the Shattered Desiccated Clays of Southern Africa," by J. E. Jennings, *First Annual Report*, International Soc. of Soil Mechanics and Foundation Eng., 1950, Section VI*d*.

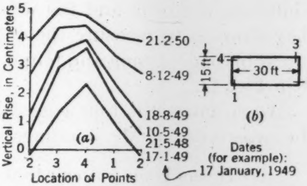

FIG. 18.—DETAILED FOUNDATION MOVEMENTS OF WASHHOUSE (REINFORCED CONCRETE BLOCK ON STRIP FOOTINGS), ST. HELENA GOLD MINE, ORANGE FREE STATE

Mr. Wooltorton has quoted successful practice based on making the foundation pressure approximately equal to the swelling pressure of the soil. It is believed that swelling pressure, measured by the method proposed by L. F. Cooling, will vary, depending on the initial moisture content of the sample. In South Africa swelling pressures are found to be of the order of from 3 tons per sq ft to 5 tons per sq ft, and in one area several successful buildings have been erected using a foundation pressure of approximately 3 tons per sq ft. The method seems open to some doubt, however, for two reasons: (1) The swelling pressure will depend on the moisture content applying at the time of construction; and (2) shearing tests in these soils have indicated that care should be exercised before allowing too high a bearing pressure to be used. Shear tests of soils in South Africa have indicated that bearing pressures should be limited to not more than 1.5 tons per sq ft; but, in view of the success with the aforementioned buildings, the entire testing procedure is being examined. Until this procedure has been developed and fully tested in the field, it is probable that this approach will not find general application in South Africa.

In conclusion, experience in South Africa has shown that the movements vary considerably, even on the same site, and also that the movements experienced by a building are radically different from those on open ground. It is believed that any field observations must be conducted on full-scale structures. Detailed level observations, such as those given in Figs. 17 and 18, have considerably assisted in clarifying a problem that appears fundamentally different from problems experienced in the countries where the principles of soil mechanics have been most generally applied.

F. L. D. WOOLTORTON.[25]—The contributions by Messrs. Winterkorn and Jennings are of particular value in emphasizing the importance of pedological engineering studies in conjunction with soil mechanics. This especially applies to desiccated soils and, more particularly, to such soils under tropical conditions. As world development gets under way so will the need for, and the importance of, pedological engineering be increasingly felt.

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[25] Constr. Engr., Road Constr. Div., Public Works Dept., Nakuru, Kenya Colony, East Africa.

Some of the constructional difficulties experienced in the expansive desiccated clays of Texas together with some of the design methods found successful in that state (including the use of suspended floors and foundation soil ventilation) have been described by Willard E. Simpson,[26] M. ASCE. Incidentally, while Mr. Simpson and the writer were motoring through San Antonio, Tex., they stopped to measure the distance between the crests of a length of waved concrete road curbing and found them to be 62 ft against the value of CE = 67.5 ft in Fig. 11.

Apparently, the following extract from Mr. Simpson's paper[26] agrees with the experience of Mr. Jennings:

> "It has been noticed that the point of maximum swelling is not near the surface, but at a point three to six feet below the surface. The swelling of the soil immediately on the surface is not so forceful but it is equally troublesome."

In the absence of further data, the following possibilities may feature in the explanation:

a. The presence of a higher clay content or of more active clay minerals at this depth (which is not uncommon).

b. The presence of certain residues of organic matter, in the upper 5 ft, acting as protective colloids masking or changing the physico-chemical properties of the clay fraction.

c. The Texas and South African[24] swelling problem may be aggravated, in some way, by the phenomena of swelling, "softening," and sometimes creep, which characterize some shattered clays after exposure by excavation. These aspects have been discussed by Karl Terzaghi,[27] Hon. M. ASCE, and A. W. Skempton.[28] The two phenomena have similarities but should not be confused. The swelling phenomenon under consideration is not necessarily accompanied by partial softening. It is a result of the release of internal stress and not of a reduction in external loading. It is a property of the clay material modified by the soil structure within its natural elements (which may be infinite in size); whereas the swelling and softening of stiff-fissured clays is a property of the clay modified by the structure of the entire profile rather than that of the individual elements into which the profile is fissured or fractured.

In the former instance, the swelling pressure may be represented as the loading to be applied to a desiccated soil to prevent swelling when in contact with water. It is determinable from the loading curve.

In the latter instance, the swelling or rebound occurs in a clay, already site saturated, as a result of the removal of a known external loading. It is often determined from the appropriate unloading curve.

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[26] "Problems in the Design of Foundations," by Willard E. Simpson, *Refiner and Natural Gasoline Manufacturer*, September, 1935, p. 426.

[27] "Stability of Slopes of Natural Clay," by Karl Terzaghi, *Proceedings*, International Conference on Soil Mechanics and Foundation Eng., Vol. I, Harvard Univ., Cambridge, Mass., June, 1936, p. 161.

[28] "The Principles and Application of Soil Mechanics," by A. W. Skempton, Inst. C. E., London, 1948, p. 51.

Presumably, when the stress is reduced in saturated stiff-fissured clays, water may move comparatively fast through the profile, by way of cleavage planes. The result is to produce some swelling and possibly some slaking of the natural element surfaces. This moisture movement cannot occur with the same rapidity and in such quantities in the nonshattered Burma soils where upward moisture movement must be of the thermo-osmotic type rather than in the capillary state.

It was estimated, in Mandalay (Burma), that no soil movement would occur at a depth of 26.5 ft. This compared with the 25 ft found by Henry C. Porter[29] to apply in parts of Texas. In consequence, no light building constructed on shallow piles could be expected to be generally satisfactory. Some such buildings had been constructed in Mandalay on 9-in. by 9-in. underreamed piles, but where the piles transmitted less than about 10 tons (that is, about 2.5 tons per sq ft) there was cracking in the superstructure.

Successful designs, developed in Mandalay, included both heavy buildings (transmitting loads in excess of 1.5 tons per sq ft) and light flexible buildings, although there was considerable distortion of the latter. Aids to reduce the movements in, and damage to, light flexible buildings included:

1. Tanking the floors and sides of the excavations with a 3-in. layer of selected soil treated with calcium chloride at the rate of 1.5 lb per sq yd and backfilling with selected nonexpansive material;
2. Placing the foundations below the depth of soil weathering in the layer of maximum apparent density (which was also the depth of maximum pH, minimum moisture-content change, and minimum rate of moisture change) and always below any lenses of sandy clay loam; and
3. Framing with timber posts bolted to tie beams in the floor and in the eaves and to reinforced concrete strip foundation beams.

Mr. Jennings, who recorded that observation in South Africa, suggests that an asymptotic value of the heaving may be reached in about eight years, after which seasonal cyclic movement may be found to occur. The many Mandalay buildings examined partly support this belief. It is obvious that heaving had been incrementally accumulative, but as all the observed buildings were at least ten years old (some of the order of a hundred years) this asymptotic condition might be considered to have been reached prior to observations. Nevertheless there was evidence, for certain buildings, that any condition in equilibrium with the average climate had been frequently disrupted by unusual climatic conditions and a new series of movements initiated.

It would not be expected, from the point of view of pure vertical movement, that an uncovered level ground surface, in equilibrium with the climate, would indicate any appreciable relative movements other than those caused by micro-changes in the topography. These are only to be expected under abnormal conditions of moisture distribution, as when evaporation is hindered over a limited area.

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[29] "Observations of the Texas State Highway Department on the Subsequent Effects of the Uniformity and the Non-Uniformity of Foundation Soil-Types on Pavements; and Also the Effects of Uniformity and Non-Uniformity of Moisture Content Fluctuations in Soil Foundations of High Volumetric Change," by Henry C. Porter, *Proceedings*, International Conference on Soil Mechanics and Foundation Eng., Vol. II, Harvard Univ., Cambridge, Mass., June, 1936, p. 256.

Mr. Winterkorn's remarks and data on permeability not only illustrate the care that may be necessary in applying the results of permeability experiments to a design problem but the important role played by pedological processes on the resultant properties of a soil profile.

The writer accepts with gratitude Mr. Winterkorn's correction of "distillation effect" to "thermo-osmotic flow." The phenomenon described by Mr. A. F. Lebedev[6] in 1927 would appear identical with that under discussion but unfortunately he was at a loss for a more suitable descriptive term. To illustrate how engineering pedology has advanced the knowledge and understanding of soil phenomena in the engineering world, and has even led in the agricultural field, it may be added that when the writer discussed the possibility of appreciable moisture flow in the vapor or film phase with authorities in 1936 his data were greeted with considerable doubt. It is now known, thanks to Mr. Winterkorn and the Road Research Laboratory (England),[30,31] that considerable moisture may move through desiccated soils under a temperature gradient, especially when the average moisture content approaches the plastic limit as in Burma. This moisture flow may result in a change in the soil's physical properties; and, when evaporation is hindered, it may create a moisture condition and shearing resistance very different from what might have been expected. In two instances free water was found in the soil material under the floors of buildings examined in Burma during the dry weather.

This phenomenon is of considerable importance to the highway engineer.

Mr. Winterkorn refers to soil swelling pressure as being normally of greater importance to the highway engineer than to the structural engineer. The effect on highway design has been discussed by the writer[32] and investigated by Mr. Winterkorn,[22] who suggests that one reason why building engineers have not given more attention to certain soil-physical phenomena may be that the type of buildings adversely affected by such phenomena represent too small a capital outlay to merit detailed studies and protective measures. This attitude, however, should not apply to organizations controlling a large number of such structures. It is noteworthy that the Great Indian Peninsula Railway adopted this policy in 1948.

Is it generally true that soil movements are only detrimental to light structures? Table 7 gives some of the information available. The values of the swelling pressure given should be treated with caution as the conditions under which they were determined are unknown. It may be noted that, in general, the procedure for selecting a suitable foundation bearing pressure is not the same for the A-6 sodium illite-montmorillonite soils of Burma (FME = 18) as it is for the more usual A-7 calcium-bentonite soils of Texas (FME = 35) and that the procedure in South Africa, where the soils appear to be A-7 or A-6 illite-montmorillonite soils (FME = 35 to 20?) is somewhat undecided, sometimes corresponding to the Texas method and sometimes to the Burma method.

Whereas, functionally, the Mandalay soils are structurally homogeneous, the South African and Texas soils are generally shattered.

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[30] "Moisture Movements Occurring in Soil Due to the Existence of a Temperature Gradient," Road Research Laboratory (England), *Note RN 761 DJM*, May, 1946.

[31] "Soil Moisture and the Factors Governing Its Movement," Road Research Laboratory (England), *Note RN 944 DC*, February, 1948.

[32] "Relation Between the Plastic Index and the Percentage of Fines in Granular Soil Stabilization," by F. L. D. Wooltorton, *Proceedings*, Highway Research Board, National Research Council, Vol. 27, 1947, p. 479.

It appears reasonable that, in principle, the design pressure for light buildings, constructed on desiccated swelling soils, should be sufficient to overcome the effective swelling pressure if differential wall movements are to be avoided; and when this is not possible, because of a limiting soil shear resistance or cost considerations, the design must provide for distortion stresses. Massive buildings—other than their ground or basement floors—may possibly be able to resist distortion effects through their inertia even though the pressure transmitted is less than the effective swelling pressure.

TABLE 7.—Relative Effect of Soil Movements on Light and Heavy Structures

Location (1)	Clay minerals (2)	Desiccation (uncovered state) (3)	Swelling pressure[a] (4)	Clay shattered (5)	Field moisture equivalent (6)	Type of building affected (7)
San Antonio, Tex.	Calcium bentonitic	Yes; to 17 ft	4 to 5	Yes	35	Light
Cisco, Tex.	Generally bentonitic	Yes	0 to 7	Yes ?	20	Light to medium heavy (?)
London, England blue clay	Illitic (?)	Occasionally, to about 5 ft during droughts	1.2	Yes[b]	30 (?)	Light
South Africa	Illite[c]	Yes; to at least 10 ft	3 to 5	Yes	20 to 30 (?)	Light
Mandalay, Burma	Illite[c]	Yes; to at least 12 ft	1.75	No	18	Light to medium heavy

[a] Conditions under which the swelling pressures (in tons per square foot) in Col. 4 were determined are not generally known. Remarks pertinent to each item in Col. 4 are as follows:

San Antonio, Tex.: Suitable bearing pressure was controlled by the low shear resistance of the soil when it was saturated.

Cisco, Tex.: Buildings transmitting pressures of 3.5 tons per sq ft have been affected by soil swelling.

London, England: The swelling pressure was 1.2 tons per sq ft under normal conditions. The design was based on the shearing resistance of the soil.

South Africa: Apparently, the safe bearing pressure was generally determined by shear considerations. Some satisfactorily designed buildings transmit pressures of the same order as the swelling pressures.

Mandalay, Burma: Under extreme site conditions, the swelling pressure was 1.75 tons per sq ft. Buildings that transmitted as much as 1.5 tons per sq ft were affected by soil swelling. Since the shear resistance is high, suitable design pressures are limited to a minimum value equal to the swelling pressure.

[b] The clay was fissured or shattered. [c] Illite, with small quantities of montmorillonite.

Where the nature of the clay minerals is important in influencing the maximum value of the swelling pressure, the amount of desiccation, at the time of construction and possibly during periods after construction, will determine the effective swelling pressure and is thus, in effect, of greater importance.

The selection of a suitable design pressure, and frequently of the design type, would therefore appear to involve in its general solution a consideration of two factors:

a. A maximum value based on shear considerations involving a forecast of the maximum moisture content by capillary or thermo-osmotic flow, or both, toward the foundations. For predictory work, this value may perhaps be safely taken as the field moisture equivalent (FME).

b. A minimum value determined by the effective swelling pressure involving a forecast of the minimum moisture content, or provision for resisting distortion stresses. This minimum moisture content would appear to be (in the worst cases of desiccation, although not generally) about equal to the shrinkage limit.

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ELASTIC RESTRAINT EQUATIONS FOR SEMI-RIGID CONNECTIONS

By J. E. Lothers,[1] M. ASCE

With Discussion by Messrs. Lloyd T. Cheney, Thomas C. Kavanagh, Dean F. Peterson, Jr., and Jack E. Cermak, and J. E. Lothers.

Synopsis

The restraining effects of semi-rigid, girder-column connections have received considerable attention within recent years in both the United States and Great Britain. Excellent literature on the subject is now available, and methods of analyses which take into account the restraining effects of these connections have been proposed.[2,3,4] On the whole, these proposed methods would be conducive to economy in metal structures except for one fact. Unfortunately, expensive and time-consuming laboratory tests must be performed, in advance, on any given semi-rigid connection in order to obtain a constant which measures the extent of its rigidity and which is necessary for analysis. Proposed equations for the computation of this constant, and a table of computed values for standard beam connections, are presented in this paper.

In addition, a proposed new method is advanced for the analysis and design of one type of steel bracket which is based on the bending strength of the connecting angles as opposed to the present method of basing its strength on the rivets. This proposed method was evolved as a by-product of the main problem of finding elastic restraint equations for semi-rigid connections.

The notation used in the paper is introduced directly in the text and presented alphabetically in the Appendix.

Introduction

The application of the methods of slope deflections and moment distribution to the analysis of rigid building frames is now accepted practice. In the case

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Note.—Published in February, 1950, as *Proceedings-Separate No. 5.* Positions and titles given are those in effect when the paper or the discussion was received for publication.

[1] Prof. of Architecture, Oklahoma Agri. and Mech. College, Stillwater, Okla.

[2] "Analysis of Building Frames with Semi-Rigid Connections," by Bruce Johnston and Edward H. Mount, *Transactions*, ASCE, Vol. 107, 1942, p. 993.

[3] "Elastic Properties of Riveted Connections," by J. Charles Rathbun, *ibid.*, Vol. 101, 1936, p. 524.

[4] "First, Second, and Final Reports," Steel Structures Research Committee, Dept. of Scientific and Industrial Research of Great Britain, H. M. Stationery Office, London, 1931–1936.

of continuous reinforced concrete frames with properly reinforced joints, this practice is justified. It is also justified in welded steel buildings where the ends of the beam flanges are welded directly to the column, provided the column flanges are reinforced against any strain.

On the other hand, where the girder-column connection is made through the medium of angles or tees, whether riveted or welded, the deflection due to bending in their outstanding legs gives rise to a rotation, ϕ, of the beam which is in addition to the rotation, θ, of the entire joint. This type of connection has become known among structural engineers as a semi-rigid connection. To apply one of the methods of so-called exact analysis to a semi-rigid connection, without some correction, results in an incorrect distribution of moment along the beam. To ignore the restraint provided, on the other hand, is wasteful. If a correction based on ϕ be incorporated in the usual methods of exact analysis, it is possible to determine a proper distribution of moment between the ends and the intermediate points of the beam and thus take intelligent advantage of the economy provided by these semi-rigid connections.

Literature on the subject[3] makes use of a connection constant, Z, such that,

$$Z = \frac{\phi}{M} \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

Bruce Johnston, M. ASCE, and Edward H. Mount[2] have defined Z (Eq. 1) as the angle change for unit moment. It is a factor which, when used in conjunction with the E and I of a member connected by the semi-rigid joint, becomes the necessary correction agent to render the usual exact methods of analysis applicable. The reciprocal, $1/Z$, is the slope of the moment-rotation curve[3] introduced by J. Charles Rathbun, M. ASCE. The symbol ϕ may be defined as the angle of strain in the semi-rigid connection. It is the number or radians by which the angle between the beam and the column differs from a right angle. For the purpose of this paper, M will be defined as the safe resisting moment of the connection (see Appendix).

ANALYSIS

Steps of Procedure.—To follow the steps in the development of elastic restraint equations as applied to standard connections, it is necessary to visualize what happens to a pair of connection angles supporting the end of a loaded beam. A horizontal pull is exerted by the top of the beam which puts bending in the legs of the angles and pulls their heels away from the column. The amount of this deflection (Δ_b, Fig. 1) is a maximum at the top of the connection angles and zero at their neutral, bending axis, y inches below the top (see Fig. 2).

The procedure for arriving at an equation for Z, as applied to standard connections, may be divided into the following six steps:

1. Find an expression for the critical moment in the legs of the connection angles due to the outward pull of the top of the beam, assuming no give in the rivets.[3,5,6]

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[5] "Permissible Stresses on Rivets in Tension," by C. R. Young and W. B. Dunbar, *Bulletin No. 8*, School of Eng. Research, Univ. of Toronto, Toronto, Ont., Canada, 1928, p. 7.

[6] "Tension Tests of Rivets," by W. M. Wilson and W. A. Oliver, *Bulletin No. 210*, Eng. Experiment Station, Univ. of Illinois, Urbana, Ill., 1930.

2. Find an expression for the deflection, Δ_b, of the upper ends of the connection angles outward from the face of the column (Figs. 1 and 2).

3. Locate the neutral, bending axis of the connection as governed by the elastic behavior of the legs of the angles, and find an expression for its distance, y, down from the top.

4. Note that the angle of strain, ϕ, equals the deflection, Δ_b, of the heels of the connection angles, divided by y; that is,

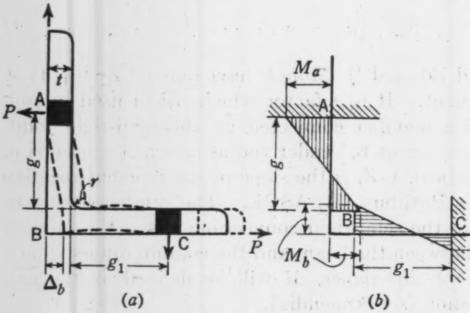
$$\phi = \frac{\Delta_b}{y} \qquad \text{(2a)}$$

5. Find an expression for the safe resisting moment, M, of the connection, based on the elastic restraint of the legs of the angles.

6. Finally

$$Z = \frac{\phi}{M} \qquad \text{(2b)}$$

in which Z will be expressed in terms of the dimensions and known functions of the connection angles.



(a) (b)

FIG. 1.—SYMBOLS

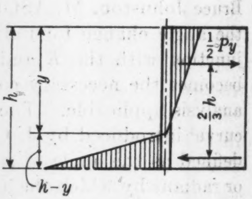


FIG. 2.—RESISTING COUPLE FOR EACH ANGLE

The Critical Moment in Connection Angles.—Expressions for the bending moments at points A, B, and C for a load, P, as shown in Fig. 1, may be derived by the method of slope deflections. The basic equations with known values substituted follow:

$$M_{ab} = \frac{2\,E\,I}{g}\,(2\,\theta_a + \theta_b - 3\,r_{ab}) = \frac{2\,E\,I}{g}\,(\theta_b - 3\,r_{ab}) \qquad (3)$$

$$M_{ba} = \frac{2\,E\,I}{g}\,(2\,\theta_b + \theta_a - 3\,r_{ab}) = \frac{2\,E\,I}{g}\,(2\,\theta_b - 3\,r_{ab}) \qquad (4)$$

$$M_{bc} = \frac{2\,E\,I}{g}\,(2\,\theta_b + \theta_c - 3\,r_{bc}) = \frac{2\,E\,I}{g_1}\,(2\,\theta_b) \qquad (5)$$

and

$$M_{cb} = \frac{2\,E\,I}{g_1}\,(2\,\theta_c + \theta_b - 3\,r_{bc}) = \frac{2\,E\,I}{g_1}\,(\theta_b) \qquad (6)$$

Referring to Fig. 1, the following relations are evident:

$$M_{ba} + M_{bc} = 0 \quad \ldots\ldots (7)$$

and

$$M_{ab} + M_{ba} = P\,g \quad \ldots\ldots (8)$$

If the proper substitutions, Eq. 3 to Eq. 5, be made in Eqs. 7 and 8, and if the latter are solved simultaneously for θ_b and r_{ab}, the following expressions result:

$$\theta_b = -\frac{P\,g^2}{2\,E\,I} \times \frac{g_1}{g_1 + 4\,g} \quad \ldots\ldots (9)$$

and

$$r_{ab} = -\frac{P\,g^2}{3\,E\,I} \times \frac{g + g_1}{4\,g + g_1} \quad \ldots\ldots (10)$$

Finally, these values of θ_b and r_{ab} are substituted in the basic equations to derive M_a, M_b, and M_c:

$$M_a = P\,g \times \frac{2\,g + g_1}{4\,g + g_1} \quad \ldots\ldots (11)$$

$$M_b = P\,g \times \frac{2\,g}{4\,g + g_1} \quad \ldots\ldots (12)$$

and

$$M_c = P\,g \times \frac{g}{4\,g + g_1} \quad \ldots\ldots (13)$$

The Deflection, Δ_b.—In Fig. 1(*b*), if the moment diagram be replaced by the $\frac{M}{E\,I}$-diagram, the deflection, Δ_b, of the heel of the angle outward from the support may be obtained by taking moments of that part of the $\frac{M}{E\,I}$-diagram on the leg AB about point B. This will give the deflection in a direction perpendicular to the tangent at point A. However, a simpler method of computing this deflection follows directly from Eq. 10. By definition, r_{ab} equals the deflection, Δ_b, divided by g. Substitute the value of r_{ab} from Eq. 10, ignore the minus sign, and solve for Δ_b—

$$\Delta_b = \frac{P\,g^3}{3\,E\,I} \times \frac{g + g_1}{4\,g + g_1} \quad \ldots\ldots (14)$$

The Angle of Strain, ϕ.—As stated in step 4 (Eq. 2*a*), ϕ is equal to the deflection, Δ_b, outward from the heels of the angles, at the top of the connection, divided by y. Therefore,

$$\phi = \frac{\Delta_b}{y} = \frac{P\,g^3}{3\,E\,I\,y} \times \frac{g + g_1}{4\,g + g_1} \quad \ldots\ldots (15)$$

An expression for P may be found from Eq. 11. The resisting moment of a 1-in. strip at the top of the connection is governed by its strength at point A—that is, $M_a = \frac{s\,t^2}{6}$, in which s is the bending stress and t is the thickness of the angle. If the value of M_a in Eq. 11 be substituted, the pull, P, on the top inch of each angle, is

$$P = \frac{s\,t^2}{6\,g} \times \frac{4\,g + g_1}{2\,g + g_1} \qquad (16)$$

In Eq. 15, substitute the value of P from Eq. 16, and for I substitute $t^3/12$:

$$\phi = \frac{2\,g^2\,s}{3\,E\,t\,y} \times \frac{g + g_1}{2\,g + g_1} \qquad (17)$$

The Resisting Moment, M, of the Connection.—Let P equal the maximum pull, in pounds per inch of depth, per angle, at the top of the connection. The average pull, per inch of depth for each angle, above the netural axis, is $\frac{1}{2}\,P\,y$. Taking moments about the lower force of the resisting couple in Fig. 2, the resisting moment is found to be $\frac{P\,y\,h}{3}$ for one angle, or $\frac{2\,P\,y\,h}{3}$ for the connection. The final expression for the total resisting moment, with the value of P substituted from Eq. 16, is

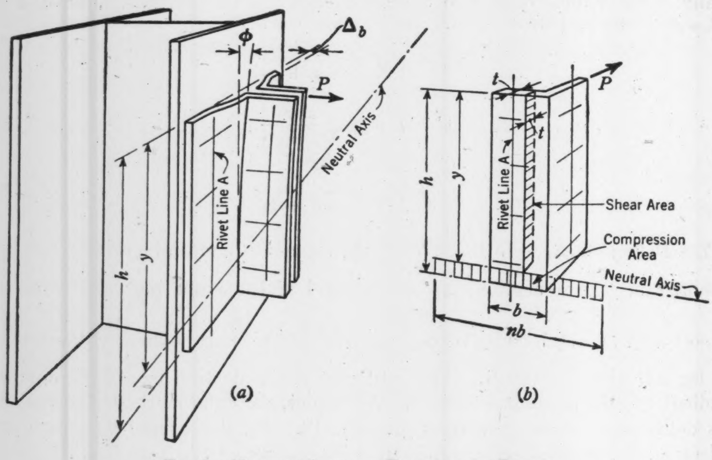


FIG. 3.—NEUTRAL AXIS AND TRANSFORMED SECTION

$$M = \frac{h\,s\,t^2\,y}{9\,g} \times \frac{4\,g + g_1}{2\,g + g_1} \qquad (18)$$

The Neutral Axis.—The problem of locating the neutral axis of the connection is analogous to that of locating the transformed section in reinforced concrete beam analysis. This follows from the fact that the compression area ($n\,b(h - y)$) below the neutral axis, as shown in Fig. 3(*b*), is opposed by a shear area ($t\,y$) above. The critical section is a right section of the column-connected leg of the angle along rivet line A at the top of the angle. This so-called critical section is subjected to a bending stress which is difficult to associate directly with the compression area below the neutral axis. On the other hand, the shear area, indicated in Fig. 3(*b*), is directly opposed to the aforementioned compression area. Fortunately, the relation between the shearing stress on this area and the critical bending stress along rivet line A can be expressed. This relationship gives rise to a stress-ratio factor, n, which is analogous to the $n = E_s/E_c$ used in establishing the transformed section in reinforced concrete beam analysis.

Eq. 16 yields the value of the pull, P, on the top inch of one angle in terms of the bending stress, s, along rivet line A. The pull P is also the total shear on the top inch of the column-connected leg of each angle. Therefore, the unit shear, v, on the top inch is P/t. Substitute the value of P from Eq. 16:

$$v = \frac{s\,t\,(4\,g + g_1)}{6\,g\,(2\,g + g_1)} \qquad (19)$$

in which s is the bending stress in the column-connected leg of the angle.

Eq. 19 establishes the desired relationship between the shear area, $t\,y$, and the critical bending section along rivet line A and, accordingly, provides the key to the desired stress-ratio factor, n. It is assumed that the ratio of bending stress to strain in the column-connected leg of the angle along rivet line A is the same as that in compression below the neutral axis. Therefore, to arrive at the properly shaped transformed area in Fig. 3(*b*), either the shear area, $t\,y$, must be multiplied by the coefficient of s in Eq. 19, or the compression area, $b\,(h - y)$, below the neutral axis, must be divided by that coefficient. The latter method will be adopted and, accordingly,

$$n = \frac{6\,g\,(2\,g + g_1)}{t\,(4\,g + g_1)} \qquad (20)$$

Equating the statical moment of the shear area, $t\,y$, above the neutral axis (Fig. 3(*b*)) to that of the compression area, $n\,b\,(h - y)$, below, taken about the neutral axis, gives rise to a quadratic equation in y which (with the plus sign before the radical discarded) becomes

$$y = \frac{h\,(n\,b - \sqrt{n\,b\,t})}{n\,b - t} \qquad (21)$$

The Semi-Rigid Connection Factor, Z.—This factor has been defined previously as the angle of strain for unit moment for any given connection; that is, $Z = \frac{\text{angle of strain}}{\text{moment}} = \frac{\phi}{M}$. Substitute in Eq. 1 the values of ϕ and M from Eqs. 17 and 18, respectively, and simplify:

$$Z = \frac{6\,g^3}{E\,h\,t^3\,y^2} \times \frac{g + g_1}{4\,g + g_1} \qquad (22)$$

Strengthening Effect of the Fillet.—Inasmuch as g and g_1 are measured from the centers of their respective rivets to the faces of the adjacent legs of the angle, the fillet functions as a haunch for both legs, and adds stiffness to the connection. To have taken the fillet into consideration in the derivation of Eqs. 18 and 22 for M and Z would have entailed the derivation of a different set of equations for each web thickness of beam and for each fillet radius involved. This follows from the fact that g changes in length with each thickness of web straddled, while the fillet remains constant for that particular connection. In addition, there are two different radii of fillet represented in the standard connections.

The effect of the fillet is not important so far as it affects Z. To include it would be inconsistent and misleading in the inferred accuracy to be obtained—not to mention the complications. There are factors that contribute to the rotation of the connection which have been neglected in the derivation of Eq. 22. Among them may be mentioned shear deflection in the column-connected leg of the angles, possible rivet slip, and strain in the column flanges. These should compensate for neglect of the fillet.

On the other hand, it is important that the true resisting moment of the connection be known for the purpose of applying the following proposed new method of steel bracket design.

The analysis leading to the derivation of Eq. 18 for the resisting moment of a connection treated the two legs of the angle as short, prismatic beams, fixed at their outer ends, A and C, and restrained at their common end, B. However, the fillet provides a comparatively long haunch at the B-end of both beams. To include the haunch in the slope deflection analysis it is necessary to rewrite Eqs. 3 to 6 in the following general forms:[7]

$$M_{ab} = \frac{E\,I_o}{g}\,[C_1\,\theta_a + C_2\,\theta_b - (C_1 + C_2)\,r_{ab}] \dots\dots\dots\dots (23)$$

$$M_{ba} = \frac{E\,I_o}{g}\,[C_2\,\theta_a + C_3\,\theta_b - (C_2 + C_3)\,r_{ab}] \dots\dots\dots\dots (24)$$

$$M_{bc} = \frac{E\,I_o}{g_1}\,[C_4\theta_b + C_5\,\theta_c - (C_4 + C_5)\,r_{bc}] \dots\dots\dots\dots (25)$$

and

$$M_{cb} = \frac{E\,I_o}{g_1}\,(C_5\,\theta_b + C_6\,\theta_c - (C_5 + C_6)\,r_{bc}] \dots\dots\dots\dots (26)$$

in which I_o, in this case, is the moment of inertia of a typical section taken along the prismatic parts of g or g_1. For a prismatic beam C_1 and C_2 are equal to 4 and 2, respectively. For a haunched beam the C-values must be computed.

As previously explained, a general equation analogous to Eq. 18, but with the fillet effects included, is impossible. By setting up equations for a typical connection, however, it is possible to establish a correction factor by the simple method of comparing the results obtained for that connection with and without fillet effects. The standard AISC (American Institute of Steel Construction)[8,9] K5 connection straddling a $\frac{3}{4}$-in. web will be chosen for this purpose. The necessary data are as follows: $g = 1.9375$ in.; $g_1 = 1.8125$ in.; $t = \frac{7}{16}$ in.; and $r = \frac{1}{2}$ in. The C-values computed by the method of L. C. Maugh,[7] M. ASCE, are, for C_1 to C_6, respectively, 2.26, 1.46, 3.04, 3.37, 1.62, and 2.43. If each of the latter values be substituted for its respective C-values in Eqs. 23 through 26 and the same procedure used as was employed with respect to Eqs. 3 through 6, the following equations for end moments may be obtained:

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[7] "Statically Indeterminate Structures," by L. C. Maugh, John Wiley & Sons, Inc., New York, N. Y., 1946, pp. 132–137.

[8] "Steel Construction," AISC Manual, New York, N. Y., 1947, pp. 252–261.

[9] *Ibid.*, p. 285.

$$M_a = P\,g \times \frac{2.64\,g + g_1}{5.84\,g + g_1} \ldots\ldots (27)$$

$$M_b = P\,g \times \frac{3.20\,g}{5.84\,g + g_1} \ldots\ldots (28)$$

and

$$M_c = P\,g \times \frac{1.54\,g}{5.84 + g_1} \ldots\ldots (29)$$

Eqs. 27 through 29 provide for the strengthening effect of the fillet and are analogous to Eqs. 11 through 13. Proceeding in a manner similar to that followed in obtaining Eq. 18, an analogous expression for the resisting moment of the standard AISC connection[8,9] K5, and one which includes the effect of the fillet, may be found:

$$M = \frac{h\,s\,t^2\,y}{9\,g} \times \frac{5.84\,g + g_1}{2.64\,g + g_1} \ldots\ldots (30)$$

Comparing Eqs. 18 and 30 it may be seen that they differ only in their coefficients of g in the second fraction. Setting $g = g_1$, the second fraction is 5/3 in Eq. 18 and 6.84/3.64 in Eq. 30. The latter is 1.13 times the former. Therefore, the correction factor by which the right-hand side of Eq. 18 should be multiplied in order to take into account the strengthening effect of the fillet is 1.13. The resulting equation is very nearly

$$M = \frac{h\,s\,t^2\,y}{8\,g} \times \frac{4\,g + g_1}{2\,g + g_1} \ldots\ldots (31)$$

Inasmuch as the constant, 1.13, was established by a comparison between the results from Eq. 18 and those taken from a fillet-based equation for one web thickness only, it might well be questioned that sufficient evidence has been presented to prove that the aforementioned constant applies throughout the entire range of web thicknesses. A set of fillet-based equations similar to Eqs. 27 through 30 was derived for each of the web thicknesses, $\frac{1}{2}$ in., $\frac{3}{4}$ in., and 1 in. Although the pair of constants, K_1 and K_2 in the $\frac{K_1\,g + g_1}{K_2\,g + g_1}$-term of the equation for M were different for all three of the web thicknesses considered, they did establish an important fact. It was found that, if g and g_1 each be set equal to unity in the foregoing term, it was constant for all three web thicknesses, as far as could be detected by slide rule, and, accordingly, bore a constant ratio to the analogous term in Eq. 18. It is believed, therefore, that the use of the 1.13-factor, as indicated, is justified.

TABLE 1.—CONSTANTS FOR STANDARD AMERICAN INSTITUTE OF STEEL CONSTRUCTION CONNECTIONS

Web thickness (in.)	Length, g (in.)	(*A*) Series A–B–H–K ($r_{A-B} = \frac{3}{8}$ in. and $r_{H-K} = \frac{1}{2}$ in.)			(*B*) Series HH–KK ($r = \frac{1}{2}$ in.)		
		Distance, y (in.)	Moment, M (in-lb)	$\frac{1}{Z \times 10^8}$	Distance, y (in.)	Moment, M (in-lb)	$\frac{1}{Z \times 10^8}$
(1)	(2)	(3)	(4)	(5)	(3)	(4)	(5)
		(*a*) CONNECTION 3: $h = 8.5$; $t = \frac{3}{8}$; and $g_1 = 1.8750$					
....	....	$y = 0.936\ h$			$y = 0.947\ h$		
$\frac{5}{8}$	2.0625	7.95	19,500	0.402	8.04	19,800	0.411
$\frac{1}{2}$	2.1250	7.96	19,000	0.371	8.05	19.300	0.379
$\frac{3}{8}$	2.1875	7.96	18,600	0.343	8.05	18,900	0.351
$\frac{1}{4}$	2.2500	7.97	18,100	0.319	8.06	18,400	0.326
		(*b*) CONNECTION 4: $h = 11.5$; $t = \frac{3}{8}$; and $g_1 = 1.8750$					
....	....	$y = 0.936\ h$			$y = 0.947\ h$		
$\frac{3}{4}$	2.0000	10.75	36,700	1.079	10.88	37,100	1.105
$\frac{5}{8}$	2.0625	10.75	35,800	0.994	10.88	36,100	1.018
$\frac{1}{2}$	2.1250	10.76	34,800	0.917	10.89	35,300	0.884
$\frac{3}{8}$	2.1875	10.77	34,300	0.850	10.90	34,500	0.870
$\frac{1}{4}$	2.2500	10.78	33,200	0.789	10.91	33,600	0.808
		(*c*) CONNECTION 5: $h = 14.5$; $t = \frac{7}{16}$; and $g_1 = 1.8125$					
....	....	$y = 0.924\ h$			$y = 0.938\ h$		
$\frac{3}{4}$	1.9375	13.39	80,900	3.692	13.58	82,000	3.798
$\frac{5}{8}$	2.0000	13.40	78,800	3.392	13.59	80,000	3.489
$\frac{1}{2}$	2.0625	13.41	76,800	3.126	13.60	77,900	3.215
$\frac{3}{8}$	2.1250	13.43	74,800	2.888	13.62	76,000	2.970
		(*d*) CONNECTION 6: $h = 17.5$; $t = \frac{7}{16}$; and $g_1 = 1.8125$					
....	....	$y = 0.924\ h$			$y = 0.938\ h$		
$\frac{3}{4}$	1.9375	16.16	117,900	6.490	16.39	119,500	6.676
$\frac{5}{8}$	2.0000	16.17	114,700	5.961	16.40	116,500	6.131
$\frac{1}{2}$	2.0625	16.19	111,800	5.499	16.41	113,500	5.650
$\frac{3}{8}$	2.1250	16.20	109,200	5.071	16.43	109,300	5.216
		(*e*) CONNECTION 7: $H = 20.5$; $t = \frac{7}{16}$; and $g_1 = 1.8125$					
....	....	$y = 0.924\ h$			$y = 0.938\ h$		
$\frac{3}{4}$	1.9375	18.93	161,700	10.433	19.20	164,100	10.733
$\frac{5}{8}$	2.0000	18.95	157,600	9.590	19.21	159,700	9.854
$\frac{1}{2}$	2.0625	18.96	152,600	8.835	19.23	155,700	9.088
		(*f*) CONNECTION 8: $h = 23.5$; $t = \frac{7}{16}$; and $g_1 = 1.8125$					
....	....	$y = 0.924\ h$			$y = 0.937\ h$		
$\frac{3}{4}$	1.9375	21.72	212,800	15.745	22.01	215,600	16.168
$\frac{5}{8}$	2.0000	21.73	207,200	14.455	22.02	209,900	14.843
$\frac{1}{2}$	2.0625	21.74	201,800	13.316	22.04	204,600	13.686

TABLE 1.—(*Continued*)

Web thickness (in.)	Length, g (in.)	(*A*) Series A-B-H-K ($r_{A-B} = \frac{3}{8}$ in. and $r_{H-K} = \frac{1}{2}$ in.)			(*B*) Series HH-KK ($r = \frac{1}{2}$ in.)		
		Distance, y (in.)	Moment, M (in.-lb)	$\frac{1}{Z \times 10^8}$	Distance, y (in.)	Moment, M (in.-lb)	$\frac{1}{Z \times 10^8}$
(1)	(2)	(3)	(4)	(5)	(3)	(4)	(5)
		(*g*) CONNECTION 9: $h = 26.5$; $t = \frac{7}{16}$; and $g_1 = 1.8125$					
....	....	$y = 0.924\ h$			$y = 0.937\ h$		
$\frac{7}{8}$	1.8750	24.45	277,900	24.528	24.80	281,900	25.236
$\frac{3}{4}$	1.9375	24.48	270,500	22.554	24.82	274,200	23.184
$\frac{5}{8}$	2.0000	24.49	263,200	20.704	24.83	266,900	21.283
$\frac{1}{2}$	2.0625	24.52	256,700	19.102	24.86	260,200	19.635
		(*h*) CONNECTION 10: $h = 29.5$; $t = \frac{7}{16}$; and $g_1 = 1.8125$					
....	....	$y = 0.923\ h$			$y = 0.936\ h$		
1	1.8125	27.18	354,600	37.058	27.57	359,600	38.129
$\frac{7}{8}$	1.8750	27.20	344,200	33.793	27.60	349,300	34.784
$\frac{3}{4}$	1.9375	27.22	337,700	31.042	27.63	339,900	31.984
$\frac{5}{8}$	2.0000	27.25	326,100	28.535	27.64	330,800	29.358

CONSTANTS FOR STANDARD CONNECTIONS

Computed Values of $1/Z$.—Table 1 includes values of $1/Z$ for all the standard AISC connections[8,9] with the exception of numbers 1 and 2 which, for obvious reasons, have been omitted. (In Cols. 5, Table 1, values are multiplied by 10^8 to obtain $1/Z$.) They were computed by Eq. 22.

The standard distance, $5\frac{1}{2}$ in.,[8,9] between the two inner rows of column rivets gives rise to a different value of g for each web thickness straddled by the connection. Therefore, since the connection constants are affected by any change in g, values for M and $1/Z$ for the complete range of web thicknesses likely to be encountered have been included in Table 1. Both the web thicknesses and the corresponding values of g are recorded. Also recorded are the web-connected leg, g_1; the height of the connection, h; the thickness of the connection angle, t; and the distance from the top of the connection to its neutral axis, y.

The tabulated values for M were computed by Eq. 31. The strengthening effect of the fillet has thus been taken into consideration and, therefore, these values for M may be used in the proposed new method of bracket design.

COMPARISON WITH PUBLISHED LABORATORY RESULTS

Computed Slopes for Moment-Rotation Curves.—Table 2 contains the data necessary for computing the slopes of the moment-rotation curves for specimens tested by Mr. Rathbun,[3] as well as the corresponding slopes as computed by Eq. 22. It will be noticed that none of the four specimens fall among the AISC standard connections. Mr. Rathbun's moment-rotation curves for these specimens[3] have been reproduced in Fig. 4 and, in order to establish a check between his laboratory measured results and those computed by Eq. 22, the latter have been plotted on the reproduced curves.

Attention is called to the consistent agreement between the computed slopes and the curves for specimens 5, 6, and 7. That the computed slope for specimen 4 is slightly too steep for the curve may be explained by the fact that specimen 4 is the weakest of those checked and that, therefore, the factors mentioned as having been neglected in the derivation of Eq. 22 might be expected to assume the greatest relative importance with respect to this specimen.

TABLE 2.—DATA FROM PUBLISHED LABORATORY RESULTS ($t = \frac{3}{8}$ IN. AND g_1 = 1.8750 IN.)

Specimen[a]	h	g	b	n	y	$1/(Z \times 10^6)$
4	9	2.1875	4	20.59	8.43	0.407
5	9	1.9375	6	18.52	8.51	0.577
6	15	2.1250	4	20.07	14.04	2.037
7	15	1.8750	6	18.00	14.17	2.906

[a] "Elastic Properties of Riveted Connections," by J. Charles Rathbun, *Transactions*, ASCE, Vol. 101, 1936, p. 527, Fig. 1.

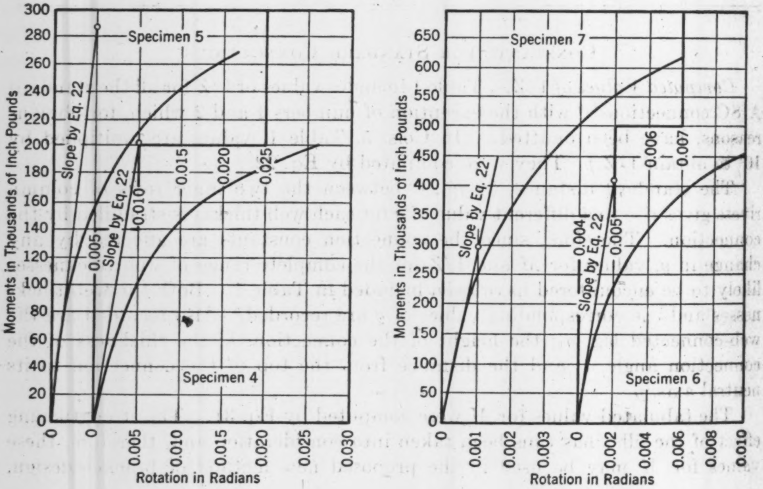

FIG. 4.—COMPUTED SLOPES PLOTTED ON MOMENT-ROTATION CURVES

In the case of specimen 7, on the other hand, the computed slope is barely steep enough. This is the strongest of the specimens checked and, in this instance, inclusion of the fillet effects in the computed slope might be expected to give it the necessary additional steepness.

APPLICATION OF EQ. 31 TO BRACKET DESIGN

It is quite generally recognized by the engineering profession that the present method of basing the design of certain types of steel brackets on the rivet strength often results in an overstress in the connection angles. Furthermore, the 1946 AISC specification[8,9] for permissible tension in rivets will heighten this tendency. Eq. 31 provides an alternate method of design for the the type of bracket indicated in Fig. 5, and one which is based on the critical bending stress in the angles.

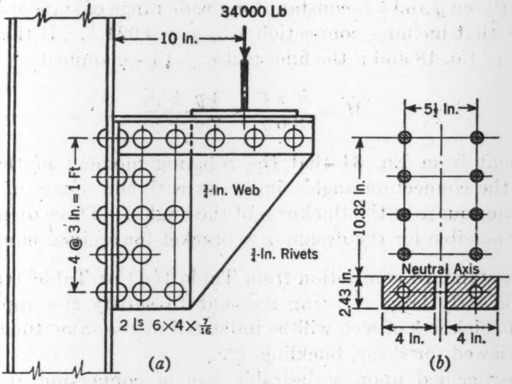

FIG. 5.—BRACKET LOADED TO FULL RIVET STRENGTH

Bracket Design Based on Rivet Strength.—In the following example the comparison to be demonstrated is that between the strength of the rivets connecting the column and the bending strength of the column-connected legs of the connection angles. The other rivets are a separate consideration and will not be treated in this example.

Although a direct approach (outside the scope of this paper) is possible, the usual procedure is to assume a bracket and then to test the sufficiency of the rivets in tension and shear, respectively, by the following equations:

$$f = \frac{M\,d}{2\,I} + \sqrt{\left(\frac{M\,d}{2\,I}\right)^2 + \left(\frac{P}{N\,a}\right)^2} \quad \text{(32)}$$

and

$$v = \sqrt{\left(\frac{M\,d}{2\,I}\right)^2 + \left(\frac{P}{N\,a}\right)^2} \quad \text{(33)}$$

in which f is the maximum tension in the topmost rivet, a distance of d inches from the neutral axis, due to combined bending and shear; and v is the corresponding maximum shear in the same rivet. These stresses are limited to 20,000 lb per sq in. and 15,000 lb per sq in., respectively. Symbol M denotes the moment from the eccentric load, P; N is the number of rivets through the column flange; a is the sectional area of one rivet shank; and I is the moment of inertia of the contact section (Fig. 5(b)) between the bracket and the column.

The shape of the contact section is found by taking static moments about the neutral axis as demonstrated by Thomas C. Shedd,[10] M. ASCE. For the bracket in Fig. 5, I is 217.8 in.^{4}; the distance, d, from the neutral axis to the topmost rivets is 10.82 in.; M is 340,000 in-lb; and N is assumed as 10.

Applying Eqs. 32 and 33, f and v are found to be 19,880 lb per sq in. and 11,430 lb per sq in., respectively, both of which are within the allowable limits specified by the AISC.[9,11] The rivets through the web of the bracket may be investigated by methods recommended by Mr. Shedd.[12]

Bracket Design Based on Angle Strength.—As may be observed from Table 1, the ratio between y and h is constant for a wide range of standard connections. For the series that includes connection K5, $y = 0.924\,h$. If this substitution for y be made in Eq. 18 and if the fillet factor, 1.13, be applied,

$$M = \frac{h^2\,s\,t^2}{8.6\,g} \times \frac{4\,g + g_1}{2\,g + g_1} \qquad (34)$$

It is evident from Eq. 34 that the resisting moment of the bracket, as governed by the connection angles, increases with the square of its height, as well as with the square of the thickness of the angles. Three direct methods of approach are possible for the design of a bracket for a given moment:

1. Select a standard connection from Table 1(A) or Table 1(B) which furnishes a sufficiently large resisting moment to satisfy the applied moment. The required thickness of web will be indicated at the same time; although, it should be reviewed for shear, buckling, etc.
2. Having decided upon a desirable size of connection angles and web thickness, compute the necessary height, h, or length of connection angles, by Eq. 34 or Eq. 35.

Eqs. 20 and 21 indicate upon what elements of the connection the ratio y/h depends. With that information at hand, the proper ratio may be found in Table 1. However, if the angles chosen are not listed in Table 1, resort must be had to a general form of Eq. 34 as follows:

$$M = \frac{h^2\,s\,t^2}{K\,g} \times \frac{4\,g + g_1}{2\,g + g_1} \qquad (35)$$

in which $K = 9/\,(y/h \times 1.13)$.

3. Having decided upon a desirable height of bracket and having made a tentative selection of the angles, as regards the length of their legs, the required thickness of the chosen angles may be computed by Eq. 35.

Applying the first method to the present example, Table 1(A) shows a resisting moment of only 80,900 in-lb for the assumed standard connection, K5. In fact, Table 1(A) indicates that, if a AISC standard connection is to be selected to resist the applied moment of 340,000 in-lb, it will require a number 10.

On the other hand, if it seems desirable to retain the assumed height of the bracket, the necessary thickness of angles may be found, in accordance with method 3, by Eq. 34. A shorter method is as follows: The strength of the connection, as indicated by Eqs. 34 and 35, increases as the square of the thickness, therefore, $\left(\frac{t}{0.4375}\right)^2 = \frac{340{,}000}{80{,}900}$; and $t = 0.897$ in.

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[10] "Structural Design in Steel," by Thomas C. Shedd, John Wiley & Sons, Inc., New York, N. Y., 1934, p. 293.

[11] "Steel Construction," AISC Manual, New York, N. Y., 1947, p. 286.

[12] "Structural Design in Steel," by Thomas C. Shedd, John Wiley & Sons, Inc., New York, N. Y., 1934, p. 305.

Concluding Remarks

Eqs. 18 througn 35 apply only to the type of connection indicated in Fig. 3(*a*); that is, any connection that is made by straddling the web of the beam or the bracket with a pair of angles, the outstanding legs of which make the connection to the column, and which are not accompanied by seat angles or clip angles. However, the same reasoning that was employed in the derivation of the aforementioned equations may be applied to the split beam type of connection, to the seat and clip angle connections, or to a combination of the latter type and the present type under consideration. In the last-named combination type of connection, the equations become rather long and tedious; but they certainly provide an improvement over the present method of laboratory tests.

One conclusion that might be drawn from the foregoing example is that the present design practice tends toward the use of weak angles in connections. It is true that, in many cases, this is deliberate, in an attempt to approach simple beam conditions. On the other hand, economy might be achieved by stiffening these connections and by taking their restraint into consideration in design. New specifications[8,9] permit additional tension in rivets. Undoubtedly, this has been brought about by the published results of research on the tensile strength of rivets.[4,5,6] On the other hand, the thicknesses of standard connection angles still remain at $\frac{3}{8}$ in. and $\frac{7}{16}$ in. Examples of angle failure in which one or both legs of the angle have been strained out of shape, with no accompanying evidence of strain in the rivets, are common.

Certain factors that contribute to the angle of strain, ϕ, have been neglected in the foregoing analysis. Within the range of permissible working loads, however, the strain in the column-connected legs of the connection angles is so great by comparison that all other factors may be considered negligible. In the interest of simplicity, in fact, the same relative assumptions have been made as is the practice in computing secondary stresses in trusses.

APPENDIX. NOTATION

The following letter symbols, used in the paper, conform essentially to American Standard Letter Symbols for Structural Analysis (ASA—Z10.8—1949), prepared by a committee of the American Standards Association, with Society representation, and approved by the Association in 1949. Subscripts *ab*, *bc*, etc., denote "end A of span AB," "end B of span BC," etc. The subscripts $A-B$, $H-K$, HH, etc., denote "of the standard connection series,[8,9] A, B, H, K, HH," etc. Symbol A denotes the fixed end of the column-connected leg of a connection angle, assumed to be at the inner rivet when there is more than one row of rivets; B is the heel end of either angle leg g or g_1; and C, the fixed end of the web-connected leg of a connection angle assumed to be at the rivet nearest the end of the beam, also, with numerical subscripts, represents constants in the general form of the slope deflection equation.

a = area of a section taken across the rivet shank; also, used as a subscript in referring to the A-end of a beam;
b = the total length of the column-connected leg of a connection angle which is used to indicate the width of the compression area between the back of the angle and the face of the column; also, b is used as a subscript in referring to the B-end of a beam;
c = a subscript used in referring to the C-end of a beam;
d = the distance from the neutral axis of a group of rivets to any given rivet of the group;
E = modulus of elasticity;
f = maximum tensile stress in a rivet due to combined shear and tension;
g = the length of the column-connected leg of the angle, measured from the center of rivet A to the near face of the web-connected leg, as shown in Fig. 1;
g_1 = the length of the web-connected leg of the angle, measured from the center of rivet C to the near face of the column-connected leg, as shown in Fig. 1;
h = the total height of the connection;
I = moment of inertia; I_o = moment of inertia of some reference section, in this case, that of the prismatic portion of the angle leg;
K = a general constant term with suitable subscripts;
$k = g_1/(4\,g + 2\,g_1)$ or corresponding term for fillet-based equations;
M = with subscript, the moment at the end of an angle leg; without subscript, the resisting moment of the connection;
N = the number of rivets connecting the angles to the column;
$n = 1/k$, the ratio of the intensity of the compressive stress below the neutral axis to the tensile stress due to bending in the outstanding leg of the angle above the neutral axis;
P = a concentrated load;
r = without subscript, the radius of the angle fillet; with suitable subscript, the deflection of the heel of the angle divided by the effective length, g, of the leg;
s = without subscript, the allowable bending stress in the legs of the connection; with suitable subscript, the stress at the indicated point;
t = the nominal thickness of the angle;
v = maximum shearing stress in a rivet due to combined shear and tension;
y = the distance from the top of the angle to the neutral axis as governed by bending in the column-connected leg of the angle;
Z = a constant such that $Z = \phi/M$ and such that $1/Z$ equals the slope of the moment-rotation curve for the connection;
Δ_b = deflection of the heel of the angle at the top of the connection;
θ = the angle of rotation of the joint, as a whole, with appropriate subscript to indicate the rotating joint; and
ϕ = the angle of strain.

DISCUSSION

Lloyd T. Cheney,[13] Jun. ASCE.—An interesting analytical approach to the problem of evaluating the rigidity of standard beam web-angle connections is presented in this paper. As stated by the author, several factors which contribute to the rotation of the connection have been neglected in the derivation of Eq. 22.

It should be noted that the comparison drawn in Fig. 4 between the experimental results of J. Charles Rathbun,[14] M. ASCE, and the slopes of the moment-rotation curves as computed by Eq. 22 can be valid only when the beam is framed to the web of a column or girder rather than to the column flanges as illustrated in Fig. 3(*a*). When the tension rivets of a web-angle connection, a top-angle and seat-angle connection, or a tee and seat-angle connection are joined with the column flanges, it is not reasonable to neglect the deformations of the column flanges. L. E. Grinter, M. ASCE, in his discussion[15] of Mr. Rathbun's paper, observed, among the data therein presented, the large contribution to the rotation of the joint that was made by "* * * the duplication of tension rivets and the deformation of the column section." Additional evidence of this column-flange distortion was observed by the writer during a program to investigate the effect of various types of building connections on the local buckling of column flanges.[16]

The question arises as to why the author has developed the analysis for these beam web-angle connections. This type of connection, of course, does offer some moment resistance but it is relatively inefficient in that respect when compared with the moment resistance obtainable with the same number of rivets in top-angle and seat-angle connections. If the engineer wishes to obtain economy in design by taking advantage of the end restraint of a connection, he would do well to employ one of the more efficient moment resistant types of connection. Robert A. Hechtman, Assoc. M. ASCE, and Bruce G. Johnston, M. ASCE, in their investigation of semi-rigid beam to column connections,[17] found the web-angle connections relatively inefficient and concluded that:

> "It was evident from the first series of web angle tests that the top and seat angle connection offered greater end restraint effectiveness and the remainder of the program was devoted to this type connection * * *."

The summary of their test results shows very strikingly the justification of this conclusion. Messrs. Hechtman and Johnston report[18] that a connection very similar to the standard KK-4 has a percentage of rigidity of 12.6 for uniform load and a load factor of 1.65 whereas a top-angle and seat-angle connection with the same number of rivets and a slightly heavier top angle has a percentage of rigidity of 31.4 under the same conditions of load. As a result of that investigation, Messrs. Hechtman and Johnston tabulated the dependable percentage of rigidity for a series of proposed standard semi-rigid connections. It may well be that in the near future these proposed connections and their experimentally determined constants will be accepted as standard.

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[13] Asst. Prof., Civ. Eng., Wayne Univ., Detroit, Mich.

[14] "Elastic Properties of Riveted Connections," by J. Charles Rathbun, *Transactions*, ASCE, Vol. 101, 1936, pp. 527 and 528.

[15] *Ibid.*, p. 571.

[16] "Steel Columns of Rolled Wide Flange Section," by Bruce G. Johnston and Lloyd A. Cheney, *Progress Report No. 1*, Committee on Steel Structures Research, AISC, 1942, p. 31.

[17] "Riveted Semi-Rigid Beam-to-Column Building Connections," by Robert A. Hechtman and Bruce G. Johnston, *Progress Report No. 1*, Committee on Steel Structures Research, AISC, 1947, p. 39.

[18] *Ibid.*, pp. 32 and 33.

In his concluding remarks Mr. Lothers comments that the same reasoning employed in the derivation of Eqs. 18 through 35 for the web-angle connections may be applied to the split-beam connection, the top-angle and seat-angle connection, etc., and that such application would constitute an improvement over the present method of laboratory tests. A connection tested by Messrs. Hechtman and Johnston is believed to be typical.[19] Since the factors considered by the author as contributing to the rotation of the connection do not involve the largest part of the total rotation as assumed, it is apparent that to ignore the contribution of the other elements will lead to results quite different from actual connection performance.

Thomas C. Kavanagh,[20] Assoc. M. ASCE.—The development, theoretically, of basic data required to realize the advantages of semi-rigid connections is a worthwhile contribution to the literature.

The need for data of this kind is evidenced in a paper by Leroy A. Beaufoy, Assoc. M. ASCE, and A. Moharram,[21] wherein the curves of M versus ϕ for standard web connections of the type used by the author have been derived from experimental data on joints connected at the flanges. Perhaps the most obvious observation in connection with this latter paper is the concern of these British investigators with the complete curve of M versus ϕ. The same was true of the Final Report of the Steel Structures Research Committee[4] wherein a general curve of the form $M = C\,\phi^{0.412}$ was proposed for several of these connections.

The paper by Mr. Lothers, in comparison, yields the initial tangent apparently with good experimental agreement. The question arises as to how adequate this initial value is for practical work. Actually, no part of the moment curve shows a linear relationship, and permanent set takes place on unloading even at low moments, which is evidence of plastic deformation. The initial tangent is then an upper limit of the stiffness, and this must be taken into account in designing with it on the semi-rigid theory.

The writer would like to question whether the fact that the connection angles are never in perfect contact with the column will not favor the assumption of the neutral axis at the center of the connection for low values of the moment, as compared with the author's procedure. Messrs. Beaufoy and Moharram [21] refer to tests showing horizontal movements on the compression side of the connection, corresponding to a "closing-in" of the initial clearance existing between the connection angle and the face of the support. Initially, this "closing-in" is almost as large as the "opening-up" on the tension side, but it disappears with larger application of moment. This is of interest, particularly because it is precisely in the low moment region that the best experimental agreement occurs.

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[19] "Riveted Semi-Rigid Beam-to-Column Building Connections," by Robert A. Hechtman and Bruce G. Johnston, *Progress Report No. 1*, Committee on Steel Structures Research, AISC, 1947, p. 99, test No. 20.

[20] Prof. of Civ. Eng., Pennsylvania State College, State College, Pa.

[21] "Derived Moment-Angle Curves for Web-Cleat Connections," by Leroy A. Beaufoy and A. Moharram, Preliminary Publication of the 3d Cong., International Assn. for Bridge and Structural Eng., Liege, Belgium, 1948.

Table 1 should be of considerable use to designers of semi-rigid structures. It is quite probable, however, that the moment M indicated therein will be higher in many practical cases than will be allowed by combined moment and shear on the standard number of rivets usually provided in these connections.

Dean F. Peterson, Jr.,[22] Assoc. M. ASCE, and Jack E. Cermak,[23] Jun. ASCE.—There can be no doubt that one of the major uncertainties in the stress analysis of building frames arises from the analyst's inability to evaluate the modifying effects of the connection on the end restraints of the members. Since 1940, fairly simple mathematical methods have been developed for introducing the effects of the strains in the connections into the analysis, and answers that consider the effects of these strains may be quite readily obtained[2,24] as far as the mathematics of analysis are concerned. The practical difficulty, however, has been that the specific elastic behavior (ratio of angle change to moment, Z) of the joint connection must be ascertained by experiment. The author has presented formulas for predicting such behavior on the basis of the physical dimensions of the angles forming the connection for a certain type of connection.

In considering the case of a beam connected to a column by a pair of vertical angles riveted to the web of the beam, the author deduces (using the slope deflection equations) an expression for the horizontal deflection, Δ_b, at the top of the connection. This deflection is expressed as a function of the pull (P) per unit vertical length of connection at the top of the connecting angles (Eq. 14). The connection is then considered analogous to a section in flexure. The angle or rotation (ϕ) and the bending moment (M) are expressed in terms of the bending stress (s) in the connecting angle, the distance (y) to the neutral axis (Fig. 3) and the physical dimensions of the angles forming the connections (see Eqs. 17 and 18.) To evaluate M and ϕ one must know s and y. Both values are statically indeterminate and some additional assumptions must be made in order to find them. In the expression for Z (Eq. 22) the value of s, however, disappears so that only y must be determined. The method for determining y is discussed under the heading, "Analysis: The Neutral Axis."

The writers experienced some difficulty in following the reasoning of the author in deriving the expression for the position of the neutral axis. The key to this derivation appears to be in the statement (four lines below Eq. 19):

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[22] Prof. and Head of Civ. Eng., Colorado Agri. and Mech. College, Fort Collins, Colo.

[23] Asst. Prof. of Civ. Eng., Colorado Agri. and Mech. College, Fort Collins, Colo.

[24] "Analysis of Frames with Elastic Joints," by Ralph W. Stewart, *Transactions*, ASCE, Vol. 114, 1949, p. 17.

> "It is assumed that the ratio of bending stress to strain in the column-connected leg of the angle along rivet line A is the same as that in compression below the neutral axis."

The writers infer that this statement, in mathematical terms, means

$$\frac{s}{\Delta_b} = \frac{s_c}{\Delta_c} \qquad \text{.............}(36)$$

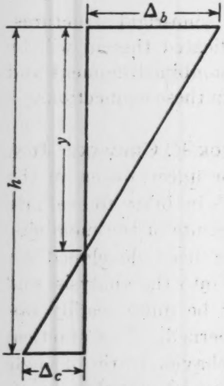

FIG. 6

in which s is the maximum bending stress in the angle; s_c is the maximum compressive stress between the column-connected angle leg and the column; and Δ_c is the corresponding strain. From Fig. 6,

$$\Delta_b = \frac{y}{h - y}\,\Delta_c \qquad \text{..............}(37)$$

Substituting from Eq. 36,

$$s = \frac{y}{h - y}\,s_c \qquad \text{.............}(38)$$

Eq. 38 states that the compressive stress at the column-connected leg is in proportion to the maximum bending stress at rivet line A as the distance from the neutral axis. As stated by the author, the total shearing force across the rivet line section A is opposed by the total compressive force below the neutral axis on the column-connected leg and, from the condition that these forces are equal,

$$\frac{v\,t\,y}{2} = \frac{s_c\,b\,(h - y)}{2} \qquad \text{.......................}(39)$$

Substituting from Eq. 38,

$$\frac{v\,t\,y^2}{2} = \frac{s\,b\,(h - y)^2}{2} \qquad \text{......................}(40a)$$

Substituting the value of s in terms of v (Eq. 19) and the value of n (Eq. 20),

$$\frac{t\,y^2}{2} = \frac{n\,b\,(h - y)^2}{2} \qquad \text{.........................}(40b)$$

which leads to Eq. 21 for y. The foregoing discussion is presented to help clarify the author's discussion leading to Eq. 21.

The writers take issue with the major premise on which the derivation of Eq. 21 is based—that is, that $\Delta_b/s = \Delta_c/s_c$. Although Δ_b is probably proportional to s, and although Δ_c is probably proportional to s_c, there is no justification for assuming that these proportions are equal. The writers would normally expect Δ_b/s to be much larger than Δ_c/s_c. It also appears illogical that the relationship between Δ_b/s and Δ_c/s_c is independent of the dimensions of the connections.

If one considers that the modulus of elasticity in bending equals the modulus in compression, an expression for y may be deduced which is not subject to the foregoing criticism. Substituting the value of P from Eq. 16 in Eq. 14, remembering $I = t^3/12$, and solving for E_b,

$$E_b = \frac{2}{3}\,\frac{s\,g^2}{\Delta_b\,t}\left(\frac{g + g_1}{2\,g + g_1}\right) \dots\dots\dots\dots \tag{41}$$

Most of the strain on the compression side occurs as the result of compression of the beam-connected leg between the rivet holes and the column, and the unit compressive strain at the extreme fiber is $\dfrac{\Delta_c}{g_1 + t}$. The modulus of elasticity in compression is expressed by

$$E_c = \frac{s'_c}{\Delta_c}\,(g_1 + t) \dots\dots\dots\dots \tag{42a}$$

in which s'_c is the maximum compressive stress in the beam-connected leg of the connecting angle.

Substituting the value of Δ_c from Eq. 37 yields

$$E_c = \frac{s'_c}{\Delta_b}\left(\frac{y}{h - y}\right)(g_1 + t) \dots\dots\dots\dots \tag{42b}$$

Equating Eqs. 41 and 42*b* yields

$$s = \frac{3}{2}\,s'_c\left(\frac{y}{h - y}\right)\left(\frac{2\,g + g_1}{g + g_1}\right)(g_1 + t)\,\frac{t}{g^2} \dots\dots\dots\dots \tag{43}$$

Substituting s from Eq. 43 into Eq. 16,

$$v = \frac{s'_c}{4}\left(\frac{y}{h - y}\right)\left(\frac{4\,g + g_1}{g + g_1}\right)(g_1 + t)\,\frac{t^2}{g^3} \dots\dots\dots\dots \tag{44}$$

Let—

$$n' = \frac{4\,g^3}{t^2\,(g_1 + t)}\left(\frac{g + g_1}{4\,g + g_1}\right) \dots\dots\dots\dots \tag{45}$$

it follows that—

$$v = \frac{s'_c}{n'}\left(\frac{y}{h - y}\right) \dots\dots\dots\dots \tag{46}$$

Since the total shear on the section through A must equal the compression in the column-connected leg below the neutral axis,

$$\frac{s'_c\,t\,y}{2\,n'}\left(\frac{y}{h - y}\right) = \frac{s'_c}{2}\,(h - y)\,t \dots\dots\dots\dots \tag{47}$$

from which

$$y = \frac{\sqrt{n'}}{1 - \sqrt{n'}}\,h \dots\dots\dots\dots \tag{48}$$

Eq. 48 is based on the proved physical property of steel, that its compression modulus is equal, or nearly so, to its bending modulus. Eq. 48, therefore, is considered to be superior to Eq. 21, which appears to be based on a rather arbitrary, if not questionable, assumption. In the development of a relationship between physical quantities, the use of any special assumption applying only to this case, indicates that either the basic principles of mechanics are not correct and are in need of restatement, or the development has been oversimplified.

The formula for Z may be simplified by substituting the value for n' into Eq. 22 to

$$Z = \frac{3}{2} \frac{g_1 + t}{E\,h\,t\,y^2} n' \quad \text{.....................} (49)$$

The values of $1/Z$ from Eq. 49 have been computed for several different sizes of connections and are listed in Tables 3 and 4 with the corresponding values obtained by the author. In all instances the values of $1/Z$ computed by Eq. 49 are definitely lower than those computed by Eq. 22.

TABLE 3.—COMPARISON OF CONSTANTS FOR STANDARD CONNECTIONS, AMERICAN INSTITUTE OF STEEL CONSTRUCTION

Connection (see Table 1) (1)	Beam web thickness (2)	$\frac{1}{Z \times 10^8}$ (Eq. 22) (3)	$\frac{1}{Z \times 10^8}$ (Eq. 49) (4)
A(a)	$\frac{5}{8}$	0.422	0.345
A(a)	$\frac{1}{2}$	0.319	0.281
A(c)	$\frac{3}{4}$	3.692	3.040
A(c)	$\frac{5}{8}$	2.888	2.455

TABLE 4.—COMPARISON OF CONSTANTS FOR PUBLISHED LABORATORY RESULTS

Specimen (1)	$\frac{1}{Z \times 10^8}$ (Eq. 22) (2)	$\frac{1}{Z \times 10^8}$ (Eq. 49) (3)
4	0.407	0.357
5	0.577	0.477
6	2.037	1.775
7	2.906	2.370

The author is to be commended for introducing a rational approach to the problem of evaluating the stiffness factors for standard connections. Introduction of this concept into structural analysis of building frames will lead to more certainty about stress conditions and perhaps greater economy in design.

J. E. LOTHERS,[25] M. ASCE.—The extent of the discussion by Messrs. Peterson and Cermak is sincerely appreciated. They carry an analysis of their own through to a final expression for Z. Also, grateful thanks are due Messrs. Cheney and Kavanagh who find the paper worthy of some careful thought, and both of whom raise some thought-provoking questions.

A comparison of the $\frac{1}{Z}$-values computed by Eq. 49 with those computed by Eq. 22 (see Tables 3 and 4) indicates that the slopes from the former are less steep; and, if plotted on the moment-rotation curves[26] proposed by Mr. Rathbun for his specimens 5 to 7, inclusive, they will fall tangent to those curves at points considerably above the origin. In the case of specimen 7, for example, the slope as computed from Eq. 49 falls tangent approximately at the 150-in-kip level. The resisting moment of the connection used in specimen 7, as computed by Eq. 31, is only 66.4 in-kips. In other words, to use Eq. 49 for design purposes in the case of specimen 7 is equivalent to permitting a bending stress of 45.3 kips per sq in. in the connection angles. Eq. 22, on the other hand, yields a slope that is tangent at approximately the 66.4-in-kip level which corresponds to a design stress of 20 kips per sq in.

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[25] Prof. of Architecture, Oklahoma Agri. and Mech. College, Stillwater, Okla.

[26] "Elastic Properties of Riveted Connections," by J. Charles Rathbun, *Transactions*, ASCE, Vol. 101, 1936, p. 539.

The writer does not mean to disparage the value of Eq. 49, however. As a matter of fact, it yields slopes that are more nearly in keeping with present design practice than does Eq. 22. In fact, a stress of 45 kips per sq in. in the connection is not considered serious. Attention is called to an illustration[27] presented by Bruce Johnston, M. ASCE, and Edward H. Mount which indicates their recommended design range for $\frac{1}{Z}$-values. It will be noted that slopes from both Eqs. 22 and 49 fall within that range.

Mr. Cheney questions the writer's judgment in selecting the web-angle type of connection for his analyses. There were two reasons for so doing: (1) The web-angle connection has been more or less standardized and, therefore, is a very commonly used connection; and (2) there is only one other type of connection for which elastic restraint equations would be justified—namely, the combined web-angle and seat-and-clip connection. The purely seat-and-clip-angle type of connection should be avoided. It is the least desirable of all the commonly used beam connections. The writer wishes to take exception to Mr. Cheney's statement to the effect that, with the same number of rivets, the top-angle and seat-angle connection offers more moment resistance than the web-angle connection. Attention is again called to the Rathbun moment-rotation curves, this time to specimens 5, 8, and 9.[28] Specimen 5 is a web-angle connection requiring twenty-four rivets. At a rotation of 0.001 radian this connection developed a resisting moment of 210 in-kips. Specimen 8 is what Mr. Cheney refers to as a top-angle and seat-angle connection. This specimen requires twenty-two rivets and at a rotation of 0.001 radian developed a resisting moment of 140 in-kips. Specimen 5 is, thus, 50% stronger than specimen 8. Specimen 9 is also a top-angle and seat-angle connection requiring twenty-seven rivets. At the same rotation as the two other specimens, specimen 9 developed a resisting moment of 160 in-kips. All three specimens were used with the same size of beam—namely, a 12 I 31.8.

The split beam type of connection, if accompanied by reinforcing plates between the column flanges (as is the practice in welded girder-column connections) is practically a rigid connection and, in the opinion of the writer, should be so treated.

The reason that the web-angle connection was chosen over the combination type for purposes of analyses is that the web-angle type is much easier to analyze. Analysis of the combination type may follow, but that for the web-angle connection will have paved the way.

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[27] "Analysis of Building Frames with Semi-Rigid Connections," by Bruce Johnston and Edward H. Mount, *Transactions*, ASCE, Vol. 107, 1942, p. 996, Fig. 1.

[28] "Elastic Properties of Riveted Connections," by J. Charles Rathbun, *ibid.*, Vol. 101, 1936, pp. 539–541.

Mr. Kavanagh poses the question as to how adequate the initial tangent is for practical design purposes. As was mentioned previously, using $\frac{1}{Z}$-values as computed by Eq. 22 is equivalent to designing the connection angles for a bending stress of 20.0 kips per sq in. This standard might be considered unnecessarily conservative by some designers. On the other hand, although the complete moment-rotation curves were available, in his examples, Mr. Rathbun chose the slope at the origin for his values of $1/Z$. Furthermore, an inspection of the Rathbun moment-rotation curves will show that the slope at the origin is approximately parallel to the average slope of the "set"[29] curves.

In answer to Mr. Kavanagh's concluding statement to the effect that the moment strengths indicated in Table 1 for the standard connections would quite probably be higher than will be allowed by combined moment and shear on the standard number of rivets usually provided in these connections, the writer wishes to refer him to the example in bracket design at the end of the paper. A standard connection was used, employing the standard number of the smaller sized rivets—namely, $\frac{3}{4}$-in. rivets. The bracket was designed to the full capacity of the rivets in combined moment and shear, and it was found that the connection angles were very much overstressed in moment.

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[29] "Elastic Properties of Riveted Connections," by J. Charles Rathbun, *Transactions*, ASCE, Vol. 101, 1936, pp. 538–544 and 564.

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ATCHAFALAYA DIVERSION AND ITS EFFECT ON THE MISSISSIPPI RIVER

BY LEO M. ODOM,[1] M. ASCE

WITH DISCUSSION BY MESSRS. E. F. SALISBURY, JAMES F. HALSEY, JERE W. HIGGS, AND LEO M. ODOM.

SYNOPSIS

The diversion of floodwaters from the Mississippi River via the Old and the Atchafalaya rivers in Louisiana is an important factor in the control of the Mississippi River floods below the point of the diversion. Diversion occurs at all stages of the river, and the only control heretofore exercised in the quantity diverted has depended on the resistance of the channel of the Atchafalaya River. These facts create a different problem from that of controlled overbank floodways like those built at Bonnet Carré.

Diversion of water from the Mississippi River to control floods has long been a disputed procedure. One of the most reiterated contentions of its opponents has been that the Mississippi River would adjust its capacity below the diversion to the lesser requirements, as the diversion increased, and that therefore there would be no net gain in total capacity.

A study of the data on discharges versus gage heights reveals that data prior to 1913 cannot be used for comparative purposes without qualification. The data collected since about 1940 must also be evaluated carefully because of changes in procedure in taking measurements.

With these facts in mind a study of the stage-discharge relationship does not indicate any progressive changes in the discharge capacity of the Mississippi River below Old River. The experiments that have been made on movable bed models of bifurcated channels give a clue as to why the normal tendency of an alluvial stream toward adjustment of its capacity to its flow requirements has not been found to operate in this case.

INTRODUCTION

The control of the Mississippi River and of its tributaries is the largest civil works project which any government of any country has ever undertaken.

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[1] Cons. Engr., Baton Rouge, La.

Works of such magnitude are involved that they dwarf any previous or contemporaneous projects of a similar nature to the extent that very few data obtained on other projects are applicable. The part of this huge undertaking assigned to the United States Mississippi River Commission (flood control and improvement of navigation on the Mississippi River below Cape Girardeau, Mo.) is the major and most important part.

Throughout practically the entire length of the Lower Mississippi River, nearly 1,000 miles, the river flows through a bed carved in its own alluvium within a broad flood plain wherein it formerly wandered at will. The funnel-shaped drainage area of the river is 1,245,000 sq miles in extent, but for the lower 600 miles of its length the river receives no major tributary.

Control of the Lower Mississippi River has been under way since 1717 when Le Blond, Sieur de la Tour, built the first levee for the protection of the City of New Orleans (La.). Flood control efforts expanded rapidly from that small beginning; but the problem has only been attacked in a scientific, organized, and adequately financed manner since the passage by the United States Congress of the Flood Control Act of 1928.

Before 1928 the paucity of funds and the lack of a central authority for planning and directing the work restricted activities to levee construction to confine the river to its channel, and to some revetment work to try to stabilize the channel in the most critical bends. Diversions and cutoffs were greatly discussed before 1927; but so much argument could be advanced on either side, because of the lack of dependable information about exactly what their effects would be, that it seemed the best policy to "let well enough alone" and follow the tried policy of building levees bigger and better. The flood of 1927 proved the inadequacy of past measures and brought cutoffs and diversions back into the picture. The act of 1928 provided sufficient funds to undertake their construction.

The most upstream point for logical diversion of water from the Mississippi is into the head of the Atchafalaya Basin, which lies approximately 300 miles by way of the Mississippi River above the Head of Passes. At this point, it is possible to divert tremendous quantities of water through undeveloped swampland to the Gulf of Mexico for the protection of the highly developed section of the valley lying in southern Louisiana below that point. Without this diversion southern Louisiana would be in far greater jeopardy than it was before the completion of levees and cutoffs upstream. While protecting upstream areas, the levees and upstream cutoffs robbed the river of natural storage basins and hastened the flow, thus increasing flood peaks in the lower reaches.

The Atchafalaya Basin (see Fig. 1) is a huge lowland lying west of the Mississippi River in south central Louisiana. It is connected with the Mississippi and Red rivers through the Old River and the Atchafalaya River and empties into the Gulf of Mexico. For thousands of years this basin has served as an outlet for overflow waters from the Mississippi and Red rivers.

The Atchafalaya River, which is leveed on each side for practically its entire length, is a short stream which has its source in Old River and flows as a well-defined stream in the middle of the basin for about half the length of the basin. It is then lost in a maze of deltaic channels which flow into the lakes at the lower end of the basin. These lakes are connected with the Gulf of Mexico through the Teche Ridge, which otherwise closes the lower end of the basin, by the Lower Atchafalaya River, and by an artificial channel constructed by the Mississippi River Commission called the Wax Lake Outlet.

The Mississippi River Commission and its predecessors in flood control work on the Mississippi have always recognized the importance of the Atchafalaya Basin as a route for diversion. Beginning with the passage of the Flood Control Act of 1928, plans have been prepared and great works have been constructed with the objectives of utilizing to the fullest practicable extent the diversion possibilities offered by the Atchafalaya Basin and, at the same time, of protecting the fertile lands and prosperous communities lying adjacent to it on each side and in its upper reaches.

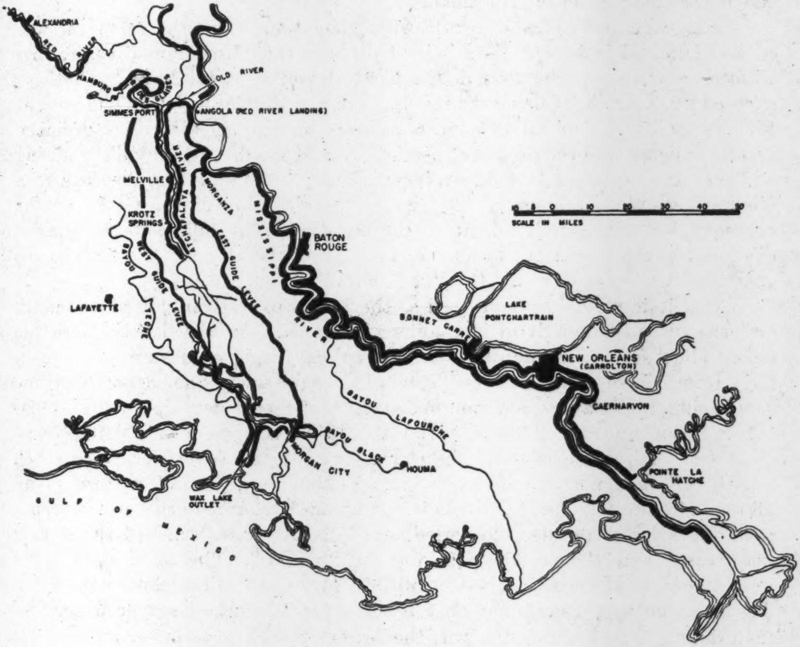
ALEXANDRIA
RED RIVER
HAMBURG
BAYOU DES GLAISES
OLD RIVER
SIMMESPORT
ANGOLA (RED RIVER LANDING)
MELVILLE
KROTZ SPRINGS
ATCHAFALAYA RIVER
MORGANZA
EAST GUIDE LEVEE
MISSISSIPPI RIVER
BATON ROUGE
WEST GUIDE LEVEE
BAYOU TECHE
LAFAYETTE
10 5 0 10 20 30 40 50
SCALE IN MILES
LAKE PONTCHARTRAIN
BONNET CARRE
NEW ORLEANS (CARROLTON)
CAERNARVON
BAYOU LAFOURCHE
BAYOU BLACK
HOUMA
MORGAN CITY
WAX LAKE OUTLET
POINTE LA HATCHE
GULF OF MEXICO

Fig. 1.—Atchafalaya Basin

The Atchafalaya River originally carried relatively little flow because of the poor development of its channel and the fact that it was blocked near its head by a raft of logs and debris. Improvement of its channel for navigation was begun by the State of Louisiana before 1850. More recently definite steps have been taken by the Mississippi River Commission to increase its discharge capacity so as to alleviate the flood heights on the Mississippi River.

GEOLOGY OF THE LOWER MISSISSIPPI AND ATCHAFALAYA RIVERS

The Atchafalaya Basin lies wholly within the deltaic plain of the Mississippi River. The Lower Mississippi River flows through a broad low area from Cape Girardeau to the Gulf of Mexico. This lowland (known as the Mississippi alluvial valley) slopes toward the Gulf and is bordered by abrupt escarpments for almost its entire length. It lies within the Central Gulf Coastal Plain. The deltaic plain of the Mississippi alluvial valley comprises the part of the valley below the head of the Atchafalaya River. It is separated from the flood plain area to the north by the natural levees of Bayou des Glaises and Old River (see Fig. 1) in Louisiana.

During the last stage of world-wide glaciation when the sea level was several hundred feet lower than it is at present, the Mississippi River and its tributaries cut deep channels in the Central Gulf Coastal Plain. The area effected by these channels marks the limits of the alluvial valley. The present Mississippi River evolved through a series of changes in the valley drainage system which were brought about by the filling of the deeply cut valleys during and since the postglacial rise in sea level. The present alluvial valley has been filled by alluvium because of the rise in the elevation of the Gulf and the consequent decrease in gradient of the streams. The filling of the ancient channels, however, did not bring the elevation of the alluvial valley up to the general elevation of the Gulf Coastal Plain.

The deltaic plain was developed as the Mississippi River shifted its mouth and the place of deposit of its sediment changed. In the deltaic plain the recent alluvium reaches depths greater than 350 ft and grades upward irregularly from coarse "gravelliferous" sands to progressively finer deposits of fine sands, silts, and clays. The minimum depth of recent deposits in the Atchafalaya Basin is not much less than 100 ft and the average depth is about 150 ft.

Slope, load, discharge, and the character of the materials forming its bed greatly affect the length, average cross section, and general pattern of an alluvial stream. When the determining factors have remained within reasonable limits for a considerable period, over-all cutting or filling of the stream bed ceases and the stream is said to be "poised." The Mississippi River long ago reached such a poised condition from Cape Girardeau to the Gulf. The local cutting and filling that occur do not affect the over-all picture materially. Below the Old River, the Mississippi River is much more stable than in its upper reaches. The channel is narrower and deeper, high-water slopes are much less, and the bed materials are much more resistant to erosion. The channel is very efficient and it carries its large burden of discharge with very small slopes to the Gulf.

The Atchafalaya River, although offering a much shorter route, has a far smaller cross section, and the resistance of its channel to flow is much greater. However, it flows through erodible material and its average cross section throughout its leveed length has been constantly increasing for many years. This enlargement of the leveed channel has been accompanied by the silting up of the lakes and swamps between the ends of the leveed channel and the lower end of the basin.

HISTORY OF THE ATCHAFALAYA RIVER

The Atchafalaya River was a distributary of the Mississippi River when the region was first visited by European explorers. Geological studies show that it became a distributary long before that time when the lower part of the former Turnbull Bend on the Mississippi River, in Louisiana, caved into an abandoned course of the Red River. This old Red River channel provided a connection from the Mississippi River into the low area that is now called the Atchafalaya Basin. The Atchafalaya River developed slowly below the old channel by forming small distributaries of typical deltaic pattern during each flood and then in later floods choosing one of them to use as its main channel as its delta was pushed out into the basin area.

At the time that the Atchafalaya River was formed, the Red River was following its present course and joined the Mississippi River in the upper limb of the old Turnbull Bend. The head of the Atchafalaya River was about 3 miles farther downstream on the Turnbull Bend. This situation existed until 1831.

In 1831 a cutoff was made in the Mississippi River across Turnbull Bend by Capt. Henry Shreve. As a result of this cutoff the lower limb of the bend, now called the Old River, has become the connection between the Mississippi and the Atchafalaya rivers, and the western part of the old bend forms an extension of the Red River, so that today the Old, the Atchafalaya, and the Red rivers join at a common point. For many years after the cutoff was completed, the upper limb of Turnbull Bend was the principal channel connecting the Red River with the Mississippi River, but it has now silted up.

For many years development of the Atchafalaya River was greatly impeded because the river was blocked by an extensive raft of logs and brush thrown into it by the Mississippi. In 1839 it could be crossed on foot on the raft near its head. The State of Louisiana began work toward the removal of this raft in 1840 so as to create a navigable channel. The reports of the state engineers of Louisiana from 1840 to 1875 contain frequent references to this work.

There was considerable argument about 1847 as to whether the Atchafalaya River should be dammed off. A strong plea for leaving it open, together with the statement that it carried 120,000 cu ft per sec of water during high floods, appeared in 1875.[2]

In 1879 the Mississippi River Commission was created with the duty of preparing plans to improve the Mississippi River channel, to protect its banks, to improve navigation, to prevent destructive floods, and to promote and facilitate commerce. About the same time large levee districts were created in Louisiana and the state engineers turned their attention almost wholly to the task of constructing levee systems along the major streams to protect the alluvial lands from overflow.

The Mississippi River Commission was thus placed in the middle of the argument as to the disposition to be made of the Atchafalaya River as the sole remaining responsible agency in the field. The original policy of the Mississippi River Commission with reference to the Atchafalaya River was summarized in its annual report for 1884 as follows:

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[2] *House Document No. 127*, 43d Cong., 2d Session, January 16, 1875.

> "The situation at the mouth of Red River is one which has engaged the attention of the Commission from the beginning of its work ***.
>
> "It is considered that any plan for this work ought to comprehend the prevention of the divergence of the Mississippi River into the Atchafalaya Basin, the closure of any depleting outlet at that point, either existing now or likely to be induced by changes which are reasonably to be anticipated, and the preservation of the navigation of the Red and Atchafalaya rivers."

It was stated in the 1884 report that practically all the flow in the Atchafalaya River came from the Red and the Black rivers and from overflow of the Mississippi River from crevasses above the Old River. The Old River was not enlarging and instead seemed to be filling up.

Efforts were made by the Mississippi River Commission to control the enlargement of the Atchafalaya River and to improve the channel in the Old River for navigation in 1889 and subsequent years in line with the policy adopted in 1884. The channel in the Old River was dredged for many years and finally became self-maintaining. Sill dams were placed across the Atchafalaya River in 1889 near its head. This control section was maintained for about 35 years. During the life of the sill dams the river below the control section enlarged greatly. This enlargement was a result of: (1) Extending the levees southward on both banks of the Atchafalaya River to confine flood flows; (2) improving levees above and below the Old River on the Mississippi River which increased gage heights and, therefore, the energy available for scour; and (3) constructing levees on the south banks of the Old River and of the Bayou des Glaises which prevented overbank flow into the basin and forced the flow down the Atchafalaya River.

The levees along the Atchafalaya River were extended gradually through the years, more or less keeping up with the development of the channel. The progress of the levee extensions is shown in Fig. 2.

The Flood Control Act of 1928, based on the recommendations of the Chief of Engineers contained in the so-called Jadwin Plan, provided for a change in policy of the federal government toward the Atchafalaya River. The Jadwin report stated that it was clearly desirable that the Atchafalaya River be utilized to the extent of its capacity at flood stages to carry waters to the Gulf, and inferred the desirability of the removal of the sills previously constructed near its head. This report also provided for the overbank floodways into the Atchafalaya Basin east and west of the Atchafalaya River.

Study of the Atchafalaya River led the president of the Mississippi River Commission to the conclusion that, without some help, the river would not develop very rapidly, if at all, the relief capacity necessary to secure flood protection below the Old River as proposed by the Jadwin Plan. The swamp section between the end of the river proper and the lakes offered considerable resistance to flow. This part of the basin is largely composed of stiff "backswamp" clays and the current was not capable of scouring an efficient channel in the material. Sedimentation was lessening the flow capacity of this section and causing the gage heights at the lower end of the river levees to increase for equal discharges.

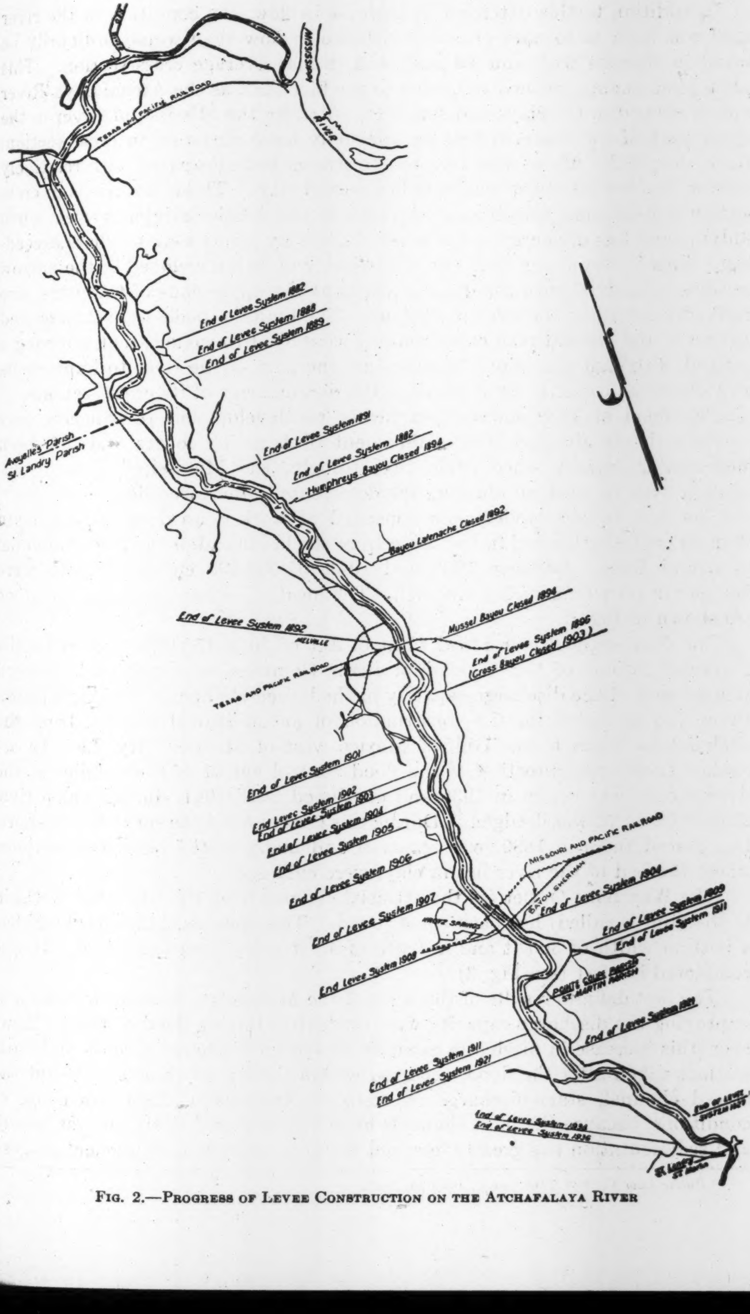
RED RIV
MISSISSIPPI
RIVER
TEXAS AND PACIFIC RAILROAD
End of Levee System 1882
End of Levee System 1888
End of Levee System 1889
Avoyelles Parish
St. Landry Parish
End of Levee System 1891
End of Levee System 1882
Humphreys Bayou Closed 1894
Bayou Latenache Closed 1892
End of Levee System 1892
MELVILLE
Mussel Bayou Closed 1894
End of Levee System 1896
(Cross Bayou Closed 1903)
TEXAS AND PACIFIC RAILROAD
End of Levee System 1901
End Levee System 1902
End of Levee System 1903
End of Levee System 1904
End of Levee System 1905
End of Levee System 1906
MISSOURI AND PACIFIC RAILROAD
BAYOU SHERMAN
End of Levee System 1904
End of Levee System 1907
KROTZ SPRINGS
End of Levee System 1909
End of Levee System 1911
End Levee System 1908
POINTE COUPEE PARISH
ST MARTIN PARISH
End of Levee System 1921
End of Levee System 1911
End of Levee System 1921
END OF LEVEE SYSTEM 1927
End of Levee System 1934
End of Levee System 1936
St. Landry
St. Martin

FIG. 2.—PROGRESS OF LEVEE CONSTRUCTION ON THE ATCHAFALAYA RIVER

In addition to this deterrent to increase in flow, the condition of the river itself was such as to offer greater resistance to flow than would ordinarily be found in streams with alluvial beds and similar average cross section. This latter phenomenon seemed to be due to the fact that, as the Atchafalaya River had developed in the clays and sands deposited by the Mississippi River in the upper part of the basin, it had an unusually large variation in cross section. Wide, deep holes where sand formed the stream bed alternated with relatively narrow, shallow stretches where its bed was in clay. These variations in cross section caused high velocities to develop in the smaller stretches with much eddying and loss of energy in the pools. The clay layers were nowhere exceedingly thick in the upper river and the velocity of flow developed in the narrow sections, combined with the digging action at the upper ends of the pools, was gradually enlarging the narrow sections. The limits of pools were thus moved upstream and the old pool cross sections were filled to some extent, showing a natural, although very slow, tendency on the part of the river to improve its own discharge capacity by decreasing the dissimilarity of its cross sections.

The flood of 1932 showed practically no development in capacity over previous floods although the confinement of flows by levees had not been increased materially since 1927. Accordingly, the Mississippi River Commission went to work eliminating the deterrents to increased flow.

The first project undertaken consisted of work toward creating a main channel centrally located in the basin from the lower end of the leveed channel to Grand Lake. Between 1932 and 1941, 91,000,000 cu yd of earth were dredged in providing such a channel. The locations where dredging was done are shown in Fig. 3.

The Mississippi River Flood Control Act of June 15, 1936,[3] based on the recommendations of the Mississippi River Commission, specifically required enlargement of the discharge capacity of the leveed channel of the Atchafalaya River and provided for the construction of an additional channel from the Atchafalaya Basin to the Gulf of Mexico west of Morgan City, La. In accordance with the directive of the flood control act of 1936 dredging in the river proper was begun in 1936 and continued until 1941, during which time 20,000,000 cu yd was dredged. During this latter period the sill at Simmesport, La., placed there in 1889, was removed and many of the restricted sections, where the bed of the river lies in clay, were enlarged.

The Wax Lake Outlet (as the artificial channel from the Atchafalaya Basin to the Gulf is called) has been constructed. This channel, 115 miles long, has a bottom width of 300 ft and a depth of 45 ft below mean sea level. It was completed in 1941 (see Fig. 3).

The first definite results of the work of the Mississippi River Commission in improving the discharge capacity were evident in the big flood of 1945. However, this increase in discharge capacity, based on discharge records and cross sections taken after the flood, occurred within the leveed channel. Below the leveed channel, stage-discharge relationships indicate a slight worsening of conditions, because the pilot channels have not increased in size as was hoped, and sedimentation has greatly lessened the cross section at equivalent stages.

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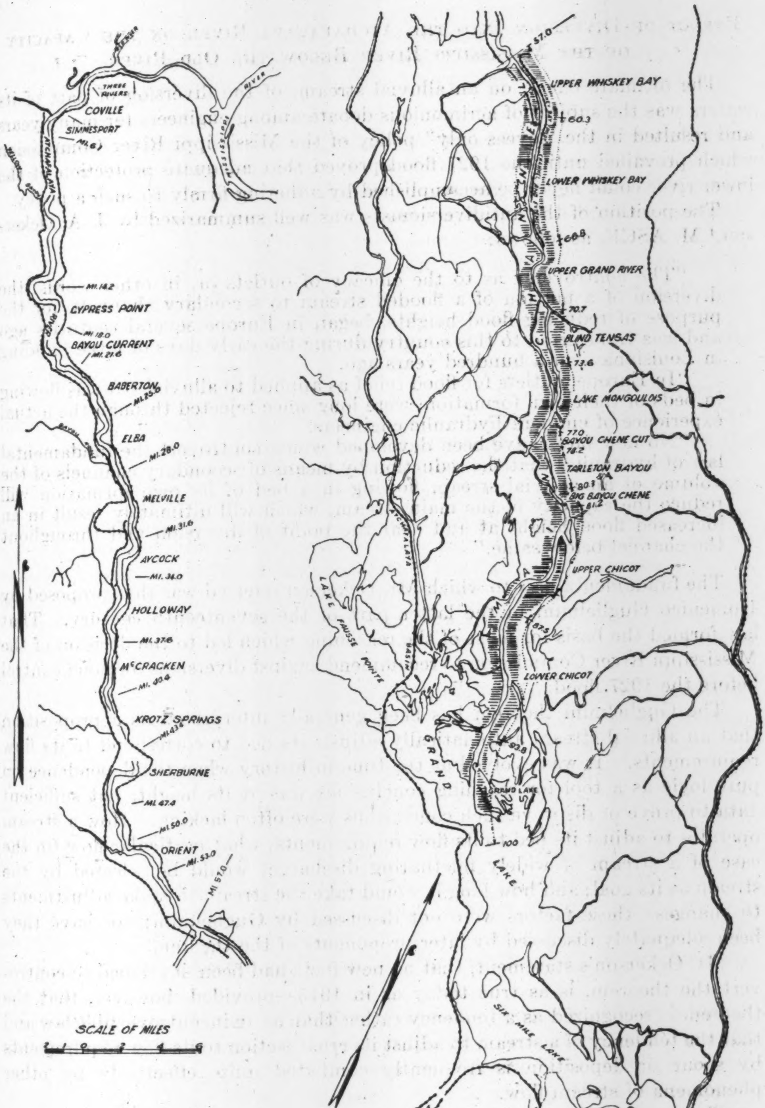
[3] *Public Law No. 678*, 74th Cong., June 15, 1936.


THREE RIVERS
COVILLE
SIMMESPORT
MI. 6.1
ATCHAFALAYA
MISSISSIPPI RIVER
MI. 14.2
CYPRESS POINT
MI. 18.0
BAYOU CURRENT
MI. 21.6
BABERTON
MI. 25.0
ELBA
MI. 28.0
MELVILLE
MI. 31.6
AYCOCK
MI. 34.0
HOLLOWAY
MI. 37.6
McCRACKEN
MI. 40.4
KROTZ SPRINGS
MI. 43.6
SHERBURNE
MI. 47.4
MI. 50.0
ORIENT
MI. 53.0
MI. 57.0
57.0
UPPER WHISKEY BAY
60.3
LOWER WHISKEY BAY
66.6
UPPER GRAND RIVER
70.7
BLIND TENSAS
73.6
LAKE MONGOULOIS
77.0
BAYOU CHENE CUT
78.2
TARLETON BAYOU
80.1
BIG BAYOU CHENE
81.2
UPPER CHICOT
87.5
LOWER CHICOT
93.8
GRAND LAKE
100
LAKE FAUSSE POINTE
ATCHAFALAYA BASIN
GRAND LAKE
SIX MILE LAKE
WAX LAKE OUTLET
PATTERSON
MORGAN CITY
SCALE OF MILES
0 2 4 6

LEGEND

45 Distance in Miles Below Head of Atchafalaya River

ORIENT Name and Limit of Reach in Which Dredging Was Done

FIG. 3.—MAIN CHANNEL DREDGING, ATCHAFALAYA BASIN

EFFECT OF DIVERSION INTO THE ATCHAFALAYA RIVER ON THE CAPACITY OF THE MISSISSIPPI RIVER BELOW THE OLD RIVER

The ultimate effect, on an alluvial stream, of the diversion of part of its waters was the subject of acrimonious debate among engineers for many years and resulted in the "levees only" policy of the Mississippi River Commission which prevailed until the 1927 flood proved that adequate protection of the lower river could never be accomplished by adhering firmly to such a policy.

The position of the antidiversionists was well summarized by J. A. Ockerson,[4] M. ASCE, as follows:

> "The controversy as to the efficacy of outlets or, in other words, the diversion of a portion of a flooded stream to secondary channels, for the purpose of reducing flood heights, began in Europe several centuries ago and was transferred to this country during the early days of levee building in Louisiana over a hundred years ago.
>
> "In Europe, outlets for flood relief as applied to alluvial streams flowing in beds of their own formation, were long since rejected through the actual experience of eminent hydraulic engineers.
>
> "No new facts have been developed which controvert the fundamental law of hydraulics that the reduction by means of secondary channels of the volume of an alluvial stream flowing in a bed of its own formation will reduce the capacity of the main stream, which will ultimately result in an increased flood height at and near the point of diversion and throughout the channel below same."

The fundamental law to which Mr. Ockerson referred was that proposed by Domenico Guglielmini in the latter part of the seventeenth century. That law formed the basis for most of the reasoning which led to the decision of the Mississippi River Commission to recommend against diversions for flood control before the 1927 flood.

The Guglielmini theorem has been generally interpreted as a proposition that an alluvial stream automatically adjusts its bed to correspond to its flow requirements. It was evolved at the time in history when the dependence on pure logic as a tool for reaching conclusions was at its height, but sufficient data to prove or disprove such conclusions were often lacking. How a stream operates to adjust its bed to its flow requirements; what particular flow (in the case of a stream of widely fluctuating discharge) would be selected by the stream as its goal; and how long it would take the stream to make adjustments to changes—these factors were not discussed by Guglielmini; nor have they been adequately discussed by later proponents of the theorem.

Mr. Ockerson's statement, that no new facts had been developed to controvert the theorem, is as true today as in 1915—provided, however, that the theorem is recognized as a tendency rather than as an incontrovertible law and that the tendency of a stream to adjust its cross section to its flow requirements by scour or deposition is frequently combated quite effectively by other phenomena of stream flow.

The main reason why the action of an alluvial river is so little understood has been expressed succinctly by D. O. Elliott,[5] M. ASCE, as follows:

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[4] *Professional Memoirs*, by J. A. Ockerson, Corps of Engrs., U. S. Army, Engr. School, Washington Barracks, D. C., September–October, 1915.

[5] "The Improvement of the Lower Mississippi River for Flood Control and Navigation," by D. O. Elliott, U. S. Waterways Experiment Station, Vicksburg, Miss., 1932.

> "The power of running water to entrain and transport solids of various degrees of comminution and density, remains, after a century and a quarter of investigation a phenomenon for which no generally recognized laws have been deduced."

It can be stated indisputably that, in spite of an additional 16 years of much more intensive investigation, generally recognized laws are yet to be deduced.

To dispute or to defend, scientifically, any theory about the action of an alluvial river, the basic requirement would be the establishment of general laws, incontestably correct, covering the basic actions of a stream in picking up and handling solids. Therefore, it would not seem to be worthwhile to waste much time disputing as to whether such theories should be accepted. An engineer, being practical, would undoubtedly require some dependable information about angels before he would argue about how many of them could stand on the point of a pin.

In permitting and encouraging the diversion of water from the Mississippi River into the Atchafalaya River, the Mississippi River Commission was not acting on theory but on necessity, since it appeared to be impossible for the main river to be confined sufficiently so that it would carry large floods to the sea unaided. An analysis of the available data has been made (without reference to general laws or theories) to determine whether the Mississippi River has lost capacity as the Atchafalaya River has gained capacity, as the "Guglielminists" have forecast for generations. If the main river did lose capacity as the diversion increased, or if such an occurrence seems probable, the navigation channel in the Mississippi River will soon be a thing of the past and the inhabitants of the valley below the Old River are in greater jeopardy from floods than they ever were.

The factors which are known to influence the capacity of a stream are its cross-sectional area, its length, the shape of its cross section, the roughness of the channel, the variability of its cross section, and its sinuosity. Changes in capacity are influenced by the character of the bed and banks of the stream and its tributaries, the variation in flow, and the existence of diversionary channels. Adequate data for a comparative study through the years of each of these elements are not available. Such data as have been collected must be scrutinized very carefully to insure that a true comparison of the various features is secured. For instance, in regard to cross sections, ranges selected as perpendicular to the river during one year are frequently not so in succeeding years and if not changed will invariably indicate increases which have not occurred.

In taking cross sections of a large stream the technique of the leadsman and the weight of the lead introduce possibilities of variation in results, which can obviate the value of the most scientific line of reasoning. Modern practice in determining depths with a supersonic echo sounding device undoubtedly provides the truest picture of the cross section that has ever been available; but, when conclusions are to be drawn from comparisons with data collected by the older practices, it is quite possible that these comparisons will lead to somewhat erroneous results.

The measurement of flow in rivers has also been subject to great improvements in technique and equipment, and again the personal equation may result in considerable error. All the older measurements were made with floats. These are now known to be very inaccurate. Metered measurements, particularly during high stages, are subject to errors and are greatly affected by the weight used with the meter and by the number of readings taken in a cross section where practice has been known to vary considerably through the years. It would appear, therefore, that too much credence cannot be given to small apparent variations and that great refinement in the comparison of available data would be useless.

Analysis of Stage and Discharge Measurements

Various studies have been made in the past in an attempt to determine whether the Mississippi River has lost capacity because of the Atchafalaya diversion. Of these, the method of analysis used by E. F. Salisbury,[6] M. ASCE, is apparently the least troublesome method and should give nearly correct answers. Mr. Salisbury assumed that the gage height which will be reached at a selected gaging station, when passing a given discharge, is a measure of the resistance to flow of the channel from that gaging station to the sea. The gage heights when this discharge was reached, compared over a number of years, should provide the answer as to whether the capacity of the channel is changing.

E. W. Lane,[7] M. ASCE, in an excellent discussion of Mr. Salisbury's paper, presented several graphs showing the stages at Red River Landing at several selected discharges for the years from 1882 to 1930, inclusive, and on each of these graphs of stages he drew a mean line which he considered would indicate the trend of the capacity of the river.

The stage graphs presented by Mr. Lane for four discharges—800,000 cu ft per sec, 900,000 cu ft per sec, 1,000,000 cu ft per sec, and 1,100,000 cu ft per sec—were prepared from available current meter measurements between 1,150,000 cu ft per sec and 750,000 cu ft per sec. The stages for each discharge within 50,000 cu ft per sec of the even 100,000 cu ft per sec were adjusted to the nearest even 100,000 cu ft per sec by using the discharge increment corresponding to that discharge. All measurements that might have been affected by crevasses were eliminated, except those of 1882. The adjusted stages were plotted against their respective years for each of the four discharges. The mean of the adjusted stages for each discharge in each year was plotted. Two other sets of elevations were also plotted showing, respectively, the lowest adjusted stage and the highest adjusted stage in respective years. Above each set of points the number of observations on which the points were based was given. For each discharge Mr. Lane drew a straight line on the graph to show the general trend through the years.

Using Mr. Lane's method, the writer has extended those graphs of stages plotted against selected discharges to 1945, as shown in Fig. 4. The trend

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[6] "Influence of Diversion on the Mississippi and Atchafalaya Rivers," by E. F. Salisbury, *Transactions*, ASCE, Vol. 102, 1937, p. 75.

[7] *Ibid.*, p. 109.

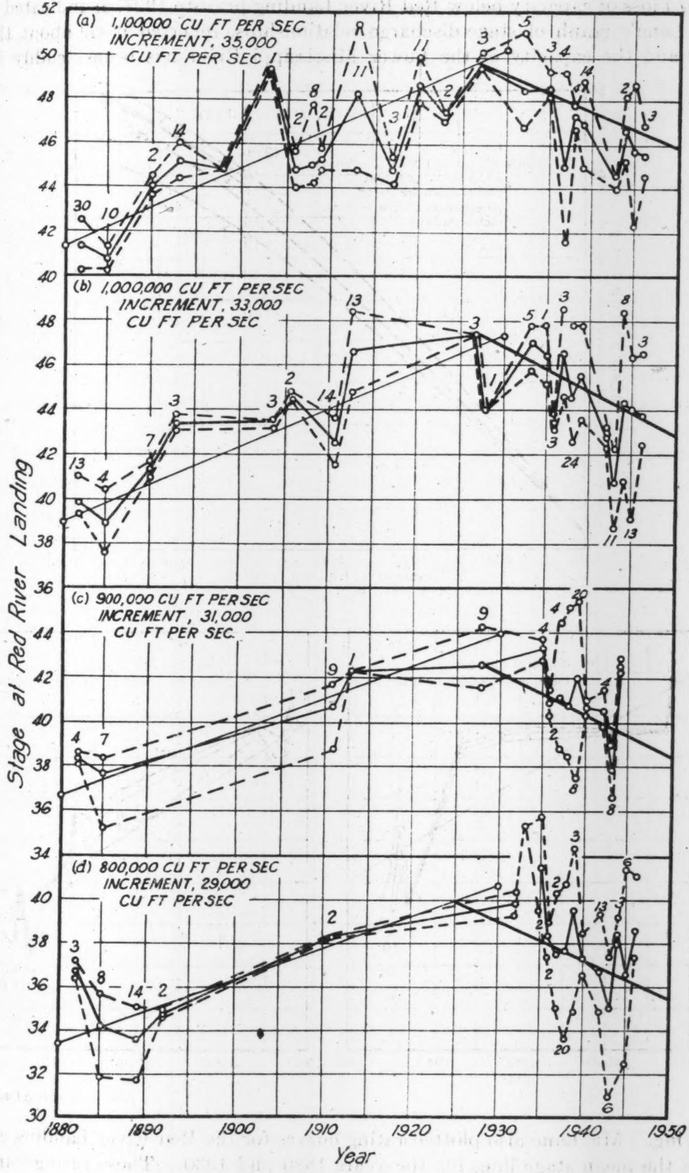


FIG. 4.—STAGE OBSERVATIONS AT RED RIVER LANDING

toward loss of capacity below Red River Landing prior to 1927, as indicated by Mr. Lane's graph of stage-discharge relationships, reversed itself about that time and the capacity of the Lower Mississippi River has been steadily improving. Mr. Lane also plotted rating curves for the Red River Landing gage from the mean stage lines for the years 1880 and 1930. These rating curves indicated a loss in capacity of 250,000 cu ft per sec at a stage of 40 ft.

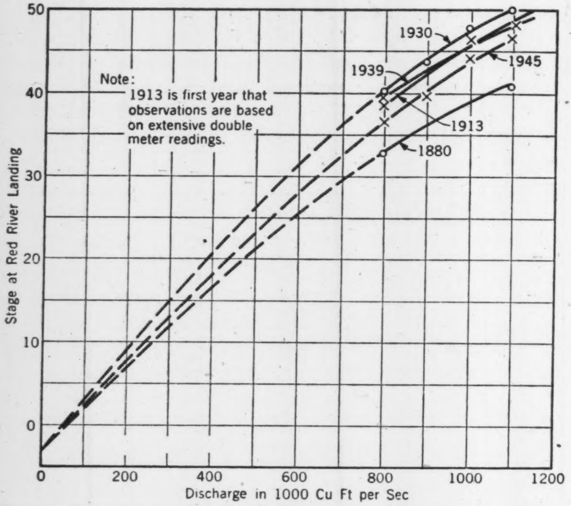


FIG. 5

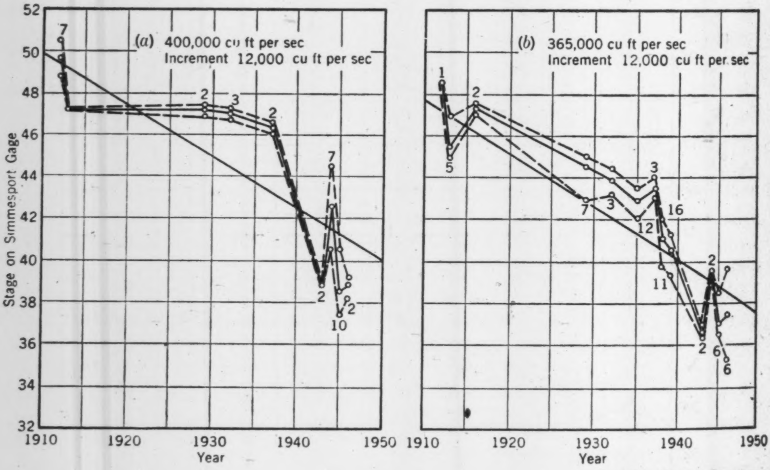


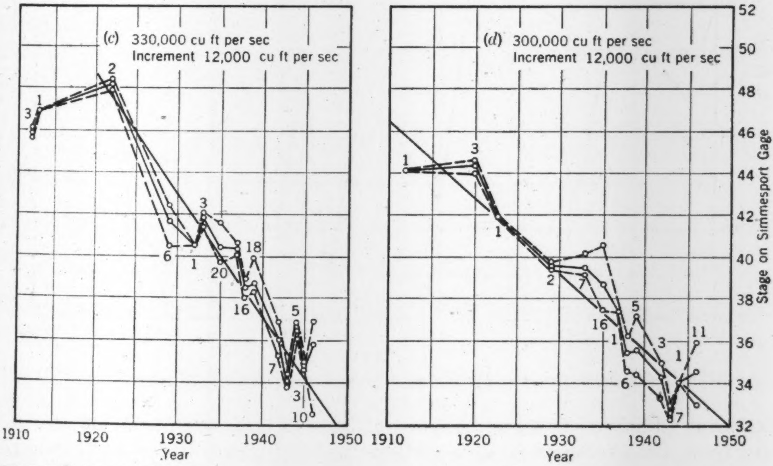
FIG. 6.—STAGE OBSERVA

The writer has added to Mr. Lane's curves rating curves plotted from the mean lines in Fig. 4 for 1913, 1939, and 1945, as shown in Fig. 5. The year 1913 is the earliest in which a sufficiently wide range of double-metered measurements is available to justify comparison with later years. The years 1939 and 1945 were selected because of the large number of measurements available within the range of the selected discharges.

A gain of 100,000 cu ft in discharge capacity at a stage of 40 ft is indicated by the rating curves between 1930 and 1945. The 1913 and 1939 curves lie quite close together, about midway between the 1930 and 1945 curves. These curves would indicate that the capacity decreased from 1913 to 1930 and then gradually increased to 1945.

Graphs of the stages at Simmesport on the Atchafalaya River for the years 1913 to 1945, inclusive, for discharges of 300,000 cu ft per sec, 330,000 cu ft per sec, 365,000 cu ft per sec, and 400,000 cu ft per sec were plotted from data prepared in accordance with Mr. Lane's method. Equivalent discharges at much lower gage readings show conclusively the great increase in the capacity of the Atchafalaya River during that period. These graphs are shown in Fig. 6.

Reference to Fig. 6(*a*) will show that gage heights for a flow of 400,000 cu ft per sec have decreased 8 ft between 1910 and 1945. Figs. 6(*b*), 6(*c*), and 6(*d*), show the decrease in stage for the respective discharges analyzed. The mean stage lowering at the four discharges is 0.37 ft per yr. In 1937, 471,000 cu ft per sec passed Simmesport on the Atchafalaya River at a gage height of 55.0 ft above mean Gulf level (MGL). In 1945 three discharge measurements (April 27, 28, and 30) at a stage of 57.2 ft MGL were obtained. These were 637,000 cu ft per sec, 661,000 cu ft per sec, and 645,000 cu ft per sec. If the peak flow in 1945 is considered to have been 640,000 cu ft per sec, an increase



TIONS AT SIMMESPORT GAGE

in flow of 36% with a stage increase of only 4% occurred between 1937 and 1945. In 1945 the peak discharge of the Atchafalaya River at Simmesport was 43.7% of that of the Mississippi River at Red River Landing on the same date, as compared to 32.2% in 1937. The measured discharges at Red River Landing on the dates of the peaks at Simmesport in 1937 and 1945 were practically equivalent.

Most of the data collected before 1913 are believed to be useless for comparison with later data. However, the period from 1913 to 1945, inclusive, provides a range of 33 years which should be long enough to determine if a trend exists. The only possible conclusion from the data is that the Mississippi River has not lost capacity as the Atchafalaya enlarged; actually it has more capacity today than it had in 1913, in spite of the fact that the Atchafalaya River is taking a much greater proportion of flood flows. Because this conclusion is exactly opposite to that reached by most previous students of the subject, several of these prior studies have been analyzed, and reasons sought to discover why this apparent increase in capacity has occurred despite the previous findings.

Variation in Annual Flow Reaching the Mouth of the Old River

A factor to be considered in the study of the effect of the diversion into the Atchafalaya River concerns the question of whether the annual runoff reaching the Mississippi during the period of flow has changed materially. If sufficient additional flow had become available to take care of the increase in diversion, the average flow in the Mississippi River past Red River Landing might remain the same as in the past. The amount and concentration of runoff are dependent on meteorological conditions. Floods occur as a result of general, intensive rains over the basins of the various tributaries, spaced in time so as to cause the flood runoff of each tributary to reach the main stem as

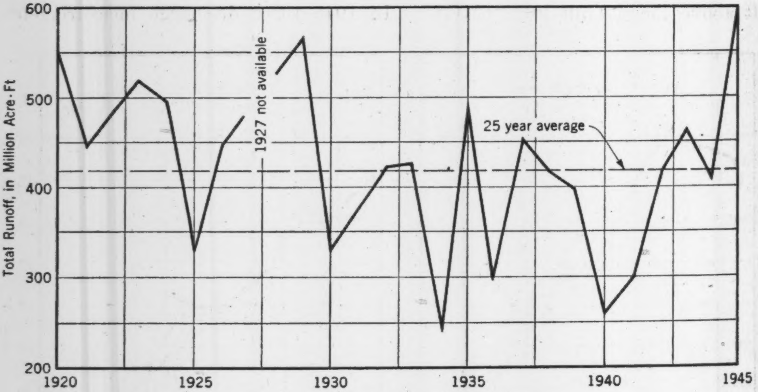


Fig. 7.—Mississippi Hydrograph at Vicksburg, Miss., 1920–1945

its flood discharge is passing that point. Man-made regulatory works on the main stem and tributaries are known to affect the timing of the concentration of runoff only slightly as far as the lower river is concerned. Fig. 7 shows the hydrograph of the Mississippi River at Vicksburg, Miss., from 1920 to 1946. The ordinate is the runoff in millions of acre-feet. (Flow entering the Mississippi River between Vicksburg and the Old River is only a small fraction of the total flow, and has been disregarded.) An inspection of Fig. 7 shows that the Mississippi River Basin experiences wet years and dry years, but the annual average volume of flow reaching the mouth of the Old River has changed very slightly.

Accuracy of Flow Measurements

The period covered by this study has seen a gradual continuing effort made by the Corps of Engineers, United States Department of the Army, and the United States Geological Survey to improve stream measuring procedures. At first, depths were measured with chains and velocities were obtained by surface floats. Depths are now measured by Fathometers; and velocities are measured by Price current meters, at one or more depths.

The writer does not question that the latest methods are the best, but he does question the unadjusted comparison of data obtained by early methods with data obtained by the latest methods. Some of the apparent decrease and the subsequent increase in discharge capacity at Red River Landing may be attributable to the comparison of data obtained by different techniques.

TABLE 1.—Stage-Discharge Relationships at Baton Rouge, La.

(Units Are in Cubic Feet per Second)

Stage (ft)	1939	1945	Change
32	895,000	975,000	+80,000
34	950,000	1,043,000	+92,000
36	1,012,000	1,106,000	+96,000
38	1,077,000	1,171,000	+94,000
40	1,140,000	1,236,000	+96,000

To try to answer this question, a gaging station in a relatively stable regimen, with minute changes below it, was sought. The discharge range at Baton Rouge, La., was thought to be best for the following reasons:[8]

> "Bed of sand clay, and silt. High-water control is affected by levee on right bank, and by high ground and levee on left bank. Stage and discharge are affected by diversion into Atchafalaya River through Lower Old River, and during low stages by tides of the Gulf of Mexico. Stage-discharge relation is fairly consistent for high and medium stages, but is affected by tides during low stages."

A comparison of stage-discharge relationships through the years at Baton Rouge should reflect any changes in results stemming from changes in stream measuring procedures. Using only meter measurements, the 1939 and 1945 stage-discharge graphs were plotted. A comparison of these years is given in Table 1. An apparent increase in discharge capacity is shown at each stage ranging from 8.9% at 32 ft to 8.4% at 40 ft.

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[8] "Stages and Discharges, Mississippi River and Its Outlets and Tributaries, 1946," Mississippi River Comm., Vicksburg, Miss., p. 51.

Inasmuch as the Mississippi River regimen downstream from Baton Rouge is considered by nearly everyone to be relatively stable, it must be concluded that most of the apparent increase in discharge capacity of the river at that point is due to the "improvements" in gaging procedures which are known to have been instituted since 1940.

DREDGING BELOW THE MOUTH OF THE OLD RIVER

It will be noticed that Fig. 4 shows the greatest improvement in channel capacity at Red River Landing after about 1938. Prior to that time,[9] a sand bar having an elevation of + 10 MGL extended from mile 299.7 to mile 298.8 above Head of Passes (AHP). Also, at Smithland, La., 3 miles below the mouth of the Old River opposite the Old Raccourci Cutoff, a towhead with its split channel had been formed. The narrow chute on the west side had depths to − 23 MGL, but the wide channel on the east side of Miles Bar had depths to only 0 MGL, or about 8 ft below mean low water. Below the towhead, at mile 296.5, there was a deep hole with an elevation of − 70 MGL. The sand bar and split channels constituted a restriction to flow, the former being a product of the action of the latter. Actually, the sand bar could be considered an extension of Miles Bar, so that the split channel existed from mile 299.7 to mile 296.7, not from mile 298.1 to mile 296.7.

To eliminate these restrictions to flow, H. B. Ferguson, M. ASCE, in 1938, directed that "corrective dredging" be undertaken in the reach opposite Smithland. The following is quoted from a report on that subject:[10]

> "The objective of the work at this location is to obliterate the split channel at the head of the old cut-off, improving the east side channel and to extend deep water upstream.
>
> "Excepting a few minor shoals, deep water extends upstream a distance of 314 miles from the river's mouth to Mile 296.5. From Mile 296.5 on upstream to Mile 301 deep water occurs intermittently with minor shoals between. To correct these conditions the deep water at Mile 300.5 down to deep water at Mile 296.5 was connected in 1938 by a series of cuts to a depth of approximately minus 25 feet M.G.L. These cuts were made in several separate operations. A careful running observation was conducted and the plan of dredging work was modified from time to time to conform to whatever natural trends the river seemed to be taking.
>
> "From Mile 300.5 down to Mile 298.5, the spoil was so disposed as to effectively exchange excessive width for depth, the majority of the spoil being placed against the east bank. Throughout the work in this vicinity many pockets of gravel were encountered and much debris was found embedded in old deposits. From Mile 298.5 down to Mile 296.5, the cuts followed down the center of the channel to the east of a middle ground. The material encountered here was much lighter than that found above, consisting principally of fine sand. Some of this material was deposited on the head of the middle ground in order to influence further deposits by the river to raise the elevation of this gently sloping sand bar which formed the east bank in this vicinity, so placing the spoil as to provide more definite banks. A cut was made along the right bank between Mile 296.5 to 296.0 to remove the extremely tough clay, which borings had disclosed

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[9] "Mississippi River Hydrographic Surveys," Mississippi River Comm., Vicksburg, Miss., October, 1937.

[10] "The History of the Improvement of the Lower Mississippi, 1932–1939," Mississippi River Comm., Vicksburg, Miss.

to exist in this vicinity. Partial removal of this point immediately provided some betterment in alignment of the channel coming down the east side of the middle ground and into the deep water at Mile 296.5."

(To conform to present-day river mileage, the mileage in the excerpt has been changed from miles below Cairo, Ill. (BC), to miles above Head of Passes—thus, mile 300 AHP = mile 772.5 BC.)

Table 2 is a comparison of depths in the Smithland reach, between miles 300.5 and 296.5 AHP, for four hydrographic surveys, beginning in 1937 and ending in 1945. The "corrective dredging" was effective, depths throughout the reach being increased and the chute on the west side of Miles Bar closing up. The closing of the chute concentrated the river energy into one channel—the east—with subsequent improvement of that channel. Whereas, prior to 1938, the Smithland reach was a flow restriction, it no longer can be considered such.

TABLE 2.—COMPARISON OF CHANNEL DEPTHS IN THE MISSISSIPPI RIVER BELOW RED RIVER LANDING (MEAN GULF LEVEL DATUM)

Mile (AHP)	October, 1937	September, 1942	September, 1944	September, 1945
300.5	−12	−15	−50	−40[a]
299.5	+10[b]	−18[c]	−25[d]	−23[e]
298.5	− 3[f]	−14[g]	−23	−20[h]
297.5	−23[i]	−20[j]	−12[k]	−29[l]
296.5	−70	−70	−40	−40

[a] At mile 300. [b] Sand bar extending from mile 299.7 to mile 298.8. [c] Sand bar eliminated. [d] Channel shifted from center to west side. [e] El. −23 on west side; slight bar in center to El. +3. [f] Split channel from mile 299.7 to mile 296.7. [g] Upper end of west channel closed to El. +13. [h] El. −5 at center; El. −20 on west side. [i] El. −23 on the west side; El. −0 on the east side. [j] El. −20 on the west side; El. −27 on the east side. [k] Chute closed. [l] West chute open between mile 297.3 and mile 296.7.

DIVERSION OF SEDIMENT AT A CHANNEL BIFURCATION

Several important laboratory studies of the diversion of sediment at a channel bifurcation have been made. Inasmuch as a complete analysis of existing data concerning the diversion of sediment would not be within the scope of this paper, only pertinent references will be cited and general conclusions drawn from them.

The problem was first studied in Europe by H. Thoma of Munich, Germany, at the request of a power company. That company, situated on the Middle Isar River, in Germany, was troubled by sedimentation at the intakes of its power plants. Later, Th. Rehbock and H. Bulle studied the problem at Karlsruhe, Germany. The early investigators (1923) used rectangular flumes, the size of the main channel and bifurcating channel being identical (0.66 ft wide). The ratios of bed load diverted for different bifurcation angles were as given in Table 3. These experiments demonstrated that for an equal division of flow the branching channel diverts the major part of the bed load. In summarizing his work Mr. Bulle noted:

(*a*) Just inside the branching channel and opposite the point of diversion, a roller or eddy was formed. This roller varied in intensity with the angle of diversion, and with the form of the diversion point. It operated to contract the section of the branching channel, and so to increase slopes at its mouth.

(*b*) Relative to the percentage of water carried by the side flumes the percentage of "geschiebe" (bed-load material) deposited was always very large.

TABLE 3.—PERCENTAGES OF TOTAL BED LOAD DIVERTED FOR DIFFERENT BIFURCATION ANGLES (50% OF THE TOTAL WATER WAS DIVERTED IN EACH CASE)

Magnitude	ANGLE OF BIFURCATION (DEGREES)				
	30	60	90	120	150
Percentage of total bed load diverted.....	97.3	96.2	90.5	87.5	92.0

(*c*) A rounded corner at the convex side of the side channel entrance (tested only for angles of 30°, 60°, and 90°) tended to decrease the side channel contraction produced by the roller, and also decreased the transverse slope at the point of diversion. More water but less "geschiebe" was carried into the side channel as the result of the rounded corner.

In 1932 K. D. Nichols and C. D. Curran, Assoc. M. ASCE, began similar studies at the Waterways Experiment Station in Vicksburg. Gerard H. Matthes, Hon. M. ASCE, writes of the Vicksburg experiments:[11]

> "Similar experiments * * *, with flumes of various types, do not confirm the high ratios above quoted [see Table 3], but indicate that the branch-channel carries, in proportion to its discharge capacity, more bed-load than the straight channel does below the point of diversion. * * * In one of the flume experiments made * * * it required 70 hours of continuous operation to bring about a condition of equilibrium. During the first 14 hours the bulk of the bed-load materials entering the branch-channel dropped and built up bars. This was followed by a period of bed-load travel in which dunes and loose materials progressed with regularity. The stabilized condition showed the branch-channel to discharge about 37 per cent of the total flow; and about 66 per cent of the total bed-load. It was found, furthermore, that the bed-load movement, although consisting of these two distinct forms, always totaled the same. In this instance the branch-channel did not fill with sand, but maintained its discharge capacity while at the same time carrying more sand in proportion to its discharge than the main river-channel."

The discrepancy between the German findings and those of the Waterways Experiment Station may be explained in this manner: In the earlier experiments by Messrs. Rehbock and Bulle, rectangular sections of equal area were employed in the two flumes, and the discharges in the two flumes were likewise maintained. Messrs. Curran and Nichols, using rectangular flumes 1.75 ft wide, for an even distribution of flow, found a deposition of 81.6% occurred in the side channel in 3 hours, 45 min. In other tests at Vicksburg, semicircular cross sections were employed, with the side channel being given an area of two thirds that of the main channel. The discharge through the side channel was less than 50% of the total flow. Also, experimental runs in Germany approximated 1 hour in duration, whereas the runs at Vicksburg were, in some instances, 70 hours long. Mr. Bulle found that the percentage of "geschiebe" moved down the side channel decreased with time. At Vicksburg, the percentage did not decrease. With an increase of time the distribution of flow varied inversely with the distribution of the bed load. After beginning a run, the percentage of "geschiebe" moved into the side channel, and the percentage of flow down the straight channel below the fork increased for a short time. A period of decrease in both percentages then existed until the vertical roller near the head of the side channel was eliminated. After extinction of the roller, both percentages increased with time. If runs as reported by Mr. Bulle had been longer, he might have noted the increase after the decrease. At Vicksburg it was noted that, once the bed had been built up, deposition and scour were negligible, thus indicating that a condition of equilibrium had been reached. Nature sought the most efficient channel; and, until such a channel was obtained, deposition and scour took place.

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[11] "Diversion of Sediment at Branching Channels," by Gerard H. Matthes, *Transactions*, Am. Geophysical Union, Vol. 14, 1933, p. 506.

Since experiments indicate a greater proportion of "geschiebe" relative to the quantity of water diverted into a bifurcation channel, one naturally looks for the cause. It should be clearly understood that material deposited (and carried through) the side channel is of bed-load nature. Material in suspension has been proved by the aforementioned investigators to divide in a ratio commensurate with the division of flow. Tests, utilizing small celluloid balls filled with water to keep them on the bottom, indicate that practically all the bottom layers of flow turned into the side channel. Current indicators also show most of the water adjacent to the bottom moving into the branching channel.

If one will visualize the vertical velocity distribution formed in most streams, a reasonable argument for the movement of the "bottom layers" into the side channel may be advanced. Immediately next to the bottom a boundary layer retards flow. The momentum of the lower stream filaments is less than that of the upper ones (for equal masses). One may conclude that the greater momentum of the upper filaments of water gives them more of a tendency to continue in a straight line, leaving the supplying of the branch channel largely to the lower stream filaments.

If the lower filaments do move into the bifurcation channel and they possess sufficient velocity to cause the bed load to move, one must conclude that the greater part of the bed load will move into the side channel, for the bed-load movement takes place in the lower filaments of the stream.

PRIOR STUDIES OF DISCHARGE CAPACITY AT RED RIVER LANDING

All the early investigators attempted to deduce the trend in discharge capacity by making comparisons of cross-sectional areas of the rivers over a period of years. Many unpublished papers on this subject are on file in the Mississippi River Commission Library at Vicksburg, but only those papers and documents which have been published will be cited.

In 1914 E. J. Thomas, assistant engineer of the Mississippi River Commission, reported[12] a comparison of the hydrographic surveys of 1895 and 1910, as shown in Table 4.[13]

It will be noted that, except for the low-water stage, the average areas for the reach increased. In the same source Kivas Tully, assistant engineer of the Mississippi River Commission, reported that:

> "From the plotted discharge curves of Red River Landing for 1882 it will be seen that the discharging capacity of the Mississippi River at that point was greater than it has been at any time since. The measured discharge on March 31 of that year was 1,595,000 cu ft per sec at a stage of 48.3 ft. The discharge on May 14, 1897, at a stage of 50.2 ft was only 1,290,000 cu ft per sec. That is, at a stage 1.9 ft higher, the discharge was 305,000 cu ft per sec less."

Mr. Tully also stated that the average area of the reach from ¼ mile above to 1¼ miles below the mouth of the Old River had decreased 18,067 sq ft, or about 8% between 1884 and 1909. Messrs. Thomas and Tully did not agree on the changes in cross-sectional area of the reach.

TABLE 4.—COMPARISON OF HYDROGRAPHIC SURVEYS OF 1895 AND 1910

Stage	Widths (ft)	Areas (sq ft)	DEPTH (FT)	
			Mean	Maximum
Low water	+ 803	−22,680	−22.9	−31.0
Medium	+1,487	+31,647	− 5.4	...
Bankfull	+1,463	+51,127	− 5.2	...

The conclusion reached in a subsequent report dated 1919 was that the decrease in channel area was responsible for a decrease in discharge capacity.[14] Continuing the same line of reasoning, in 1931 a report[15] presented the data on the comparison of cross-sectional elements from mile 771.0 to mile 837.1 BC, as shown in Table 5. The changes in average areas from 1895 to 1921 were increases and therefore could not be deemed responsible for any decrease in the discharge capacity of the reach.

TABLE 5.—COMPARISON OF CROSS-SECTIONAL ELEMENTS FROM MILE 771.0 TO MILE 837.1 BELOW CAIRO, ILL.

Stage	AVERAGE AREA (SQUARE FEET)			AVERAGE WIDTH (FEET)			AVERAGE MEAN DEPTH (FEET)		
	1895	1921	Change (%)	1895	1921	Change (%)	1895	1921	Change (%)
Bankfull	194,269	195,960	+0.9	3,599	3,503	−2.7	56.0	58.2	+3.9
Medium	151,235	154,259	+2.0	3,256	3,254	0.0	48.5	50.2	+3.5
Standard low	81,095	82,606	+1.9	2,265	2,385	+5.1	38.1	37.5	−1.6

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[12] "Separation of Red and Atchafalaya Rivers from Mississippi River," *House Resolution No. 841*, 63d Cong., 2d Session, 1914, p. 20.

[13] *Ibid.*, p. 113.

[14] *House Resolution No. 288*, 66th Cong., 1919.

[15] *House Resolution No. 798*, 71st Cong., 1931, Pt. I.

In 1932, the Corps of Engineers, presented further data[16] on changes of cross-sectional elements for the 2-mile reach below the Old River which indicated a definite channel deterioration (see Table 6).[17] However, it should be noted that this occurred between 1882–1883 and 1910. From 1910 to 1924 the average area and mean depth were relatively constant despite the fact that the width was continuing to increase. The changes from 1910 to 1924 may be considered "local, insignificant" changes.

TABLE 6.—Changes of Cross-Sectional Elements in the 2-Mile Reach Below Old River

Year	Width, in Feet			Area, in Square Feet			Mean Depth, in Feet		
	Bank-full	Medium	Low water	Bank-full	Medium	Low water	Bank-full	Medium	Low water
1882–1883	4,138	4,003	3,299	237,860	173,594	75,178	57.7	43.5	23.4
1895	4,277	4,140	3,687	232,814	163,410	59,136	54.4	39.5	16.2
1910	4,559	4,424	3,812	226,857	154,463	43,066	49.8	34.9	11.6
1922	4,631	4,513	3,738	223,235	150,147	40,729	48.2	33.3	11.0
1923	4,612	4,471	3,665	227,312	153,771	44,441	49.3	34.4	12.4
1924	4,673	4,525	3,689	224,771	151,171	42,400	48.1	33.4	11.8

Mr Salisbury[6] used the same data. He noted the change from 1882–1883 to 1924, but failed to consider the fact that most of the change occurred before 1910.

Although a change in cross-sectional elements may be determined in the manner used by early investigators, the writer considers that such a study does not lead to correct conclusions. It is relatively easy to provide for coincident hydrographic surveys over a period of years, but the relation of such a survey line to the direction of flow is subject to change. A survey line may be normal to the flow in one year, and in the following year it may depart as much as 25° from the normal. The later survey would always indicate an increase in area, but the data would be misleading. Also, a change in area at any point does not necessarily imply a change in channel capacity. The latter is determined for the main part by the resistance to flow offered by channel conditions below the reach in question. Quite often many cross sections do not carry their full capacities because of a "bottleneck" below them. Therefore, it is believed that a study of channel cross sections alone is not the correct approach to the determination of the changes in channel capacity of the Mississippi River.

Conclusions

The student of the Mississippi River today faces a set of conditions entirely different from those encountered by A. A. Humphreys, Hon. M. ASCE, and Henry L. Abbot[18] in at least one respect. Messrs. Humphreys and Abbot had few data available, and most of them were undependable. Their problem called for superior intellectual power to bridge over the gaps, and to reach rational conclusions. Today the investigator encounters an almost overwhelming amount of information; not in all cases what he would like to have, perhaps, but all of it requiring study so as to be certain that he has not neglected some vital occurrence.

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[16] "The Improvement of the Lower Mississippi River for Flood Control and Navigation," War Dept., U. S. Govt. Printing Office, Washington, D. C., May 1, 1932.

[17] *Ibid.*, p. 55, Table VI.

[18] "Report on the Physics and Hydraulics of the Mississippi River," by A A. Humphreys and Henry L. Abbot, *Professional Paper No. 13*, Corps of Engrs., U. S. Army, 1876.

In seeking to determine the effect (on the capacity of the Mississippi River) of the increasing diversion into the Atchafalaya River, the writer has seen fit to discard all measurements prior to 1913. The older data are demonstrably less accurate and cannot be used comparatively in studying the later data. Since dependable data do exist for the years following 1912 (and during that period the Atchafalaya River has increased the most and discharged the most), it seems to him a reasonable step. The older data were obtained with cruder instruments and techniques, and the labor of trying to adjust them to a comparable basis does not seem to be justified.

Prior studies of the effect of the Atchafalaya River diversion which were based on comparison of cross-sectional elements were fruitless, in that the procedure used did not provide the correct approach to the problem. The study made by Mr. Salisbury was the best one made from a technical standpoint, but he reached erroneous conclusions partly because of attaching too much importance to the older data.

The capacity of the Mississippi River at Red River Landing decreased from 1913 to 1930 because of the split channel at Smithland and the bar which formed between Smithland and Red River Landing. The removal of this bar and the corrective dredging in the Smithland main channel restored this capacity to that in 1913 as indicated by the rating curve in Fig. 5.

The increase in capacity between 1939 and 1945 indicated by the measurements taken in 1945 was probably largely a result of the change in equipment and technique by which the data were secured. This conclusion is supported by a study of the Baton Rouge gage.

There has been no noticeable increase in the total annual discharge of the Mississippi River to indicate that the increased diversion was being compensated for on that score.

The laboratory experiments on bifurcated channels afford a clue to the reason that the Mississippi River has not adjusted its bed to a smaller average discharge. These experiments proved that a much larger proportion of the bed load would move down a bifurcating channel than would proceed down the main stream, and that the suspended load would remain in approximately the same concentration in both channels as it was above the bifurcation.

The tendency of an alluvial channel to adjust itself to its average discharge requirements is not questioned. The channel of the Mississippi River, in its lower reaches, is not alluvial in the ordinary sense of the word, as is the section at Greenville, Miss., for instance. The bed and banks, having been laid down in still water, are much more resistant to erosion. The fact that the bed load is largely diverted leaves the river little opportunity to adjust its cross section to reduced discharge requirements.

DISCUSSION

E. F. Salisbury,[19] M. ASCE.—Briefly stated, the author's contentions are that:

1. Unrestricted below-bank diversion creates a problem different from that of controlled over-bank diversion.
2. Discharge measurements prior to 1913 cannot be used for comparative purposes without qualification, and data collected about 1940 must be carefully evaluated because of changes in the procedure of taking measurements.
3. A study of the stage-discharge relationship does not indicate any progressive changes in the discharge capacity of the Mississippi River below Old River in Louisiana.
4. Experiments that have been made on movable bed models of bifurcated channels give a clue as to why the normal tendency of an alluvial stream toward adjustment of its capacity to its flow requirements has not been found to operate in this case.

The writer's thoughts are not at great variance with those of the author, and his discussion will cover these four points in order.

Unrestricted diversion of flow from the Mississippi River into the Atchafalaya River at all stages (such as at Red River Landing in Louisiana) reduces the velocity in the main channel downstream from the point of diversion as compared with velocities upstream. As the volume of water diverted increases, there is a corresponding increase in the difference between the upstream and the downstream velocities. The natural result is that some of the silt load is deposited downstream, thus reducing the cross-sectional area of the channel. The direct effect of the unrestricted diversion is to increase the slope of the main river at the point of diversion, as evidenced by an increase in gage height for the same discharge.

When the flow is diverted at stages that overtop the banks (controlled or uncontrolled), the downstream reduction in cross-sectional area is not noticeable. This fact has been demonstrated at a point upstream from Arkansas City, Ark. (Cypress Creek Gap), where uncontrolled over-bank diversion occurred for a number of years. Since 1921 the levee line has been closed at this point and floodwaters have been confined except at infrequent intervals when they have been released through crevasses. The over-bank flow prior to 1921 has had no noticeable effect on the Arkansas City gage, the discharge remaining constant for the period from 1890 to 1929. (Cutoffs made in this reach of the river subsequent to this period have radically changed the gage and discharge relationship.)

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[19] Chf. Engr., The Kansas City Southern Ry. Co., Kansas City, Mo.

At Bonnet Carré spillway, about 24 miles upstream from the New Orleans (La.) gage station at Carrollton, the flow has been diverted, over bank, three times; and this controlled floodwater escape had no noticeable effect on the New Orleans gage.

Although the technique of measuring the discharge of rivers has improved, the effect on studies of the Mississippi River, using both old and new gage readings and discharges, is problematical. Nevertheless, the result of unrestricted diversion at Red River Landing is apparent by either method of analysis.

By studying records of meter readings,[6] about 1935, the writer found that, between the periods from 1867 to 1886 and from 1912 to 1933, the average flood peaks at Red River Landing were increased 7.3 ft. This was the result of raising and strengthening the levees on the Mississippi, thus confining increasingly larger quantities of water in the main river and causing correspondingly larger diversion of flow to the Atchafalaya River. Below the point of diversion, the main river adjusted itself progressively with increased flood heights and consequent increase in diversion—the river channel being partly closed by a silt deposited dam. By an analysis of discharge measurements obtained with older techniques, it was apparent that this diversion had increased the slope and thus raised the gage 6.9 ft. in order to permit the discharge to remain the same as it had been since 1882. It is interesting, therefore, to note how very closely the analysis based on recent measurements agrees with that obtained by the older data.

TABLE 7.—STUDY OF 1945 GAGE RECORD, RED RIVER LANDING, IN LOUISIANA

Date[a] (1)	Gage height, in feet (2)	Mean velocity, in ft per sec (3)	Discharge,[a] in thousands of cu ft per sec (4)	Gage height adjusted to 1,100,000 cu ft per sec (5)
March 9, 1945	41.4	5.34	1,073	42.2
March 12, 1945	43.0	5.45	1,125	42.3
May 18, 1945	49.8	4.93	1,130	48.9
May 19, 1945	49.0	5.02	1,126	48.3
May 21, 1945	48.2	5.00	1,105	48.1

[a] Maximum discharge on April 16, 1945, was 1,515,000 cu ft per sec at gage of 55.7 ft. Bonnet Carré spillway was in operation from March 24 to May 17, 1945.

The flood of 1945 provides an excellent example, selecting a discharge of 1,100,000 cu ft per sec to compute the heights at the gaging station. In Table 7, Col. 4, discharge readings within 30,000 cu ft per sec of the selected discharge are adjusted to that discharge (1,100,000 cu ft per sec) in Col. 5. During the early stages of the 1945 flood (March 9) the selected discharge passed at a low gage height of 42.2 ft, as the result of corrective dredging, in 1938, at Smithland, La. (see heading, "Dredging Below the Mouth of the Old River"). As the flood height increased, larger volumes passed into the Atchafalaya River and the gage readings in the main river increased correspondingly to reflect the existence of a constant remaining discharge.

The variation of 6.7 ft for a similar volume of water during the 1945 flood compares with a difference of 6.9 ft computed some years ago for the period from 1882 to 1933. In the earlier study, the discharge data indicated that the increase had been progressive. Analysis of the 1945 flood shows that the increase in the slope of the main river, below the point of diversion, occurred during the same flood—a result that may or may not have been caused by the corrective dredging at Smithland. As stated by Mr. Odom, the Atchafalaya River was dredged extensively between 1932 and 1941, and a new outlet was constructed at Wax Lake.

The treatment of the silt load by a silt-bearing stream is evidence of the powerful capacity of such a river to adjust itself to changed conditions. The result of this adjustment can be overcome or delayed by artificial methods, such as dredging. The changes in the slope of the Atchafalaya River, caused by diversions in the two periods from 1882 to 1923 and from 1933 to 1945, cannot be compared because of the new outlet at Wax Lake and the aforementioned dredging in the main channel.

Loss in the cross-sectional area at Red River Landing during the same flood and when maximum diversion is occurring was well illustrated during the high water of 1929 and also during the high water of 1945. During the 1929 flood the deposition of silt below the point of diversion restricted the cross-sectional area to the extent of holding it to the same area during the last 5-ft rise in flood height as follows:

Gage, in feet	Area, in square feet
47.4	234,000
48.8	234,000
49.8	236,000
51.1	230,000
52.2	230,000
52.4	232,000

The loss in cross-sectional area during the 1945 flood, shown subsequently, followed closely that of 1929 although corrective dredging had been previously carried out at Smithland:

Gage, in feet	Area, in square feet
50.3	236,500
52.4	236,700
55.8	227,800
56.4	245,200

From the foregoing data, the deposition of silt during the same flood is shown to act as a natural automatic valve, holding the cross section to about the same area during maximum diversion and against a rise of several feet in gage, thus producing a steeper slope during the same flood. This silt is removed during the same flood or during intervening waters. For some time the writer has been of the opinion that when the silt load of a heavily laden stream is subjected to a major disturbance causing either a considerable increase or a considerable loss in the silt load, such as that produced by a diversion of the flow, a material increase in roughness factor is immediately produced over that customarily used for ordinary river channels. This may or may not be a contributing factor in causing a higher gage for discharging the same volume of water below the point of diversion.

By the use of mean discharges (Fig. 4), the author has endeavored to show that the capacity of the Lower Mississippi has been improving steadily since 1927. The wide fluctuation of the gage for the same volume of water (1,100,000 cu ft per sec) during the flood of 1945 shows the misleading conclusion that can be drawn from this method.

The low gage of 42.2 ft (March 9, 1945, Table 7) closely approaches the average gage of 41.3 ft for the period from 1882 to 1885. The high gage of 48.9 ft (May 18, 1945, Table 7) exceeds the average gage of 48.2 ft for the period from 1913 to 1932 and is close to the peak gage of 50.4 ft which was registered on May 5, 1913.

The mean value of high and low gages for 1945 will indicate that the mean discharge during this year is less than that for some of the other years. Such a computation should not obscure the fact that, for a present discharge as great as that from 1913 to 1932, the gage heights must still be correspondingly high.

The writer is not questioning the results of the experiments made on movable bed models of bifurcated channels, as mentioned by the author; but the disturbance to the silt load by unrestricted diversion, such as that at Red River Landing, will result in an inefficient handling of the floodwaters in the main channel below the point of diversion. This condition can be overcome by eliminating diversion below bankfull stage, thus disturbing the greater part of the silt load, to a minor extent, and destroying the river's power of adjustment. Furthermore, waters below bankfull stage, acting for 365 days, can themselves clean out any deposition of silt left after the short duration of over-bank diversion.

A regimen is thus established by which the river is working constantly to remove any deposition of silt left after short periods of over-bank diversion. As a result, the main river, at Red River Landing, would be lowered 6 ft or 7 ft at flood heights, assuming no change in the discharge; and the backwater reservoir in the Red River would reflect correspondingly beneficial results. The quality of the New Orleans water supply, at low-water stages, would also be improved. After the Morganza Floodway project has been completed, the foregoing approach to the flood control "picture" becomes inevitable. It is possible that the operation of this floodway (which is so close to the Red River Landing) may, when it is in operation during extremely high floods, regulate the velocity through the point of diversion sufficiently to eliminate most of the disturbance caused by the present uncontrolled diversion. Before the present flood control plan was put in operation by the Corps of Engineers, United States Department of the Army, the Upper Atchafalaya Basin was flooded about once every 5 years; but no flooding has occurred since 1927.

JAMES F. HALSEY,[20] **ASSOC. M. ASCE.**—Time and patience have been required to assemble the wealth of data presented in this paper. One does not truly appreciate the time required for preparation of such a paper or the author's statement (see under the heading, "Conclusions")—

> "Today the investigator encounters an almost overwhelming amount of information; not in all cases what he would like to have, perhaps, but all of it requiring study so as to be certain that he has not neglected some vital occurrence."

—until one makes a similar study or begins preparing a discussion of such a paper. River studies are vastly different from many other types of analyses in that events at many different locations must be given a chronological order and placed in their proper position of importance before a concise picture of causes and effects is made available for a complete understanding of the particular problem.

The problem upon which Mr. Odom has focused his attention is the century old argument, "Has diversion of water from the Mississippi River into Old River and thence into the Atchafalaya River affected the carrying capacity of the Mississippi below the point of diversion?" As noted in the paper, this subject has been one of the most argumentative topics among river hydraulicians since the log raft in the Atchafalaya was removed in 1840. The literature abounds with the pros and cons of the subject as advanced by many men and Mr. Odom has presented his ideas and thoughts for further consideration and for a possible revision of some engineers' concept of the problem. The writer regrets that Mr. Odom did not see fit to draw a more concrete conclusion from the data presented. However, by reading between the lines the writer concludes that Mr. Odom firmly believes that diversion through the Atchafalaya was not responsible for the deterioration of the Mississippi River channel at, and immediately below, Red River Landing.

In support of this "between lines" conclusion the author has offered seven subconclusions. The writer agrees with each of the seven statements and proposes the following three points for further study.

1. What is the effect upon the trend lines in Figs. 4 and 6 for Red River Landing and the Simmesport (La.) gage by the addition of the data for 1947–1949?

2. Has the Lower Atchafalaya River improved its stage-discharge relationship as much as the Upper Atchafalaya River, as shown by the Simmesport gage?

3. What caused the split channel at Smithland Landing in the Mississippi River to develop? Was it due to diversion or some other cause?

As stated earlier, a complete chronological order of events is necessary for a true and complete understanding of seemingly unrelated occurrences. This problem of diversion has its origin with the initial construction of levees on the

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[20] Research Asst. Prof. of Fluid Mechanics, Dept. of Theoretical and Applied Mechanics, Univ. of Illinois, Urbana, Ill.

Mississippi. The "squeezing" of the main river into a comparatively small channel provided extra head at the latitude of Angola (La.) to assist in forcing water from the Mississippi down the Atchafalaya. As the levee system was lengthened and strengthened, the frequency of crevasses below Angola was diminished so that this extra head was available for longer periods each succeeding year. However, until the log raft in the Atchafalaya was removed, that stream acted somewhat as an orifice—that is, the extra head at Red River Landing could not change the coefficient of discharge through the Atchafalaya. Immediately after the removal of the raft improvement of the Atchafalaya began. With a continued strengthening of the levees on the Mississippi and the building of levees on the Atchafalaya the latter river had no alternative except to enlarge and convey more water to the Gulf.

In 1890[21] a discharge of 480,000 cu ft per sec was reported for Simmesport; and, discounting 35% as being due to the pull affected by the crevasses downstream, there was still a discharge of 312,000 cu ft per sec, or 2.6 times the quantity reported for 1875. Thus the rate of change in the early stages of development was much greater than the rate of change since the turn of the century. At about this time (1890), according to Fig. 4, the Mississippi River began showing higher stages for equivalent discharges. The effect of this change should be apparent at once for more "head" was provided, thus causing more flow down the Atchafalaya. Unfortunately, from a purely scientific study of physical actions, man's scope of work on these two streams continued to enlarge so that man's influence was playing the major role in the development of the Atchafalaya. Continued levee building, raising of levee lines, dredging since 1933, and the Carr Cutoff in 1944—all contributed very materially to the improvement of the carrying capacity of the Atchafalaya. The writer ventures to guess that had not these man-made works been built, the Atchafalaya would not carry more than 300,000 cu ft per sec today. (This statement is contradictory to the thoughts of the pro-diversion group and will be explained subsequently.)

Inasmuch as the writer questioned the advisability of extending the method of adjusted stages (proposed by E. W. Lane,[22] M. ASCE) over a long period of record, he made a study of the method before applying it to the Krotz Springs gaging station record. (Krotz Springs is located at mile 41.7 on the Atchafalaya.) Essentially, Mr. Lane's method assumes that, even though a general rising or falling trend in the stage-discharge relationship may exist at a gaging station, a 1-ft increment of stage at a base discharge produces the same differential of discharge over a period of years. Mr. Odom used Mr. Lane's method in the preparation of data for Figs. 4 and 6, and the writer has used it in preparing the additional data.

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[21] "The Improvement of the Lower Mississippi River for Flood Control and Navigation," by D. O. Elliott, U. S. Waterways Experiment Station, Vicksburg, Miss., 1932, p. 52.

[22] Discussion by E. W. Lane of "Influence of Diversion on the Mississippi and Atchafalaya Rivers," by E. F. Salisbury, *Transactions*, ASCE, Vol. 102, 1937, p. 109.

At a base discharge of 200,000 cu ft per sec, the discharge per foot of stage in the Atchafalaya River at Krotz Springs is as follows:

Year	Discharge
1936	14,500
1937	14,000
1938	15,000
1939	13,000
1940	13,000
1941	14,000
1942	14,000
1943	15,000
1944	14,500
1945	15,000
1946	16,000
1947	15,000
1948	16,000
1949	14,500
Average used	15,000

These data convinced the writer that Mr. Lane's method is valid and may be used in a study of a changing stage-discharge relationship for a stream. Table 8 contains data for 1947 through 1949, for the Red River Landing and Simmesport gages by which the curves in Figs. 4 and 6 can be extended. When the curves are thus plotted, it will be seen that the trend toward an increasing channel capacity for the same stage is continuing. In trying to deduce just what importance should be attached to a trend line such as Mr. Odom has drawn, the writer tried to fit several types of curves—straight line, logarithmic, and hyperbolic—to the field data. It would be possible to fit a curve to the points in Figs. 4 and 6 which, in future years, would show an increasing stage for equivalent discharges or to fit a curve that would show a flattening out of the trend lines; but the danger of extrapolation into the future from past physical data led the writer to conclude that, regardless of what he believes may happen in years to come, the only safe approach to the problem was to show what has happened and let the reader conjecture as to what will happen. Table 9 is a comparison of slopes of trend lines (slope being equal to change in stage, in feet per year) given by the author and those found by the writer in fitting a straight-line curve ($y = m\,x + b$) to the data by the least squares method.

In the case of Red River Landing, the addition of 3 years of record causes the slope of the average trend line to flatten out, thus indicating a slowing of the rate of change of capacity. Nevertheless, the stage-discharge relationship is continuing to improve. Since it is physically impossible (without crevasses or overtopping of levees) for the Lower Mississippi River to convey much more water to the Gulf of Mexico than it now does, the writer predicts a leveling off of the trend line for the Red River Landing gage so that in time the channel capacity of the Mississippi at that point will be 950,000 cu ft per sec (between the 1880 and 1945 rating curves in Fig. 5).

With reference to the data for the Simmesport gage on the Atchafalaya the analyst is confronted with a somewhat different problem for the physical characteristics—that is, the width, depth, sinuosity, etc., of this stream were greatly altered by man during the period from 1912 to 1949. Whereas the only change in channel configuration of the Mississippi affecting the Red River Landing gage was the minor dredging at Smithland, La., millions of cubic yards of material were dredged out of the Atchafalaya. Some 85,000,000 cu yd was removed between Krotz Springs (mile 41.7) and mile 100 in Grand Lake between 1933 and 1943.[23] Also, in 1944, Carr Cutoff just above the head of Old River removed the reverse curve entry of Old River and provided a smooth curved approach for the waters of the Mississippi into Old River and thence into the Atchafalaya.

TABLE 8.—EXTENSION OF STAGE OBSERVATIONS, FIGS. 4 AND 6

Year	(a) MISSISSIPPI RIVER AT RED RIVER LANDING					(b) ATCHAFALAYA RIVER AT SIMMESPORT, LA.				
	Base discharge (cu ft per sec)	No. of measurements	Adjusted Stages			Base discharge (cu ft per sec)	No. of measurements	Adjusted Stages		
			Highest	Mean	Lowest			Highest	Mean	Lowest
(1)	(2)	(3)	(4)	(5)	(6)	(2)	(3)	(4)	(5)	(6)
1947	1,100,000	none	....	....	....	400,000	none	....	....	....
1948		none	....	....	....		3	39.35	38.39	36.97
1949		9	48.13	46.20	42.37		7	37.34	36.82	35.77
1947	1,000,000	none	....	....	....	365,000	5	38.50	37.60	36.18
1948		1	....	42.37	....		13	39.32	37.72	35.77
1949		6	45.45	43.94	41.34		4	35.93	35.02	34.37
1947	900,000	6	44.36	43.53	42.36	330,000	4	36.58	35.72	34.33
1948		19	45.59	43.43	41.23		6	37.13	35.58	34.70
1949		7	42.08	41.37	39.21		2	32.68	32.43	32.18
1947	800,000	2	39.70	39.50	39.30	300,000	10	34.30	33.14	31.88
1948		8	40.23	38.36	36.51		5	34.55	33.73	33.02
1949		4	39.27	38.00	35.33		2	30.49	30.36	30.32

With such an "assist" from mankind one naturally would expect the Atchafalaya to show the tremendous increase in channel capacity that it did. A study of Table 9 shows plainly that the greatest increase (other than the initial increase after removal of the log raft) occurred after the aforementioned dredging because the average slope of the trend lines (Table 9) for the Simmesport gage in the period after 1935 is twice as much as for the total period of 1912–1949.

One may wonder as to whether the change in channel capacity throughout the Atchafalaya was as great as the indicated change at Simmesport. A study of the Krotz Springs gage[23] and the extension of that data on a calendar year basis through 1949 indicates that the increase in channel capacity was not as great in the lower reaches as in the upper reaches. For discharges of 200,000 cu ft per sec, 300,000 cu ft per sec, and 400,000 cu ft per sec, the mean stage lowering at Krotz Springs from 1935 to 1949 was 0.342 ft per yr, or only 50% of that evinced by the Simmesport gage. This fact can mean only one thing—the lower Atchafalaya is proving to be a serious deterrent to the over-all improvement of that stream. This action will be welcomed by those who have been worried about whether, and how soon, the Atchafalaya will "capture" the Mississippi. Even if the Atchafalaya River were to continue to increase its carrying capacity at the 1935–1949 pace, as indicated by the Simmesport gage, some 180 years would be required for the "capture." It is the feeling of the writer that regardless of what structures may be built on Old River (a navigation lock and diversion control being planned) the physical make-up of the Lower Atchafalaya Basin at Morgan City, La., and Wax Lake would preclude any such capture. The situation is somewhat analogous to that of trying to put more than 1 lb of sugar into a 1-lb container—it cannot be done without spilling some of it over the top.

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[23] Discussion by James F. Halsey of "Solids in Stream Flow," by Gerard H. Matthes, *Transactions*, Am. Geophysical Union, Vol. 31, Pt. 1, April, 1950, p. 284.

TABLE 9.—COMPARISON OF SLOPES OF TREND LINES

(a) MISSISSIPPI RIVER AT RED RIVER LANDING			(b) ATCHAFALAYA RIVER AT SIMMESPORT, LA.			
Discharge (cu ft per sec) (1)	1927 to 1946 (author) (2)	1927 to 1949 (writer) (3)	Discharge (cu ft per sec) (4)	1912 to 1946 (author) (5)	1912 to 1949 (writer) (6)	1935 to 1949 (writer) (7)
1,100,000	−0.170	−0.210	400,000	−0.246	−0.312	−0.781
1,000,000	−0.266	−0.170	365,000	−0.253	−0.311	−0.541
900,000	−0.187	−0.031	330,000	−0.580	−0.489	−0.746
800,000	−0.177	−0.205	300,000	−0.338	−0.369	−0.696
Average	(−0.200)	(−0.154)	Average	(−0.354)	(−0.370)	(−0.691)

An engineer may spend a lifetime studying stage-discharge relationships and know every moment just what a stream is doing; but does he know why it is doing what it is? Mr. Odom has gone beyond the stage-discharge study phase in attempting to produce evidence that would explain the vagaries of the diversion of the Mississippi River flow into the Atchafalaya River. The writer believes he has done so, with a slight possible exception, and he feels that, given enough time, Mr. Odom would have uncovered the evidence the writer has that seems to point to the Raccourci Cutoff in 1848 as the cause for the declension of carrying capacity of the Mississippi at Red River Landing from 1883 to 1938.

A thorough study of hydrographic surveys for that part of the Mississippi from above Old River to below Smithland, and the writings of H. B. Ferguson,[24] M. ASCE, have convinced the writer that the Raccourci Cutoff—and not the diversion of flow into the Atchafalaya—was responsible for the decline in the carrying capacity of the Mississippi at Red River Landing. (Surveys dated 1883, 1921, 1922, 1923, and 1924 were furnished to the writer by the Mississippi River Commission.) The 1883 hydrographic survey shows a well-defined channel throughout the reach cited, with shallow water being present adjacent to the left descending bank at mile 296.5 AHP (above Head of Passes). (All mileage quoted has been adjusted to the 1945 datum.) A small sand bar was present there and later became part of the extended bank line when the east bank built up and out toward the west as shown by the 1895–1896 bank line survey. However, not far from the sand bar, channel depths extended to El. −80 MGL (mean Gulf level). (Mean low water is now at about +8 ft MGL.) Table 10 compares the depths throughout this reach for the years 1883 and 1921.

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[24] "The History of the Improvement of the Lower Mississippi, 1932–1939," Mississippi River Comm., Vicksburg, Miss., p. 16.

In the 38-year interval shown by Table 10 the overflow through what was the bed of the river prior to 1848 was causing the current to slacken and deposit some of its bed load opposite Smithland at mile 298. In this case it required more than the 10 years cited by General Ferguson to develop the split channel which is always the aftermath of a cutoff (regardless of where it occurs), for in 1921 the split channel was beginning to be quite definite and was causing the deposition upstream which eventually extended from mile 299.7 AHP to mile 296.7 AHP. It was during this period (1883 to 1921) that Mr. Lane and E. F. Salisbury,[6] M. ASCE, noted the great decrease in channel capacity of the Mississippi at Red River Landing. Regardless of where it is, however, any gaging station would reflect a decrease in channel capacity if, immediately downstream, the channel was being choked by sand bars.

TABLE 10.—COMPARISON OF CHANNEL BOTTOM ELEVATIONS IN THE MISSISSIPPI RIVER DOWNSTREAM FROM RED RIVER LANDING; MEAN GULF LEVEL (MGL) DATUM

Mile (AHP)[a]	January, 1883	October, 1921	Footnotes
300.5	−20	−10	[a] Above Head of Passes.
299.5	−17[b]	−11	[b] Deepest on west side.
298.5	−20	−29[c]	[c] On west side, very shallow on east side.
297.5	−20[d]	−24[e]	[d] West bank noted as caving.
296.5	−80	−65	[e] In chute on west side of island, −6 on east side.

The chronology of this diversion problem, as stated previously, purposefully omitted the effect of the Raccourci Cutoff so as to show that every single one of the events in the general vicinity of Red River Landing should be put in its proper niche in time. The aftereffects of this cutoff caused gage heights at Red River Landing to rise; it was decidedly not the diversion into the Atchafalaya which caused the deposition opposite Smithland and thence affected higher gage readings at Red River Landing. Without the sand bar opposite Smithland, gage heights at Red River Landing would not have been raised 7 ft from 1883 to 1928 (see Fig. 4) and this extra head would have not been made available to the Atchafalaya for its push through the tough back-swamp clays. (In a case of free discharge, a head of 7 ft provides a velocity of 21 ft per sec.) Without the extra head there would have been a less rapid development of the Atchafalaya and, furthermore, without the dredging since 1933, the Atchafalaya would be physically unable to discharge more than about 300,000 cu ft per sec without crevasses or levee overtopping affecting the discharge.

It is felt that the field data, presented by the author and by the writer, tend to refute the theorem of Domenico Guglielmini. Mr. Guglielmini laid down his concept of flow below a diversion when field or experimental data were not to be had. Doubtless his concept of the normal stream velocity profile, as stated by Robert Manning[25]—

> "He [Guglielmini] believed that the velocities increased from the surface towards the bottom as the ordinates of a parabola, and that motion at the surface was produced by the effects of the fluid molecules imparting a velocity to those above them. In the early part of the 18th century Pitot * * * proved by actual experiment the fallacy of Guglielmini's theory."

—was the basis of his diversion theory.

Even though a few engineers may still rely on the Guglielmini theorem as an explanation of the flow phenomena occurring in the Mississippi River at Angola, the writer believes that the ideas presented by Mr. Odom will pass the acid test of time and, in due course, fully supplant the Guglielmini theorem.

JERE W. HIGGS, JR.,[26] JUN. ASCE.—Mr. Odom is to be commended for his thorough study of a subject that has received surprisingly little attention in the technical press. Certainly the diversion of water through the Atchafalaya River presents engineering problems of sufficient magnitude to warrant the most detailed study by all river engineers.

The author has presented data which indicate the trends in the development of the Atchafalaya River and the effect of the increased diversion on the Mississippi River. He has conducted his analysis (under the heading, "Effect of Diversion Into the Atchafalaya River on the Capacity of the Mississippi River Below the Old River") "* * * without reference to general laws or theories * * *."

Only by conducting the study under such conditions can it be considered truly objective. Engineers, however, must insist upon some logical explanation for natural phenomena. If no action dependent on basic physical laws may be shown, eventually, to be responsible for the observed changes in river regimen, the modern conception of engineering is in need of revision.

Data presented in the text and in Figs. 4 and 5 reveal no long-term deterioration in the channel of the Mississippi River below Old River. If it is admitted that the total flow in the Mississippi downstream from Old River has decreased an appreciable amount as the Atchafalaya has become more efficient, these data seem to be in direct contradiction to the commonly accepted theory that an alluvial river will adjust its bed to its discharge. As pointed out by the author, however, these data do not disprove the theory but serve to emphasize the importance of considering local conditions in its application.

The question of how much change has occurred in the quantity of flow passing Red River Landing immediately presents itself if an attempt is made to explain the failure of the Mississippi River channel to deteriorate. It seems quite possible that flow past Red River Landing has not as yet been decreased sufficiently to permit channel deterioration, or that the decrease in flow passing Red River Landing has not been maintained for a period long enough to permit channel deterioration.

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[25] "On the Flow of Water in Open Channels and Pipes," by Robert Manning, *Transactions*, Institution of Civ. Engrs. of Ireland, Vol. XX, 1891, p. 161.

[26] Engr., Leo M. Odom, Cons. Engr., Baton Rouge, La.

The Atchafalaya River has shown a marked increase in efficiency from 1920 to 1945, but the total quantity of flow passing Red River Landing has not as yet shown any marked or long continued decrease (see Fig. 8). A comparison of Fig. 8 with Fig. 7 shows that there is no trend apparent at Red River Landing which might not be considered equivalent to "normal" annual variation.

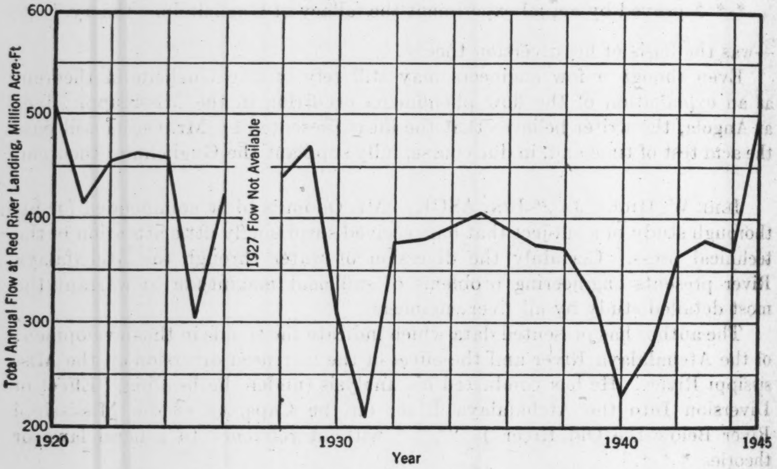

FIG. 8.—TOTAL ANNUAL FLOW PASSING RED RIVER LANDING, IN LOUISIANA

The 1920–1945 period also includes major floods which, through scour, doubtless served to retard any channel deterioration in the Mississippi which might have taken place. It is significant that the trend toward channel deterioration at Red River Landing, as shown in Fig. 4, reversed itself in about 1927. Even the 1927 flood might have resulted in a minor break in the trend line had it not been followed by a large total flow in 1928, and then the 1929 flood. Breaks traceable to the large 1935 flow, the 1937 flood, and the Smithland Reach dredging are also apparent in Fig. 4.

The author has suggested the diversion of sediment at the mouth of Old River as a clue to the failure of the Mississippi River channel to deteriorate as flow diversion increased. This clue should be subjected to further analysis if an understanding of Atchafalaya diversion is to be attained. It is believed that, to a large extent, the sediment diversion characteristics at Old River will govern the rate at which any channel deterioration might take place.

Data on sediment movement in a stream as large as the Mississippi River are difficult to obtain, and once obtained are of doubtful value. The only attempt known to the writer to measure sediment in the Mississippi River in the vicinity of Old River was made in 1929 under the direction of the Mississippi River Commission.[27]

Two discharge ranges on the Mississippi River were selected as sampling ranges—one at Tarbert's Landing, which in 1929 was about 3½ miles above the mouth of Old River; and one at Red River Landing about 1½ miles below the mouth of Old River. Eight points at approximately equal intervals across the stream's width were selected as sampling stations. Three samples were obtained at each station—one sample at the surface, one at middepth, and one at the bottom. Samples were secured by a standard Mississippi River Commission "slip bottle" sampling device, consisting essentially of a hollow iron cylinder, 6 in. high and 8 in. in diameter, closed at its upper end and sliding on a vertical steel rod. Two steel disks are fixed on the vertical rod at a distance apart equal to the height of the cylinder. As the device is lowered to the sampling point the cylinder is in a raised position so that the upper disk seals the cylinder. Once the sampler has been placed the cylinder is dropped so that the bottom disk forms the seal, the water and sediment sample being trapped within the cylinder.

Samples obtained at each range were combined into three composite samples —one composite sample for the surface, one for the middepth, and one for the bottom. Analyses were run to determine the sediment concentration in each composite sample and the total sediment load was then computed.

The reliability of the data obtained by the method of sampling and analysis just outlined is open to question. It is believed, however, that these data should at least be indicative of the action taking place.

As shown in Table 11 there is a considerable variation in the quantity of sediment recorded from day to day at each station. There is an even more marked difference between the quantity of sediment observed at the Tarbert's Landing range and that observed at the Red River Landing range.

Day-to-day variations or even minute-to-minute variations in the sediment load at either discharge range might preclude the value of these data were it not for the fact that marked trends are apparent.

An inspection of Table 11 will show that there is apparently no relation between the quantity of water diverted and the quantity of sediment diverted. In one instance there was even an increase in the total sediment observed at Red River Landing over that observed at Tarbert's Landing. A close inspection of Table 11 will reveal that in all instances where there was a large concentration of sediment in the bottom sample there was a large apparent sediment diversion. This trend is presented graphically in Fig. 9.

The data presented in Table 11 and in Fig. 9 indicate that the diverted water is drawn largely from the lower part of the stream cross section. These data offer a considerable degree of support to the view expressed by the author and based solely on laboratory experiments.

Sediment obtained with a "slip bottle" sampler is probably suspended sediment. It is the writer's belief that very little, if any, bed-load material would be obtained with the sampling device and methods used in the study described in the foregoing paragraphs. Even if a sample containing bed-load material were obtained in the stream thalweg, where the largest bed load would probably be found, its significance would be obscured by the compositing of all bottom samples.

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[27] "Sediment Investigations on the Mississippi River and Its Tributaries Prior to 1930," *Paper H*, U. S. Waterways Experiment Station, Vicksburg, Miss., July, 1930.

If it is admitted that the data presented in Table 11 and Fig. 9 are applicable to the suspended sediment rather than to the total sediment load, the author's statement concerning the nature of the diverted material may be questioned.

TABLE 11.—Comparison of Sediment Loads, Red River Landing and Tarbert's Landing, in Louisiana[a]

Date (1929)	Red River Landing gage,[b] (ft)	Parts Sediment per 100,000 Parts of Water, Tarbert's Landing			Sediment Discharge (Lb per Sec)		Water discharge Tarbert's Landing (1,000 cu ft per sec)	Apparent Percentage Diversion[c]	
		Surface	Mid-depth	Bottom	Tarbert's Landing	Red River Landing		Sediment[c]	Water
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
March 23..	41.4	53.2	69.2	108.2	54,763	32,517	1,171	40.7	19.4
March 28..	44.5	41.6	54.8	58.6	40,823	25,796	1,248	36.8	18.4
March 30..	44.5	44.8	50.4	56.0	39,265	28,401	1,248	27.6	18.4
April 3....	46.7	34.5	50.3	55.1	38,260	27,504	1,288	28.0	12.7
April 9....	47.4	26.0	30.2	34.0	25,917	20,337	1,379	21.5	21.8
April 13...	48.1	28.3	32.4	37.0	28,120	21,170	1,385	24.7	20.5
April 16...	48.4	30.2	32.0	46.4	30,662	24,556	1,395	19.9	20.6
April 20...	49.0	31.8	36.4	35.4	31,123	24,555	1,424	21.1	20.8
April 23...	49.4	34.6	40.2	42.6	35,371	23,903	1,438	32.5	20.5
April 27...	49.7	32.2	36.6	41.6	33,304	26,354	1,454	20.8	21.4
April 30...	49.8	31.2	32.6	52.6	33,800	19,064	1,455	43.7	21.1
May 4.....	49.9	21.2	22.2	27.4	21,174	21,644	1,462	2.2	21.4
May 7.....	49.8	35.4	38.0	39.0	33,826	25,774	1,441	23.8	22.1
May 11....	49.7	28.2	40.2	37.7	32,446	30,606	1,420	5.7	19.5
May 14....	49.4	30.6	35.4	56.2	34,387	23,464	1,398	31.8	19.5
May 22....	49.7	27.2	37.9	37.2	30,503	25,567	1,396	16.2	19.7
May 24....	49.9	30.0	32.4	139.2	50,103	26,665	1,372	46.8	17.9
May 28....	50.5	31.6	42.2	49.8	38,572	30,349	1,489	21.2	21.9
June 3.....	51.8	32.4	36.2	46.0	36,002	27,954	1,530	22.3	24.0
June 8.....	52.3	24.0	30.0	37.8	29,810	27,669	1,565	7.2	20.9
June 18....	52.0	19.8	20.3	26.6	19,144	17,508	1,407	8.6	19.9

[a] "Sediment Investigations on the Mississippi River and Its Tributaries Prior to 1930," *Paper H*, U. S. Waterways Experiment Station, Vicksburg, Miss., July, 1930, pp. 59–60. [b] Bankfull stage is 43 ft. [c] Calculated from the difference in observed discharges at Tarbert's Landing and Red River Landing.

The following statements are quoted from the paper (under the heading, "Diversion of Sediment at a Channel Bifurcation"):

> "It should be clearly understood that material deposited (and carried through) the side channel is of bed-load nature. Material in suspension has been proved by the aforementioned [laboratory] investigators to divide in a ratio commensurate with the division of flow."

These statements may be quite true for laboratory channels in which a uniform suspended sediment distribution may be approached, but no relation between the quantity of suspended sediment diversion and the quantity of flow diversion is evident in Table 11. The division of suspended material is dependent on its distribution at the point of diversion. Laboratory investigations notwithstanding, there are at present no methods for predicting, accurately, the distribution of suspended sediment in the cross section of a river; nor may the division of suspended load at the point of diversion be predicted.

If suspended sediment in the lower part of the stream cross section is diverted, it seems self-evident that most of the bed load will be diverted. Although the writer knows of no successful measurements of bed-load movement in natural streams, it seems reasonable to believe that at certain river stages the bed load will comprise a substantial part, if not the larger part, of the total sediment load.

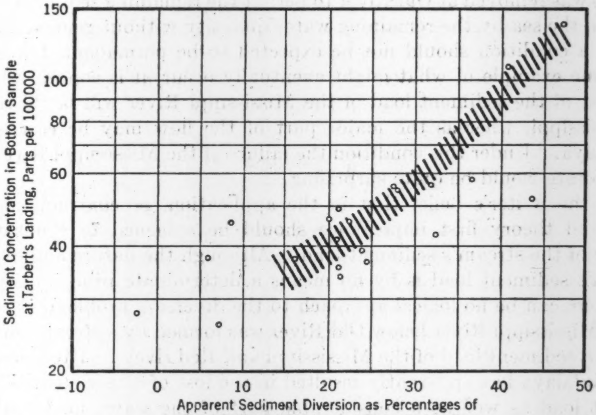

FIG. 9.—RELATION BETWEEN SEDIMENT DISTRIBUTION AND SEDIMENT DIVERSION AT TARBERT'S LANDING, IN LOUISIANA

Another factor that should not be overlooked in considering the effect of sediment diversion at Old River is the influence exerted by the decrease in flow reaching the Mississippi from Red River. The Red River carries an extremely high sediment load during high water. For example, at Alexandria, La., on December 13, 1930, a sediment discharge of 19,830 lb per sec was observed at a water discharge of 62,300 cu ft per sec.[28] Although this value is the maximum observed sediment load and discharge during about a one-year period when observations were made every second or third day, it is interesting to note that the maximum observed sediment load at Red River Landing on the Mississippi during the same high-water period was only 23,377 lb per sec at a discharge of 281,000 cu ft per sec.[29]

The channel of the Mississippi below Old River was formed by a stream carrying not only the total sediment load of the Mississippi River, but also the total sediment load of the Red River. The fact that development of the

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[28] "Sediment Investigations on the Mississippi River and Its Tributaries, 1930–31," *Paper U*, U. S. Waterways Experiment Station, Vicksburg, Miss., December, 1931, p. 87.

[29] *Ibid.*, p. 22.

Atchafalaya has eliminated the Red River as a source of sediment for the Mississippi downstream from Red River Landing will certainly emphasize any effects which might be realized from the diversion of sediment at Old River.

Diversion of sediment at Old River is believed to be a major factor affecting the stability of the Mississippi River channel below Red River Landing. It must be realized, however, that the sediment diversion characteristics are continually changing. The failure of the Mississippi River channel to deteriorate between 1927 and 1945 indicates that the sediment diversion occurring at Old River during that period was such that the competence of the Mississippi River below Red River Landing was not overtaxed. Apparently enough sediment was removed at Old River to permit the remaining sediment load to be carried to the sea by the remaining water quantity without general deposition.

Such a condition should not be expected to be permanent, however. As an extreme example of what might eventually occur, it is conceivable that in time most of the sediment load of the Mississippi River will be "diverted" to the Mississippi, whereas the major part of the flow may be carried by the Atchafalaya. Under this condition the failure of the Mississippi River channel to deteriorate would be quite surprising.

It is the writer's belief that in the application, or contradiction, of the Guglielmini theory first importance should be attached to the nature and quantity of the stream's sediment load. Although the nature and quantity of a stream's sediment load is by no means a determinate problem, without its study there can be no logical approach to the diversion problem.

The Mississippi River below Old River was formed by a stream carrying the water and sediment load of the Mississippi and Red rivers. Diversion through the Atchafalaya has apparently resulted in the loss of a large part of the total sediment load as well as a considerable part of the water load. At present the diversion of sediment has occurred in sufficient quantity to prevent long-term channel deterioration in the Mississippi. It is believed that subsequent developments will depend on the variations in the ratio of water and sediment diversion and the occurrence of "abnormal" river flows, such as floods or very low water.

Mr. Odom states that the channel of the Mississippi River below Old River is much more stable than in its upper reaches (under the heading, "Geology of the Lower Mississippi and Atchafalaya Rivers")—

> "The channel is narrower and deeper, high-water slopes are much less, and the bed materials are much more resistant to erosion [than in the upper reaches]."

The author further states, with regard to the Atchafalaya, that:

> "* * * it flows through erodible material and its average cross section throughout its leveed length has been constantly increasing for many years. This enlargement of the leveed channel has been accompanied by the silting up of the lakes and swamps between the ends of the leveed channel and the lower end of the basin."

Any marked variation in the nature of the bed materials at corresponding latitudes on the Atchafalaya and Mississippi rivers is somewhat surprising.

Both streams flow through the deltaic plain of the Mississippi and Red rivers and the areas traversed by both rivers are of similar if not identical geologic origin.

Also, it is believed that the significance of the erodibility of a particular bank material is somewhat obscured when considering the characteristics of the Atchafalaya River. Erodibility at the site is dependent on many other factors as well as on the nature of the bank material.

The upper Atchafalaya is in an early stage of geologic development. The water surface slopes are greater than is necessary to carry the stream's water and sediment load to the sea, and scour and channel enlargement result. This condition has been emphasized by the construction of levees, which serve to increase the water surface elevations, concentrate the current, and raise water surface slopes.

In the lower Atchafalaya, downstream from the ends of the levees, water surface slopes are insufficient to carry the sediment load and large-scale deposition results. Natural channels in the lower basin exhibit a braided stream pattern, and fill rather than scour generally occurs.

The Mississippi River downstream from Old River (as stated by the author) is in a poised condition. Its slope has been so adjusted that no general fill or scour takes place. Its relative stability in comparison with the Mississippi above Old River is doubtless due, in a large part, to the fact that the bank materials on the lower river are very resistant to erosion, but the effect exerted by the sediment diversion at Old River should not be overlooked. The relatively low sediment load presumably existing downstream from Old River would serve to deprive the Mississippi of material which might be used to initiate bed changes.

The Atchafalaya River and its relation to the Mississippi afford a rigorous testing ground for many of the theories of alluvial rivers. Mr. Odom has rendered a valuable service in presenting such a comprehensive study.

Leo M. Odom,[30] M. ASCE.—The material presented in this paper was accumulated while the writer was chief engineer of the Department of Public Works of Louisiana. The contents were drawn from a much more comprehensive study which was made to obtain "ammunition" for convincing Congress that the diversion from the Mississippi River into the Atchafalaya River should be controlled. This effort was made because of the great increase in capacity of the Atchafalaya River which was indicated by the flood of 1945. That flood caused the writer to reach the decision that it was time to do something about the diversion before the Atchafalaya actually replaced the Mississippi as the main stream.

The writer would have been happy to have offered the loss of capacity of the main river as an argument for control of the diversion, and it was while investigating this possibility that he came to the conclusions set forth in the paper.

Mr. Salisbury states that the velocity below the diversion is less than the velocity above and that, therefore, a portion of the silt load is deposited downstream. It is true that the measured velocity at Tarbert's Landing discharge range (mile 304.1 AHP) above Old River is nearly always greater than it is at the Red River Landing range (mile 299.5 AHP) immediately below Old River. This relation was obtained throughout the rise and fall of the major flood of 1945 or until the diversion through Old River dropped below 100,000 cu ft per sec. Had the velocity at Red River Landing not remained less than that at Tarbert's Landing during the time when Old River was diverting nearly 25% of the flow of the Mississippi above, the automatic adjustment in cross section posited by Mr. Salisbury would have been easier to visualize. Reduction in discharge accompanied by reduction in velocity does not indicate a reduction in cross section.

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[30] Cons. Engr., Baton Rouge, La.

As proof of his contention of automatic adjustment in cross section at Red River Landing due to the diversion, Mr. Salisbury gives several readings of the gage and discharge measurements taken during the 1945 flood. It is to be noted that the first two measurements given in Table 7 were taken on a rising river and the latter two on a falling river. The writer has tried a similar comparison for the same flood using the records of the Natchez (Miss.) gage and readings taken near bankfull at Natchez (mile 359.2 AHP), at about the same time that the readings at Red River Landing were taken in Table 7. Table 12 indicates a variation of 8.3 ft for the Natchez gage at the same discharge. (The first two measurements were on rising stages and the last one on the falling stage.)

A check of several gages farther upstream will reveal that, in the 1945 flood, the elevation for approximately bankfull discharge at those gages was from 6 ft to 9 ft greater on the falling stage than on the rising stage.

TABLE 12.—READINGS AT NATCHEZ GAGE, TO BE COMPARED WITH THOSE IN TABLE 7 FOR RED RIVER LANDING

Date (1945) (1)	Gage height, in ft (2)	Mean velocity, in ft per sec (3)	Discharge, in thousands of cu ft per sec (4)
March 9 . . .	41.1	6.33	1,329
March 12 . .	43.2	6.54	1,326
May 14 . . .	49.4	5.71	1,328

Gerard H. Matthes,[31] Hon. M. ASCE, notes this phenomenon in his paper on Mississippi River cutoffs, using only the rising stages in comparing the effects of cutoffs. In this connection, he states that reductions in channel capacity of from 200,000 cu ft per sec to 250,000 cu ft per sec were found between rising and falling stages and he attributes this fact to the building up of the crossing bars. Since reduction in capacity between rising and falling stages is as great (possibly greater) at stations far removed from diversions as it is at Red River Landing this fact could scarcely be attributed to the diversion at Old River.

With reference to Mr. Salisbury's conclusion that the Red River Landing range lost cross-sectional area during the last 5 ft of rise because of the diversion, it is noted that many other stations on the river exhibited similar loss of cross-sectional area. At Natchez, in 1945, the cross section on the rising river at a gage reading of 50.8 was 233,100 sq ft, whereas at a gage reading of 55.2 it was 227,200 sq ft. The long flood of 1929 (during which overbank stages lasted for 102 days at Red River Landing) had several minor crests, and the cross sections at most stations on the river varied considerably for equivalent gage heights. Unfortunately the Natchez discharge measurements are very scanty for that flood. The Vicksburg station records of the first rise, for a gage height of 49.3, show a cross section of 219,000 sq ft whereas on the same rise, for a gage reading of 52.7, the cross section was 217,700 sq ft.

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[31] "Mississippi River Cutoffs," by Gerard H. Matthes, *Transactions*, ASCE, Vol. 113, 1948, pp. 35–36.

The absence of a long-term trend toward loss of cross section at the Red River Landing range is shown by the fact that, on the rise of the 1927 flood, the cross section at a gage reading of 47.8 was 225,200 sq ft, whereas at 47.4 on the gage in 1945, the cross section was 229,000 sq ft.

The reasons for changes in cross section during floods are numerous. During 1945 the right bank below Old River was caving rapidly and adding a great burden to the river at the Red River Landing range. This caving was greatest on the rise. Sand waves along the bottom of the river are said to cause the lessening of the cross section during their passage. The study of sand movements made in 1880–1881 at Plum Point reach (report of the United States Mississippi River Commission, November 25, 1881) revealed sand wave crests as high as 13 ft above the channel bottom. The diversion may have had some effect but the data seem to be very inconclusive.

Mr. Salisbury apparently would prefer to have Old River controlled by an overbank floodway. This possibility was discarded because the cities along the southern rim of the Atchafalaya Basin depend on the low-water diversion from the Mississippi River for their water supply since the low-water flow of Red River alone is negligible. It was also feared that, if the flow in the Atchafalaya River was restricted for long periods of time to volumes that reached its head from Red River, the latter (which is a much greater silt carrier than the Mississippi) would succeed in partly filling the channel of the Atchafalaya with silt. When the Atchafalaya River was needed for a relief outlet for the Mississippi River, it would then be incapable of furnishing the required capacity. The present project for closure of Old River provides for year-round diversion of flow but would control the flow by a fixed weir, at a low elevation, for stages below dangerous flood heights and would furnish movable gate control for overbank flows when the Mississippi required relief.

Mr. Higgs presents data on sediment measurements which were available to the writer when his study was made. The writer did not note any particular tendency from his study of these data and is glad that Mr. Higgs was able to determine that the sediment measurements bear out his more theoretical conclusions. Mr. Higgs' suggestion that the fact that Red River no longer flows into the Mississippi at any stage except the very lowest stages, and that this fact has changed conditions in regard to silt load below Red River Landing, is of interest. However, this condition has existed for a long time. The maximum discharge of Red River was about 250,000 cu ft per sec in 1945, whereas the maximum measured discharge of the Atchafalaya was 661,000 cu ft per sec. Mr. Halsey's discussion contains data about the growth of the Atchafalaya River which show that the Atchafalaya has long had a much larger capacity than the Red River.

Mr. Higgs is correct in his statement that, unless enlargement of the Atchafalaya River is controlled, it will become the main channel and that the Mississippi will become the diversion channel. At that time, the Mississippi will lose capacity rapidly.

There is no great variation in bank and bed materials between the Atchafalaya River and the Mississippi River below Red River Landing as Mr. Higgs points out. In fact, the Atchafalaya bed and banks are tougher. The bed and banks of both rivers are erodible, as is easily seen from inspection. The writer merely pointed out that the banks of the Lower Mississippi are tougher than those found farther upstream.

Mr. Halsey's discussion enlarges the scope of the paper and his critical investigation supports the assumptions made in preparing it. His extrapolations of the curves to include later data are welcome and his careful plotting of the trend lines, although it does not change the picture materially, has caused the writer to redraw them on his own copies of the exhibits.

Mr. Halsey's study of the cause of the divided channel at Smithland, and subsequent loss of capacity there until the corrective dredging was done, adds to the value of the paper. His discussion of the enlargement of the Atchafalaya River and the relative danger of its capture by the Mississippi River is outside the scope of this paper and a complete analysis of his conclusions in this regard would require more space than is available. However, the writer has studied this problem in considerable detail. Mr. Higgs gives a brief statement as to the forces involved with which the writer agrees. The enlargement of the Atchafalaya River has progressed downstream as its levees were extended. The soil in its basin is composed of clays, silts, and sands.

Enlargement of its cross section by scour was great where sands were encountered, so that extreme depth was also accompanied by abnormal width. As the scour progressed, clay blankets in the bed above and below the large holes were undermined. The channel was formerly a succession of large holes and smaller reaches. Flow was very turbulent, therefore, and the great loss of energy was reflected in the fact that a much higher roughness factor is indicated than is usual for a stream of its size. The dredging by the United States Mississippi River Commission in the early 1930's was done for the purpose of improving the discharge capacity by dredging through a great many of the clay layers in the bed. Once the alternate hole and choke pattern was broken up, the river did as was expected and filled the larger holes to some extent while it enlarged the choke sections. The capacity was greatly improved—much more than the growth in average cross section would indicate.

In the earlier stages of development of the Atchafalaya River most of the enlargement occurred at the upper end. The upper 3 miles had an average bankfull cross section of 58,000 sq ft as compared to an average of 15,000 sq ft for the lower 5 miles and an average of 44,000 sq ft for the entire 58.0 mile-length of the main river. In later years the locus of maximum enlargement has shifted to the lower end. After the 1945 flood, the upper 3 miles had an average cross section of 88,200 sq ft as compared to 72,400 sq ft for the lower 5 miles and an average throughout the entire length of 79,800 sq ft.

Prior to the 1945 flood the gain in discharge capacity and the gain in cross section had increased with each major flood; but the increase had been relatively slow and, if the average cross-sectional areas for the stream were plotted against the years, a straight line would closely approximate the increase. A straight-line relationship also applied to the bankfull and overbank discharges. However, the 1945 flood caused enlargements which departed greatly from the previous straight-line development and the discharge increased in an even greater proportion. Prior to 1945 the responsible authorities had concluded that the enlargement would proceed at the relatively uniform rate which had been in vogue for 40 years. With that rate of increase, it would have taken many years for piracy of the Mississippi to be threatened. The U. S. Mississippi River Commission, in formulating its flood control program, needed considerably greater high-water discharge down the Atchafalaya than was occurring and much time lapse seemed likely before the desired cross section would be developed. The large jump from the straight-line increase during the 1945 flood resulted in seeking Congressional authorization for control works.

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Paper No. 2439

TREATMENT OF FOUNDATIONS FOR LARGE DAMS BY GROUTING METHODS

By A. W. Simonds [1] and Fred H. Lippold,[2] Members, ASCE, and R. E. Keim,[3] Assoc. M. ASCE

With Discussion by Messrs. A. Mayer; V. L. Minear; Byram Steele; James B. Hays; and A. W. Simonds, Fred H. Lippold, and R. E. Keim.

Synopsis

Present-day methods used in grouting foundations of large dams were developed for the purpose of reducing seepage under and around the structures and for improving the load-carrying properties of the foundation rock. These methods received impetus in the early 1930's during the construction of Hoover Dam in Arizona and Nevada. Since that time the foundation designs for nearly all large dams constructed in the United States have included extensive grouting programs. Although foundation grouting is still in the process of development, a more or less rudimentary report of the present equipment, materials, and procedures used in foundation treatment seems pertinent.

The information contained in this paper, for the most part, is based on experiences gained in grouting the foundations of twenty or more large dams. This information is presented under three main headings—foundation grouting equipment, grouting materials, and foundation grouting procedures. In the main, the paper is concerned with the application of neat-cement grout. The use of other foundation grouting materials, such as asphalt, bentonite, and clay, is also discussed. One important method of foundation treatment, chemical grouting, is not discussed in this paper since it is not as yet generally used by the Bureau of Reclamation (USBR), United States Department of the Interior.

Introduction

The proper foundation treatment of large dams is one of the major problems of modern construction. In order that the superimposed structure may

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[1] Engr., Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo.

[2] Engr., Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo.

[3] Engr., J. Gordon Turnbull, Inc., Richland, Wash.; formerly Engr., Bureau of Reclamation, Denver, Colo., and TVA, Knoxville, Tenn.

function as designed, it is essential that the foundation support the maximum loads applied to it. If the foundation rock in its natural state is inadequate, it may be possible to remedy the existing defects and improve the rock so that it will provide suitable support. The injection of grout into defective foundations is a practical method of improvement.

Pressure grouting is the process by which mixtures of cement, water, or other materials are forced into confined and inaccessible spaces for the purpose of consolidating the mass as a whole. Pressure grouting is being used by the USBR primarily for the elimination of seepage and the reduction of uplift pressure in the foundation beneath the structure. It also increases the bearing strength of the foundation rock by filling any voids or fissures that may be present.

The reduction of seepage and uplift pressures beneath a masonry dam is generally effected by grouting the foundation through deep holes spaced at close intervals in one or more rows parallel to the axis of the structure. This procedure (known as "curtain grouting") is supplemented by a row of drainage holes drilled a short distance downstream and parallel to the grout curtain. The function of these drainage holes is to relieve any hydrostatic pressure that may develop from seepage water passing through the grouted zone.

Before a foundation grouting design can be prepared for a dam site, the foundation should be explored thoroughly. Data obtained from the exploration of the foundation are used to determine the following:

1. The type and the size of dam suitable for the site;
2. The properties of the foundation rock;
3. The location of faults, fractures, seams, and cavities; and
4. The suitable grouting plans.

GROUTING PLANS

In the construction of large dams, the USBR uses three classes of grouting—known as low-pressure, intermediate-pressure, and high-pressure grouting. They may be used collectively or singly as required for a particular foundation. A typical arrangement of grouting pressure zones is shown in Fig. 1. In a normal construction program, low-pressure grouting is done first. This consists of blanket grouting to seal and consolidate the part of the foundation near the surface. Intermediate-pressure grouting is used to effect a deeper seal in the foundation along the upstream edge of the structure. High-pressure grouting is used to form the main cutoff curtain to reduce seepage beneath the structure. In addition, special grouting may be required to seal spring drains, to consolidate the rock around tunnels, and to stop occasional leaks within the foundation rock.

In the construction of concrete dams, low-pressure blanket grouting is performed in the surface rock in the upstream third of the foundation area. If necessary, this type of grouting may be used in the entire foundation area. Blanket grouting for earth dams, if needed, is usually confined to the near-surface part of the foundation rock adjacent to the cutoff walls. It is utilized under any type of dam to seal the near-surface bedrock where zones of weakness occur.

Low-pressure grout holes are usually drilled to depths varying from 10 ft to 50 ft. The axis of the hole is taken normal to the foundation surface except where it is parallel to a seam; when this condition occurs, the hole is drilled at an angle with the foundation surface so that the hole intersects the seam. The pressures used depend on the size and type of the structure and on the foundation conditions. In general, pressures are maintained as high as possible without producing detrimental uplift of the foundation rock. The range of pressures used varies between 20 lb per sq in. and 150 lb per sq in.

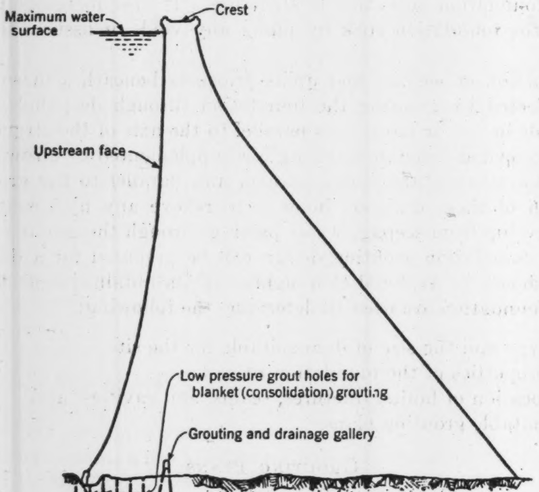
Maximum water surface
Crest
Upstream face
Low pressure grout holes for blanket (consolidation) grouting
Grouting and drainage gallery
Foundation material
Drain holes
100 50 0 100
Scale of feet
Intermediate pressure grout holes for auxiliary cutoff curtain
High pressure grout holes for principal cutoff curtain

Fig. 1.—Zones of Pressure Grouting Beneath a Large Dam

Intermediate-pressure grout holes in concrete dams may be drilled from the surface of the foundation rock just upstream from the edge of the structure, or through pipes placed in the upstream fillet at the base of the dam. The depths generally range from 50 ft to 100 ft depending on conditions encountered. The pressures used depend on the weight of the concrete on the foundation at the time of grouting, and also on the geologic structure and strength of the foundation rock. Normally, these pressures range from 75 lb per sq in. to 400 lb per sq in.

High-pressure grouting to form cutoff curtains is used in the construction of both concrete and earth dams. In constructing large concrete dams, this grouting is generally done from a gallery adjacent to or slightly downstream from the axis and close to the foundation rock. Pipes extending from bedrock to the gallery are placed in the concrete during construction. The holes are drilled and grouted with high pressure through the pipes after concrete has been placed to a reasonable height above the foundation. The depths of these holes vary since they depend on geologic and topographic conditions. For some dams the following criterion has been used for determining the depth of holes in the cutoff curtains:

$$d = \tfrac{1}{3} h + C \qquad (1)$$

in which d is the depth of the hole, in feet; h is the height of dam above the hole, in feet; and C is the constant, varying from 25 ft to 75 ft.

In the case of large dams, the pressures used for high-pressure grouting normally range from 100 lb per sq in. to 500 lb per sq in. In a few exceptional cases, pressures up to 1,000 lb per sq in. have been used in grouting the bottom sections of deep holes. These pressures have been localized by packers in the holes. In order to prevent damage to the foundation caused by heaving or uplifting as a result of the grout pressure, high pressure should be used with caution and never in a haphazard manner.

Special grouting refers to any grouting program that supplements the aforementioned classes of grouting and is performed prior to or during the construction of the dam. It includes: (*a*) Grouting an additional curtain to reduce uplift; (*b*) consolidating shattered rock to permit cutting of steep slopes; (*c*) consolidating weathered joints or fault zones; (*d*) grouting off springs and other leakage in the foundation; and (*e*) consolidating rock to increase its bearing capacity. Special grouting is frequently necessary for diversion tunnels or other tunnels appurtenant to the dam. This form of grouting may include any or all of the following: (1) Grouting ahead of the bore to prevent caving or to shut off flows of water; (2) filling the void between the arch of the tunnel lining and the inside of the bore; (3) grouting a curtain in the rock surrounding the tunnel intake; (4) sealing joints in the tunnel plugs; and (5) consolidating the surrounding rock. When the tunnel is driven through an abutment, it is sometimes advantageous to grout parts of the abutment from locations within the bore.

In some cases after a structure has been completed special grouting is required to supplement the foundation grouting program. The purpose of this grouting is to reduce uplift pressures and to cut off excessive seepage. To accomplish this it is occasionally necessary to grout fractures that have been opened by saturation of unstable materials or by deformation of the foundation by superimposed loads.

FOUNDATION GROUTING EQUIPMENT

Equipment used in pressure grouting operations has undergone considerable development in recent years. Essentially, it consists of grout pumps, grout mixers, water meters, agitator sumps, pressure gages, packers, pipe lines and fittings, and miscellaneous supplies. Fig. 2 shows the grout plant used by the contractor at Davis Dam on the Colorado River in Arizona and Nevada.

For injecting grout a duplex type, positive displacement, reciprocating pump, patterned after the slush pumps developed for use in oil well drilling, has proved satisfactory. These pumps are driven either by compressed air or steam engines or by electric or gasoline motors. A pump properly equipped for cement-grouting service should have special hardened steel liners, rubber pistons, combination steel and rubber valves, and hardened steel valve seats. With a properly designed and equipped grout pump, it is possible to control the pressures closely, to provide a variable rate of injection, and to minimize clogging of valves and feed lines.

FIG. 2.—CONTRACTOR'S GROUT PLANT AT DAVIS DAM ON THE COLORADO RIVER

Reciprocating grout pumps are of three general types: The line type, the side pot type, and the divided fluid end type. These pumps can be obtained in sizes varying from 5¼ in. by 3½ in. by 5 in. to 10 in. by 7¼ in. by 10 in. Sizes commonly used are 6 in. by 4 in. by 6 in., 7 in. by 3 in. by 10 in., and 10 in. by 4½ in. by 10 in. In the line type (shown in Fig. 3) the discharge valves are located directly over the suction valves so that both can be removed through the same opening in the top. The advantage of this type lies in the accessibility of the valves, as only one cover plate needs to be removed to reach two valves in the event that the valves require cleaning. In the side pot type of grout pump each valve is located in a separate pot or chamber. Each valve pot has an independent cover which must be removed before the valve can be reached. The advantage of this type of pump is that all valves and seats are interchangeable. Furthermore, each valve has a separate cover plate which permits the suction valves to be removed and cleaned without disturbing the discharge valves as in the case of a line type of pump. The divided fluid end type of pump is a special variety of the side pot pump which has separate removable cylinders. Each of the foregoing types is designed to pump grout at any required pressures to a maximum of 1,000 lb per sq in. Their capacities range from 20 gal per min to 100 gal per min.

Since cement grout is highly abrasive, component parts of a grout pump are designed to resist abrasion. These component parts include the valves, liners, pistons, piston rods, packing, and oilers. Fig. 3 shows the locations of these parts in a pump properly fitted for cement-grouting service. The valves are of special design and are drop-forged from alloy steel and have steel bumpers and rubber inserts. The seal is formed by the rubber insert and the metal valve seat. Balata and fiber inserts are not satisfactory.

The cylinder liners are removable sleeves that fit into the main body of the pump. They may be fabricated from separate shells, the outer one machined from cast iron or steel and the inner one from casehardened steel, or from a single shell of casehardened steel. In either type, the inner surface should be honed to a mirror-like finish. Liners in new pumps should be removed and inspected for defects before using the pump. When inserting the liner, a copper-bound asbestos gasket or a gasket cut from sheet cork should be fitted between the shoulders of the liner and the cylinder block. Some pumps are factory equipped with soft rubber sleeves between the liner and the block. Sleeves of this type have not proved effective in preventing grout from passing between the liner and the block. A satisfactory seal can be obtained by installing several courses of hydraulic packing painted with either white lead or a mixture of grease and graphite. The outer surface of the liner should be coated with waterproof grease to prevent it from seizing to the block, and to facilitate its removal during disassembly of the pump.

Pistons for cement-grouting service are made of a medium-soft rubber, molded onto a steel core so the unit may be fastened to the end of the piston rod. They are nearly cylindrical in shape with each end slightly flared or enlarged. The flare sweeps the inner surface of the liner clean with each stroke, thereby preventing grout from collecting between the piston and the cylinder.

Pump rods are important parts of grout pumps. Untreated steel rods are a continued source of trouble whereas rods that are properly casehardened and ground, and have a polished surface, give excellent service. For air-driven or steam-driven pumps, it is desirable to use piston rods made in two sections. The air end section transmits the force to the pump from the piston in the engine, and is subject to little wear. The liquid end section drives the piston in the pump and is subject to considerable wear where it passes through the packing gland. Piston rods for the liquid end section should be made of casehardened or stainless steel.

Proper packing around the pump rods is necessary for efficient operation. For the air end, a good grade of steam packing should be used. For the liquid end, braided graphite packing has been used for low-pressure grouting. For intermediate-pressure and high-pressure grouting, a chevron type packing will

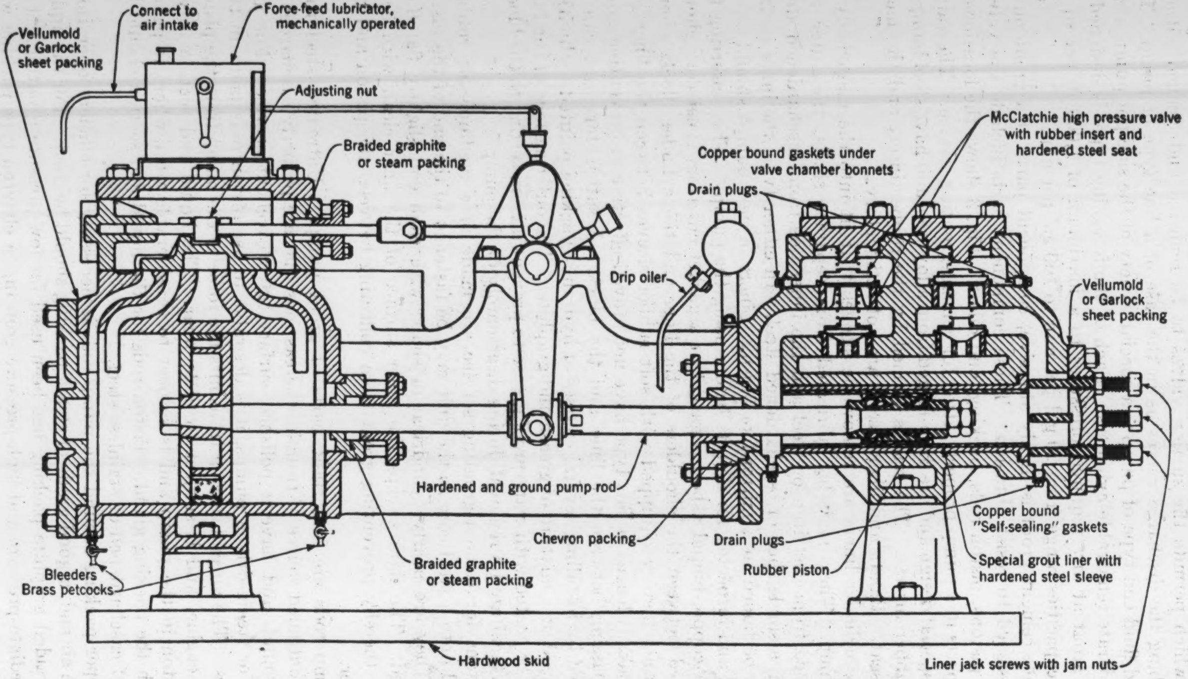
Connect to air intake
Force-feed lubricator, mechanically operated
Vellumold or Garlock sheet packing
Adjusting nut
Braided graphite or steam packing
Copper bound gaskets under valve chamber bonnets
Drain plugs
McClatchie high pressure valve with rubber insert and hardened steel seat
Drip oiler
Vellumold or Garlock sheet packing
Hardened and ground pump rod
Chevron packing
Drain plugs
Copper bound "Self-sealing" gaskets
Special grout liner with hardened steel sleeve
Rubber piston
Bleeders
Brass petcocks
Braided graphite or steam packing
Hardwood skid
Liner jack screws with jam nuts

FIG. 3.—LINE TYPE OF SLUSH PUMP EQUIPPED FOR CEMENT-GROUT SERVICE

be needed to hold the higher pressures. This packing consists of several cup-shaped disks with a hole at the center of each disk for the shaft. Each disk has a V-shape cross section, except the two end disks which have one flat side to fit against the bearing surfaces. When installed in the packing gland, the convex side of one disk nests into the concave side of the adjoining disk.

The engines of compressed-air pumps are usually lubricated by single-compartment, single-feed type lubricators. These supply oil to the power cylinders of the pump. In compressed-air, steam-driven, and motor-driven pumps, drip type oilers are used to lubricate the liquid end section of the pump rod.

During the operation of a grout pump, care must be taken to prevent the valves from being fouled with partly-set grout, particularly when pumping slowly. Fouling of the valves may be prevented by flushing the pump with water at specified intervals. For dam sites where extensive grouting operations are in progress, a stand-by pump should be provided to prevent interruptions in grouting operations.

Pneumatic type grout injectors or "air pots" have been displaced largely by the duplex type grout pumps. When compared with the pump, the chief objection to the pneumatic injector is the lack of flexibility of the flow of grout from the machine. Unless a battery of several machines is available, the flow of grout is intermittent between charges. The pressure also is limited to that of the available air supply. These machines are useful for injecting a grout containing coarse ingredients such as sand, mill feed, sawdust, excelsior, or shavings for the purpose of stopping excessive flows of water. They are also useful for backfill grouting in tunnels and behind metal penstock liners where small quantities of grout are injected at low pressures.

A grout mixer is a machine designed or adapted for mixing the constituents of grout to form a fluid mixture. For cement grout, various machines have been used in the past. Hand-operated mixers, concrete mixers, and pneumatic type mixers have been used. On modern construction projects where extensive grouting operations are required, grout mixers are designed and fabricated specifically for the job. In general, they are of the barrel type, either horizontal or vertical, with an axial shaft to which the mixing blades are attached. The shaft is driven by a suitable motor which is sufficiently flexible in speed to meet the variable mixing requirements. A grout mixer of the horizontal barrel type is suitable where fairly large quantities of grout are needed; and, where smaller quantities of grout are involved, a mixer of the vertical barrel type is useful.

Grout mixers are controlled by varying the water-cement ratio. For convenience in mixing; cement-grout constituents are proportioned by volume. In order to measure the quantity of water used, a meter giving volumetric readings in cubic feet and tenths of a cubic foot is satisfactory. Water meters should have a direct-reading totalizer which provides an indirect check on the number of separate batches of grout that have been mixed during a shift or any interval of time.

During the period between the time the grout is mixed and the time it enters the pump, it must be agitated continually to prevent settling. This is accomplished by an agitator sump between the mixer and the grout pump. An agitator sump is usually a cylindrical tank approximately 30 in. high and 36 in. in diameter. Within the tank is an agitating mechanism consisting of a vertical shaft to which horizontal blades are connected. The agitator is operated at a speed of approximately 30 rpm. As the grout is discharged from the mixer into the agitator, it is passed through a wire screen (No. 4 mesh) to remove pieces of sack, strings, and other foreign matter.

Pressure gages graduated in pounds per square inch are used for controlling and regulating the grout pressure. Reliable pressure gages are essential in pressure grouting systems, since a knowledge of grout pressures at the collar of the hole affords a method of determining the behavior of the hole and the uplift pressures in the foundation rock. At least two gages should be connected in the grout system, one at the pump to register pumping pressure and the other at the hole to indicate grouting pressure. The mechanism of the gage must be protected from direct contact with the grout in the system. This may be accomplished by installing a gage saver or an oil filled siphon between the gage and the grout line.

Grout line accessories include pipe, fittings, hose, and valves. Usually, standard black iron pipe and fittings are suitable for grout lines. For most foundation grouting operations, 1.5-in. pipe is used. Grout hose must be a pliable, soft rubber hose which will withstand abrasion and the maximum grouting pressures used. Rubber hose is often used to connect the supply line to the manifold (header) at the hole to facilitate shifting of the grout line from one hole to another. Stopcock valves are used on grout lines. In the piping system at the grout pump, the control valves, which are subject to heavy service, are frequently lubricated plug stopcocks. This type of valve is thoroughly lubricated under pressure and is easily operated since it does not bind or stick. Gate and globe valves should never be used in grout lines because they are difficult to clean, they wear out rapidly, and they plug easily.

Packers or expanding plugs are devices inserted into holes to localize the pressure applied to certain zones within the hole. Many types of packers have been developed. In general, they are classed as removable and nonremovable. A removable packer, shown in Fig. 4, is installed in the grout hole for washing and grouting the part of the hole below the packer. Nonremovable packers are used if there is danger of their being grouted to the wall of the hole, or if a permanent plug is desired.

Dye is used to find the paths and, occasionally, the source of water through cofferdams and parts of the foundation where leaks occur. The dyes generally used are fluorescein or potassium permanganate.

Grout leaks are plugged by the use of many different calking materials, including oakum, strips of burlap, lead wool, and various sizes of wooden wedges. The wedges should be cut from dry, well-seasoned white pine, or any other wood that expands appreciably when wet.

GROUTING MATERIALS

Material suitable for consolidating foundation rock by grouting methods must: (1) Be sufficiently fluid to permit pumping; (2) be sufficiently fine grained

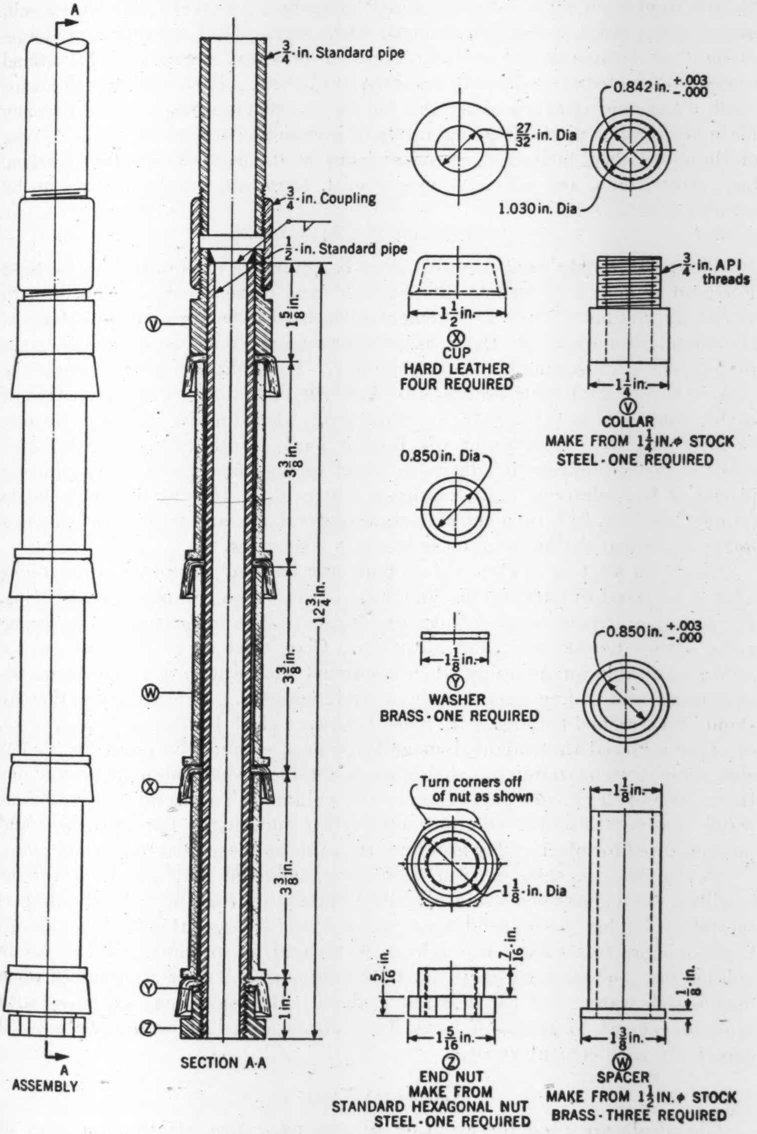
A
$\frac{3}{4}$-in. Standard pipe
$\frac{3}{4}$-in. Coupling
$\frac{1}{2}$-in. Standard pipe
V
W
X
Y
Z
$1\frac{5}{8}$ in.
$3\frac{3}{8}$ in.
$3\frac{3}{8}$ in.
$3\frac{3}{8}$ in.
$12\frac{3}{4}$ in.
1 in.
A
ASSEMBLY
SECTION A-A
$\frac{27}{32}$-in. Dia
0.842 in. $^{+.003}_{-.000}$
1.030 in. Dia
$1\frac{1}{2}$ in.
X
CUP
HARD LEATHER
FOUR REQUIRED
$\frac{3}{4}$-in. API threads
$1\frac{1}{4}$ in.
V
COLLAR
MAKE FROM $1\frac{1}{4}$ IN.φ STOCK
STEEL-ONE REQUIRED
0.850 in. Dia
$1\frac{1}{8}$ in.
Y
WASHER
BRASS-ONE REQUIRED
0.850 in. $^{+.003}_{-.000}$
Turn corners off of nut as shown
$1\frac{1}{8}$-in. Dia
$\frac{5}{16}$-in.
$\frac{7}{16}$-in.
$1\frac{5}{16}$ in.
Z
END NUT
MAKE FROM
STANDARD HEXAGONAL NUT
STEEL-ONE REQUIRED
$1\frac{1}{8}$ in.
$\frac{1}{8}$ in.
$1\frac{3}{8}$ in.
W
SPACER
MAKE FROM $1\frac{1}{2}$ IN.φ STOCK
BRASS-THREE REQUIRED

FIG. 4.—REMOVABLE PACKER FOR GROUT HOLES

to pass, under pressure, into very small openings and cracks; (3) form a substance with adequate bearing strength when set; and (4) set with a minimum amount of shrinkage. When placed under pressure, a mixture of Portland cement and water normally satisfies these conditions. For grouting cofferdams, small dikes, and reservoir rims, and for sealing off large flows of water, many materials may be used either separately or in combination with cement. Some of these are clay, bentonite, chemical compounds, asphalt, sawdust, cinders, hay, straw, oats, and silicas. Once placed, these are usually backed up by cement grout.

CEMENT-GROUT MIXTURES

The most widely used form of grout is a mixture of standard or modified Portland cement and water. For convenience, these substances are proportioned by volume. The water-cement ratio used depends on the tightness of the foundation rock. If the rock contains large voids which accept grout readily, the water-cement ratio may be 1:1. For tight seams or cracks in the rock, ratios of 20:1 have been used. The usual grout mixes are proportioned within the limits of 1:1 and 10:1. Thin grout placed under sufficient pressure to force out the excess water will form a hard, durable film. Neat-cement grout is easily pumped; it adheres to clean rock surfaces; and, when properly placed, it has relatively high compressive strength. It is used to consolidate foundation rock, to form a cutoff curtain under dams, and to grout contraction joints contained within large mass-concrete structures.

Clay or a mixture of clay and cement may be used as grout when a sealing filler is required in parts of the foundation that do not require strengthening. For example, it may be used to grout cavities and to form blankets or sealing mats within porous rock and alluvium. Clay used as grout must form a mixture suitable for pumping when combined with water, have minimum adhesiveness, and be free from organic or foreign matter. Before mixing, the clay should be screened through ¼-in. mesh hardware cloth to remove lumps, rocks, or other material that might damage the pump, or plug the grout lines. For clay-cement grout, minimum shrinkage is obtained when the mixture contains three parts of clay to one part of cement by volume. To this mixture, sufficient water is added to make the grout suitable for pumping. The equipment and methods used for placing clay grout are the same as those used for cement grout.

Bentonite is a soft, moisture-absorbing, colloidal clay found in various localities throughout the western United States. It has the following properties which, under special conditions, make it useful as grout: (1) It will absorb three or more times its own weight of water and, in so doing, will increase in volume seven or more times its dry bulk volume; (2) it is easily pumped when mixed with water; and (3) it is fine grained Bentonite may be mixed with sand or cement to grout off large flows of water, which cannot be grouted effectively with cement grout.

CHEMICAL GROUT

Chemicals are used for grouting by mixing, either prior to pumping or within the void, two or more solutions of mineral salts. These combine chemically to form another compound which crystallizes into a solid mass.

A number of different chemicals have been used for this purpose; but the most common are solutions of sodium silicate in combination with calcium chloride, sodium bicarbonate, or aluminum sulfate. Since these chemicals form a colloidal solution, it will penetrate tighter seams than cement grout. Two methods of chemical grouting are used—the single injection method in which the chemicals are mixed before being pumped, and the double injection method in which the individual chemicals are pumped separately into the foundation.

Asphalt Grout

Hot asphalt has been used to grout cavities or joints which could not be grouted by cement or cement-clay mixes because of the presence of percolating water. Asphalt grouting is expensive, and for that reason it should be used only to grout rock that cannot be sealed by other methods. The equipment required and the methods used for placing asphalt grout are described subsequently.

Special Grouting Mixtures

A mixture of Portland cement, fine well-rounded sand, and water has been used as a grout mixture to economize on cement. Grout mixtures containing a highly abrasive material, such as sand, usually are not satisfactory for grouting foundations because: (1) The sand grains are too large to penetrate into the small voids; (2) the mixture causes premature plugging within the grout hole; (3) the sand may settle out of the mixture and thereby plug supply lines; and (4) the mixture causes excessive wear of the pump and of the valves in the supply headers. However, sand-cement grout has been used successfully in tunnel-arch grouting, and also for filling large voids in the foundations.

An admixture is a material added to cement grout to control, to a limited degree, the physical properties of the mix. By supplying admixtures to the grout the following properties of the mix may be controlled: (*a*) Workability, (*b*) shrinkage, (*c*) high early strength, and (*d*) setting time. Materials used as admixtures, and the purposes for which they are used, are as follows:

Admixtures	Purpose
Pumicite, or diatomaceous earth	Improves workability of grout mixture
Calcium chloride, very high alumina cement (not Portland), or gypsum plaster	Decreases setting time
Clay, sand, or rock flour	Produces economical mix when filling large voids

Some of the materials are used to set nipples in nearly all grouting operations. They are high alumina cement, plasters, and calcium chloride. Very high alumina cement is used to make a quick-setting mortar for grout nipples. A satisfactory mortar may be obtained by blending 35% of very high alumina cement with 65% of Portland cement. If this blend is proportioned with water in a 0.5:1 water-cement ratio, the resulting mix will have a flash set. Gaging, molding, and other plasters with a gypsum base are used with cement for grouting nipples and patching leaks. A 1:1 mixture of plaster and cement, when mixed with water, will produce a grout that has a flash set. The setting time can be increased by reducing the proportion of plaster in the mixture. Calcium chloride, when added to neat-cement grout, can be used also to grout nipples and to patch leaks. The mixture should not contain more than 5% of calcium chloride by dry weight. On some jobs, calcium chloride has been added to the grout mix as an accelerator to plug leaks that occur at long distances from the hole. Accelerators should be used only under the supervision of experienced personnel; otherwise, some difficulties may occur due to a flash set of the mixture.

Foundation Grouting Procedures

Basic grouting procedures apply for all foundations requiring treatment by grouting methods. The manner in which these procedures are adapted depends on the judgment of the engineer in charge of the foundation treatment program. When making decisions, he must rely on his experience and on precedent established by the successful treatment of foundations at other dams. From time to time throughout the grouting program, the engineer must be able to decide pumping pressures, consistency of grout mix, location and spacing of grout holes, depth of hole to be grouted in one stage, rate of grout injection, and probable location of grout leaks. In making these decisions, he is aided by information gained from preliminary exploration data, water tests, and "grout-take" of the first series of holes.

The procedure followed in grouting foundations for concrete and earth dams is similar, the main difference being the location of the lines of holes. In the construction of earth dams, the plans for blanket grouting normally provide for lines of holes to be located between the cutoff walls, on either side of a cutoff wall, if only one wall is to be constructed, or over the entire area of the foundation beneath the impervious zone of the dam. The curtain grouting consists of one or more lines of fairly deep holes along or close to the axis of the cutoff wall footing.

In the construction of concrete dams, there are two major grouting operations—blanket grouting and curtain grouting. The blanket grouting plan usually provides for holes to be drilled on 10-ft to 20-ft centers along three or more lines parallel to the axis in the upstream part of the foundation area. The lines are about the same distance apart as the minimum distance between holes. For the curtain grouting, grout holes are normally spaced on 5-ft to 10-ft centers along one or more lines parallel to, and a short distance downstream from, the axis of the dam.

There are no fixed rules for establishing the minimum distance between grout holes. For the most part, this spacing depends on the type and extent of consolidation required in the foundation rock. Some bedrock contains bedding planes and fractures filled with weathered material, which should be removed by washing methods if practicable; that is, grout holes should be drilled close enough so that compressed air and water can circulate between the holes in order to remove the material effectively.

Several empirical formulas have been proposed for determining the depths for drilling grout holes. Although such formulas are satisfactory for rock which is sound and not affected by deep weathering, they cannot be applied dogmatically for estimating depths to which all grout holes must be drilled. When determining the depth for drilling grout holes, the engineer must have information on the purpose of the hole, the character of the foundation rock, the height and type of structure, and the position of the hole in the grout plan.

The grout supply line is connected to a grout hole through a manifold and nipple. The grout nipple is usually 1.5-in. black iron pipe, with one end threaded and the other end flared for calking, or both ends are threaded and a coupling is screwed on the lower end. The length of the nipple depends on the type of rock to be grouted and on the pressure to be used. In very poor rock, the nipple is extended as much as 5 ft into the hole to prevent excess grout leakage through rock fractures. If the surface rock is sound, a shorter nipple can be used. Usually, grout nipples are bonded to the wall of the hole with a thick grout made from cement with high early strength. They may also be anchored by being calked in place with lead wool.

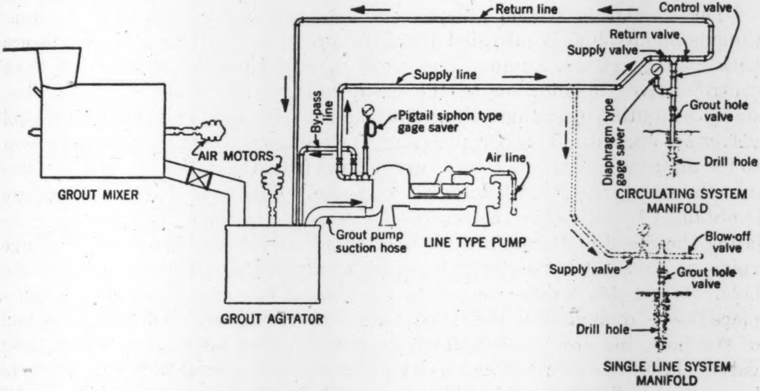
Return line
Control valve
Return valve
Supply valve
Supply line
By-pass line
Pigtail siphon type gage saver
Grout hole valve
Diaphragm type gage saver
AIR MOTORS
Air line
Drill hole
GROUT MIXER
CIRCULATING SYSTEM MANIFOLD
Grout pump suction hose
LINE TYPE PUMP
Blow-off valve
Supply valve
Grout hole valve
GROUT AGITATOR
Drill hole
SINGLE LINE SYSTEM MANIFOLD

FIG. 5.—SCHEMATIC DIAGRAM OF CIRCULATING AND SINGLE-LINE GROUTING SYSTEMS

The USBR uses two principal systems of piping for supplying grout from the pump to the grout holes. These systems, shown in Fig. 5, are designated as the single-line system (shown dotted) and the circulating system. In the single-line system one line extends from the pump to the grout hole. The connecting manifold has two valves, a supply valve and a blowoff valve. The supply valve is used to hold grout in the supply line when moving the manifold from one hole to another. The blowoff valve is used for clearing or washing the supply line. Beneath the manifold and attached to the pipe nipple is the grout hole valve. This valve is used to retain the grout within the hole when disconnecting the manifold at the union or to prevent water from entering the hole when washing the supply line.

The procedure for grouting with a single-line system is as follows: At the beginning of the grouting operation, when grout is first introduced into the supply line, the grout hole valve is closed and the supply and blowoff valves are opened. As soon as grout of the proper consistency is ejected from the outlet end of the blowoff valve, the grout hole valve is opened and the blowoff valve closed. When grout is being pumped into the hole, the pressure gage on the manifold must be observed continually to insure that the desired grout pressures are being used. The desired pressure is obtained by varying the speed of the pump, or in an emergency by opening the blowoff valve. When the pressure differential between the hole and the pump becomes excessive, it is an indication that the supply line is becoming partly plugged. This condition can be remedied by closing the valve to the grout hole and opening the blowoff valve. Grout can then be pumped through the supply line at increased velocity to scour out any deposits within it; or the supply line can be flushed with water. The blowoff valve should never be opened or closed in a manner that produces a water hammer in the supply line, with the grout valve open. Such a procedure might plug the fractures in the foundation rock that are penetrated by the grout hole. When the grouting is completed, the grout hole valve is closed to retain the grout within the hole until the grout has set.

The circulating system differs from the single-line system in that a continuous supply line is provided from the pump to the manifold and thence back to the agitator sump. The grout manifold for this system has three control valves in addition to the grout hole valve on the nipple. When starting grouting operations, the grout hole valve is closed and the three control valves are opened. This permits grout to circulate through the line and return to the agitator sump. As soon as grout flows from the outlet of the return line into the sump, the grout hole valve is opened. The desired grouting pressure is obtained by adjusting the return-line valve, or in some instances, by regulating the speed of the pump. It is often advantageous, when grouting large voids, to close the return valve in order to direct the full grout flow into the hole. When this is done the return valve must be opened periodically to replace the residual grout in the return lines with fresh grout. On the completion of the hole, the grout hole and main control valves are closed. With these valves closed the manifold can be disconnected and moved to the next hole to be grouted while the grout is still in the circulating system, unless it is necessary to add additional lengths of pipe to the system.

A brief comparison of the advantages of the two systems follows. The advantages of a single-line system are: (1) It is more economical to install since only one grout line is required; (2) less cement is wasted by cleaning the supply line when operations are temporarily discontinued; and (3) fewer valves and other replacement parts are required. A circulating system is advantageous because: (1) Less tendency exists for the grout lines to become plugged due to settling of cement out of the mix; (2) less cement is wasted when moving the supply line from one hole to another; (3) close control of grouting pressures is possible at the hole; (4) waste grout need not be deposited on the surface of the foundation in the vicinity of the grout holes; and (5) the grout pump remains cleaner because of rapid circulation of grout within the pump chambers.

Several modifications of the single-line and circulating systems are used. One modification is to install a return line to the agitator sump from a point on the supply line close to the pump. This line is equipped with a valve which is opened during periods when the pump is cleaned. Another modification includes the installation of a blowoff valve on a manifold connected to a circulating system. By this arrangement, the crew can begin grouting the hole as soon as the grout reaches the manifold instead of waiting until it returns to the agitator sump. Furthermore, by regulating the return valve, this system can be converted to a circulating system when the rate of injection decreases or grout leaks occur.

When foundation rock contains open fractures, the pervious and loose material in the fractures should be washed out (when possible) and brought to the surface. Water pumped into the hole, followed by air under pressure, causes a boiling action that breaks up the weathered material, penetrates fractures which connect with other holes, and carries loose material to the surface through adjacent holes. Separate supply lines are connected to a two-way manifold so that air and water can be applied alternately until clean water comes to the surface. The pressure used in washing fractures and seams should be slightly less than the maximum pressure used in grouting.

Grout holes are usually pressure tested with water to determine the consistency of the grout mix to be used, or to locate seams or other openings in the rock through which grout can leak to the surface. To pressure test a grout hole, water is forced through the supply line by the grout pump. When practicable, the pressure maintained should be the maximum that will be used when grouting the hole. The hole is subjected to this pressure for at least 10 min. If the leaks are sufficiently large so that the desired pressure cannot be attained, they must be calked.

GROUTING METHODS

Four general methods used in foundation grouting procedure are single-stage grouting, grouting by packers, successive-stage grouting, and multiple grouting. The choice of methods suitable for foundation treatment for any one site depends on the type and character of the foundation rock.

The single-stage grouting method consists of drilling the grout hole to full depth, installing the nipple, washing the hole, and grouting the hole in one operation. This method should be used only in grouting shallow holes in sound rock containing only small cracks and joints. Since relatively high pressures are required to attain the proper amount of grout penetration for single-stage grouting, this method should not be used where surface leaks develop in the foundation rock.

The use of packers affords a method of grouting the foundation rock at any required depth. The hole is drilled to full depth, and a packer is set at any desired depth within the hole. The hole can be washed either before or after the packer is placed. When it is necessary to confine washing operations to a limited part of the foundation traversed by the grout hole, washing may be performed after the packer is placed. If the foundation rock contains seams, these may be washed and grouted more effectively by placing a packer at the elevation where the hole intersects the upper limits of the seam. Although the packer-grouting method may be more costly than the single-stage grouting method in some cases, many foundations may be grouted more effectively when the former is used.

In successive-stage grouting the first stage requires: (1) Drilling grout holes to a predetermined depth; (2) washing this stage of the hole prior to grouting, then water testing the hole and grouting it; and (3) washing the grout out of the hole as soon as the cement has received a partial set. For the next stage the procedure is repeated after the grout in the surrounding rock has fully set. For each stage, as the hole is extended to its final depth, the grout pressures are usually increased.

When compared to single-stage grouting, successive-stage grouting has two principal advantages: (*a*) Increasing grout pressures can be used for increasing depths of hole; and (*b*) the amount of grout wasted at surface leaks is reduced. Successive-stage grouting is a more expensive method of foundation treatment than single-stage grouting.

Occasionally, when attempting to seal off leaks, a thick grout is used during the first stage; and the hole is washed or redrilled as soon as the grout in the rock has attained a partial set. The hole is then regrouted with a thinner mix. This operation is sometimes called multiple grouting.

When grouting a foundation, the grout inspector must decide the water-cement ratio of the initial grout mix and a suitable grout pressure for the particular hole. As the maximum allowable pressure is governed by foundation conditions, it is necessary for the inspector to determine an efficient rate of injection (number of sacks of cement per hour). This is done by varying the grout mix and the pressure without exceeding the maximum allowable pressure.

It has been found convenient to base grout mixtures on volumetric water-cement ratios. The consistency of the mix used depends on geologic conditions of the rock, and on the rate of injection. The range of water-cement ratios for grout mixtures may extend from 20:1 for thin grout to 0.5:1 for thick grout. As a general rule, the grout mixture used during the initial injection period for any hole should be thin, ranging from 10:1 to 5:1. By using a thin mix at the beginning of grouting operations, the inspector is able to observe the behavior of the grout hole and to minimize the chances of prematurely plugging the hole by using too thick a grout.

Accepted foundation grouting procedure specifies that the optimum rate of grout injection is that which permits the greatest quantity of cement to be forced into the foundation rock without causing appreciable displacement of the surface. With this in mind, the engineer should control the mix or pressure to achieve this purpose. When a thin grout mixture can be pumped readily without developing the maximum permissible pressures, the quantity of cement being injected can be increased by decreasing the water-cement ratio and maintaining the resulting higher pressures. Grout that is too thick may fill the rock fractures partly; however, grout that is too thin may shrink when it sets, unless the pressure is sufficient to force the excess water out of the mixture into the surrounding rock. The optimum grout mix, therefore, is one that contains the minimum volume of water necessary to carry the cement into all rock voids and fractures.

In order to determine the allowable pressure, the engineer must know the rock structure, depth of zone being grouted, and permissible magnitude of foundation movement, if any. In general, sedimentary rocks having horizontal bedding cannot be grouted with pressures as high as those used for grouting steeply dipping sedimentary or badly fractured igneous or metamorphic rocks. If it is necessary to place a rigid structure on a part of the foundation before it has been grouted, measurements of surface movement near the structure should be made while grouting is in progress.

The conditions to be met before grouting of any hole can be terminated are usually stated in the construction specifications. Present specifications of the USBR require:

> "* * * that the grouting of any hole shall be continued until the hole or grout connection takes grout at a rate of less than one cubic foot of the grout mixture in 20 minutes if pressures of 50 pounds per square inch or less are being used, in 15 minutes if pressures between 50 and 100 pounds per square inch are being used, in ten minutes of pressures between 100 and 200 pounds per square inch are being used, and in five minutes if pressures in excess of 200 pounds per square inch are being used."

Other specifications state that grouting shall be continued until "refusal." This means that grout is applied to the hole at maximum pressure until it sets within the hole or pipe nipple. Obviously, this condition applies only when the circulating grout system is used for supplying grout to the hole.

ASPHALT-GROUTING METHODS

Large subsurface flows of water are difficult to stop by grouting with cement or clay grout. For this condition, asphalt grouting has been used successfully. This method of grouting has proved particularly effective in sealing watercourses in underground rock channels. In addition, it has been used to plug leaks in cofferdams and in foundation rock.

The principal items of equipment required for asphalt grouting are heating tank, pump, supply lines, and gages. A suitable arrangement of equipment is shown in Fig. 6. A commercial asphalt heater similar to those used by roofing contractors has been found satisfactory. The heater should be provided with a baffle near the outlet to prevent lumps of asphalt from entering the supply line. Gear pumps, reciprocating pumps with ball valves, or 1-in. boiler-feed piston pumps have been used to pump asphalt grout. The supply lines are fabricated from 1.5-in. standard black iron pipe. The grout pressure gage is connected to the supply line by a 1-in. nipple and "pigtail" siphon. In order to prevent asphalt grout from plugging the gages the 1-in. nipple is filled with grease and the siphon is filled with oil.

Often, when grouting during cold weather, it is necessary to provide a method for heating the supply lines. One successful method is to insert an insulated, heavy, iron wire within the supply line. One end of this wire is connected to the bottom of the supply line; and the other end, to a high-amperage, low-voltage generator, such as is used in commercial welding machines. The machine is grounded to the supply line at a point where the supply line joins the pump. Steam has also been used to heat the supply line, but has proved cumbersome and costly.

Asphalt with a melting point between 165° F and 175° F has been found suitable for grouting. This material is heated to temperatures ranging from 400° F to 500° F before it enters the pump. The asphalt should be clean—that is, free from trash, gravel, or other debris. If the grout contains such materials, they will settle to the bottom of the heater and cause the formation of "hot spots" which result in premature deterioration of the heater. Coal-tar pitch is not a desirable material for grouting because it melts more slowly and chills more quickly than asphalt grout. Moreover, when pitch is heated above its melting point, it emits fumes which are dangerous to personnel.

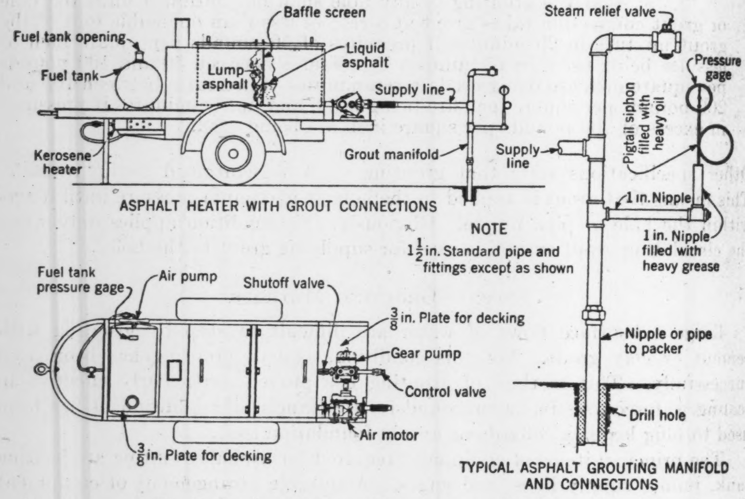
Baffle screen
Fuel tank opening
Liquid asphalt
Fuel tank
Lump asphalt
Supply line
Kerosene heater
Grout manifold
ASPHALT HEATER WITH GROUT CONNECTIONS
Steam relief valve
Pressure gage
Pigtail siphon filled with heavy oil
Supply line
1 in. Nipple
1 in. Nipple filled with heavy grease
NOTE
1½ in. Standard pipe and fittings except as shown
Fuel tank pressure gage
Air pump
Shutoff valve
⅜ in. Plate for decking
Gear pump
Control valve
Air motor
⅜ in. Plate for decking
Nipple or pipe to packer
Drill hole
TYPICAL ASPHALT GROUTING MANIFOLD AND CONNECTIONS

Fig. 6.—Asphalt-Grouting Equipment and Connections

Most grout holes contain some water; and, when hot asphalt is pumped into the hole, steam will form. Some of this steam is caught in the supply line and rises to the manifold. To prevent steam from blowing back into the heater, a relief valve connected to the top of the manifold is opened periodically during the initial injection of grout into each hole.

Extensive precautions should be taken to protect workmen engaged in asphalt grouting operations. Gloves, goggles, and ointment on exposed skin should be used by all grouting personnel. Occasionally, accidents happen in this type of grouting and precautions must be taken to prevent them.

Summary

This paper describes the present-day practices of the treatment of foundations of large dams by the USBR. Most of the future dams to be built on rock foundations will require some form of foundation treatment. Experience has proved that, when properly executed, pressure grouting effectively serves its purpose of improving foundations. Positive displacement pumps and mechanical mixers guarantee a continuous injection, and packers add to the control of pressures, thereby increasing the efficiency of the operation. Neat cement is still the most effective grouting material where permanence and strength are prerequisites. Certain admixtures are sometimes used under special conditions; but, generally, they offer no advantage in foundation grouting. Hot asphalt has proved effective in stopping large flows of water that could not be controlled by cement grout. Chemicals will penetrate smaller openings than cement but as yet chemical grouting is in the development stage and has not been proved on a large scale. The basic procedure used in executing the grouting program must be adapted to local foundation conditions, and the experience and the judgment of the engineer in charge are important factors in successful treatment.

DISCUSSION

A. MAYER,[4] M. ASCE.—The "Synopsis" of this paper contains the statement:

> "One important method of foundation treatment, chemical grouting, is not discussed in this paper since it is not as yet generally used by the Bureau of Reclamation (USBR) * * *."

and, in the "Summary," the authors state:

> "Chemicals will penetrate smaller openings than cement but as yet chemical grouting is in the development stage and has not been proved on a large scale."

Testimony presented at the 1948 meeting on large dams in Stockholm, Sweden, does not support the latter statement as applied to practice outside the USBR. At that meeting the writer described the work on the Genissiat Cofferdam and on the Lac Noir Dam in France.[5] E. Ischy,[6] general manager of the Soletanche Company, discussed the grouting procedure at the Castillon Dam, in France, where chemicals were used in combination with cement or clay. Because of these cases and other cases which have occured since the writer sincerely believes that grouting with chemicals is no longer in the experimental stage and can be applied reliably wherever necessary.

Aside from the question of chemical grouting, there is a general method to be followed in grouting procedure which, in the writer's opinion, the authors did not make quite clear. The problem, as they state it, is to stop percolation in the foundation material, be it rock or pervious alluvial material. In order to stop the flow, one must use a grouting material that will remain in the voids without being washed away either during the grouting process or afterward, when the reservoir has been filled. The nature of the grout will thus depend on the nature of the voids or cavities to be filled, and on the velocity of the percolating water. The most important quality of the grout will be its stability—the suspension must remain as a suspension until it has arrived at its setting point. This is particularly true when Portland cement is used because cement suspensions are generally not stable.

The writer, therefore, always recommends an addition of clay (bentonite, if possible) to the cement grout to insure the stability of the suspension during the pumping process. If the voids are very large, the proportions of cement, clay, and (eventually) sand are a matter of circumstance, to be determined in each separate case according to existing conditions. In the case of the Castillon Dam, the object of the grout was to enable a fissured, calcareous mass to be used as an abutment for an arch dam. The cavities were so large that cement alone would not have filled them. The percentages of cement, sand, and clay necessary to obtain a stable suspension were determined in the laboratory. The resulting product was so stable that it could be pumped into a regular water stream, remain there until the cement had set, and still yield the desired bearing power to the mass.

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[4] Inspecteur Général des Mines, Ministère de l'Industrie et du Commerce, Paris, France.

[5] "Dispositions les Plus Recentes pour Empecher la Formation des Renards," by A. Mayer, *Question No. 10, Report No. 22*, International Comm. on Large Dams, Paris, France, 1948.

[6] "Barrage de Castillon, Lutte contre les Erosions Souterraines," by E. Ischy, *Question No. 10, Report No. 36*, International Comm. on Large Dams, Paris, France, 1948.

Stable suspensions can also be obtained with cement and small admixtures of clay by commercially patented processes involving pressure grouting of sand-cement suspension. If resistance is not required, chemically treated clay can also be used, the object of the treatment being to obtain a progressive increase of the viscosity of the grout that will thus be stable against water pressure. The tests made during the grouting process at Genissiat, between two lines of clay grouted holes, 12 ft apart, showed that both screens were watertight up to a pressure of 9 kg per sq cm (128 lb per sq in.), the maximum that could be applied without breaking through the alluvial strata.

A choice between different methods will be determined by the size of the voids: (1) In alluvial sand (from 0.1 mm to 0.2 mm) it will be necessary to grout with chemicals or an asphalt solution; (2) in sand larger than 0.2 mm, a chemically treated clay can be used with or without Portland cement; and (3) if the voids are unusually large, sand can be incorporated in the grout to advantage.

Great care must be given to the nature of the chemical additions necessary to keep the clay in suspension, and to the chemicals used for the grouting process. If mineral wells, natural salts, or sea water are encountered, the conditions of set are completely changed and awkward incidents may occur. Most of these types of incidents have already been encountered and experience has shown that close cooperation between the laboratory and the field will solve most of the problems that arise.

V. L. MINEAR,[7] M. ASCE.—The literature on foundation grouting is enhanced by the publication of this paper, which summarizes what has been found to be good practice in the construction of twenty or more large dams since 1930. The paper is full of many small details each of which has been learned "the hard way" on some project. The difference between an excellent job and a mediocre job depends on the observance of these small details.

Great progress has been made in grouting equipment and methods since 1930. It is "a far cry" from the old "air pot" pneumatic injector to the modern grout pump with its special pistons, liners, valves, valve seats, packing, etc. Likewise, there is a great difference between the results accomplished by some temporarily idle concrete inspector, who mixed up some cement and water and blew it down a hole in the ground, and today's foundation engineer who, with a clear understanding of subsurface conditions given him by the engineering geologist, proceeds to correct existing defects rationally.

No comparable progress has been made in the selection and improvement of materials for grouting during this period. Many engineers question the durability of neat cement grout placed with the extremely high water-cement ratios (as much as 150 gal per bag) currently used, even if all the surplus water actually were extruded by pressure. The experience of the USBR with uplift on some of its older dams, the foundations for which were grouted, should be enlightening. One of the primary purposes of curtain grouting is to reduce uplift. Conversely, if the curtain deteriorates through the solution of the hardened cement, then uplift should increase progressively with age as the grout deteriorates.

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[7] Engr., Corps of Engrs., Office, Chf. of Engrs., Washington, D. C.

In 1935, D. W. Cole,[8] M. ASCE, stated:

> "The suggested limitation of benefits by the cementation process alone, opens a further field of usefulness for cement chemists and testing laboratories. A safe and reliable means of sealing the voids in otherwise completed masonry structures is needed, in order to make them virtually impermeable by water. This problem involves the production, at moderate cost, of a substance of demonstrable durability, having the property of setting or hardening in place without loss of volume or injurious reaction to any masonry constituent, and which can be carried in solution in water under pressure while maintaining a fluidity equal to, or greater than, water itself."

In the opinion of the writer the need expressed by Mr. Cole still exists for foundation grouting where the use of grouts with excessively high water content is considered a makeshift of doubtful value.

Byram W. Steele,[9] Assoc. M. ASCE.—The data on grouting equipment, materials, and procedures, presented in this paper, are interesting and valuable. The paper serves to revive curiosity as to the why's and wherefore's of ungroutable places that should be filled with grout of low water-cement ratio.

High water-cement ratios are not conducive to impermeability, durability, or satisfactory strengths in concrete. The same statement can be made relative to high water-cement ratios in Portland cement grout, except that the principal objectionable feature in such a grout is that it is pervious. Leaching will eventually take place and gradually transform an impervious area into a pervious one.

Under the heading, "Cement-Grout Mixtures," the authors state:

> "For tight seams or cracks in the rock, ratios of 20:1 have been used. The usual grout mixes are proportioned within the limits of 1:1 and 10:1. Thin grout placed under sufficient pressure to force out the excess water will form a hard, durable film."

The last sentence of this quotation caught the writer's eye on first reading and reminded him of some grouting experience in connection with mass concrete research for Hoover Dam (Arizona-Nevada) about 1930, in which it was reported that:[10]

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[8] "Stabilizing Constructed Masonry Dams by Means of Cement Injections," by D. W. Cole, *Transactions*, ASCE, Vol. 101, 1936, p. 743.

[9] Special Asst., Civ. Works Div., Office, Chf. of Engrs., Dept. of the Army, Washington, D. C.

[10] "Mass Concrete Research for Hoover Dam," by Byram W. Steele, *Proceedings*, A.C.I., Vol. 29, 1933, p. 313.

> "* * * For cracks of the order of .05 in. a grout in which the ratio of water to cement is as 75 to 100 by volume gives the most desirable film. A water-cement ratio of 1.00 contains too much free water that cannot be forced into the concrete or otherwise dissipated."

It is admitted that, if the excess water can be forced out and the resulting grout consolidated, a film could be produced that would be hard and durable, but this does not occur in too many instances and the result is a chalky grout that is permeable.

Development work in grout mixtures has been conducted over the intervening years in connection with oil well drilling, and especially in connection with repairs to tunnel linings and similar instances of deterioration in concrete. Newer patented processes have marked a very definite contribution to the science and technique of better grouting methods, but some basic research work still remains to be done before it will be possible to make use of the low water-cement ratios that are necessary to produce impermeability, even in grout.

In the early days of grouting contraction joints in dams, it was found necessary to screen all the cement through a 200-mesh sieve to produce a grout that would be devoid of large particles of cement and sufficiently fluid to fill contraction joints with from $\frac{1}{16}$-in. to $\frac{1}{8}$-in. openings. It is an extremely expensive and laborious matter to move cement through a 200-mesh sieve even if the operation is all done by machinery and more or less automatically. There are various methods of disposing of the oversize particles of cement, but reliable information relative to ways and means of producing flowable grout with a low water-cement ratio, in fine cracks, is lacking. Therefore, it appears that it would be desirable to do some extensive basic research on grouting materials of various kinds, including Portland-pozzolanic blends of cement. Particle size and gradation of all material should also come in for a thorough investigation. This statement is made because of the obvious advantages relative to nonsegregation that accrue to the grout mixtures that have been perfected by at least one commercial concern; but the so-called oversize particles in the cement and in the fly ash still present the same old difficulties found in straight Portland cement grouts. It is extremely difficult to do a satisfactory job of grouting fine seams of the order of 0.02 in. or less under moderate pressures such as can be used in contraction joints or in foundation areas immediately adjacent to the surface. Clusters of coarse particles (± 60 microns) plug the cracks and prevent the creation of a grout film of acceptable consistency.

It thus appears that the fineness and gradation of a type III[11] (high early strength) cement which has relatively few particles of + 60 microns might be more satisfactory for grouting fine seams than cements of types I and II.[11] If it can be proved that there is a definite advantage in the use of finenesses similar to, or finer than, the fineness of a type III cement, it is suggested that the cement and the admixture used in the commercially produced grout be ground to a fineness similar to, or finer than, that of type III cement, and that further tests be conducted with this particular grouting procedure.

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[11] "Long-Time Study of Cement Performance in Concrete," by William Lerch and C. L. Ford, *Proceedings*, A.C.I., Vol. 44, 1948, p. 763

To summarize: It is believed that to produce relatively impermeable grout films under moderate pressure (50 lb per sq in.), and especially for seams as fine as about 0.02 in., it is mandatory that water-cement ratios of the order of 10 to 1 and more be reduced to as near 1 to 1 as is practicable. To accomplish this objective, finer-ground cementing materials and blends should be substituted for those normally used and this refined grout should be tested in comparison with the product now in common use.

James B. Hays,[12] **M. ASCE.**—All the basic information, as well as the operational details, for treating, tightening up, or "firming" foundations for dams, is contained in this paper. The same procedure is also suitable for other structures. The authors have carefully described certain features of the mechanical system, particularly the pumps, that are a "must."

During the late 1920's, the writer was confronted with instructions to do some high-pressure grouting on the Calderwood (Tenn.) dam site. The literature of that day contained very little information except on a certain patented process. At that time the engineers had been using both the vertical and horizontal grout machines with an air pressure of 100 lb. A pump was tried and the old horizontal mixer type of grout machine was used to mix the batches which were dumped into a large tub and stirred by hand, thus supplying a steady flow for the pump. The early type of valves, liners, and packing provided caused no end of trouble.

When the writer went to Boulder Dam (Arizona-Nevada), it was found that the contractors had learned how to handle the pump troubles by rubber valves and pistons. However, they had a cumbersome method of mixing and measuring water. By a combination of the Boulder Dam and the Calderwood Dam methods, together with some other improvements, a really efficient and compact unit was evolved. Later similar outfits were built and used on many of the projects of the Tennessee Valley Authority where they were known as the "Boulder grout machines."

There have been instances of drainage holes drilled from a gallery, after grouting, directly into a grouted zone; as a result there was no seepage of water from the holes. However, not far downstream and continuing for some distance, several holes drilled for uplift testing showed considerable pressure under full reservoir. In general, the writer would suggest that drainage holes (see Fig. 1) be inclined more in a downstream direction, say, as much as 30°. In this way the holes should tap an ungrouted zone and relieve the upward pressure.

Under the heading, "Cement-Grout Mixtures," the use of bentonite as an admixture is mentioned. Bentonite is a dispersing agent. If it is mixed first and the cement added later there will be a minimum loss of swelling; but if the cement is mixed first and the bentonite added later the swelling will be greatly reduced.

In conclusion, the writer highly recommends this clear and factual paper to all who are concerned with grouting.

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[12] Formerly Project Mgr., Morrison-Knudsen Afghanistan Co., San Francisco, Calif.

A. W. SIMONDS[13] AND FRED H. LIPPOLD,[14] MEMBERS, ASCE, AND R. E. KEIM,[15] ASSOC. M. ASCE.—The discussion by Mr. Mayer presents valuable information regarding grouting with chemicals and other materials. It is true that chemical grouting has been accomplished successfully on some projects, chiefly in Europe; however, its use for grouting foundations of dams has not become widespread in the United States. The chief objections to chemical grouting, using sodium silicate as one of the ingredients, are the high cost of the materials and the tendency of the grout to shrink and disintegrate if it is allowed to dry out during periods of low reservoir level. The grout does not return to its original condition when it is again wetted. Certain resin plastics may have more stable qualities, but they are also expensive. Until a low cost chemical grout having foolproof qualities is available, the writers feel that there is opportunity for further development in chemical grouting materials.

The writers agree with Mr. Mayer's statement that the addition of clay or bentonite to cement grout will improve the pumpability of the mixture by keeping the cement in suspension. The use of clay or bentonite is particularly beneficial when grouting against flowing water and when various bulking materials are needed in the grout for stopping leaks. On one occasion, when grouting against a leak beneath the cofferdam during the construction of Grand Coulee Dam in Washington, a grout mixture of the following composition was used:

Ingredients	Quantity
Portland cement (cubic feet)	2
Sand (cubic feet)	6
Lump bentonite (pounds)	50
Sawdust (cubic feet)	8
Shavings (cubic feet)	10 to 16

The foregoing ingredients were mixed thoroughly in the dry state and the following were added:

Ingredients	Quantity
Water (cubic feet)	8 to 10
Pulverized bentonite (pounds)	30

After the second part of the batch was added and the whole thoroughly mixed, the resulting product had a volume of about 14 cu ft and resembled wet cement plaster. This grout was effective in sealing off flowing water. It is doubtful if a grout mixture containing solids of this type could have been injected without the addition of bentonite.

In the case of grouting cavities where large volumes are to be placed, the use of various filler materials with the cement is profitable. The fillers most commonly used with Portland cements are clay, rock flour, sand, and (recently) certain pozzuolanic materials. If sand alone is used as a filler, it should be sufficiently fine so as not to settle out of the grout stream. A fine sand from wind-blown deposits is preferred. The usual mixes for filling large voids which have been used consist of one part of Portland cement to two parts or three parts of sand or clay.

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[13] Engr., Bureau of Reclamation, U.S. Dept. of the Interior, Denver, Colo.

[14] Engr., Bureau of Reclamation, U.S. Dept. of the Interior, Denver, Colo.

[15] Engr., J. Gordon Turnbull, Inc., Richland, Wash.; formerly Engr., Bureau of Reclamation, Denver, Colo., and TVA, Knoxville, Tenn

Both Mr. Minear and Mr. Hays call attention to the improvements in grouting equipment which have taken place since about 1930. Equipment failures are costly as they may result in the loss of valuable time, the loss of an expensively drilled hole, or the loss of an embedded grouting system. With modern equipment, such failures are less frequent. It is gratifying to have dependable pumps and mixers which will stand up for a reasonable period of time before needing extensive repairs.

Mr. Hays' suggestion that drainage holes be inclined more in a downstream direction is worthy of consideration. Mr. Minear asks whether leaching of grout causes high uplift pressures under dams after several years of service. To date (1950), at USBR dams on which uplift pressures have had to be reduced, additional drain holes intercepting only slight quantities of water have been effective. The small quantities of water collected by the new drains and the recleaned original drains lead the writers to believe that high uplift under older dams is not caused by leaching of the cement grout as much as it is caused by the ineffectiveness of the drain holes in tight rock. At Hoover Dam it was found that the original grout curtain had not been deep enough to cut off the water under that extremely high head. Additional deeper grouting and a new drainage system corrected the uplift pressure there.[16]

Mr. Steele questions the use of grout mixtures having high water-cement ratios. Inasmuch as a low water-cement ratio suitable for a mass concrete mix would not be pumpable under most foundation grouting conditions, the writers believe that thin mixtures are no more weakened than moderate mixtures if most of the excess water is forced out of the grout. In injecting grout at high pressures into a dry formation, some of the excess water is taken up by the absorptive properties of the ground and some is squeezed out of the grout into progressively smaller seams by the pressure applied from the pump. It has been noticed on many occasions that grout emerging from surface leaks is thicker than that being pumped into the hole. Obviously, if the foundation is wet, or if grout is being used to shut off the flow of water, there is no value in using a thin grout mixture.

Cements now available commercially are greatly improved over the cements manufactured in about 1930. The greatest difficulties encountered formerly were the presence of tramp iron and relatively coarse particles of unground clinker in the cement. Such material not only was harmful to pumping equipment, but also plugged fine seams and cracks. Cement from modern mills equipped with air separators is superior to the cement made in old style mills, in that this coarse material is rarely present.

The writers agree with Mr. Steele that further research and development are needed for materials which can be used to advantage in grouting where the maximum permissible pressure is about 50 lb per sq in.

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[16] "Uplift Pressures in Concrete Dams," by Kenneth B. Keener, *Proceedings-Separate No. 25*, June, 1950.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2440

PUBLIC UTILITY CONDEMNATION CASES IN THE STATE OF WASHINGTON

By Henry L. Gray,[1] M. ASCE

With Discussion by Messrs. H. D. Southerland, Jr., and Henry L. Gray.

Synopsis

No instructions or suggestions as to methods will be found in this paper, which is devoted entirely to certain special uses for valuation data, and to an attempted analysis of the results. A simple narration of the problems entailed in the condemnation cases in the State of Washington is the purpose of this paper. It must not be taken as a brief in favor of public ownership, or against it; but as an argument for the improvement of engineering techniques in this field and for a more objective participation by engineers. This paper was presented at the 1948 ASCE Annual Convention at Seattle, Wash., and has been brought up to date as of December 1, 1949.

Enabling Legislation

In the State of Washington there is a very definite and formidable plan for the public ownership of light and power utilities. An appreciable part of the plan already has been completed. The plan originated with the State Grange, which sponsored a measure known as Initiative Number One, which was adopted by popular vote at the general election of November 4, 1930. The initiated measure was sent to, and passed by, the 1931 Legislature[2] as an act authorizing:

> "* * * the establishment of Public Utility Districts for the purpose of conserving the water and power resources of the State of Washington for the benefit of the people, and to supply public utility service, including water and electricity for all purposes."

Throughout the campaign, the measure was commonly known as "the public power bill." For quite a while after its passage, the possibility of water service was overlooked; but presently several public utility districts began to

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Note.—Published in February, 1950, as *Proceedings-Separate No. 2.* Positions and titles given are those in effect when the paper or the discussion was received for publication.

[1] Cons. Engr., Seattle, Wash.

[2] "Remington's Revised Statutes of Washington," Vol. 11, Chapter 3, Title 84, Sections 11605 and 11616, Olympia, Wash., 1931.

provide water service. In this paper, "public utility districts" and "districts" have the same meaning, and the terms are used interchangeably.

PUBLIC UTILITY DISTRICTS

The act provides for the creation of public utility districts, and for the election of boards of three commissioners, which are given broad powers. Light and power and water systems can be constructed, purchased, or acquired through condemnation; revenue bonds can be issued in any amount; general obligation bonds up to 1.5% of the assessed valuation can be issued; taxes up to 2 mills can be levied annually—all possible without a vote of the people. The only authority required for the construction or acquisition of a property is a resolution of the board, which must recite the program, describe the property to be built or acquired, state the probable cost, outline the plan of financing, and specify the type of security to be sold. With a few minor exceptions, all the financing has been with revenue bonds, which in most instances have had a ready sale at low rates of interest. (General obligations in excess of 1.5% of the assessed valuation require a popular vote.) A public utility district can acquire property outside its borders—even in another county. Thus far, the business of public utility districts has been limited to light and power service and domestic water service; but it is generally believed that they can engage in irrigation, although none has gone into that field.

The property of public utility districts is not on the tax rolls. In lieu of general property taxes, the districts pay the following percentages of their annual gross electric revenues: (1) 5% from sales of energy generated in plants owned by the district; and (2) 2% from sales of purchased energy. There is no corresponding tax on gross revenues of district-owned water systems. In common with all other utility business in the state, public utility districts pay state excise taxes of 3% on gross sales of electricity and water. The total light and power taxes paid by such districts, therefore, will range from 5% to 8% of their gross revenues—considerably less than the taxes paid by private light and power companies. Federal income taxes account for most of the difference.

Following the passage of the legislative act, the public utility district program was principally one of organization. Nothing much was done about acquiring properties until the late nineteen thirties. At that time there were about twenty private companies, large and small, engaged in the light and power business in the state. When the campaign of acquisition got under way, the public utility districts bought out twelve of those companies, and condemned two others; so that only six private light and power companies remain in business in the State of Washington today. Three of those are quite small. The three large companies remaining together served thirty-four of the thirty-nine counties of the state; but they have been subjected to repeated condemnation proceedings instituted by the public utility districts, and have lost fifteen counties to the districts—ten through condemnations, and five through negotiated sales. There are sixteen city-owned light and power systems in the state, which cannot be condemned.

There are thirty-nine counties in the State of Washington. There are thirty public utility districts in twenty-nine counties—one county has two districts. With that exception, the districts are coextensive with the counties. Of the thirty public utility districts, seven have no utilities. One district has a water system only. Twenty-two districts are in the light and power business and, in some instances, also have water systems. Of the twenty-two districts, seven have some generating facilities; but the greater part of the energy sold by the districts is purchased from the Bonneville Power Administration. These twenty-two districts have instituted sixteen condemnation suits. In addition, three other districts went through condemnation proceedings but did not accept the awards. To date, there have been nineteen condemnation cases carried through by public utility districts, all involving light and power properties. Thirteen different properties in ten counties, were taken through condemnation, and eight others were acquired through negotations stemming from possible condemnation suits. There were six cases in which the districts did not accept the awards; but, later, three of the properties involved were purchased by the respective districts. These condemnation suits are the subject of this paper.

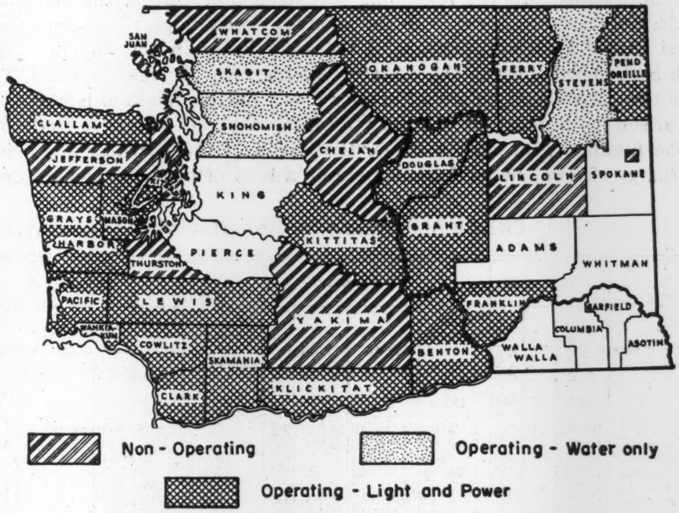
WHATCOM
SAN JUAN
SKAGIT
OKANOGAN
FERRY
STEVENS
PEND OREILLE
CLALLAM
SNOHOMISH
CHELAN
JEFFERSON
DOUGLAS
LINCOLN
SPOKANE
KING
GRAYS HARBOR
MASON
GRANT
KITTITAS
ADAMS
PIERCE
THURSTON
WHITMAN
PACIFIC
LEWIS
FRANKLIN
GARFIELD
YAKIMA
WAHKIAKUM
COLUMBIA
COWLITZ
BENTON
WALLA WALLA
ASOTIN
SKAMANIA
CLARK
KLICKITAT
Non - Operating
Operating - Water only
Operating - Light and Power

FIG. 1.—PUBLIC UTILITY DISTRICTS IN THE STATE OF WASHINGTON

ELEMENTS OF VALUE

The counties that contain districts are shown in Fig. 1. Mason County has two districts, both operating.

The various elements of value and final value claimed by each petitioner (district) and by each respondent (company), in each of the nineteen cases, together with the awards, and other data are listed in Table 1. The abbreviations in Col. 2 signify the following companies:

Abbreviation	Full name
WGE	Washington Gas and Electric Company
PSP	Puget Sound Power and Light Company
NWE	Northwestern Electric Company
PPL	Pacific Power and Light Company
WWP	Washington Water Power Company
PGE	Portland General Electric Company

Of the nineteen court proceedings, two were heard by the same judge, who fixed the awards. In the remaining seventeen proceedings, the verdicts, or awards, were fixed by juries. In the nineteen proceedings, nine were heard in United States District Courts, and ten were heard in the Superior Court, which is the local court of the State of Washington. In cases 3 and 4, Table 1 (the two cases heard before a Superior Court judge), the awards were low in comparison with most of the jury awards. As the attorneys and courts became familiar with such matters, the length of the hearings was reduced materially. These condemnations became a kind of business, and frequently the same men—lawyers, engineers, and economists—would appear in different cases. The attorney who handled case 1, Table 1, for the company was in a number of the later cases. Another attorney handled the greater number of cases for the districts.

TABLE 1.—STATISTICS RELATING TO CONDEMNATION CASES

Reference	Public utility district No. 1 of the county of: (1)	versus Private company[a] (2)	COURT[b]		Case opened (5)	VERDICT		Actual cost of investment in property (8)	REPRODUCTION COST NEW		
			U. S. District (3)	State Superior (4)		Date (6)	Dollars (7)		New Condition		Less De
									District (9)	Company (10)	District (11)
1	Cowlitz	WGE	✓		2/ 4/40	4/19/40	$6,011,506	....	$5,721,994[c]	$6,482,975	$3,596,146[c]
2	Whatcom	PSP	✓		5/12/40	8/10/40	5,000,000	....	4,387,180[c]	5,356,711	2,664,603[c]
3	Skamania	NWE		✓[b]	1/26/42	1/31/42	68,000	170,936	100,051	102,608	60,889[c]
4	Skamania	PPL		✓[b]	1/26/42	1/31/42	32,000	64,418	42,937	42,503	25,964[c]
5	Lewis	PSP	✓		4/ 6/42	5/10/42	2,100,000	....	2,231,219[c]	2,604,278	1,361,141[c]
6	Thurston	PSP	✓		4/ 6/42	5/10/42	4,400,000	....	3,598,349[c]	4,195,251	2,501,139[c]
7	Cowlitz	PSP	✓		4/ 6/42	5/10/42	1,100,000	....	1,250,435[c]	1,437,060	765,236[c]
8	Okanogan	WWP		✓	11/ 5/42	1/ 2/43	2,227,531	....	2,674,411[c]	3,056,267	1,883,477[c]
9	Douglas	WWP		✓	11/ 5/42	1/ 2/43	139,165	....	166,689[c]	186,293	105,953[c]
10	Snohomish	PSP	✓		2/ 2/43	3/19/43	9,500,000	....	5,968,111[c]	7,437,869	3,631,589[c]
11	Clallam	PSP		✓	11/15/43	12/ 1/43	600,000	445,851	718,214	814,495	449,224
12	Grant	WWP		✓	11/13/44	12/ 8/44	546,985	509,737	595,985	691,630	428,674
13	Clark	PGE	✓		10/ 8/45	10/17/45	801,000	706,419	978,960	1,225,045	728,993
14	Benton	PPL	✓		6/18/46	6/29/46	1,265,000	622,825	844,560	1,227,432[c]	561,760
15	Cowlitz	NWE		✓	8/12/46	8/20/46	493,215	279,110	429,546	600,754	285,632
16	Klickitat	PPL	✓		11/12/46	11/21/46	670.070	477,582	669,016	881,714	468,021
17	Franklin	PPL		✓	5/12/47	5/22/47	1,200,000	421,405	651,614	830,850	461,818
18	Clark	PPL		✓	11/12/47	11/25/47	4,837,500	3,476,997	5,701,306	6,905,640	3,747,109
19	Stevens	WWP		✓	5/24/48	6/11/48	2,500,000	1,002,033	1,765,574	2,640,603	896,029

[a] Symbolic names for bre vity; for full title of company, see text. [b] All cases decided by jury, except cases 3 and 4

Reproduction Costs.—In each proceeding, the usual elements of value were presented. Actual cost of construction (when it could be ascertained) was introduced, generally by the districts. Each side presented estimates of reproduction cost new; and in the past few years it has not been easy to keep the reproduction cost estimates abreast of current material prices and wage scales. The matter of deliveries received no consideration, as it was assumed that materials could be obtained. In cases involving much property, it was the tendency of the engineers preparing the estimates of reproduction cost for the districts to consider that all material would be furnished by the owner, but would be put in place by a contractor who would furnish all labor and construction equipment. Although this was a departure from methods frequently used, it was justified by the fact that several local light and power concerns were doing a substantial part of their expansion work precisely that way. In the reproduction cost estimates prepared for the power companies, the labor construction overhead, including contractor's profit, generally ranged from 40% to 50% of the cost of labor. In such estimates, the cost of labor was from 28% to 35% of the total cost of labor and material.

In estimating reproduction costs, all prices used were spot; and frequently there was an agreement on the inventory, and on the land values; but nearly

INVOLVING PUBLIC UTILITIES IN THE STATE OF WASHINGTON

preciation	GOING VALUE		Operating revenue, last twelve months	MEAN MARKET VALUE		MEAN ADMITTED SEVERANCE DAMAGES		TOTAL, COLS. 16 PLUS 18 AND 17 PLUS 19		Reference
Company (12)	District (13)	Company (14)	(15)	District (16)	Company (17)	District (18)	Company (19)	District (20)	Company (21)	
$5,677,365[c]	$347,500[c]	$615,502[c]	$ 976,108	$3,778,333	$7,400,000	....	....	$3,778,333	$7,400,000	1
4,322,246	262,000[c]	600,000	1,089,214	2,867,000	5,033,333	$118,687	$2,100,000	2,985,687	7,133,333	2
77,648	....	....	21,898	51,000	95,000	1,272	5,507	52,272	100,507	3
32,939	....	....	13,341	25,500	40,000	710	1,608	26,210	41,608	4
2,125,774	....	210,000	330,927	1,100,000	2,033,333	40,667	566,667	1,140,667	2,600,000	5
3,646,544	340,175[c]	450,000	750,691	2,606,667	4,667,778	157,500	1,544,444	2,764,167	6,212,222	6
1,185,331	105,004[c]	150,000	245,767	820,000	1,455,555	21,333	466,667	841,333	1,922,222	7
2,651,531	....	420,000	396,524	1,100,000	3,750,000	29,004	1,000,000	1,129,004	4,750,000	8
158,442	....	30,000	31,495	53,333	265,000	775	93,333	54,108	358,333	9
6,116,514	....	....	1,828,518	3,850,000	9,083,333	65,000	3,583,333	3,915,000	12,666,667	10
655,469	....	80,000	220,437	402,500	883,233	10,000	275,000	412,500	1,158,233	11
608,994	....	110,000	125,248	392,500	812,500	10,000	200,000	402,500	1,012,500	12
999,924	....	....	457,490	682,500	1,225,000	39,500	250,000	722,000	1,475,000	13
1,032,282[c]	....	....	304,049	537,500	1,375,000	27,500	350,000	565,000	1,725,000	14
521,573	....	....	100,998	312,500	625,000	12,250	97,500	324,750	722,500	15
796,202	....	....	168,169	465,000	850,000	14,000	124,000	479,000	974,000	16
746,361	....	....	294,249	625,000	1,212,500	31,500	317,500	656,500	1,530,000	17
5,949,805	....	....	1,308,539	3,350,000	7,125,000	50,033	698,750	3,400,033	7,823,750	18
2,241,446	....	....	287,656	777,500	2,200,000	6,400	328,333	783,900	2,528,333	19

which were decided by judge. [c] The mean of two or more.

always there was a substantial difference in the estimates of the two sets of engineers, which was accounted for principally by the assumed performances of work crews. After the end of price control, following World War II, there was considerable variation in material prices as used.

The reproduction cost estimates were then reduced by deducting accrued depreciation, which was arrived at by various methods, such as inspection, mortality tables, and efficiency tests. After the valuation experts completed their work, the carefully prepared reproduction costs and depreciation studies were buried beneath an avalanche of opinions as to final value. These details were very necessary and involved much work; but, in the end, they disappeared from sight.

One law firm engaged on these cases had a low opinion of reproduction cost estimates, and considered them of doubtful value. Apparently the decision of the United States Supreme Court in the Hope Natural Gas Case (320 US 591)[3] gave considerable support to actual cost as the prime element of value. Although there is a decided tendency for regulatory bodies to favor actual cost as the rate base, it does not play much of a part in sales or condemnations. In the nineteen cases in Table 1, actual cost was given in eleven cases.

Engineering Estimates.—Engineers may have "lost caste" in this handling of valuation problems, because of the wide differences in engineering estimates. In the nineteen cases in Table 1, the aggregate reproduction cost new, as estimated by the engineers representing the companies was about 20% in excess of the aggregate estimate made by the engineers for the districts. This is a substantial difference where inventories are practically the same. In these nineteen cases, all the estimates of reproduction cost new, introduced by the districts, were made by local engineers who also took the inventories, were intimately familiar with field conditions, and spent much time on the job. The reproduction costs prepared for the companies were made by engineers from other parts of the United States, who generally had not spent much time on the job; and, invariably, their estimates were based on inventories taken by someone else. Although the writer implies no criticism of nonresident engineers, he wishes to sound a note of warning to all engineers. Since about 1938 engineers have lost much ground to the accountants and the economists. It cannot be regained when engineers are unable to agree among themselves on simple construction details and costs. Naturally, differences are to be expected, and some allowances must be made for excessive zeal, or bias, and for occasional incompetence; but there is nevertheless room for improvement in engineering cost estimates.

Going Value.—In the cases cited (Table 1, Cols. 13 and 14) "going value" appeared from time to time. In this field of estimating, engineers also are weak. Nothing exists more definitely than that factor called "going value" and yet engineers have been unable to devise a rational method for fixing its dimensions. In Table 1, there were thirteen claims for going value; and not a single logical suggestion was advanced for determining it. Most of the claims represented the opinions of the witnesses, and had no basis. Not the slightest attention was paid to any such claims. Unless engineers can develop some method of arriving at the proper amount of going value, they should stop talking about it.

Gross Revenue.—In Table 1, Col. 15 gives "operating revenue, last twelve months." This is "gross revenue" for the most recent 12-month period available. There were practically no disputes about this item, although some companies that had paid so-called "customer dividends" by forgiving one month's bill contended that the forgiven bills should be included as revenue. Revenue is a very important element. Any purchaser of utility shares will want to know first how much do they pay in dividends—that is, how much do they earn? The stock purchaser does not attempt to arrive at reproduction cost per share, or at similar valuation details. He does not have any such data. He is concerned principally with earnings; they fix the value to the investor. Unfortunately, in the nineteen cases of Table 1, there were disagreements as to net earnings—because of disagreements as to the proper methods of arriving at the cost of power and of properly apportioning federal income taxes. If there had been agreement on net earnings, some real information might have become available.

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[3] *Public Utilities Reports*, New Series, Vol. 51, 1944, p. 193.

Market Value.—In the nineteen cases, there were approximately sixty-five different estimates of market value. The testimony of the market value experts (most of whom were engineers) literally filled hundreds of pages of transcript. In every case, with every witness, the technique was exactly the same. The witness would begin by giving the standard legal definition of market value; then he would mention the various elements of value assembled by his side to which he had given consideration; and then he would wander from a field of Chamber of Commerce statistics into a realm of conjecture; and, finally when everyone in the court had begun to doze, he would give his estimate of market value, which was his opinion, based on his study of the facts. To the writer, these constant performances were tiresome—they must have been even more so to the witnesses. Each side would have two or more market value witnesses; and, although the market value estimates introduced by the districts differed widely from those introduced by the companies, there was a definite "meeting of minds" among the witnesses for any one side. The entire market value show was depressing, and yet it is typical of what happens all over the United States in every valuation case. Surely, engineers can do better than that.

Purposely, of course, the writer is severe. Witnesses are not free to speak their minds. They must work with the lawyers who, in turn, must consider the client. As the subject is controversial, naturally the lawyers want to give their client "a break." Value cannot be determined by formula, and engineers cannot put in too much argumentative testimony; but it seems to the writer that they might at least explain their mental processes in an attractive manner. Unhappily, they are "stymied" on one particularly important point—capitalizing net earnings. Public utility rates that produce the earnings are subject to regulation; therefore the capitalization of net earnings is not conclusive.

ANALYSIS OF COURT RECORDS

In the condemnation cases in the State of Washington, a district is not obliged to accept the award. This would seem unfair to the companies, but actually it is not. The companies do not want to lose their properties; and, if the award is such that the deal can be financed, the districts will take over.

By paying the amount of the award to the court, a district can take over even with an appeal pending. This procedure was not always permitted.

In thirteen of the nineteen condemnation cases listed, the districts accepted the awards, arranged a financing, took possession of the properties, and generally are doing quite well. In five cases, the properties were not taken over. In one instance, case 2, Table 1, the award of the jury was reasonable enough; but the company appealed and the final verdict was delayed. Condemnation awards bear interest until paid or finally disposed of. In case 2, the interest accumulation was large, and the district could not secure the kind of financing it wanted and finally dropped the matter. Then came World War II, and the earnings of the property increased substantially—so that the deal would have been advantageous for the district—but by that time it was too late.

Cases 5, 6, and 7, Table 1, were consolidated in court, with the understanding that all verdicts must be accepted or rejected by all three districts. Later, it was found that, in case 5, the verdict was too high for financing; hence all three verdicts were rejected. In cases 10 and 19, the awards were too high to permit financing.

TABLE 2.—APPRAISAL OF CONDEMNATION CASES IN TERMS OF COURT AWARDS (COLUMN NUMBERS IN HEADINGS REFER TO TABLE 1)

Case No. (see Table 1)	VERDICT (COL. 7) DIVIDED BY THE SUM OF MEAN MARKET VALUE AND ADMITTED MEAN SEVERANCE DAMAGES (COLS. 20 AND 21)			Verdict (Col. 7) divided by operating revenue (Col. 15)	VERDICT (COL. 7) DIVIDED BY REPRODUCTION COST (COLS. 9 AND 10)		
	District (1)	Company (2)	Mean (3)	(4)	District (5)	Company (6)	Mean (7)
1	1.59	0.81	1.08	6.16	1.05	0.93	0.99
2	1.67	0.70	0.99	4.59	1.14	0.93	1.03
3	1.30	0.68	0.89	3.11	0.68	0.66	0.67
4	1.22	0.77	0.94	2.40	0.75	0.75	0.75
5	1.84	0.81	1.12	6.35	0.94	0.81	0.87
6	1.59	0.71	0.98	5.86	1.22	1.05	1.13
7	1.31	0 57	0.80	4.48	0.88	0.77	0.82
8	1.97	0.47	0.76	5.62	0.83	0.73	0.78
9	2.57	0.39	0.67	4.42	0.83	0.75	0.79
10	2.43	0.75	1.15	5.20	1.59	1.28	1.42
11	1.45	0.52	0.76	2.72	0.84	0.74	0.78
12	1.36	0.54	0.85	4.37	0.92	0.74	0.85
13	1.11	0.54	0.73	1.75	0.82	0.65	0.73
14	2.24	0.73	1.10	4.16	1.50	1.03	1.16
15	1.52	0.68	0.94	4.88	1.15	0.82	0.96
16	1.40	0.69	0.92	3.98	1.00	0.76	0.88
17	1.83	0.78	1.10	4.08	1.84	1.44	1.62
18	1.42	0.62	0.86	3.70	0.85	0.70	0.77
19	3.19	0.99	1.51	8.69	1.42	0.95	1.18
Average....	(1.74)	(0.67)	(0.96)	(4.55)	(1.07)	(0.87)	(0.95)

In Table 2, the evidence is resolved in terms of the awards. Presumably, the jury would listen to the testimony, weigh the various claims, and arrive at a market value. The jury, presumably, would find the amount of severance damages and add that to market value; and the sum would be the award. This procedure is a customary one. The results may be examined by comparing the award in each case with the sum of market value and admitted severance damages claimed by each side as in Cols. 1, 2, and 3, Table 2. The word "admitted" is used to modify "severance damages" because some witnesses refused to admit that there would be severance damages, and their estimate of severance damages was zero. The "mean" used is the mean of amounts greater than zero. Table 2 shows that the awards ranged from 111% to 319% of the sums of the amounts for market value and severance damages claimed by the witnesses for the districts, and from 39% to 99% of the corresponding amounts claimed by the witnesses for the companies. For the nineteen cases, the straight average percentage, based on district claims, was 174%; and, when based on company claims, the straight average was only 67%.

Table 2 also shows, for each case, the corresponding percentages based on the means of the claims—that is, market value plus severance damages—made by the districts and the companies. In the nineteen cases, the awards ranged from 67% to 151% of the mean claim, the straight average percentage being 96%. This result looks something like a split difference—but it is a coincidence. In the individual cases, such a division just was not made.

Operating Revenues.—As previously stated, there was an agreement on "operating revenues" (which are gross earnings) but the various claims for "net operating revenues" were far apart, the difference being due to disagreements as to the cost of power and as to the consequent differences in federal income taxes. The awards ranged from 1.75 to 8.69 times the operating revenues, the straight average being 4.55. However, eighteen of the nineteen awards presumably contained some provision for severance damages, which have no connection with revenues. It is probable that, eliminating severance damages, the average multiplier would be about 4. Although there would have been no immediate change in operating revenues due to the proposed change of ownership, there would have been an immediate increase in net operating revenues of about 25%, due principally to the elimination of federal income taxes and other taxes which the districts would not pay. There was testimony to that effect, and presumably it was considered in connection with market value—but the matter was not given much prominence.

On public utility condemnation cases, most of the engineering work goes into the estimate of reproduction cost new, and the estimate of accrued depreciation. It is difficult to compare these elements of value with the awards—which presumably (except in case 1) contain some provision for severance damages. Table 2 (Cols. 5, 6, and 7) shows the relationships between the awards and the corresponding estimates of reproduction cost new. It also shows that the awards ranged from 68% to 184% of the reproduction costs of the district, with a straight average of 107%. There was a corresponding range of from 66% to 144% in the reproduction costs claimed by the companies, with a straight average of 87%. Taking the straight average of the percentages obtained by dividing the awards by the district and company estimates, it was found that the average percentage was 95.

It does not seem worthwhile to make comparisons with the respective reproduction cost less depreciation estimates, except to state that in every instance the award was in excess of the reproduction cost less depreciation; and generally it was considerably in excess.

In nine cases, estimates of "going value" were given (Cols. 13 and 14, Table 1). In all those cases, the companies claimed going value, whereas the engineers for the public utility districts declared "going value" in five instances only. No one denied the existence of going value, but it was difficult to determine; so, generally, it was absorbed in "market value," which presumably is purely a matter of judgment.

Severance Damages.—The item of "severance damages" was the subject of much debate. Case 1 was a detached property, taken in its entirety, and there were no severance damages; but, in all other cases, severance damages were claimed and presumably were included in the awards. In cases 3 and 4, which were decided by a Superior Court judge, the inclusions for severance damages were definitely stated—as 7.35% of the total award in case 3 and as 6.25% in case 4. These cases involved very small properties, which were not detached. After the close of a case, it sometimes is possible to talk with jurors and discover what the jury considered proper for severance damages. This was done in six cases (cases 10 to 16), and the known inclusions for severance damages, compared with cases 3 and 4, were as shown in Table 3. In the remaining eleven cases, the sums allowed for severance damages are unknown.

TABLE 3.—COMPARISON OF JURY ALLOWANCES FOR SEVERANCE DAMAGES

Case	Total award	Severance damages	Percentage
3	$ 68,000	$ 5,000	7.35
4	32,000	2,000	6.25
10	9,500,000	2,000,000	21.05
11	600,000	10,000	1.67
13	801,000	54,000	6.74
14	1,265,000	165,000	13.04
16	670,070	15,000	2.24
18	4,837,500	120,000	2.48

Severance damage generally was defined about as follows: "A reduction in the value of the property not to be taken, caused by the severance of the property being taken." The calculations, however, introduced other features. According to one witness who appeared for the companies in a number of cases, severance damages are made up of three elements:

1. Physical severance, or the cost of binding together the physical property remaining after severance;
2. Operating severance, or that part of the operating and general expense chargeable to the property taken, which would continue after severance; and
3. General severance, or the reduction in value of the remaining property caused by severance.

The witnesses for the districts generally would admit the legitimacy of items 1 and 2, although in smaller amounts than those claimed by the companies; but in no instance did a witness for a district admit item 3 or general severance.

Item 1, or physical severance, is clear. It is the cost of rearranging the remaining property after the severance of a part of the original whole. Item 2 covers the cost of carrying those members of the company who were employed on the part severed, and who would remain with the organization after severance; and it also includes that part of the company general expense ordinarily charged against the severed part, which expense would continue. The general scheme adopted by both sides was: (*a*) To estimate the total annual amount of item 2; (*b*) to fix a readjustment period (generally 3 years) after which all such expense would be redistributed, and would then become properly chargeable to the operations of the remaining property; and (*c*) to provide some plan for gradually redistributing such expenses during the readjustment period. Until such expenses could be redistributed, they would be of the nature of unnecessary expense, and would make up item 2.

There was only one affirmative argument for item 3, which ran something like this:

(*a*) When a number of disconnected small operating units are consolidated into one large integrated property, the operation becomes more efficient, with consequent operating economies and other advantages; hence the value of the integrated property is greater than the sum of the individual values of the several operating units of which it is composed; and

(*b*) When any one operating unit is severed from a consolidated property, there is a loss in value to the remaining property greater than the value of the unit severed.

The argument was all conversation—no demonstrations were offered. Somehow, it seemed to the writer that something more might have been done.

Jury Reaction.—Like everyone else, jurors are impressionable. A well-prepared and capably presented case is certain to have a favorable effect; however, juries are unpredictable. In the courts of the State of Washington, ten of the twelve jurors can decide a civil case, whereas the federal courts require the verdict to be reached by all jurors. In the nineteen cases cited, there were fourteen juries. (Certain cases were consolidated.) It is difficult to compare a group of twelve people with society as a whole; and the writer did not see all the juries. Most of those that he saw were representative, but one was definitely below par; and later it developed that at least two members of that jury were strongly prejudiced against the company, and were unwilling to make a fair award—yet the award was reasonably fair. The verdict was a result of the fact that two other members of the jury were determined to get an adequate award, and were "smart" enough to get it. Exactly the reverse might happen, and sometimes does.

Case 13, Table 1, is an example. The property was very desirable, notwithstanding a competitive situation. The company and the district were friendly, and the company unquestionably would have sold to the district had it not been tied up with another company in a gentleman's agreement not to sell. The utmost good feeling prevailed throughout the inventory and appraisal work. The writer is reasonably certain that the district would have been willing to pay at least $1,250,000 for the property. Definitely, the district wanted the company to get a fair price. The jury was above the average, and the presiding judge was capable and fair. The company claimed

$1,475,000; and the district admitted $722,000. Four times the gross revenue was $1,829,960—yet the award was only $801,000!

Exactly the reverse happened in case 19, Table 1. The company put up a stiff fight. The district was well represented, its lawyer having acted in case 13 and other cases. Unquestionably the district's case was well handled. The company claimed an award (market value plus severance damages) of a mean sum of $2,528,333. The district admitted a corresponding mean sum of only $783,900. Four times the gross revenue was $1,150,624. The award of the jury was $2,500,000!

In the jury room, all kinds of things happen. The jury just does not "spit at a crack" or split the difference, as is commonly supposed. Any jury may contain partisans who will fight for their preference; and sometimes they win. That possibility must be taken into account.

CONCLUSIONS

The foregoing evidence derived from practice leads to five broad conclusions as follows:

1. In fixing awards, juries are likely to be more liberal than judges. This is contrary to the popular idea.
2. With jury cases, as far as the award is concerned, it makes little difference whether the case is in a federal district court or in a local state court.
3. The range of awards may be expected to be from 75% to 115% of the mean of the final claims of the opposing sides.
4. The part of an award that concerns market value, excluding severance damages, may be expected to run from three to six times the current gross revenue. In a number of cases, the average multiplier probably will be about four.
5. The amount of severance damages in an award will depend on the particular conditions, and is difficult to predict; however, it may be expected to range from 2% to 7% of the total award. It is not so important as is commonly supposed.

The question naturally arises: Would it not have been better to have purchased outright than to have condemned? Early in the game, the districts started to purchase outright, and bought a number of properties. There are three large companies remaining. Two of them show no inclination to sell piecemeal; but the largest company recently sold its properties in eight counties, at reasonably good prices, even though it was practically forced to sell. It still operates in ten counties.

Another question arises: How have the awards compared with what an engineer might have fixed as a fair purchase price, plus some allowance for severance? Some of the awards have been high, and some have been low. As an average, the awards have been less than the sums that a willing buyer, wanting the property, would have paid.

In condemnation cases, there is considerable expense for preparation and trial. In two cases known to the writer, the awards were $1,265,000 and $1,200,000. In each case, the total district expense was approximately $60,000, or 5% of the awards. In very small cases, with awards of approximately $100,000, district expense might run as much as 15%. On fairly large cases, with awards of about $5,000,000, the expense probably would run around 3%. Even after adding these expenses, it may be concluded that, in general, the districts fared better than the companies.

These valuation and condemnation cases require far more engineering talent than is commonly supposed. Sometimes they lead to other work; engineers representing the districts frequently continue as consultants. The future for this kind of work is difficult to predict; but there may be considerable opportunity, and it behooves the profession to give this field some thought.

ACKNOWLEDGMENT

This paper is presented in lieu of a report of the Committee of the ASCE Power Division on Power Cost, through its Subcommittee on Valuation of Power Plants and Properties. Members of the subcommittee are John P. Hogan, Past-President, ASCE, William P. Creager, Hon. M. ASCE, and Joel D. Justin, Ross M. Riegel, and F. W. Scheidenhelm, Members, ASCE, with the writer as chairman. For their assistance in contributing useful advice and data the writer is deeply grateful to three attorneys—Paul J. Coughlin, Archie E. Blair, and Henry S. Gray—and to Frank A. Stewart, managing director, Public Utility Districts Research and Information Service, Inc., and C. P. Johnson, vice-president, Puget Sound Power and Light Company.

DISCUSSION

H. D. Southerland, Jr.,[4] Assoc. M. ASCE.—The engineering literature available in this specialized phase of valuations has been enhanced by this excellent contribution of basic information. Although the paper has been centered primarily on the engineering phases and (apparently intentionally) has not covered previous court decisions, it seems appropriate to comment briefly on a few of the court decisions that may have influenced these condemnation proceedings. Even though some of the decisions were made in connection with valuations for rate making purposes, the principles laid down probably have been influential in valuations for condemnation purposes as well.

The author states (under the heading, "Elements of Value: Engineering Estimates") that: "Since about 1938 engineers have lost much ground to the accountants and the economists." It would be interesting to know the events which led the author to select this date, 1938, as this point of departure. In 1935, Mr. Justice Harlan F. Stone, dissenting in the case of West versus Chesapeake and Potomac Telephone Company (295 U. S. 662,690) stated that:

> "* * * an examination of the present record will disclose that the results obtained by the application of price indices to the historial cost of plant are far less variable than engineers' valuations and in general are more trustworthy."

He then cited from the record in the case of New York Telephone Company versus Prendergast (36 Fed. (2d) 54, D. C., 1929) as an example of the variations in appraisals of telephone property, summarizing the estimates of fair value as of July 1, 1926, as follows:

	Valuation	Increase over the Commission Valuation
"Majority of Commission	$366,915,493	
Statutory Court	397,207,925	8.2%
Minority of Commission	405,502,993	10.5%
Master's Report	518,109,584	41.2%
Company claim based on Whittemore appraisal	528,753,738	44.1%
Company claim based on Stone and Webster appraisal	615,000,000	67.1%"

Concerning "going value" the author states (under the heading, "Elements of Value: Going Value") that: "Not the slightest attention was paid to any such claims." It could be that the decisions in the Denver Union Stock Yards (304 U. S. 470, 478 (1938)) line of cases were exerting their cumulative influence. This line of cases which have disallowed the necessity of expressing the going value of a public utility as a separate amount includes the following: Dayton Power and Light Company versus Public Utilities Commission (292 U. S. 290, 308 (1934)); Columbus Gas and Fuel Company versus Public Utilities Commission (292 U. S. 398, 411 (1934)); Driscoll versus Edison Light and Power

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[4] Attorney, Birmingham, Ala.

Company (307 U. S. 104, 117 (1939)); and Federal Power Commission versus Natural Gas Pipeline Company (315 U. S. 575, 586 (1942)). In the Denver Union Stock Yards case the court ruled out the necessity for expressing the going value of a public utility as a separate amount, stating that the plant without business would be worth much less than the value of the land plus the cost of reproduction new, less the depreciation of the structures, plus certain other allowances, and plus working capital, as allowed by the Secretary of Agriculture.

The prudent investment rule for rate making, which is based on cost, was expressed well in the case of State ex rel. Southwestern Bell Telephone Company versus Public Service Commission of Missouri (262 U. S. 276, 306 (1923)) in which Mr. Justice Louis Brandeis stated that:

> "The adoption of the amount prudently invested as the rate base and the amount of the capital charge as the measure of the rate of return would give definiteness to these two factors involved in rate controversies which are now shifting and treacherous, and which render the proceedings peculiarly burdensome and largely futile."

It is interesting to note that, despite the foregoing opinion, the actual cost had little effect in these condemnation cases.

In connection with the vicissitudes that continue to shroud valuation problems the words of Mr. Justice Brandeis in the Southwestern Bell Telephone Company case (262 U. S. 310 (1923)) appear to be appropriate—"Value is a word of many meanings."

Henry L. Gray,[5] M. ASCE.—There have been no more light and power condemnation cases in the State of Washington since the paper was published; but there has been a definite trend toward municipal ownership of light and power properties by small cities within the state. One such city announced its intention of taking over the light and power system within its borders, which system already is owned and operated by a public utility district. This raises an interesting legal question and certainly adds to the general confusion.

In addition to the condemnation cases referred to in the paper, there has been a number of outright purchases and sales, the details of which bear a marked similarity. Those were not discussed in the paper.

The writer is grateful to Mr. Southerland for his citations and comments on leading court cases covering valuation matters. As stated in the paper, the writer did not attempt to raise that phase of the matter, but tried to confine the scope closely to the procedure and the results in the cases mentioned.

Mr. Southerland inquires as to why the writer had selected 1938 as marking the beginning of the period of eclipse for engineers engaged in valuation work. The year 1938 has no special significance except that, from personal observation, supplemented by news reports, that year was about the time when the economists and accountants surged to the front and to a considerable extent supplemented the engineers. Definitely, that was the case locally and it is pleasing to note that the local engineers seem to be coming back. It is the writer's opinion that the economists failed to produce anything but words, and that the accountants were too meticulous for work of this kind.

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[5] Cons. Engr., Seattle, Wash.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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Paper No. 2441

NUMERICAL COMPUTATION OF BUCKLING LOADS BY FINITE DIFFERENCES

By Mario G. Salvadori,[1] Assoc. M. ASCE

With Discussion by Messrs. I. K. Silverman, Bruno A. Boley, G. S. Ramaswamy, Chi-Teh Wang, George Herrmann, and Mario G. Salvadori.

Synopsis

After introducing the concept of central differences of a function to approximate its derivatives by simple numerical expressions, the writer shows how a problem involving a linear differential equation of the homogeneous type, with homogeneous boundary conditions, may be reduced to a problem involving the solution of a system of simultaneous linear algebraic equations. In this manner solutions of buckling problems may be obtained by purely numerical computations, using a procedure of successive approximations, which is greatly enhanced by a simple method of extrapolation. The errors involved in the procedure are discussed and complete tables are given to be used in connection with extrapolation. The procedure is then applied to various cases of buckling of beams, plates, and shells to demonstrate the technique of its application and to solve a few problems not appearing in the literature.

1. INTRODUCTION

The evaluation of buckling loads of beams and plates often presents a difficult analytical problem, whose rigorous solution may become practically impossible when the shape of the plate is irregular or when the variation of the moment of inertia of the beam or plate does not follow a simple law.

For these reasons, approximate methods of computation have been devised, the best known among them perhaps being Rayleigh's energy method,[2] which gives upper bounds for the buckling loads. With the help of other methods giving lower bounds,[3,4,5] it is possible to squeeze the approximate load between given limits.

Note.—Published in December, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or the discussion was received for publication.

[1] Associate Prof. of Civ. Eng., Columbia Univ., New York, N. Y.

[2] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936.

[3] "Ein Gegenstück zum Ritzchen Verfahren," by E. Trefftz, *Proceedings*, 2d International Cong. for Applied Mechanics, Zürich, 1926.

[4] "On a Minimal Problem in Theory of Elasticity," by A. Weinstein, *Journal*, London Mathematical Soc., Vol. 10, 1935.

[5] "Étude des équations des plaques encastrées," by A. Weinstein, *Mémoires des Sciences Mathématiques*, Vol. 88, 1937.

Purely numerical methods for the solution of structural problems have become very popular among engineers, particularly since 1919, because of their simplicity and because of the wider availability of calculating machines. Hardy Cross,[6] Hon. M. ASCE, and R. V. Southwell[7,8] have pioneered in this field.

The numerical and graphical computation of buckling loads by successive approximations has been introduced by L. Vianello and A. Stodola, in connection with struts, and their method has been greatly improved by N. M. Newmark,[9] M. ASCE. Mr. Southwell[7,8] has applied his relaxation procedure to the same problem. Since 1910, however, L. F. Richardson[10] has suggested a finite difference procedure for the determination of characteristic values of differential equations, which was investigated independently by L. Collatz.[11] This procedure seems to present a very wide range of applications as well as great simplicity. It is the aim of this paper to give a comprehensive presentation of the finite difference method, together with some extrapolation procedures, greatly enhancing its usefulness.

2. CENTRAL DIFFERENCES

Let the values of a Taylor expansible function $y = f(x)$ be known at evenly spaced points of the x-axis of abscissas: $\cdots, x - 3h, x - 2h, x - h, x, x + h, x + 2h, x + 3h, \cdots$ (Fig. 1) and also, for the time being, at the middle point of the intervals of width h so defined. The "first central difference" of $f(x)$ at x, indicated by the operational symbol δf (first introduced by W. F. Sheppard[12]) is defined by

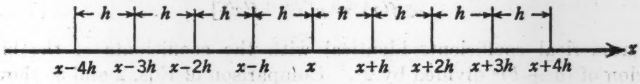

FIG. 1

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) \qquad (1)$$

The "second difference" of $f(x)$ at x is defined as the difference of the first difference and is indicated by the symbol $\delta^2 f(x)$:

$$\delta^2 f(x) = \delta[\delta f(x)] = f(x + h) - 2f(x) + f(x - h) \qquad (2)$$

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[6] "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, *Transactions*, ASCE, Vol. 96, 1932.

[7] "Relaxation Methods in Engineering Science," by R. V. Southwell, Clarendon Press, Oxford, England, 1940.

[8] "Relaxation Methods in Theoretical Physics," by R. V. Southwell, Clarendon Press, Oxford, England, 1947.

[9] "Numerical Procedure for Computing Deflections, Moments, and Buckling Loads," by N. M. Newmark, *Transactions*, ASCE, Vol. 108, 1943.

[10] "The Approximate Arithmetical Solution by Finite Differences of Physical Problems Involving Differential Equations with an Application to the Stresses in a Masonry Dam," by L. F. Richardson, *Philosophical Transactions*, Royal Soc. of London, Vol. 210, 1911.

[11] "Zusammenfassender Bericht: Genäherte Berechnung von Eigenwerten," by L. Collatz, *Zeitschrift für angewandte Mathematik und Mechanik*, Vol. 19, 1939.

[12] "Central Differences Formulae," by W. F. Sheppard, *Proceedings*, London Mathematical Soc., Vol. 31, 1899.

"Higher order differences" are defined by the recurrence equation:

$$\delta^n f(x) = \delta[\delta^{n-1} f(x)] \quad \text{(3)}$$

which leads to

$$\delta^3 f(x) = f\left(x + \frac{3h}{2}\right) - 3f\left(x + \frac{h}{2}\right) + 3f\left(x - \frac{h}{2}\right) - f\left(x - \frac{3h}{2}\right) \quad \text{(4a)}$$

and

$$\delta^4 f(x) = f(x + 2h) - 4f(x + h) + 6f(x) - 4f(x - h) + f(x - 2h) \quad \text{(4b)}$$

The numerical coefficients of $\delta^n f(x)$ are the coefficients of the binomial expansion of $(a - b)^n$, and even order differences involve the values of $f(x \pm i h)$ $(i = 0, 1, 2, \cdots)$, whereas odd order differences require the knowledge of $f(x \pm i h/2)$. To eliminate these last values of $f(x)$, Mr. Sheppard[12] introduced a new operator μ, called the "averager" and defined by the equation:

$$\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right] \quad \text{(5a)}$$

whose "square" is

$$\mu^2 f(x) = \mu[\mu f(x)] = \frac{1}{4}[f(x + h) + 2f(x) + f(x - h)] \quad \text{(5b)}$$

and whose nth power—

$$\mu^n f(x) = \mu[\mu^{n-1} f(x)] \quad \text{(5c)}$$

—has numerical coefficients identical with the coefficients of the binomial expansion of $(a + b)^n$ divided by 2^n. Comparison of Eqs. 2 and 5b shows that $\mu^2 f(x) - \frac{1}{4}\delta^2 f(x) = f(x)$, or, writing this equation in purely operational form by dropping the symbol $f(x)$ on both of its sides, that

$$\mu^2 = 1 + \frac{1}{4}\delta^2 \quad \text{(6)}$$

The successive derivatives of the function $f(x)$ are represented subsequently by the differential operator D:

$$D f(x) = \frac{df}{dx} \quad \text{(7a)}$$

whose nth "power" is

$$D^n f(x) = D[D^{n-1} f(x)] = \frac{d^n f}{dx^n} \quad \text{(7b)}$$

It may be easily shown that the three operators δ, μ, and D satisfy, formally, the fundamental laws of algebra and that hence they can be used as if they were algebraic quantities, provided that the result of the operations is interpretable in terms of their definition.

Thus, the "products" $\mu\,\delta$ and $\mu\,\delta^3$ are defined as

$$\mu\,\delta[f(x)] = \mu[\delta f(x)] = \frac{1}{2}[f(x + h) - f(x - h)] \quad \text{(8a)}$$

and

$$\mu\,\delta^3[f(x)] = \mu[\delta^3 f(x)]$$
$$= \frac{1}{2}\,[f(x + 2\,h) - 2\,f(x + h) + 2\,f(x - h) - f(x - 2\,h)] \quad \text{.....(8b)}$$

and the product $\mu\ \delta^{2m+1} f(x)$ $(m = 0, 1, 2, \cdots)$ involves only the values of $f(x)$ at $x + i\,h$ $(i = 0, 1, 2, \cdots)$.

Some important relationships between the operators δ, μ, and D may be easily obtained from the Taylor expansion of $f(x)$:

$$f(x + h) = f(x) + \frac{h}{1!}\,D\,f(x) + \frac{h^2}{2!}\,D^2 f(x) + \frac{h^3}{3!} D^3 f(x) + \cdots$$

which, "factoring" the function $f(x)$ and remembering the power series expansion of e^x, may be written symbolically as

$$f(x + h) = e^{hD} f(x) \quad \text{....................(9a)}$$

Changing h into $-\,h$, Eq. 9*a* gives

$$f(x - h) = e^{-hD} f(x) \quad \text{....................(9b)}$$

and, by Eqs. 9, Eq. 8*a* becomes: $\mu\,\delta\,f(x) = \frac{1}{2}\,[f(x + h) - f(x - h)]$ $= \frac{1}{2}\,[e^{hD} f(x) - e^{-hD} f(x)] = \frac{e^{hD} - e^{-hD}}{2} f(x) = \sinh\,(h\,D)\,f(x)$; or, in purely operational form,

$$\mu\,\delta = \sinh\,(h\,D) \quad \text{....................(10a)}$$

Expanding $\sinh\,(h\,D)$ into a power series, the operator $\mu\,\delta$ can finally be represented by the following series involving all the derivatives of $f(x)$:

$$\mu\,\delta = h\,D + \frac{h^3\,D^3}{3!} + \frac{h^5\,D^5}{5!} + \cdots \quad \text{...............(10b)}$$

Similarly, by Eqs. 9, $\delta^2 f$ becomes $\delta^2 f(x) = 2\left(\frac{e^{hD} + e^{-hD}}{2} - 1\right) f(x)$; or, in symbolical form,

$$\delta^2 = 2[\cosh\,(h\,D) - 1] = h^2\,D^2 + \frac{h^4\,D^4}{12} + \frac{h^6\,D^6}{360} + \cdots \quad \text{........(11)}$$

Multiplying Eq. 10*b* by Eq. 11,

$$\mu\,\delta^3 = h^3\,D^3 + \frac{h^5\,D^5}{4} + \frac{h^7\,D^7}{45} + \cdots \quad \text{..............(12a)}$$

and, squaring Eq. 11,

$$\delta^4 = h^4\,D^4 + \frac{h^6\,D^6}{6} + \frac{h^8\,D^8}{80} + \cdots \quad \text{..............(12b)}$$

Thus, the operators $\frac{\delta^{2m}}{h^{2m}}$ and $\mu\,\frac{\delta^{2m+1}}{h^{2m+1}}$ (m being an integer) may be expressed, in terms of the derivatives of $f(x)$, by power series in h involving only the even powers of h.

The inverse relationships, giving the successive derivatives of $f(x)$ in terms of its differences, are obtained as follows: Solving Eq. 10a for $h\,D$ and remembering the power series expansion of $\sinh^{-1} x$,

$$h\,D = \sinh^{-1} \mu\,\delta = \mu\,\delta - \frac{\mu^3\,\delta^3}{6} + \frac{3\,\mu^5\,\delta^5}{40} - \cdots$$

or, making use of Eq. 6 to eliminate all the even powers of μ,

$$h\,D = \mu\left(\delta - \frac{\delta^3}{6} + \frac{\delta^5}{30} - \cdots\right) \dots\dots\dots \tag{13}$$

Similarly, squaring Eq. 13 and again using Eq. 6,

$$h^2\,D^2 = \delta^2 - \frac{\delta^4}{12} + \frac{\delta^6}{90} - \cdots \dots\dots\dots \tag{14a}$$

multiplying Eq. 13 by Eq. 14a,

$$h^3\,D^3 = \mu\left(\delta^3 - \frac{\delta^5}{4} + \frac{7\,\delta^7}{120} - \cdots\right) \dots\dots\dots \tag{14b}$$

and, squaring Eq. 14a,

$$h^4\,D^4 = \delta^4 - \frac{\delta^6}{6} + \frac{7\,\delta^8}{240} - \cdots \dots\dots\dots \tag{14c}$$

3. DIFFERENCE OPERATORS IN CARTESIAN COORDINATES

A. Ordinary Operators.—In the approximate solution of differential equations by finite differences it is often assumed that all the terms at the right-hand members of Eqs. 13 to 14c beyond the first are negligible and that, hence, the derivatives of $f(x)$ may be approximated by

$$h\,D\,f(x) \doteq \mu\,\delta\,f(x) = \frac{1}{2}\,[f(x+h) - f(x-h)] \dots\dots\dots \tag{15a}$$

$$h^2\,D^2\,f(x) \doteq \delta^2\,f(x) = f(x+h) - 2\,f(x) + f(x-h) \dots\dots\dots \tag{15b}$$

$$h^3\,D^3\,f(x) \doteq \mu\,\delta^3\,f(x) = \frac{1}{2}\,[f(x+2\,h) - 2\,f(x+h) + 2\,f(x-h) - f(x-2\,h)] \dots \tag{15c}$$

$$h^4\,D^4\,f(x) \doteq \delta^4\,f(x) = f(x+2\,h) - 4\,f(x+h) + 6\,f(x) - 4\,f(x-h) + f(x+2\,h) \dots \tag{15d}$$

in which $\doteq$ represents "approximately equal to." By Eqs. 10b to 12b, these relations imply that the higher derivatives of $f(x)$ are not large in comparison with its lower derivatives and that the higher powers of h are negligible in comparison with its lower powers. (The error inherent in this approximation is evaluated in Section 5.)

The right-hand members of Eqs. 15 are called the "first order central differences" of $f(x)$ and may be represented graphically by the convenient schemes of Fig. 2.

By taking into account two terms of the series of Eqs. 13 and 14 the derivatives of $f(x)$ may be represented in terms of "second order central differences,' as follows:

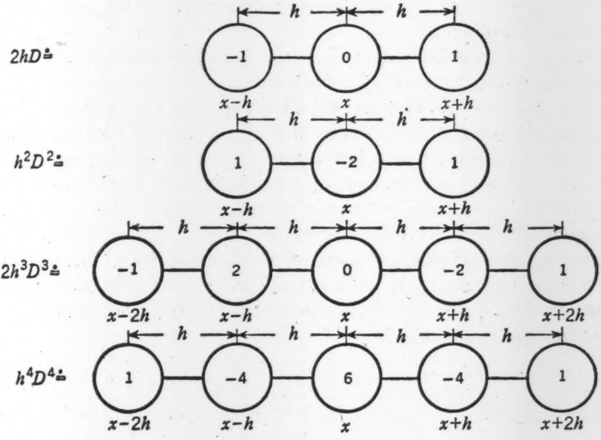


FIG. 2.—FIRST ORDER DIFFERENCES

$$h\,D f(x) \doteq \mu\left(\delta - \frac{\delta^3}{6}\right) f(x)$$

$$= \frac{1}{12}[- f(x + 2\,h) + 8\,f(x + h) - 8\,f(x - h) + f(x - 2\,h)] \quad ..(16a)$$

$$h^2\,D^2 f(x) \doteq \left(\delta^2 - \frac{\delta^4}{12}\right) f(x) = \frac{1}{12}[- f(x + 2\,h) + 16\,f(x + h)$$
$$- 30\,f(x) + 16\,f(x - h) - f(x - 2\,h)] \quad ..(16b)$$

$$h^3\,D^3 f(x) \doteq \mu\left(\delta^3 - \frac{\delta^5}{4}\right) f(x) = \frac{1}{8}[- f(x + 3\,h) + 8\,f(x + 2\,h)$$
$$- 13\,f(x + h) + 13\,f(x - 2\,h) + f(x + 3\,h)] \quad ..(16c)$$

$$h^4\,D^4 \doteq \left(\delta^4 - \frac{\delta^6}{6}\right) f(x) = \frac{1}{6}[- f(x + 3\,h) + 12\,f(x + 2\,h) - 39\,f(x + h)$$
$$+ 56\,f(x) - 39\,f(x - h) + 12\,f(x - 2\,h) - f(x - 3\,h)] \quad ..(16d)$$

The right-hand members of Eqs. 16 are represented graphically in Fig. 3.

Second order difference operators are more accurate than first order operators, but involve the values of $f(x)$ at a larger number of points. The

Nth order difference operators are similarly obtained. Since difference ratios approach the corresponding derivatives as h approaches zero, the accuracy of the numerical approximations may be increased either by making h smaller or by taking higher order differences. Which procedure is more advantageous depends on the type of problem at hand, but it is seldom that the additional labor required by higher order differences is warranted in practice by the results obtainable.

(Expressions for the derivatives in terms of differences may also be obtained by passing nth order parabolas through the points of abscissa $x \pm i\,h$ and of ordinate $y = f(x \pm i\,h)$, and by substituting the derivatives of the interpolated parabola for the derivatives of $f(x)$. When the interpolation points are symmetrically located with respect to x, the formulas thus obtained are identical with the central difference operators given in Eqs. 15 and 16. When the interpolation points are not symmetrically located with respect to x, the formulas give the derivatives in terms of "lateral differences." This latter type of formula is conveniently used on the boundaries of beams and plates, but will not be considered in this paper. Mr. Southwell lists[8] most of the central and lateral difference formulas needed in the applications, under the name of "Bickley's formulas.")

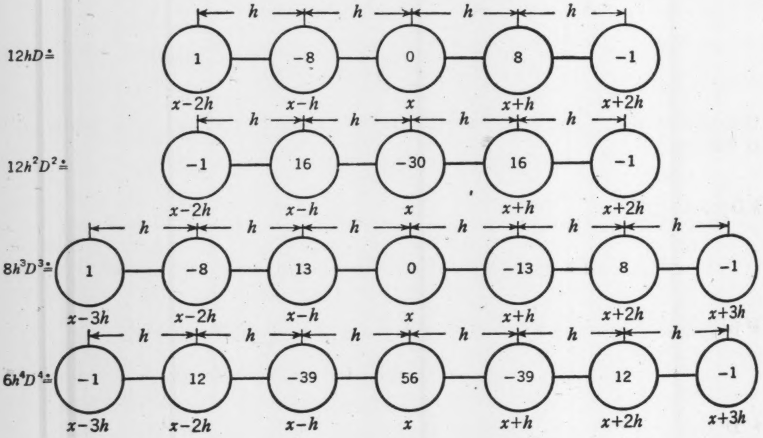


FIG. 3.—SECOND ORDER DIFFERENCES

B. Partial Operators.—The first order and second order difference expressions, Eqs. 15 and 16, may be used to evaluate the partial derivatives of a function $z = f(x,y)$ with respect to x (or y) by substituting for $D = \dfrac{d}{dx}$ the partial derivative symbol $D_x = \dfrac{\partial}{\partial x}$ $\left(\text{or } D_y = \dfrac{\partial}{\partial y}\right)$ and by holding y (or x) constant in computing the differences. Thus, using first order differences and calling $r\,h$ the spacing of points in the y-direction,

$$h\,D_x\,z \doteq \frac{1}{2}\,[f(x+h,y) - f(x-h,y)] \ldots\ldots\ldots\ldots (17a)$$

and

$$r\,h\,D_y\,z \doteq \frac{1}{2}\,[f(x,y+r\,h) - f(x,y-r\,h)] \ldots\ldots\ldots\ldots (17b)$$

whereas the second mixed derivative of z with respect to x and y, $\frac{\partial^2 z}{\partial x\,\partial y} = D_{xy}\,f(x,y)$, is obtained by applying consecutively the operators at the right-hand members of Eqs. 17:

$$\begin{aligned} r\,h^2\,D_{xy}\,z \doteq \frac{1}{4}\,[f(x+h,\,y+r\,h) &- f(x+h,\,y-r\,h) \\ &- f(x-h,\,y+r\,h) + f(x-h,\,y-r\,h)] \,.\,.\, (18a) \end{aligned}$$

and the fourth mixed derivative $\frac{\partial^4 f}{\partial x^2\,\partial y^2}$ by squaring the operator D_{xy}:

$$\begin{aligned} r^2\,h^4\,D^2_{xy}\,z \doteq \frac{1}{16}\,\{f(x+h,\,y+r\,h) &+ f(x-h,\,y+r\,h) + f(x+h,\,y-r\,h) \\ &+ f(x-h,\,y-r\,h) - 2[f(x,\,y+r\,h) + f(x+h,\,y) \\ &+ f(x-h,\,y) + f(x,\,y-r\,h)] + 4\,f(x,\,y)\} \ldots (18b) \end{aligned}$$

Similarly, the first order difference operator corresponding to the Laplacian $\nabla^2 z = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ becomes

$$\begin{aligned} h^2\,\nabla^2 z \doteq f(x+h,\,y) + f(x-h,\,y) &+ \left(\frac{1}{r^2}\right)[f(x,\,y+r\,h) + f(x,\,y-r\,h)] \\ &- 2\left(1+\frac{1}{r^2}\right)f(x,\,y) \ldots (19a) \end{aligned}$$

and, in particular for $r = 1$,

$$h^2\,\nabla^2 z \doteq f(x+h,y) + f(x-h,y) + f(x,y+h) + f(x,y-h) - 4\,f(x,y) \,.\,.\, (19b)$$

Squaring Eq. 19*b*, the first order operator corresponding to $\nabla^4 z = \frac{\partial^4 f}{\partial x^4} + 2\,\frac{\partial^4 f}{\partial x^2\,\partial y^2} + \frac{\partial^4 f}{\partial y^4}$ for the case $r = 1$ is found to be

$$\begin{aligned} h^4\,\nabla^4 z \doteq\; &[f(x+2\,h,y) + f(x,y+2\,h) + f(x-2\,h,y) + f(x,y-2\,h)] \\ &+ 2\,[f(x+h,\,y+h) + f(x-h,\,y+h) + f(x+h,\,y-h) + f(x-h,\,y-h)] \\ &- 8\,[f(x+h,y) + f(x-h,y) + f(x,y-h) + f(x,y+h)] + 20 f(x,y) \,.\,.\, (19c) \end{aligned}$$

Using second order differences (Eqs. 16) the expressions for second order partial difference operators may be similarly obtained.

The schemes of Fig. 4 and Fig. 5 give some of the first order and second order operators most commonly encountered, showing how partial difference operators involve the values of the function z at the corners of a rectangular or square grid. Partial operators in Cartesian coordinates are well adapted to the solution of problems involving rectangular or square domains.

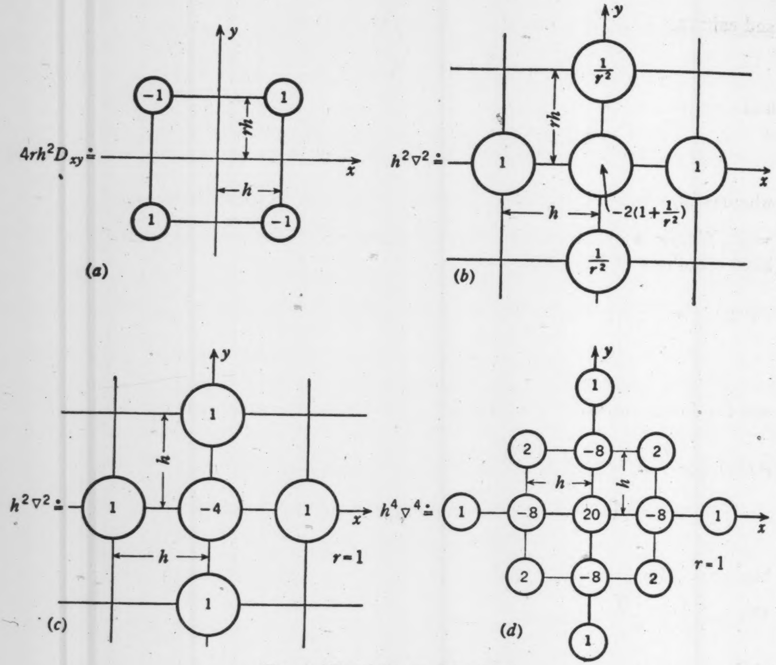
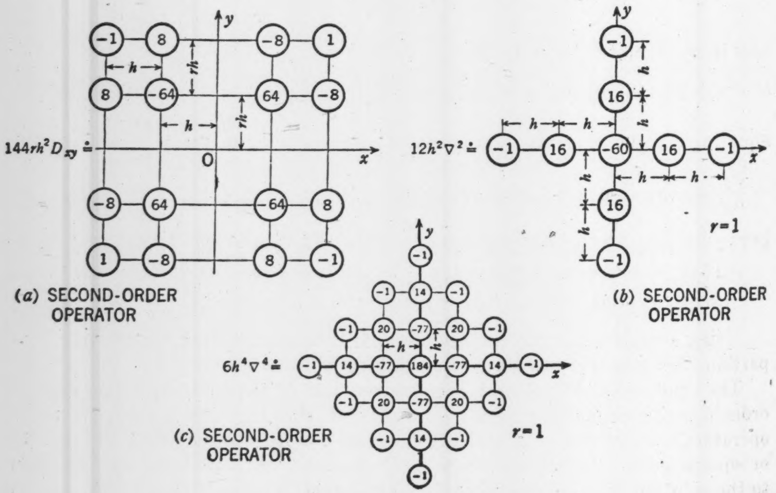

(a)
$4rh^2D_{xy} \doteq$
-1, 1, 1, -1
rh
h
x
y
(b)
$h^2\nabla^2 \doteq$
$\frac{1}{r^2}$
1, 1
$-2(1+\frac{1}{r^2})$
rh
h
(c)
$h^2\nabla^2 \doteq$
1, 1, -4, 1, 1
h
r=1
(d)
$h^4\nabla^4 \doteq$
1, 2, -8, 2, 1, -8, 20, -8, 1, 2, -8, 2, 1
h
r=1

FIG. 4.—FIRST ORDER OPERATORS


$144rh^2D_{xy} \doteq$
-1, 8, -8, 1
8, -64, 64, -8
-8, 64, -64, 8
1, -8, 8, -1
h
rh
O
x
y
$12h^2\nabla^2 \doteq$
-1, 16, -60, 16, -1
h
r=1
$6h^4\nabla^4 \doteq$
-1, 14, -1
-1, 20, -77, 20, -1
-1, 14, -77, 184, -77, 14, -1
r=1

(a) SECOND-ORDER OPERATOR

(b) SECOND-ORDER OPERATOR

(c) SECOND-ORDER OPERATOR

FIG. 5.—SECOND ORDER OPERATORS

4. DIFFERENCE OPERATORS IN SKEW COORDINATES

The solution of problems involving rhombic domains (for instance, skew plate problems) is obtained more conveniently by oblique Cartesian, or "skew," coordinates.[13,14]

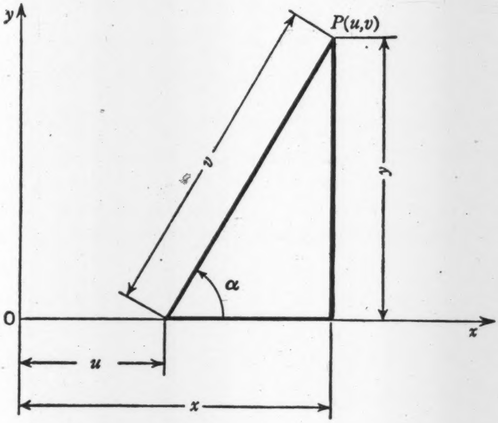


FIG. 6

Fig. 6 indicates the relationship between the Cartesian coordinates x,y of a point P and its skew coordinates u,v:

$$x = u + v\cos\alpha \quad \text{and} \quad y = v\sin\alpha \ldots\ldots\ldots\ldots (20a)$$

(in which α is the angle of skew) from which it is easy to derive the inverse relationships:

$$u = x - \frac{y}{\tan\alpha} \quad \text{and} \quad v = \frac{y}{\sin\alpha} \ldots\ldots\ldots\ldots (20b)$$

Given a function $z = f(x,y)$, its derivatives with respect to x and y may be written in terms of $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ using Eq. 20b:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} \ldots\ldots\ldots\ldots\ldots\ldots (21a)$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = -\frac{1}{\tan\alpha}\frac{\partial z}{\partial u} + \frac{1}{\sin\alpha}\frac{\partial z}{\partial v} \ldots\ldots\ldots (21b)$$

---

[13] "Le calcul des plaques obliques par la méthode des équations aux differences," by H. Favre, *Mémoires*, Association Internationale des Ponts et Charpentes, Vol. 7, Zürich, 1943–1944.

[14] "Theoretische Untersuchungen über die Eigenfrequenz parallelogramförmiger Platten," by R. Bereuter, Publication du Laboratoire de Photo-Élasticité de l'École Polytechnique Fédérale, Zürich, 1946.

The expressions for the higher derivatives of z are obtained by repeated application of Eqs. 21:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} \qquad \text{(22a)}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{\sin^2 \alpha}\left(\cos^2 \alpha \frac{\partial^2 z}{\partial u^2} - 2 \cos \alpha \frac{\partial^2 z}{\partial u \, \partial v} + \frac{\partial^2 z}{\partial v^2}\right) \qquad \text{(22b)}$$

$$\frac{\partial^2 z}{\partial x \, \partial y} = \frac{-1}{\tan \alpha} \frac{\partial^2 z}{\partial u^2} + \frac{1}{\sin \alpha} \frac{\partial^2 z}{\partial u \, \partial v} \qquad \text{(22c)}$$

$$\nabla^2 z = \frac{1}{\sin^2 \alpha}\left(\frac{\partial^2 z}{\partial u^2} - 2 \cos \alpha \frac{\partial^2 z}{\partial u \, \partial v} + \frac{\partial^2 z}{\partial v^2}\right) \qquad \text{(22d)}$$

The corresponding difference operators are derived by substitution of central difference ratios taken in the u-direction and the v-direction for the derivatives in the same directions, according to Eqs. 15 or 16, depending upon whether first order or second order operators are required. Thus, letting h and $r\,h$ be the spacing of the points in the u-direction and the v-direction, then, with first order differences,

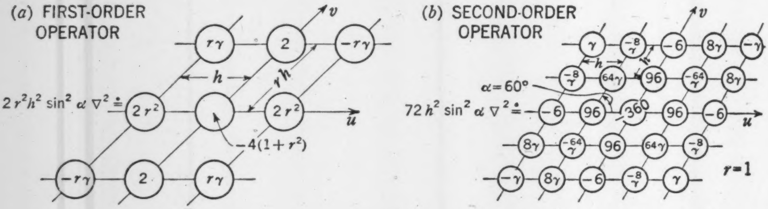


FIG. 7

$$\begin{aligned} 2\,r^2\,h^2 \sin^2 \alpha \, \nabla^2 z \doteq {} & 2\,r^2 \left[f(u + h,v) + f(u - h,v)\right] \\ & + 2\left[f(u, v + r\,h) + f(u, v - r\,h)\right] - 4\,(1 + r^2)\, f(u,v) \\ & - r \cos \alpha \left[f(u + h, v + r\,h) + f(u - h, v - r\,h)\right. \\ & \qquad \left. - f(u + h, v - r\,h) - f(u - h, v + r\,h)\right] \qquad \text{(23)} \end{aligned}$$

which is represented in Fig. 7, where $\gamma = \cos \alpha$.

5. DIFFERENCE OPERATORS IN TRIANGULAR COORDINATES

When the problem to be solved involves a domain coverable by an equilateral triangular grid (equilateral triangular plate, regular hexagonal plate, etc.), it is convenient to use the "equitriangular coordinates" shown in Fig. 8, in which a point is located by coordinates measured along three axes (x, u, and v) making 60° angles between them.

To obtain the operator ∇^2 of $z = f(x,y)$ in this new set of coordinates, notice that: $\dfrac{\partial x}{\partial u} = -\dfrac{\partial x}{\partial v} = \cos 60° = \dfrac{1}{2}$ and $\dfrac{\partial y}{\partial u} = \dfrac{\partial y}{\partial v} = \sin 60° = \dfrac{\sqrt{3}}{2}$. Therefore:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{1}{2}\frac{\partial z}{\partial x} + \frac{\sqrt{3}}{2}\frac{\partial z}{\partial y} \quad \text{(24a)}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = -\frac{1}{2}\frac{\partial z}{\partial x} + \frac{\sqrt{3}}{2}\frac{\partial z}{\partial y} \quad \text{(24b)}$$

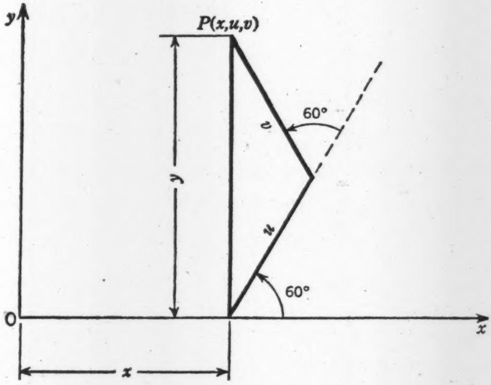


FIG. 8

Squaring Eqs. 24, the second derivatives of z with respect to u and v become

$$\frac{\partial^2 z}{\partial u^2} = \frac{1}{4}\frac{\partial^2 z}{\partial x^2} + \frac{3}{4}\frac{\partial^2 z}{\partial y^2} + \frac{\sqrt{3}}{2}\frac{\partial^2 z}{\partial x\,\partial y} \quad \text{(25a)}$$

and

$$\frac{\partial^2 z}{\partial v^2} = \frac{1}{4}\frac{\partial^2 z}{\partial x^2} + \frac{3}{4}\frac{\partial^2 z}{\partial y^2} - \frac{\sqrt{3}}{2}\frac{\partial^2 z}{\partial x\,\partial y} \quad \text{(25b)}$$

from which, by summation, $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = \frac{1}{2}\frac{\partial^2 z}{\partial x^2} + \frac{3}{2}\frac{\partial^2 z}{\partial y^2}$. Adding $\frac{\partial^2 z}{\partial x^2}$ to both sides of the latter equation, the expression for $\nabla^2 z$, in equitriangular coordinates, is

$$\nabla^2 z = \frac{2}{3}\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right) \quad \text{(26)}$$

To derive the corresponding difference operator, substitute in Eqs. 26 for the derivatives in the x-direction, the u-direction, and the v-direction the corresponding difference ratios in the same directions. If h is the equal spacing of the corners of the grid in the three directions, using first order differences,

$$\begin{aligned} h^2 \nabla^2 z \doteq \tfrac{2}{3}\,[&f(x+h,u,v) + f(x-h,u,v) + f(x,u+h,v) \\ &+ f(x,u-h,v) + f(x,u,v+h) + f(x,u,v-h) - 6\,f(x,u,v)] \quad \text{(27)} \end{aligned}$$

The graphs of Fig. 9 give the expressions of $\nabla^2 z$ by first order and second order differences and of $\nabla^4 z$ by first order differences.

V. P. Jensen,[15] M. ASCE, has given the expressions for $\nabla^2 z$ and $\nabla^4 z$ in "nonequiangular triangular coordinates," by which finite difference operators can be applied to domains coverable by a grid of equal irregular triangles. Mr. Jensen's finite difference operators are first order operators.

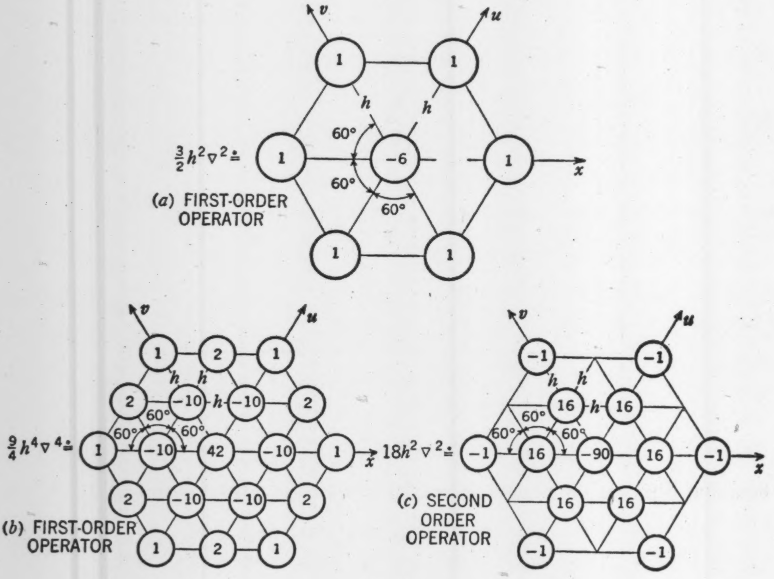

v
u
1
1
h
h
60°
$\frac{3}{2}h^2\nabla^2 \doteq$
1
−6
1
x
60°
60°
(a) FIRST-ORDER
OPERATOR
1
1
v
u
1
2
1
h
h
2
−10
h
−10
2
60°
60°
60°
$\frac{9}{4}h^4\nabla^4 \doteq$
1
−10
42
−10
1
x
2
−10
−10
2
(b) FIRST-ORDER
OPERATOR
1
2
1
v
u
−1
−1
h
h
16
h
16
60°
60°
60°
$18h^2\nabla^2 \doteq$
−1
16
−90
16
−1
x
(c) SECOND
ORDER
OPERATOR
16
16
−1
−1

Fig. 9

6. ERROR IN THE DERIVATIVES DUE TO THE SUBSTITUTION OF DIFFERENCE RATIOS

As noted in Section 3, in the solution of differential equations by finite differences the derivatives of the unknown function are approximated by the corresponding difference ratios.

When first order differences are used, the error e'_1 in the value of the first derivative is immediately found from Eq. 10*b*:

$$e'_1 = \left(D - \frac{\mu\,\delta}{h}\right) f(x) = -\frac{D^3 f}{3!}h^2 - \frac{D^5 f}{5!}h^4 - \cdots$$

---

[15] "Analyses of Skew Slabs," by V. P. Jensen, *Bulletin Series No. 332*, Univ. of Illinois Eng. Experiment Station, Urbana, Ill., 1941.

Similarly, the error in the second derivative becomes by Eq. 11:

$$e'_2 = \left(D^2 - \frac{\delta^2}{h^2} \right) f(x) = - \frac{D^4 f}{12} h^2 - \frac{D^6 f}{360} h^4 - \cdots$$

and, more generally, the error in the nth derivative is of the type:

$$e'_n = \phi'_1(x)\, h^2 + \phi'_2(x)\, h^4 + \phi'_3(x)\, h^6 + \cdots \ldots\ldots\ldots\ldots (28)$$

in which the functions $\phi'_i(x)$ are independent of h.

Eq. 28 shows that the error due to first order differences depends on the even powers of h only and that, when the Taylor series of $f(x)$ is converging rapidly (small values of h and rapidly decreasing values of the successive derivatives of $f(x)$), the series at its right-hand member will also converge rapidly—the more so as n becomes greater.

To evaluate the error due to second order differences, note that by Eq. 16*a* the first derivative is approximated by the difference ratio:

$$D f(x) \doteq \left(\frac{\mu\, \delta}{h} - \frac{\mu\, \delta^3}{6\, h} \right) f(x).$$

Substituting in this latter equation the expression $D f$ obtained from Eq. 10*b* and the expression $\mu\, \delta^3$ obtained from Eq. 12*a*, the error e''_1 becomes

$$e''_1 = \left(D - \frac{\mu\, \delta}{h} + \frac{\mu\, \delta^3}{6\, h^3} \right) f(x) = \frac{D^5 f}{30} h^4 + \frac{159\, D^7 f}{45{,}360} h^6 + \cdots$$

Similarly, by Eqs. 16*b*, 11, and 12*b*, the error in the second derivative is

$$e''_2 = \left(D^2 - \frac{\delta^2}{h^2} + \frac{\delta^4}{12\, h^4} \right) f(x) = \frac{23\, D^6 f}{2{,}520} h^4 + \frac{D^8 f}{21} h^6 + \cdots$$

and, more generally, the error in the nth derivative is of the type:

$$e''_n = \phi''_1(x)\, h^4 + \phi''_2(x)\, h^6 + \phi''_3(x)\, h^8 + \cdots \ldots\ldots\ldots (29a)$$

When the Taylor series of $f(x)$ is rapidly convergent, the series on the right-hand member of Eq. 29*a* is also rapidly convergent, and more so than the series of Eq. 28.

The derivations of this section show that the error due to the approximation of the nth derivative by mth order difference ratios is of the type:

$$e_{nm} = \phi_{1m}\, h^{2m} + \phi_{2m}\, h^{2(m+1)} + \phi_{3m}\, h^{2(m+2)} + \cdots \ldots\ldots\ldots (29b)$$

in which the functions ϕ_{im} are independent of h.

On the basis of this result (due to Mr. Sheppard[12]) Mr. Richardson[10] was able to prove that the error in the solution of a problem defined by a linear differential equation and linear boundary conditions, due to the substitution of difference ratios for derivatives, is also of the "h^{2m}-type," as expressed by Eq. 29*b*.

7. ERROR IN THE CHARACTERISTIC VALUES DUE TO THE SUBSTITUTION OF DIFFERENCE RATIOS FOR DERIVATIVES

The solution of buckling and vibration problems requires the determination of the values of a certain parameter K for which the linear homogeneous differential equation of the problem has solutions, satisfying linear homogeneous boundary conditions, and not identically equal to zero.

Calling Y one such solution and indicating by $L_1\,Y$ and $L_2\,Y$ two linear combinations of derivatives of Y, a buckling equation, for instance, may be simply written as

$$L_1\,Y + K\,L_2\,Y = 0 \qquad \text{(30)}$$

Next indicate by l_1 and l_2 the difference operators corresponding to L_1 and L_2; by y the approximate value of Y; by k the approximate value of K obtained by solving the difference rather than the differential problem; and by e the error of k, that is, the difference:

$$e = K - k \qquad \text{(31)}$$

With these symbols the approximating difference equation becomes

$$l_1\,y + k\,l_2\,y = 0 \qquad \text{(32)}$$

Indicate by e_1 and e_2 the errors due to substitution of finite differences for derivatives in $L_1\,y$ and $L_2\,y$:

$$L_1\,y - l_1\,y = e_1 \qquad \text{(33a)}$$

and

$$L_2\,y - l_2\,y = e_2 \qquad \text{(33b)}$$

and substitute Eqs. 31 and 33 in Eq. 32, thus: $L_1\,y - e_1 + (K - e)\,(L_2\,y - e_2) = 0$, from which

$$e = \frac{(L_1\,y + K\,L_2\,y) - (e_1 + K\,e_2)}{L_2\,y - e_2} \qquad \text{(34)}$$

In his classical paper on finite differences applied to the solution of differential equations, Mr. Richardson[10] makes the assumption that the error e, of K, is also of the h^{2m}-type. Eq. 34 proves this assumption generally untrue. In fact, by Mr. Richardson's theorem e_1 and e_2 (and hence $e_1 + K\,e_2$) are of the h^{2m}-type, whereas in general, $L_1\,y + K\,L_2\,y$ is not. Therefore, performing the division indicated in Eq. 34 and assuming, for instance, first order difference operators, e takes the form:

$$e = \phi_0(h) + \phi_1(h)\,h^2 + \phi_2(h)\,h^4 + \cdots \qquad \text{(35)}$$

which is not of the h^2 type, both because of the first term ϕ_0 and because of the dependence of the other ϕ_i on h. (This dependence is due to the fact that e_1 and e_2 are obviously functions of the spacing h.)

8. THE EVALUATION OF BUCKLING LOADS BY FINITE DIFFERENCES

When the differential equation—

$$L_1 f + K\, L_2 f = 0 \quad\text{.....................}(36)$$

—of a buckling problem is known, together with the corresponding boundary conditions $L_3 f = 0$, the characteristic values K may be determined by the following method of successive approximations, based on the use of finite differences:

The domain under consideration is covered by one of the grids considered in Sections 3, 4, and 5, some of whose corners will fall on the boundary of the domain (boundary points) and some of which will fall inside the boundary (body points). The values of f at the boundary points and at some neighboring points will be determined by the boundary conditions of the problem, directly or in terms of the body points, as will be noted in detail from the applications of Section 10. The values f_i of f at the N body points of the mesh ($i = 1, 2, \cdots, N$) are unknown instead.

The difference equation—

$$l_1 f + k\, l_2 f = 0 \quad\text{.....................}(37a)$$

—corresponding to the differential Eq. 36 is satisfied at each body point. Hence the application of Eq. 37*a* at each of the N body points gives rise to a set of N linear, algebraic, homogeneous equations in the N unknowns f_i:

$$l_1 f_i + k\, l_2 f_i = 0 \quad (i = 1, 2, \cdots, N) \quad\text{............}(37b)$$

and this system has solutions f_i not identically zero, if and only if its determinant Δ, which is a function of k, is equal to zero:

$$\Delta(k) = 0 \quad\text{.........................}(38)$$

If p is the degree in k of the determinantal Eq. 38 (k-equation), its p roots give approximate values of the first p characteristic values, K, of the original differential equation, Eq. 36. In problems of elastic stability the designer will be mainly interested in the lowest buckling load, that is, in the lowest characteristic value, and hence will solve Eq. 38 for its smallest root k. Once this is done for a value h_i of the mesh size h, obtaining a value k_i for k, the computations may be repeated for a new mesh size h_j smaller than h_i, to obtain a better approximation k_j of k. By reduction of the mesh size and corresponding increase in the number N of the body points considered, K may be evaluated, theoretically, to any degree of accuracy.

Instead, an increase in the number of body points actually increases very rapidly the labor involved in writing out and solving the determinantal equation and it becomes practically impossible to better the accuracy of the solution beyond certain limits. For this reason it is important to use a practical method of evaluation for the determinantal equation and a simple procedure for its solution.

Once the determinant has been obtained by the schemes of Sections 3, 4, and 5, the simplest procedure for writing the determinantal equation is by the method of "pivotal condensation." It can be easily proved[16] that the value of a determinant of order n—

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \text{..................} (39a)$$

—is equal to the value of the following determinant of order $n - 1$:

$$\Delta = \frac{1}{a^{n-2}_{11}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix} \quad \text{........} (39b)$$

in which each element is a second order determinant with the same leading element a_{11}, called the "pivot." Any other element a_{ij} of Δ may be used as pivot, but in this case the determinant of Eq. 39*b* must be given the sign $(-1)^{i+j}$.

By successive applications of Eq. 39*b* to Eq. 38 the order of the determinant $\Delta(k)$ is reduced to 1 and the k-equation is thus obtained. In this evaluation the multiplying factor $\frac{(-1)^{i+j}}{a^{n-2}_{ij}}$ may be dropped at each step, since the determinant must be equated to zero. All the elements of any row or column may also be divided by a common factor for the same reason, thus simplifying the numerical coefficients of the final k-equation.

When $\Delta(k)$ is a determinant of high order, each term of the successive determinants obtained by pivotal condensation is a high degree polynomial in k and writing out the k-equation may become cumbersome. It is easier, in such cases, to substitute an approximate numerical value of k in $\Delta(k)$ and to evaluate by pivotal condensation the corresponding numerical determinant. Two or three trials will usually give an acceptable result after interpolation.

The most practical method for the determination of the lowest root of the determinantal equation is Newton's method or the method of the tangents,[17] in which k_i is used as a starting value for the computation of k_j $(j > i)$.

9. EXTRAPOLATION PROCEDURES FOR THE EVALUATION OF BUCKLING LOADS

As shown in Section 8, the approximate values k_i approach the true value of K as the mesh size h_i approaches zero. The mode of this approach depends on the particular problem at hand—that is, on the differential equation and the boundary conditions, and it is not known, offhand.

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[16] "Determinants and Matrices," by A. C. Aitken, Oliver and Boyd, London, 1939, p. 45.

[17] "Higher Mathematics for Engineers and Physicists," by I. E. Sokolnikoff and E. S. Sokolnikoff, McGraw-Hill Book Co., Inc., New York, N. Y, 1st Ed., 1934, p. 32.

On the basis of extensive experimental investigations on the numerical evaluation of buckling loads by finite differences, the following qualitative statements may be made as to the behavior of k:

A. In many buckling problems, for which exact solutions are known, the approximations (k) of K approach the correct value monotonically (Fig. 10(*a*))—that is, either continuously decreasing or continuously increasing.

B. In some cases the approach of k to K is not monotonic for large values of h, but becomes rapidly monotonic as h approaches zero (Fig. 10(*b*)).

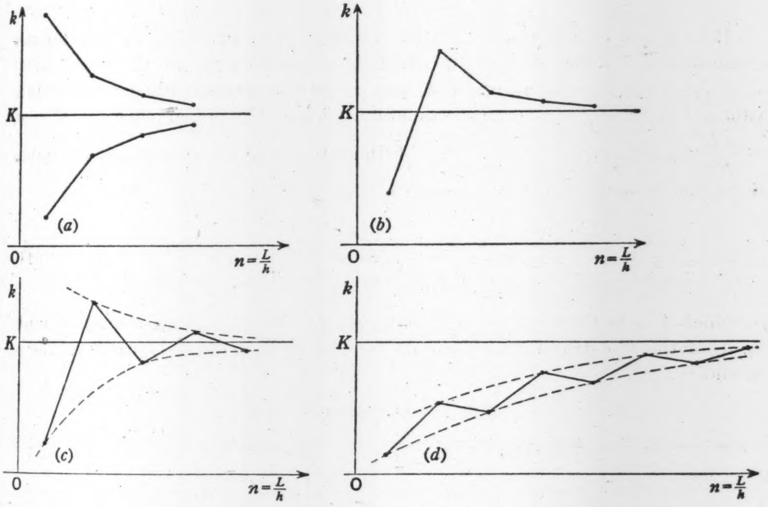


FIG. 10.—APPROACH OF k TO K

C. In a substantial number of cases the approach of k to K is oscillatory with decreasing amplitude (Fig. 10(*c*))—that is, k approaches K alternatively from above and from below.

D. In some cases the approach of k to K is one sided (from above or below), but not monotonic (Fig. 10(*d*)).

E. No other kinds of approach have been encountered thus far.

F. By choosing a proper sequence of values for h it is possible in all cases to obtain a monotonic sequence of values of k.

G. The error corresponding to a monotonic sequence is found, empirically, to be of the h^{2m}-type—that is, for a monotonic sequence the functions $\phi_{im}(h)$ of Eq. 35 are practically independent of h, and ϕ_0 is practically zero. The error e', due to first order differences, may thus be written as

$$e' = a\,h^2 + b\,h^4 + c\,h^6 + \cdots \qquad (40a)$$

in which $a, b, c, \cdots$ are constants.

This last result makes it possible to evaluate the characteristic value by extrapolation procedures, first suggested by Mr. Richardson,[10] which enhance the powerfulness of the method to a considerable extent.

By Eq. 40a, when h is sufficiently small for its higher powers to be negligible in comparison with h^2, e' is approximately given by

$$e' \doteq a\,h^2 \qquad (40b)$$

If n_i and n_j represent the number of subintervals in which a characteristic dimension L of the domain is divided, corresponding to the mesh sizes $h_i = L/n_i$ and $h_j = L/n_j$, and if k_i and k_j are the corresponding approximate values of K and e'_i, e'_j are the corresponding errors, Eq. 40b gives: $e'_i = K - k_i \doteq \dfrac{a\,L^2}{n^2_i}$ and $e'_j = K - k_j \doteq \dfrac{a\,L^2}{n^2_j}$. Eliminating $a\,L^2$ between these two equations, the "h^2-extrapolation formula" is obtained:

$$k'_{i,j} = \frac{n^2_j\,k_j - n^2_i\,k_i}{n^2_j - n^2_i} = a_j\,k_j - a_i\,k_i \qquad (41)$$

in which $k'_{i,j}$ is the extrapolated value of K. Table 1 gives the coefficients a_j, a_i of the h^2-extrapolation formula for the most commonly used values of n_i and n_j.

TABLE 1.—h^2-EXTRAPOLATIONS

n_j/n_i	a_j		a_i	
(1)	(2)	(3)	(4)	(5)
2/1	4/3	1.33333	1/3	0.33333
3/2	9/5	1.8	4/5	0.8
4/3	16/7	2.28571	9/7	1.28571
5/4	25/9	2.77778	16/9	1.77778
5/3	25/16	1.5625	9/16	0.5625
6/5	36/11	3.272727	25/11	2.272727
7/5	49/24	2.041667	25/24	1.041667
7/6	49/13	3.769234	36/13	2.769234

Since Eq. 41 is homogeneous in n, the h^2-extrapolation coefficients of Table 1 are only functions of the ratios n_j/n_i; for instance, the (2/1)-extrapolation formula holds also for (4/2).

If, in Eq. 40a, the first two terms of the series are taken into consideration and the formula is applied to three approximations k_i, k_j, and k_k, obtained by subdividing the characteristic length L into $n_i < n_j < n_k$ parts, elimination of $a\,L^2$ and $b\,L^2$ gives the "(h^2,h^4)-extrapolation formula":

$$k'_{i,j,k} = \frac{n^4{}_k(n^2{}_j - n^2{}_i)k_k - n^4{}_j(n^2{}_k - n^2{}_i)k_j + n^4{}_i(n^2{}_k - n^2{}_j)k_i}{n^4{}_k(n^2{}_j - n^2{}_i) - n^4{}_j(n^2{}_k - n^2{}_i) + n^4{}_i(n^2{}_k - n^2{}_j)}$$

$$= a_k\,k_k - a_j\,k_j + a_i\,k_i \quad \ldots\ldots\ldots\ldots\ldots\ldots (42)$$

in which k' is the extrapolated value of K.

Table 2 gives the (h^2,h^4)-extrapolation coefficients for the most commonly used values of n_i, n_j, and n_k. Again the values of n may be multiplied by a common factor without changing the formulas.

TABLE 2.—(h^2,h^4)-EXTRAPOLATIONS

n_i,n_j,n_k	a_k		a_j		a_i	
(1)	(2)	(3)	(4)	(5)	(6)	(7)
3/2/1	243/120	2.025	128/120	1.06667	5/120	0.04167
4/3/2	1,280/420	3.04762	972/420	2.31429	112/420	0.26667
5/4/3	4,375/1,008	4.34028	4,096/1,008	4.06349	729/1,008	0.72321
6/5/4	11,664/1,980	5.89091	12,500/1,980	6.31313	2,816/1,980	1.42222
7/6/5	26,411/3,432	7.69551	31,104/3,432	9.06294	8,125/3,432	2.36742
5/3/1	5,000/3,072	1.62760	1,944/3,072	0.63281	16/3,072	0.00521
7/5/3	38,416/15,360	2.50104	25,000/15,360	1.62760	1,944/15,360	0.12656

The h^2-extrapolation formula and the (h^2,h^4)-extrapolation formula are used in connection with first order differences.

Making the same type of simplifying assumptions in the formula for the error e'' due to second order differences,

$$e'' = a\,h^4 + b\,h^6 + c\,h^8 + \cdots$$

the "h^4-extrapolation formula" is

$$k''_{i,j} = \frac{n^4{}_j\,k_j - n^4{}_i\,k_i}{n^4{}_j - n^4{}_i} = b_j\,k_j - b_i\,k_i \quad \ldots\ldots\ldots\ldots (43a)$$

and the "(h^4,h^6)-extrapolation formula" is

$$k''_{i,j,k} = \frac{n^6{}_k(n^2{}_j - n^2{}_i)k_k - n^6{}_j(n^2{}_k - n^2{}_i)k_j + n^6{}_i(n^2{}_k - n^2{}_j)k_i}{n^6{}_k(n^2{}_j - n^2{}_i) - n^6{}_j(n^2{}_k - n^2{}_i) + n^6{}_i(n^2{}_k - n^2{}_j)}$$

$$= b_k\,k_k - b_j\,k_j + b_i\,k_i \quad \ldots\ldots (43b)$$

Table 3 and Table 4 give the h^4-extrapolation coefficients and the (h^4,h^6)-extrapolation coefficients most often needed. All these formulas are homogeneous in n and hold when all the values of n are multiplied by the same number.

The following points should be kept in mind in using the extrapolation formulas of Section 9:

(*a*) The h^2-extrapolation formula and the h^4-extrapolation formula assume that the successive approximations k approach K from the same side. When the approach to K is of the oscillating type, the formulas may be used separately in connection with the approximations from above or from below. The two extrapolated values thus obtained may be usefully compared. It is unsafe to use extrapolation without having at least three approximate values of K.

(*b*) Even if the approximate values of K approach the true value monotonically (either from above or from below), it is not known offhand whether the extrapolated value is above or below the true value.

TABLE 3.—h^4-EXTRAPOLATIONS

n_j/n_i	b_j		b_i	
(1)	(2)	(3)	(4)	(5)
2/1	16/15	1.06667	1/15	0.06667
3/2	81/65	1.24615	16/65	0.24615
4/3	256/175	1.46286	81/175	0.46286
5/4	625/369	1.69377	256/369	0.69377

TABLE 4.—(h^4,h^6)-EXTRAPOLATIONS

n_i,n_j,n_k	b_k		b_j		b_i	
(1)	(2)	(3)	(4)	(5)	(6)	(7)
3/2/1	2,187/ 1,680	1.301786	512/ 1,680	0.304762	5/ 1,680	0.002976
4/3/2	20,480/12,180	1.681445	8,748/12,180	0.718227	448/12,180	0.036782
5/4/3	109,375/50,400	2.170139	65,536/50,400	1.300317	6,561/50,400	0.130179

(*c*) It is true that, in general, the (h^2,h^4)-extrapolation gives better results than the h^2-extrapolation, but it is impossible to determine offhand whether an h^2-extrapolation based on larger values of n will give better results than an (h^2,h^4)-extrapolation based on smaller values of n or even on three values of n the largest of which is the same as the largest value of n in the h^2-extrapolation. The same statement holds true for the h^4-extrapolation and the (h^4,h^6)-extrapolation.

10. APPLICATIONS

In this section the method of finite differences will be used to compute the lowest buckling load of beams, plates, and shells under various load and boundary conditions. The following applications have been chosen: (1) Because they illustrate the technical details and the properties of the method; (2) because they indicate the asymptotic approach of the successive approximations to the true value of the buckling load; or (3) because, to the best knowledge of the writer, the solution of the problem had not been previously attempted.

To simplify and shorten the treatment, the differential equations and boundary conditions of each problem, as well as the true or the best available approximate value of the buckling load, have been taken from the fundamental monograph by S. Timoshenko on "Theory of Elastic Stability"[2] unless otherwise stated.

A. EULER'S BUCKLING PROBLEM

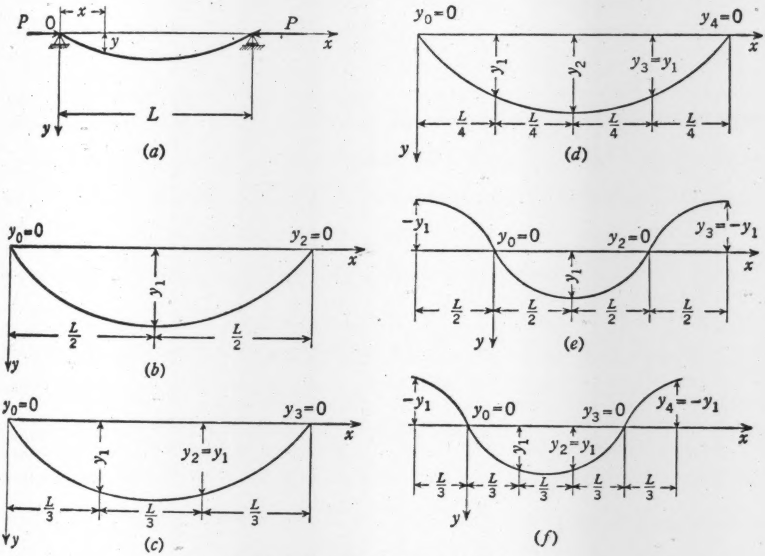
A simply supported, slender beam, of length L and constant moment of inertia I, buckles under the action of two equal longitudinal loads P (Fig. 11(a)). Calling E Young's modulus and $y = f(x)$ the normal deflection, and choosing the origin of the longitudinal x-axis at the left support, the differential equation and boundary conditions become[18]

$$y'' + \frac{P}{E\,I}\,y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (44a)$$

and

$$y(0) = 0; \quad y(L) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (44b)$$

in which the primes indicate differentiation with respect to x.


P
0
x
P
x
y
L
y
(a)
$y_0=0$
$y_4=0$
x
y_1
y_2
$y_3=y_1$
$\frac{L}{4}$
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y
(d)
$y_0=0$
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y
(b)
$-y_1$
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x
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y
(e)
$y_0=0$
$y_3=0$
x
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x
y_1
$y_2=y_1$
$\frac{L}{3}$
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$\frac{L}{3}$
$\frac{L}{3}$
$\frac{L}{3}$
y
(f)

FIG. 11.—EULER'S BUCKLING PROBLEM

(1) *First Order Differences.*—Dividing the length L into n equal parts of length $h = L/n$ and letting the ordinates $y_i = f(x_i)$ of the deflection curve at the $n + 1$ separation points x_i—$y_0, y_1, y_2, \cdots, y_{ll}, y_l, y_i, y_r, y_{rr}, \cdots, y_{n-2}, y_{n-1}, y_n$—the second derivative of y at x_i is approximated by Eq. 15b; thus: $y''_i \doteq \frac{y_l - 2\,y_i + y_r}{h^2}$.

Substitution of the latter expression in Eq. 44a gives the statement of the boundary problem in terms of differences:

$$y_l + b_n\,y_i + y_r = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (45a)$$

and

$$y_0 = 0; \qquad y_n = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (45b)$$

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[18] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, paragraph 12.

in which

$$b_n = \frac{P_n\, h^2}{E\, I} - 2 = \frac{P_n\, L^2}{n^2\, E\, I} - 2 \text{.................} (46)$$

and P_n is the approximate value of P_{cr} obtained in the approximation with n subintervals.

Approximation $n = 2$ (Fig. 11(b)).—Eq. 45a at $x = \frac{L}{2}$ yields: $0 + b_2\, y_1 + 0 = 0$. Therefore, $b_2 = 0$, from which, letting—

$$k_n = \frac{P_n}{E\, I/L^2} \text{........................} (47)$$

—the first approximation k_2 and the corresponding error become

$$k_2 = n^2(b_n + 2)]_{n=2} = 8 \quad (-\ 19\%).$$

Approximation $n = 3$.—Noticing that the deflection curve is symmetrical with respect to the middle point of the beam (Fig. 11(c)) and applying Eq. 45a at $x = \frac{L}{3}$: $0 + b_3\, y_1 + y_1 = 0$. Therefore, $b_3 = -\ 1$ and $k_3 = 9$ ($-$ 9%).

Approximation $n = 4$.—Eq. 45a at $x = L/4, L/2$ becomes: $0 + b_4\, y_1 + y_2 = 0$ and $y_1 + b_4\, y_2 + y_1 = 0$. The determinantal equation is: $\begin{vmatrix} b_4 & 1 \\ 2 & b_4 \end{vmatrix} = b^2_4 - 2 = 0$. Therefore, $b_4 = \pm\sqrt{2}$. Choosing the value of b_4 that makes k_4 smaller, $k_4 = 9.37258$ ($-$ 5%).

Approximation $n = 5$.—Eq. 45a at $x = L/5, 2\, L/5$ becomes: $0 + b_5\, y_1 + y_2 = 0$ and $y_1 + b_5\, y_2 + y_2 = 0$. The corresponding determinantal equation is: $\begin{vmatrix} b_5 & 1 \\ 1 & b_5 + 1 \end{vmatrix} = b^2_5 + b_5 - 1 = 0$. Choosing the root of this quadratic equation, which makes k_5 smaller: $b_5 = -\ 1.62$ and $k_5 = 9.54918$ ($-$ 3.2%).

Approximation $n = 6$.—Eq. 45a at $x = L/6, L/3, L/2$ becomes: $0 + b_6\, y_1 + y_2 = 0$; $y_1 + b_6\, y_2 + y_3 = 0$; and $y_2 + b_6\, y_3 + y_2 = 0$. The corresponding determinantal equation is: $\begin{vmatrix} b_6 & 1 & 0 \\ 1 & b_6 & 1 \\ 0 & 2 & b_6 \end{vmatrix} = b_6(b^2_6 - 3) = 0$, from which $b_6 = -\ 1.73$ and $k_6 = 9.64616$ ($-$ 2.3%).

Approximation $n = 7$.—Eq. 45a at $x = L/7, 2\, L/7, 3\, L/7$ becomes: $0 + b_7\, y_1 + y_2 = 0$; $y_1 + b_7\, y_2 + y_3 = 0$; and $y_2 + b_7\, y_3 + y_2 = 0$. The determinantal equation is: $\begin{vmatrix} b_7 & 1 & 0 \\ 1 & b_7 & 1 \\ 0 & 1 & b_7 + 1 \end{vmatrix} = b^3_7 + b^2_7 - 2\, b_7 - 1 = 0$, from which $b_7 = -\ 1.80$ and $k_7 = 9.70506$ ($-$ 1.7%). The correct value of K is $\pi^2 = 9.86960$.

A large number of successive approximations has been computed in the foregoing example to investigate the mode of approach of k_n toward K, but it should be noticed that, since the approximations $n = 3,5,7$ involve determinants of the same order as the approximations $n = 2,4,6$, in practice only the odd approximations would be computed.

The h^2-extrapolation formula (Eq. 41) with Table 1 and the (h^2,h^4)-extrapolation formula (Eq. 42) with Table 2 give the results in Table 5.

The extrapolated values are so near the true value that in practice two approximations and an h^2-extrapolation would give a result more than sufficiently accurate.

TABLE 5.—EXTRAPOLATIONS FOR THE CASE OF A SIMPLY SUPPORTED BEAM WITH CONSTANT MOMENT OF INERTIA

n	h^2-extrapolations	$e(\%)$	n	(h^2,h^4)-extrapolations	$e(\%)$
2,3	9.80000	−0.705	2,3,4	9.86881	−0.0080
3,4	9.85161	−0.182	3,4,5	9.86962	+0.0002
4,5	9.86314	−0.065	4,5,6	9.86932	−0.0028
5,6	9.86657	−0.031	5,6,7	9.86974	+0.0014
6,7	9.86817	−0.014	....	......	......

(2) *Second Order Differences.*—Approximating y'' by Eq. 16b the boundary value problem (Eqs. 44) in terms of second order differences becomes

$$- y_{ll} + 16\, y_l + b_n\, y_i + 16\, y_r - y_{rr} = 0 \qquad \text{(48}a\text{)}$$

and

$$y_0 = 0; \qquad y_n = 0 \qquad \text{(48}b\text{)}$$

in which

$$b_n = \frac{12\, P_n\, L^2}{n^2\, E\, I} - 30 = \frac{12\, k_n}{n^2} - 30 \qquad (49)$$

Since the difference equation (Eq. 48a) involves the values of y two points to the left and two points to the right of x_i, the values of y_{-1} and y_{n+1} must be known before the equation can be applied at x_1 and x_{n-1}. These values are easily determined by using first order differences in expressing the boundary conditions at x_0 and at x_n. At these points the beam is simply supported and, hence, $y_0 = y_n = 0$ and $y''_0 = y''_n = 0$. Expressing y'' by first order differences (Eq. 15b): $h^2\, y''_0 = y_{-1} - 2\, y_0 + y_1 = y_{-1} + y_1 = 0$ and $h^2\, y''_n = y_{n-1} - 2\, y_n + y_{n+1} = y_{n-1} + y_{n+1} = 0$. Hence,

$$y_{-1} = -\, y_1; \qquad y_{n+1} = -\, y_{n-1} \qquad (50)$$

These relations state the geometrical fact that x_0 and x_n are inflection points of the y-curve.

Proceeding as was done before with first order differences, the following results are obtained:

Approximation $n = 2$.—By Eq. 48a at $x = L/2$: $y_1 + 0 + b_2\, y_1 + 0 + y_1 = 0$; $b_2 = -\, 2$; and $k_2 = 9.33$ ($-$ 5.47%).

Approximation $n = 3$.—By Eq. 48a at $x = L/3$: $y_1 + 0 + b_3\, y_1 + 16\, y_1 + 0 = 0$; $b_3 = -\, 17$; and $k_3 = 9.75$ ($-$ 1.21%).

Approximation $n = 4$.—By Eq. 48a at $x = L/4, L/2$: $y_1 + 0 + b_4\, y_1 + 16\, y_2 - y_1 = 0$ and $0 + 16\, y_1 + b_4\, y_2 + 16\, y_1 + 0 = 0$, from which $\begin{vmatrix} b_4 & 16 \\ 32 & b_4 \end{vmatrix} = b^4 - 2 \times 16^2 = 0$ and $b_4 = -\, 22.63$. Therefore, $k_4 = 9.83$ ($-$ 0.40%).

Extrapolations.—Using the h^4-extrapolation formulas and the (h^4,h^6)-extrapolation formulas (Eqs. 43 and Tables 3 and 4), the following extrapolated values are obtained: $k_{2,3} = 9.85(-\ 0.15\%)$; $k_{3,4} = 9.87(-0.02\%)$; and $k_{2,3,4} = 9.8695(-0.0009\%)$.

The evaluation of the second order approximations and extrapolations does not require an increased amount of labor, but the second order approximations are much nearer to the true value (π^2) of K than are the first order approximations. In problems of this type it may well pay to use second order differences directly.

B. SIMPLY SUPPORTED BEAM WITH VARIABLE MOMENT OF INERTIA

In the previous problem, if the moment of inertia I varies according to the law (in which I_o is a constant),

$$I(x)\begin{cases} = I_o\left(1 + \dfrac{2\,x}{L}\right) & 0 \leqq x \leqq \dfrac{L}{2} \\[2ex] = I_o\left(3 - \dfrac{2\,x}{L}\right) & \dfrac{L}{2} \leqq x \leqq L \end{cases}$$

Eq. 44*a* becomes, by Eq. 15*b*,

$$y_l + \left(\frac{a_n}{1 + 2\,x/L} - 2\right) y_i + y_r = 0 \left(0 \leqq x \leqq \frac{L}{2}\right) \ldots\ldots\ldots (51)$$

in which

$$a_n = \frac{P_n\, L^2}{n^2\, E\, I_o} = \frac{k_n}{n^2} \ldots\ldots\ldots (52)$$

—whereas the boundary conditions remain $y_0 = 0$ and $y_n = 0$. The successive approximations are obtained as before, remembering that y is symmetrical about the middle point of the beam. Table 6 gives the results of the computations.

TABLE 6.—EXTRAPOLATIONS FOR THE CASE OF A SIMPLY SUPPORTED BEAM WITH VARIABLE MOMENT OF INERTIA

n	k_n	$e_n(\%)$	h^2-EXTRAPOLATIONS			(h^2,h^4)-EXTRAPOLATIONS		
			n	k	$e(\%)$	n	k	$e(\%)$
2	16.0000	−3.23	2,4	16.0000	−3.23	2,4,6	16.5114	−0.13
3	15.0000	−9.27	3,5	16.3933	−0.85	3,5,7	16.4964	−0.22
4	16.0000	−3.23	4,6	16.4546	−0.48	....	....	....
5	15.8917	−3.83	5,7	16.4774	−0.34	....	....	....
6	16.2526	−1.70	....	....	....	....	....	....
7	16.1786	−2.15	....	....	....	....	....	....

The approach of k_n to K is one sided, but not monotonic, whereas the separate sequences of even and odd k_n are both monotonic. The extrapolations of Table 6 are obtained by these sequences.

The energy method applied to this problem with an assumed sinusoidal deflection gives an upper bound $k = 16.5342$. Hence it is probable that the average of the two (h^2,h^4)-extrapolations is no more than 0.2% off.

C. LATERAL BUCKLING OF A SIMPLY SUPPORTED BEAM OF NARROW CROSS SECTION

A beam of narrow rectangular cross section is simply supported at $x = \pm L/2$, while the rotation of its ends around the axis of the beam is prevented. The beam buckles laterally under the action of a vertical load P at $x = 0$. If B is the smaller of the two flexural rigidities of the beam in its principal planes, C is the torsional rigidity, and β is the rotation of a section x, the boundary problem becomes[19]

$$\beta'' + \frac{P^2 L^2}{4 B C} \left(\frac{1}{2} - \frac{x}{L} \right)^2 \beta = 0 \quad \ldots\ldots\ldots\ldots\ldots (53a)$$

and

$$\beta \left(\frac{L}{2} \right) = \beta \left(- \frac{L}{2} \right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (53b)$$

The corresponding difference problem becomes, by Eq. 15*b*,

$$\beta_l + \left[a_n \left(\frac{1}{2} - \frac{x}{L} \right)^2 - 2 \right] \beta_i + \beta_r = 0 \quad \ldots\ldots\ldots\ldots (54a)$$

and

$$\beta_0 = 0; \qquad \beta_n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (54b)$$

in which

$$a_n = \frac{P^2{}_n L^4}{4 n^2 B C} = \frac{k^2{}_n}{(2 n)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (55)$$

Table 7 gives the results of the computations:

TABLE 7.—EXTRAPOLATIONS FOR THE CASE OF LATERAL BUCKLING OF A NARROW SIMPLY SUPPORTED BEAM

n	k_n	$e_n(\%)$	h^2-EXTRAPOLATIONS			(h^2, h^4)-EXTRAPOLATIONS		
			n	k	e (%)	n	k	e (%)
2	11.3137	−33.2	2,4	16.21	−4.3	2,4,6	16.905	−0.09
3	18.0000	+ 6.32	3,5	16.62	−1.8	3,5,7	16.933	+0.02
4	14.9827	−11.82	4,6	16.83	−0.6	....	....	....
5	17.1190	+ 1.11	5,7	16.88	−0.3	....	....	....
6	16.0075	− 5.40	....	....	....	....	....	....
7	17.0000	+ 0.41	....	....	....	....	....	....

The true value of K is 16.93. In this particular problem the approach of k_n to K is of the oscillatory type, the even k_n being smaller, the odd k_n being larger than K, and the extrapolations have been obtained by the even and odd sequences.

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[19] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, p. 250.

D. Lateral Buckling of a Cantilever Beam of Narrow Cross Section

If the beam of the previous example is built in at $x = 0$ and loaded by a transverse load P at $x = L$, the corresponding boundary value problem becomes[20]

$$\beta'' + \frac{P^2\,L^2}{B\,C}\left(1 - \frac{x}{L}\right)^2 \beta = 0 \quad \text{(56a)}$$

and

$$\beta\,(0) = 0; \qquad \beta'\,(L) = 0 \quad \text{(56b)}$$

and, by Eq. 15*b*, the finite difference problem reduces to

$$\beta_l + \left[a_n\left(1 - \frac{x}{L}\right)^2 - 2\right]\beta_i + \beta_r = 0 \quad \text{(57a)}$$

and

$$\beta_0 = 0; \qquad \beta_{n+1} = \beta_{n-1} \quad \text{(57b)}$$

in which

$$a_n = \frac{P^2{}_n\,L^4}{n^2\,B\,C} = \frac{k^2{}_n}{n^2} \quad \text{(58)}$$

Table 8 gives the results of the computations.

TABLE 8.—Extrapolations for the Case of Lateral Buckling of a Narrow Cantilever Beam

n	k_n	e_n (%)	h^2-Extrapolations			(h^2,h^4)-Extrapolations		
			n	k	e (%)	n	k	e (%)
2	4.000	−3.2	3,4	3.992	−0.52	2,3,4	4.030	+0.42
3	3.933	−1.9	4,5	4.006	−0.17	3,4,5	4.014	+0.025
4	3.959	−1.3	....	....	....	....	....	....
5	3.976	−0.92	....	....	....	....	....	....

The correct value of K is 4.013 and the convergence of k_n to K is monotonic after $n = 3$. The error is of the type $e = -\,a\,h^2 + b\,h^4$. The h^2-extrapolation may be applied for $n = 3{,}4$ and $n = 4{,}5$, whereas the (h^2,h^4)-extrapolation may be applied to any sequence, and in particular to $n = 2{,}3{,}4$ and $n = 3{,}4{,}5$.

E. Buckling of a Simply Supported Square Plate Under Uniform Compression

A square plate with sides L and flexural rigidity D, simply supported along its boundary, buckles under a uniform compression N per unit of boundary length. Taking the x,y-axes parallel to the plate sides with origin at one of its corners, and calling $w(x,y)$ the deflection of the plate, the corresponding boundary value problem becomes[21]

$$\nabla^4 w + \frac{N}{D}\,\nabla^2 w = 0 \quad \text{(59a)}$$

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[20] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, p. 245.

[21] *Ibid.*, p. 333.

and, on the boundary,

$$\nabla^2 w = 0 \quad \ldots\ldots (59b)$$

or, letting $\nabla^2 w = z$,

$$\nabla^2 z + \frac{N}{D} z = 0 \quad \ldots\ldots (60a)$$

and, on the boundary,

$$z = 0 \quad \ldots\ldots (60b)$$

First Order Differences.—Making use of first order differences (Eq. 19*c* and Fig. 4(*c*)) the corresponding difference problem becomes

$$z_r + z_l + z_a + z_b + c_n z_i = 0 \quad \ldots\ldots (61a)$$

and, on the boundary,

$$z_i = 0 \quad \ldots\ldots (61b)$$

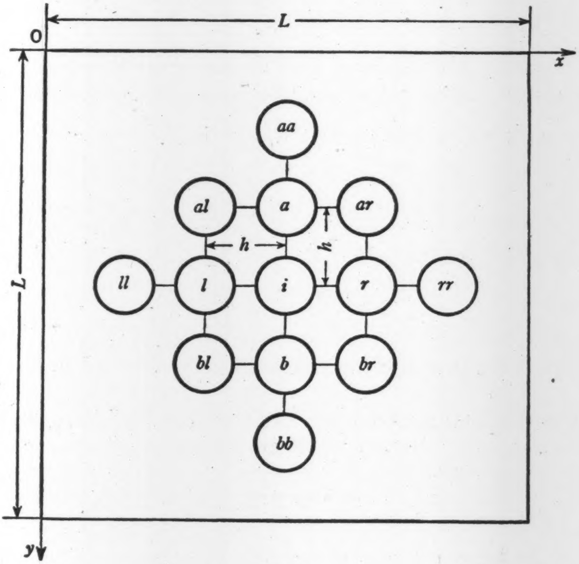

FIG. 12

in which the subscripts indicate the grid points according to Fig. 12, and

$$c_n = \frac{N_n L^2}{n^2 D} - 4 = \frac{k_n}{n^2} - 4 \quad \ldots\ldots (62)$$

Approximation $n = 2$.—Applying Eq. 61*a* at $x,y = L/2$: $c_2 = 0$ and $k_2 = 16.00$ (− 19%).

Approximation $n = 3$.—Applying Eq. 61*a* at $x,y = L/3$: $z_1 + 0 + z_1 + 0 + c_3 z_1 = 0$ and $c_3 = -2$ and $k_3 = 18.00$ (− 8.8%).

Approximation $n = 4$.—Applying Eq. 61*a* at $x,y = L/2, L/4$: $\begin{vmatrix} c_4 & 2 & 0 \\ 2 & c_4 & 1 \\ 0 & 4 & c_4 \end{vmatrix} = 0.$

Therefore, $c_4 = -2.8284$ and $k_4 = 18.75$ (-5.1%). By extrapolation: $k_{2,3} = 19.60$ (-0.71%); $k_{3,4} = 19.71$ (-0.15%); and $k_{2,3,4} = 19.75$ $(+0.051\%)$. The true value of K is 19.739.

F. BUCKLING OF A SQUARE PLATE WITH CLAMPED EDGES UNDER UNIFORM COMPRESSION

If the square plate of Example E has clamped edges, the boundary value problem becomes[22]

$$\nabla^4 w + \frac{N}{D} \nabla^2 w = 0 \ldots\ldots\ldots\ldots\ldots\ldots (63a)$$

and, on the boundary,

$$w = 0; \qquad \frac{\partial w}{\partial \nu} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (63b)$$

ν being the normal to the boundary. By the first order difference operators of Figs. 4(*b*) and 4(*d*), the corresponding difference problem becomes

$$w_{rr} + w_{ll} + w_{aa} + w_{bb} + 2\,(w_{br} + w_{bl} + w_{ar} + w_{al}) \\ + c_n\,(w_r + w_l + w_a + w_b) - (4\,c_n + 12)\,w_i = 0 \ldots (64a)$$

and

$$\left.\begin{matrix} w_i = 0, & w_l = w_r & \text{on } x = 0, & x = L \\ w_i = 0, & w_a = w_b & \text{on } y = 0, & y = L \end{matrix}\right\} \ldots\ldots (64b)$$

in which

$$c_n = \frac{N_n L^2}{n^2 D} - 8 = \frac{k_n}{n^2} - 8 \ldots\ldots\ldots\ldots\ldots\ldots (65)$$

Table 9 gives the results of the computations.

TABLE 9.—EXTRAPOLATIONS FOR THE BUCKLING OF A SQUARE PLATE WITH CLAMPED EDGES

n	k_n	e_n (%)	h^2-EXTRAPOLATIONS			(h^2,h^4)-EXTRAPOLATIONS		
			n	k	e (%)	n	k	e (%)
2	24.000	−56.0	2,3	45.600	−12.97	2,3,4	52.129	−0.49
3	35.000	−31.0	3,4	50.497	− 3.61	3,4,5	52.156	−0.44
4	43.342	−17.4	4,5	51.558	− 1.58	4,5,6	53.243	+1.64
5	45.660	−12.8	5,6	52.494	− 0.21	2,4,6	52.525	+0.21
6	47.748	− 9.0	....	....	....	....	....	....

The literature on the buckling of rectangular built-in plates is extensive, but no rigorous solution of the problem has been given. For the case of the present example, A. Weinstein[4] gives very close upper and lower bounds whose average, 52.385, has been assumed as a true value for the evaluation of the errors in the extrapolations. (The average of the bounds given by Mr. Timoshenko[23] is 52.455 and differs by 0.13% from the average of Mr. Weinstein's closer bounds.) It is interesting to notice that all the h^2-extrapolations are from below, whereas some (h^2,h^4)-extrapolations overshoot the mark. The computation of the higher approximations is very cumbersome, but the (h^2,h^4)-extrapolation gives excellent results by the first three approximations.

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[22] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, pp. 305, 298, and 364.

G. Buckling of a Rectangular Plate with Clamped Edges Under Uniform Compression

The problem of Example F can easily be extended to the case of a rectangular plate with sides of lengths 2 L and L parallel to x and y, respectively. By Eq. 19b for $r = 1/2$ and its square, the following equation for the difference problem is obtained:

$$w_{rr} + w_{ll} + 16\,(w_{bb} + w_{aa}) + 8\,(w_{br} + w_{bl} + w_{ar} + w_{al}) + c_n\,(w_r + w_l) + 4\,c_n\,(w_b + w_a) - (10\,c_n + 66)\,w_i = 0 \quad \text{(66)}$$

with

$$w_i = 0, \quad w_l = w_r \qquad \text{on } x = 0, \quad x = 2\,L \quad \text{(67a)}$$

and

$$w_i = 0, \quad w_a = w_b \qquad \text{on } y = 0, \quad y = L \quad \text{(67b)}$$

in which

$$c_n = \frac{4\,N_n\,L^2}{n^2\,D} - 20 = \frac{4\,k_n}{n^2} - 20 \quad \text{(68)}$$

TABLE 10.—Extrapolations for the Buckling of a Rectangular Plate with Clamped Edges

n	$4\,k_n$	h^2-Extrapolations			(h^2,h^4)-Extrapolations		
		n	$4\,k_n$	e (%)	n	$4\,k_n$	e (%)
2	67.200	2,3	137.400	−11.5	2,3,4	153.906	−0.84
3	106.200	3,4	149.779	− 3.53	3,4,5	154.768	−0.32
4	125.266	4,5	152.972	− 1.47	....	....	....
5	135.240	....	....	....	....	....	....

The results of the computations are presented in Table 10.[24] The errors have been computed assuming as true value the buckling load obtained by the energy method with a deflection function—$w = \frac{w_0}{4}\left(1 - \cos\frac{2\,\pi\,x}{\rho\,L}\right)\left(1 - \cos\frac{2\,\pi\,y}{L}\right)$—in which ρ is the ratio of the side parallel to x to the side parallel to y. The expression for this upper bound of the load is $\frac{4\,N\,L^2}{\pi^2\,D} = \frac{16}{3}\frac{3\,\rho^4 + 2\,\rho^2 + 3}{\rho^4 + \rho^2}$ which, for $\rho = 2$, gives $4\,K = 155.269$.

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[23] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, p. 365.

[24] "Buckling Loads for Built-In Rectangular Plates by Finite Differences," by J. Gold, thesis submitted to Columbia Univ., at New York, N. Y., in 1947, in partial fulfilment of the requirements for the degree of Master of Science.

H. BUCKLING OF A SIMPLY SUPPORTED SKEW PLATE UNDER UNIFORM COMPRESSION

A simply supported 45° skew plate, with sides L parallel to x and 45° sides $\sqrt{2}\,L$, buckles under a uniform compression N per unit of boundary length. The corresponding boundary value problem is identical with the problem of Example G.

A 45° skew plate may be covered by both a rectangular and a skew grid. Using a skew grid with $r = \sqrt{2}$ and $\alpha = 45°$, Eq. 23, or Fig. 7, Section 4, yields the difference problem:

$$2\,z_r + 2\,z_l + 4\,z_a + 4\,z_b - z_{ar} - z_{bl} + z_{br} + z_{al} + c_n\,z_i = 0 \ldots (69a)$$

and, on the boundary,

$$z_i = 0 \ldots\ldots (69b)$$

in which

$$c_n = \frac{2}{n^2}\frac{N_n L^2}{D} - 12 = \frac{2}{n^2}k_n - 12 \ldots\ldots (70)$$

Table 11(a) gives the results of the computations. The successive values of k_n obtained by a square grid are given in Table 11(b), together with the h^2-extrapolations and the (h^2,h^4)-extrapolations.

TABLE 11.—EXTRAPOLATIONS FOR THE CASE OF THE BUCKLING OF A 45° SKEW PLATE

n	(a) SKEW COORDINATES					(b) SQUARE GRID				
	k_n	h^2-Extrapolations		(h^2,h^4)-Extrapolations		k_n	h^2-Extrapolations		(h^2,h^4)-Extrapolations	
		n	k	n	k		n	k	n	k
2	24	2,3	28.73	2,3,4	27.51	16.00	2,3	25.76	2,3,4	26.68
3	26.63	3.4	27.82	3,4,5	27.21	21.42	3,4	26.45	3,4,5	26.72
4	27.15	4,5	27.43	....	....	23.62	4,5	26.62	....	....
5	27.25	....	....	....	....	24.70	....	....	....	....

No rigorous solution of this problem is known and a value of $k = 27$ may be accepted as plausible.

It must be noticed that, to obtain the $n = 5$ approximation in Cartesian coordinates, it is necessary to solve a determinantal equation of the eighth order. Skew coordinates are much better adapted to the problem and lead to the final result with a considerably smaller amount of labor.

I. BUCKLING OF A SIMPLY SUPPORTED, EQUILATERAL, TRIANGULAR PLATE UNDER UNIFORM COMPRESSION

A simply supported, equilateral, triangular plate, of sides L, buckles under a uniform compression N per unit of boundary length. The boundary value problem to be solved is identical with the problem of Example E, but in this case the plate may be covered by a triangular grid. By Eq. 27 or Fig. 9(a), the difference problem becomes

$$\Sigma z_s + c_n z_i = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (71a)$$

and, on the boundary,

$$z_i = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (71b)$$

in which

$$c_n = \frac{N\,L^2}{n^2\,D} - 6 = \frac{3}{2\,n^2}\,k_n - 6 \ldots\ldots\ldots\ldots\ldots (72)$$

and Σz_s is the sum of the values of z at the vertices of the hexagon surrounding the ith point.

TABLE 12.—EXTRAPOLATIONS FOR THE CASE OF THE BUCKLING OF A SIMPLY SUPPORTED, EQUILATERAL, TRIANGULAR PLATE UNDER UNIFORM COMPRESSION

n	k_n	e_n (%)	h^2-EXTRAPOLATIONS			(h^2, h^4)-EXTRAPOLATIONS		
			n	k	e (%)	n	k	e (%)
3	36.00	−31.6	3,4	51.25	−2.64	3,4,5	52.56	−0.15
4	42.67	−19.0	4,5	52.09	−1.04	4,5,6	52.67	+0.06
5	46.06	−13.7	5,6	52.41	−0.44	5,6,7	52.641	+0.002
6	48.00	− 8.8	6,7	52.52	−0.23	....	....	....
7	49.20	− 6.5	....	....	....	....	....	....

Table 12 gives the results of the computations. The correct value of K is 52.64.[25] Making use of finite difference operators in irregular triangular coordinates[15] a triangular plate of any shape may be similarly solved.

J. BUCKLING OF A SIMPLY SUPPORTED HEXAGONAL PLATE UNDER UNIFORM COMPRESSION

The buckling load of a regular hexagonal plate, simply supported and subject to uniform compression, may be computed by Eqs. 71 and 72 of Example I.

TABLE 13.—EXTRAPOLATIONS FOR THE CASE OF THE BUCKLING OF A SIMPLY SUPPORTED HEXAGONAL PLATE UNDER UNIFORM COMPRESSION

n	m	k_n	h^2-EXTRAPOLATIONS		(h^2, h^4)-EXTRAPOLATIONS	
			n	k	n	k
1	1	4	1,2	7.04	1,2,3	7.18
2	2	6.28	2,3	7.16	2,3,4	7.25
3	4	6.77	3,4	7.23	....	....
4	6	6.97	....	....	....	....

The values of the successive approximations are given in Table 13, in which m is the number of body points considered and hence represents the order of the determinant to be solved; 7.20 may be taken as a plausible value for k.

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[25] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, p. 371.

K. SYMMETRICAL BUCKLING OF AN INFINITELY LONG CYLINDRICAL SHELL UNDER UNIFORM AXIAL COMPRESSION

An infinitely long, thin, cylindrical shell of radius R, thickness h, flexural rigidity D, and Young's modulus E, buckles under a uniform axial compression N per unit length. Assuming that the shell buckles into half waves of length L, the corresponding boundary value problem is[26]

$$y'''' + \frac{N}{D} y'' + \frac{E\,h}{D\,R^2} y = 0 \quad \text{.................(73a)}$$

and

$$y(0) = y(L) = 0; \qquad y''(0) = y''(L) = 0 \quad \text{..........(73b)}$$

in which y is the radial displacement.

By Eqs. 15, the corresponding difference problem becomes

$$y_{ll} + (a_n - 4)\,y_l + (6 - 2\,a_n + b_n)\,y_i + (a_n - 4)\,y_r + y_{rr} = 0 \quad \text{..(74a)}$$

and

$$y_{-1} = -\,y_1; \qquad y_{n+1} = -\,y_{n-1} \quad \text{...............(74b)}$$

in which

$$a_n = \frac{N_n\,L^2}{n^2\,D}; \qquad b_n = \frac{L^4}{n^4}\,\frac{E\,h}{D\,R^2} \quad \text{................(75)}$$

Approximation $n = 2$.—Applying Eq. 73a at $x = L/2$: $4 - 2\,a_2 + b_2 = 0$ or $N_2 = D\left(\frac{8}{L^2} - \frac{E\,h}{D\,R^2}\,\frac{L^2}{8}\right)$.

Since the shell is infinitely long, it will buckle into an infinite number of half waves, whose length L is such as to make N a minimum. Letting

$$\frac{8}{L^2} = z; \qquad \frac{E\,h}{D\,R^2} = c \quad \text{.......................(76)}$$

the function, $N_2 = D\left(z + \frac{c}{z}\right)$, will be minimum with respect to L, and hence with respect to z, when $\frac{dN_2}{dz} = D\left(1 - \frac{c}{z^2}\right) = 0$. Therefore,

$$z = \sqrt{c} \quad \text{.........................(77a)}$$

Thus

$$(N_2)_{\min} = 2\,D\,\sqrt{c} = \frac{2}{R}\,\sqrt{D\,E\,h} \quad \text{..............(77b)}$$

This value of N_{cr} is rigorously correct and is obtained in all the successive approximations as the value of $(N_n)_{\min}$.

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[26] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc. New York, N. Y., 1st Ed., 1936, p. 441.

By Eqs. 76 and 77*a*, the half-wave length L_2 corresponding to $(N_2)_{\min}$ is: $L_2 = 8/\sqrt{c}$; and, letting

$$k_n = L_n \bigg/ \sqrt[4]{\frac{D\,R^2}{E\,h}} \qquad (78)$$

the corresponding value of k_2 is: $k_2 = 2.8284$ and $e_2 = -9.96\%$.

Approximations $n = 3$, 4, and 5.—For $n = 3$, applying Eq. 74*a* at $x = L/3$: $1 - a_3 + b_3 = 0$ or $N_3 = D(9/L^2 + c\,L^2/9)$. The half-wave length corresponding to N_3 gives: $k_3 = 3$ and $e_3 = -4.51\%$. Similarly, for $n = 4$ and $n = 5$: $k_4 = 3.06$, $e_4 = -2.6\%$; $k_5 = 3.09$; and $e_5 = -1.64\%$. By extrapolation: $k_{2,3} = 3.1373$ (-0.14%); $k_{3,4} = 3.1371$ (-0.14%); $k_{4,5} = 3.1433$ $(+0.055\%)$; $k_{2,3,4} = 3.1370$ (-0.15%); and $k_{3,4,5} = 3.1468$ $(+0.17\%)$. The correct value of k is π.

11. CONCLUSIONS

In this paper (which is of an essentially experimental character) the method of finite differences is applied mostly to problems with a known solution to check the rapidity and the type of convergence, as well as the validity of the suggested extrapolation procedures. However, the scope of the method is such as to allow the attack of a large variety of buckling problems. Among the advantages of the method are the following:

(*a*) Problems with different types of boundary conditions can be solved by the same basic procedures with a comparable amount of labor in all cases.

(*b*) The variability of the moment of inertia of beams and plates does not require special procedures and in general does not increase materially the amount of labor involved.

(*c*) Beams with compressive loads P variable along the axis of the beam and plates with known variable distribution of stress in the plane of the plate may be considered without increased difficulty.

(*d*) Anisotropic plates fall under the scope of the method.

(*e*) Plates of irregular shapes may be dealt with by a suitable set of coordinates, using the equations that best fit the problem in each case.

(*f*) The method does not require a knowledge of higher mathematics. It is interesting to notice that, with the exception of relaxation procedures, all other methods of solution, both approximate and rigorous, require instead the use of integral calculus, differential equations, or series expansion.

(*g*) The method is applicable to all characteristic value problems involving linear differential equations, ordinary or partial, with constant or variable coefficients.

There are also limitations and difficulties inherent in the method:

(1) The method is not applicable when the distribution of longitudinal stress in the beam or of plane stress in the plate is unknown. This limitation is common to all other methods, except the energy method.

(2) There is no way of knowing whether the extrapolated values are greater or smaller than the true value.

(3) The literal evaluation of the determinantal equation for the higher approximations may be cumbersome.

This difficulty may be avoided as explained in Section 8, but it is interesting to notice that all series solutions, including those used in connection with the energy method, lead to determinantal equations and hence involve the same amount of labor in the final step of the computations, although sometimes a larger amount of labor is required in the derivation of the determinant itself. Moreover, most of the other methods do not lend themselves easily to extrapolation procedures. (Relaxation procedures, also involving finite differences, seem to be more cumbersome than the present method, since they require the simultaneous evaluation of the deflection function, which seldom needs to be known in practice. The third book by Mr. Southwell on relaxation methods, which is planned to cover characteristic value problems, has not been published up to the time of this writing (1949) and it is therefore impossible to make a final comparison between the two methods.)

The width of its scope and the simplicity of its application should recommend the finite difference method to the attention of the engineer and of the applied mathematician.

12. ACKNOWLEDGMENTS

The writer is highly endebted to J. L. Bogdanoff, instructor in civil engineering, Columbia University at New York, N. Y., for calling his attention to Mr. Richardson's paper,[10] to his colleagues in the Department of Civil Engineering at Columbia University, and particularly to R. D. Mindlin, Assoc. M. ASCE, professor of civil engineering, Columbia University, for encouragement and suggestions in the writing of this paper. The values of the buckling loads for Example G of Section 10 are taken from a thesis by J. Gold, Jun. ASCE, entitled "Buckling Loads for Built-in Rectangular Plates by Finite Differences." The value of k_n (= 27.25) for $n = 5$ in Table 11(a) was computed by A. J. Del Mastro, Jun. ASCE.

DISCUSSION

I. K. SILVERMAN,[27] Assoc. M. ASCE.—In this very interesting paper Mr. Salvadori has shown that the solution of a differential equation is actually the simultaneous solution of an infinite number of algebraic equations and, conversely, for many practical cases, the simultaneous solution of a finite number of algebraic equations may be accepted in lieu of the solution of a differential equation. The practical consequences of this conclusion are apparent and Mr. Salvadori has solved some problems that arise in the theory of structures. The Rayleigh energy method,[2] widely used for the approximate solution of problems in elastic stability and vibrations, depends on the assumption of the deflected shape of the structure. There are some cases (the curved bar, for example) in which the shape of the deflected bar cannot be estimated and the solution of the differential equation is required.

The absence of any physical explanation of the method of finite differences is somewhat detrimental to a full appreciation and understanding of the method. Physically, the method of finite differences is equivalent to substituting a less complex structure for the ideal mathematical structure. In 1920 Heinrich Hencky[28,29] proposed that, in studying the elastic stability of structures, the continuous elastic structure be replaced by one consisting of rigid sections connected with hinges capable of resisting elastic deformation. For example, the column in Fig. 11(*a*) is to be replaced by Fig. 13, in which parts AB, BC, etc., are considered rigid and the elasticity is concentrated at the hinges. From a study of this model the designer can decide whether the substitute structure is more rigid or less rigid than the original continuous elastic structure, and if the critical load is expected to be greater than or less than that for the continuous elastic structure. The designer is also interested in knowing if an approximate solution gives values on the safe side or the unsafe side. The problem is thus no longer in the realm of pure mathematics and enters the domain of physics.

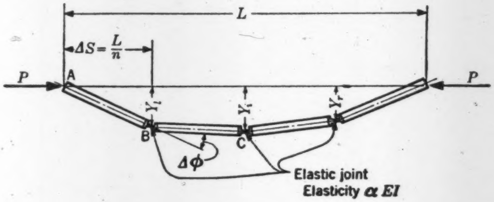

FIG. 13

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[27] Engr., U. S. Bureau of Reclamation, Denver, Colo.

[28] "Über Die Angenäherte Lösung Von Stabilitätsproblemen Im Raum Mittels Der Elastischen Gelenkkette," by Heinrich Hencky, *Der Eisenbau*, Vol. 11, 1920, pp. 437–452.

[29] "Drang Und Zwang," by August Föppl and Ludwig Föppl, R. Oldenbourg, München and Berlin, Germany, 1920, Vol. 2, p. 368.

That Mr. Salvadori's application of finite first order differences is mathematically identical with Mr. Hencky's "bar chain" method is apparent from a comparison of the mathematical computations involved. The bar chain method for the problem of Fig. 11(a) furnishes a set of linear algebraic equations by expressing the relation between curvature and moment. Thus,

$$\frac{1}{\rho} = \frac{\Delta\phi}{\Delta s} = \frac{n\,\Delta\phi}{L} = \left(\frac{n}{L}\right)^2 (2\,y_i - y_l - y_r) = \frac{M}{E\,I} = \frac{P\,y_i}{E\,I} \tag{79a}$$

and

$$2\,y_i - y_l - y_r = \frac{P\,L^2}{n^2\,E\,I}\,y_i \tag{79b}$$

The following determinant is obtained—

$$\begin{vmatrix} (2-w) & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & (2-w) & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & (2-w) & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & (2-w) & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & (2-w) & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & (2-w) \end{vmatrix} = 0 \tag{80}$$

in which

$$w = \frac{P\,L^2}{n^2\,E\,I} = b_n + 2 = \frac{k}{n^2} \tag{81}$$

The determinantal equations become

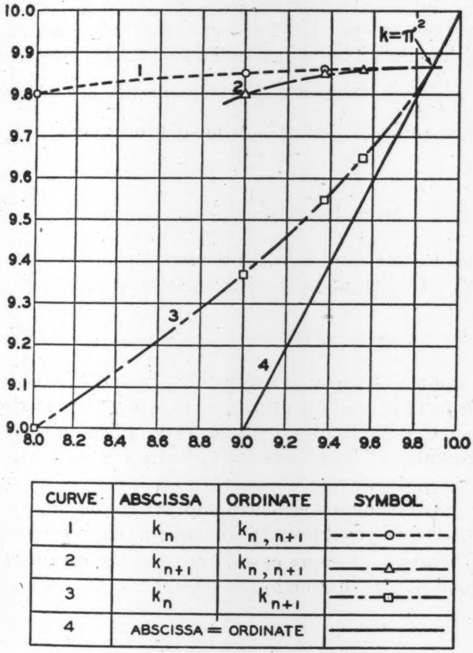
For n	Equations	w	k
2	$(2 - w) = 0$	2	8
3	$(2 - w)^2 - 1 = 0$	1	9
4	$(2 - w)^3 - (2 - w) - (2 - w) = 0$	$2 - \sqrt{2}$	9.37258

Further applications[28] have been made to torsional equilibrium, curved bars, and to the elastic equilibrium of frames. A point to be emphasized is that the solution for the substitute structure is as "exact" as the "exact" solution of the mathematical continuous structure and it only remains for the designer to decide whether the substitute structure is sufficiently similar to the actual structure for his purposes. From the results given in the paper it can be inferred that the bar chain column subjected to axial loads is a less rigid structure than the continuously elastic column and that results obtained are on the safe side. On the other hand, for case C (Section 9) the bar chain beam is less rigid or more rigid depending on whether elastic resistance is concentrated in an odd or even number of hinges. The behavior of k is thus tied in with the stiffness of the substitute structure compared with the ideal mathematical structure.

Bruno A. Boley.[30]—A clear presentation of the finite differences approach, and a very logical development of a valuable method for buckling load calculations, are given in this paper. One of the chief advantages of methods such as this is that, although sometimes the mathematics is lengthy, no complicated mathematical manipulation is required which might overshadow the physical significance of the various steps. In fact, once the governing differential equation for a particular problem is established, the buckling load may be obtained with operations not more complicated than those of algebra.

TABLE 14.—VARIATION OF k WITH n

n	2	3	4	5	6
k_n........	8.000	9.000	9.373	9.549	9.646
$k_{n,n+1}$....	9.800	9.852	9.863	9.867	9.868



CURVE	ABSCISSA	ORDINATE	SYMBOL
1	k_n	$k_{n,n+1}$	----○----
2	k_{n+1}	$k_{n,n+1}$	— —△— —
3	k_n	k_{n+1}	—·—□—·—
4	ABSCISSA = ORDINATE		———

FIG. 14.—EXAMPLE OF GRAPHICAL EXTRAPOLATION

The method suggested is useful when it is possible, without too much difficulty, to write a differential equation (or at least a system of differential equations) governing the problem under consideration, and the corresponding boundary conditions. In some cases, however, it is difficult to obtain such a system of equations; examples of such problems are the buckling of reinforced plates or shells, or the instability of frameworks. In these problems some other methods probably could be more easily applied—in particular, some of the methods developed during the investigations in 1947 at the Polytechnic Institute of Brooklyn,[31] in Brooklyn, N. Y. These methods are also finite difference methods in the sense that the structure under investigation is considered to be composed of several units of finite dimensions.

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[30] Eng. Specialist, Goodyear Aircraft Corp., Akron, Ohio.

[31] "Numerical Methods for the Calculation of Elastic Instability," by Bruno A. Boley, *Journal of the Aeronautical Sciences*, June, 1947, p. 337.

The usefulness of the method suggested by Mr. Salvadori is greatly increased, of course, by the extrapolation procedures suggested in the paper. The examples in the paper show that these procedures provide a very powerful tool. Their chief drawback is (as Mr. Salvadori points out) that it is not possible to know beforehand whether the value they yield for the buckling load is higher or lower than reality. It may be mentioned here that the methods referred to previously[31] will yield an exact answer if the analysis of the units is sufficiently rigorous. In general, the units may be chosen sufficiently simple so that an accurate analysis is possible. If an assumption is made (as those methods allow) concerning the deflected shape of the structure, it may be shown that the resulting calculated buckling load will be too high.

To obtain a better idea of the accuracy afforded by the extrapolation procedures, it is necessary that additional problems be solved involving different types of structures. In any one problem, however, a good estimate of the correct buckling load may probably be obtained by some graphical method of extrapolation, such as the rather obvious one (for the case of the Euler buckling of a straight column) outlined herein. The values in Table 14 were given by Mr. Salvadori in which $k_n = \frac{P_{cr}}{E\,I/L^2}$ for the case in which the bar is assumed divided into n equal segments, and $k_{n,\,n+1}$ is the value obtained by the h^2-extrapolation—that is,

$$k_{n,\,n+1} = \frac{(n+1)^2\,k_{n+1} - n^2\,k_n}{(n+1)^2 - n^2} \qquad (82)$$

As n becomes very large, the values of k_n and $k_{n,\,n+1}$ approach the correct value, k; in this case $k = \pi^2 = 9.8696$. In fact, at the limit, it may be stated that

$$k_n = k_{n,\,n+1} = k_{n+1} \qquad (83)$$

Thus, if plots are made of k_n against $k_{n,\,n+1}$, k_{n+1} against $k_{n,\,n+1}$, and k_n against k_{n+1}, the common intersection of the three resulting curves will represent the correct value, k. Furthermore, this intersection will occur on a line on which the abscissa is equal to the ordinate. These plots are given in Fig. 14 for the present example.

A number of such schemes may be devised either to improve the accuracy of the method or to estimate the error introduced by considering the structure comprising a finite number of points. In many cases, however, the extrapolation procedure leads to such good results that no improvement may be required for engineering accuracy.

G. S. RAMASWAMY,[32] JUN. ASCE.—The use of difference equations to obtain approximate solutions to boundary value problems in science and engineering is not new;[33] but the author's clear exposition is certain to make the method familiar to engineers.

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[32] Prof. of Civ. Eng., Annamalai Univ., Annamalainagar, South India.

[33] "Mathematical Theory of Elasticity," by I. S. Sokolinkoff, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1946.

The writer has undertaken to check the validity of the extrapolation procedures discussed by the author and to study the effect of varying boundary conditions on convergence. The example chosen is the classical Euler problem, the strut being hinged at one end and restrained in position and direction at the other end. The exact value[34] of P_{cr} for this case is 20.16 $(E\,I)/L^2$. Table 15 shows the computations based on first order differences.

TABLE 15.—CHECK COMPUTATIONS

n	$\frac{P_n\,L^2}{E\,I}$	e_n (%)	h^2-extrapolations	
2	12.00	40.5	2, 3	19.31304
3	16.0628	20.4	3, 4	19.9421
4	17.760	11.9	4, 5	20.1722
5	18.614	7.72	....	....

It is seen that the mode of approach in this case is monotonic. Comparison with Example A of Mr. Salvadori's paper reveals that convergence is very much slower for the boundary conditions assumed by the writer. However, h^2-extrapolations give surprisingly accurate results even with very few trials.

CHI-TEH WANG.[35]—The power of the Richardson method[10] of extrapolation in the computation of buckling loads is demonstrated in this paper. Although the solution of elastic stability problems by finite differences has been a subject of much study by many eminent workers, without the use of the method of extrapolation, the computations become very tedious when an accurate solution is required. In applying the method of extrapolation, Mr. Richardson assumed that the error in the buckling load is also of the h^2-type which the author has argued as not necessarily true. However, by solving many numerical examples, Mr. Salvadori has shown that the assumption did work; but, because of the lack of proof, he had to limit his conclusion by adding the phrase "of experimental characteristic." Nevertheless, the fact that the assumption did work out so well in all these cases appears more than a mere coincidence.

Mr. Richardson's assumption can actually be proved to be rigorous at least for the cases where the moment of inertia is constant. Consider the case of the buckling of a column with hinged ends under axial compression. The differential equation is

$$\frac{d^2Y}{dx^2} + K\,Y = 0 \dots\dots\dots\dots\dots\dots\dots (84)$$

for which the boundary conditions are $Y = 0$ at $x = 0$ and $x = L$. The corresponding difference problem is

$$y\,(x + h) - 2\,y\,(x) + y\,(x - h) + k\,h^2\,y\,(x) = 0 \dots\dots\dots (85)$$

and $y = 0$ at $x = 0$ and $x = L$. In Eq. 85, y is the approximate value of Y and k is the approximate value of K obtained by solving the difference equation (Eq. 85) rather than the differential equation (Eq. 84).

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[34] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, p. 89.

[35] Associate Prof. of Aeronautical Eng., Daniel Guggenheim School of Aeronautics, College of Eng., New York Univ., New York, N. Y.

The finite difference equation (Eq. 85) can be solved by assuming the solution to be of the form:

$$y = c\, e^{mx} \quad \text{(86)}$$

Substituting Eq. 86 in Eq. 85,

$$c\, e^{mx+mh} - 2\, c\, e^{mx} + c\, e^{mx-mh} + c\, k\, h^2\, e^{mx} = 0 \quad \text{(87)}$$

or

$$[(e^{mh} + e^{-mh}) + (k\, h^2 - 2)]\, c\, e^{mx} = 0 \quad \text{(88)}$$

The form of Eq. 88 requires that

$$(e^{mh} + e^{-mh}) + (k\, h^2 - 2) = 0 \quad \text{(89)}$$

or

$$\cosh m\, h + \left(\frac{k\, h^2}{2} - 1\right) = 0 \quad \text{(90)}$$

Three forms of the solution for m may be obtained depending on whether $\left|\frac{k\, h^2}{2} - 1\right| > 1$, $\left|\frac{k\, h^2}{2} - 1\right| = 1$, or $\left|\frac{k\, h^2}{2} - 1\right| < 1$. In the first two cases, it can be shown that the resulting solution for y cannot satisfy the boundary conditions and therefore that these cases will not be considered. When $\left|\frac{k\, h^2}{2} - 1\right| < 1$, $\cosh m\, h < 1$ and m is a pure imaginary number. Writing $m = i\, \lambda$, Eq. 90 becomes

$$\cos \lambda\, h = 1 - \frac{k\, h^2}{2} \quad \text{(91)}$$

therefore,

$$\lambda = \pm \frac{1}{h} \cos^{-1}\left(1 - \frac{k\, h^2}{2}\right) \quad \text{(92)}$$

Substituting values of λ from Eq. 92 in Eq. 86, the values of k can then be determined from the boundary conditions. As this procedure proves rather laborious, the solution is simplified by first determining the value of λ from the boundary conditions and then solving k from Eq. 91. Since λ can be either positive or negative, the general solution for y is

$$y = c_1 \sin \lambda\, x + c_2 \cos \lambda\, x \quad \text{(93)}$$

To satisfy the boundary conditions, it is necessary that

$$\lambda = \frac{n\, \pi}{L} \quad \text{(94)}$$

in which n is any integer, and $n = 1$ at the instant of buckling. Since $1 - \cos 2\, \theta = 2 \sin^2 \theta$, Eq. 93 can be rewritten, substituting the value of λ obtained from Eq. 94; thus:

$$k \left(\frac{h}{2}\right)^2 = \sin^2 \frac{\pi\, h}{L\, 2} \quad \text{(95)}$$

Since $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$, expanding the sine function in Eq. 95 yields

$$k\left(\frac{h}{2}\right)^2 = \left[\frac{\pi}{L}\frac{h}{2} - \frac{1}{3!}\left(\frac{\pi}{L}\frac{h}{2}\right)^3 + \cdots\right]^2$$

$$= \frac{\pi^2}{L^2}\left(\frac{h}{2}\right)^2 - \frac{2}{3!}\left(\frac{\pi}{L}\right)^4\left(\frac{h}{2}\right)^4 + \cdots \quad \ldots(96)$$

Dividing Eq. 96 by $\left(\frac{h}{2}\right)^2$,

$$k = \frac{\pi^2}{L^2} - \frac{2}{3!}\left(\frac{\pi}{L}\right)^4\left(\frac{h}{2}\right)^2 + \cdots = K - \alpha_1 h^2 - \alpha_2 h^4 + \cdots \quad \ldots(97)$$

Eq. 97 results from the fact that $K = \frac{P}{E\,I} = \frac{\pi^2}{L^2}$, thus proving that

$$e = K - k = \alpha_1 h^2 + \alpha_2 h^4 + \cdots \quad \ldots(98)$$

is of the h^2-type. It may be noted that for columns with other boundary conditions, if the differential equation is of the form as given by Eq. 84, Eq. 93 shows that λ is always equal to $\sqrt{K}$ and therefore the proof holds equally true in these cases. In general, if the governing differential equations are those with constant coefficients, the corresponding finite difference equations can be solved by the same symbolic method as in the case of the differential equations and the final solutions for y have the same form as those for Y. Therefore λ will always be equal to $\sqrt{K}$ and the relation given by Eq. 98 can always be proved to be correct.

GEORGE HERRMANN.[36]—The consideration of stability problems in different fields of engineering is steadily becoming of greater importance. The value of this paper lies in two facts: (1) The finite differences method (which had not been thoroughly investigated) is presented in a very broad and clear manner; and (2) through his simple extrapolation formulas, the author increases the power of the method to a large extent.

The worker engaged in the investigation of characteristic value problems will be thankful to the author for considering so many aspects of the method (such as different coordinate systems, different order of operators, procedures for numerical computations, types of asymptotic approach, etc.), and will notice that the method does not require any guessing of the deflection line (or surface), such as is required by the energy method and other approximate methods.[37] Since complicated boundary conditions (for example, elastically supported edges[38]) can be expressed by finite differences, the method is applicable to a larger variety of problems than indicated by the paper. The extrapolation formulas show an amazing power, as can be seen by the many illustrative examples. By their use the numerical computations are made less cumbersome, with the saving of time and energy.

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[36] Instr. in Civ. Eng., Columbia Univ., New York, N. Y.

[37] "Berechnung der Beulspannungen gedrückter Rechteckplatten," by F. Stüssi, *Abhandlungen*, Internationale Vereinigung für Brückenbau und Hochbau, Zürich, Bd. 8, 1947, p. 237.

[38] "Knickung der rechteckigen Platte bei veränderlicher Randbelastung," by Shizuo Ban, *ibid.*, Zürich, Bd. 3, 1935, p. 1.

There are, however, many problems of elastic stability in which the energy method leads to good results in a shorter time than any other approach. For example,[39,40,41,42] the plate in Fig. 15 is exposed to a thrust of triangular shape along the simply supported edges b. The load-free edges, a, are built in and free, respectively. Apply the energy method, assuming a sinusoidal deflection function in the direction of the thrust and a deflection function of the type—

$$Y_n = A\left(\cos\frac{P_n y}{b} + \cosh\frac{P_n y}{b}\right) + B\left(\cos\frac{P_n y}{b} - \cosh\frac{P_n y}{b}\right) + C\left(\sin\frac{P_n y}{b} + \sinh\frac{P_n y}{b}\right) + D\left(\sin\frac{P_n y}{b} - \sinh\frac{P_n y}{b}\right) \quad \text{..(99)}$$

in the direction perpendicular to the thrust. For the ratio $a/b = 1.635$, Eq. 99 yields $k = 1.636$ in the first approximation and practically the same value in the second approximation, the difference being noticeable only in the fifth digit. Using the finite differences method with first order operators, $k_2 = 1.14$ for $n = 2$, and $k_4 = 1.35$ for $n = 4$. With the extrapolation formula (Eq. 41), $k_{2,4} = 1.42$; the error is still 13% and to find k_4 it was necessary to solve a fourth degree equation.

Although this example is not typical, as far as convergence is concerned, in general, the investigator dealing with stability of plates, is interested in finding the ratio a/b of the edges of the plate for which k is minimum. The buckling equation being established, it is, in general, easier to find k (minimum) by the energy method, because the ratio a/b appears only in the diagonal members of the determinant, whereas, in using finite differences, this ratio appears in most elements of the determinant.

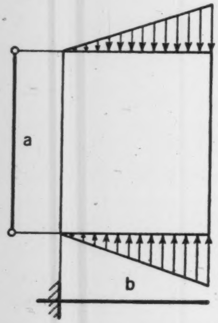

FIG. 15

The extrapolation formulas can be applied in solving not only the characteristic value problems, but also the boundary (or initial) value problems. For a given problem the type of approach depends on the choice of the type of coordinates. Thus in the problem of a skew plate with equal sides 2 a, under uniform load p, and simply supported along all four edges, the true value for the deflection can be squeezed, using different types of coordinates.

Using skew coordinates for the analysis of a 60° skew plate, H. Favre[13] obtained a central deflection of $f_2 = 0.0351\,\frac{a^4 p}{D}$ for $n = 2$, and $f_4 = 0.0385\,\frac{a^4 p}{D}$

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[39] "Elastische Beulung von auf einseitigen, ungleichmässigen Druck beanspruchten Platten," by C. F. Kollbrunner and G. Herrmann, *Mitteilung No. 1*, Technische Kommission des Verbandes Schweizerischer Brückenbau- und Stahlhochbau-Unternehmungen, Verlag Leemann, Zürich, 1948.

[40] "Reine Biegungsbeulung rechteckiger Platten in elastischen Bereich," by C. F. Kollbrunner and G. Herrmann, *Mitteilung No. 2*, Technische Kommission des Verbandes Schweizerischer Brückenbau-und Stahlhochbau-Unternehmungen, Verlag Leemann, Zürich, 1949.

[41] *Schweizerische Bauzeitung*, March 13, 1948, p. 146.

[42] *Ibid.*, May 28, 1949.

for $n = 4$, D being the flexural rigidity of the plate. Applying the extrapolation formula probably a better value $f_{2,4} = 0.0396 \frac{a^4 p}{D}$ results.

Using triangular coordinates, F. L. Ehasz[43] computed the values $f_4 = 0.0453 \frac{a^4 p}{D}$ for $n = 4$, and $f_8 = 0.0419 \frac{a^4 p}{D}$ for $n = 8$, which have been rounded off here to three figures. In this case the extrapolation formula (Eq. 41) gives $f_{4,8} = 0.0408 \frac{a^4 p}{D}$. The average of the two extrapolated values is $0.0402 \frac{a^4 p}{D}$ and, plausibly, does not differ more than approximately 1% from the true value of the deflection of the center. This example illustrates, at the same time, how important it is to choose an appropriate type of coordinates, as emphasized by the author.

Extremely good approximations are obtained, using extrapolation formulas, in computing the deflections of uniformly loaded square plates. In the case of simple supports,[13] $f_2 = 0.06250 \frac{a^4 p}{D}$ and $f_4 = 0.06445 \frac{a^4 p}{D}$. The extrapolated value (using Eq. 41) equals $f_{2,4} = 0.06510 \frac{a^4 p}{D}$ and differs by only 0.2% from the exact value $f = 0.06496 \frac{a^4 p}{D}$. In the case of built-in edges,[44] $f_4 = 0.00180 \frac{a^4 p}{D}$ and $f_8 = 0.00143 \frac{a^4 p}{D}$. In spite of the fact that these values differ from the exact value $\left(f = 0.00128 \frac{a^4 p}{D}\right)$ by 41% and 12%, respectively, the extrapolation formula (Eq. 41) leads to $f_{4,8} = 0.00131 \frac{a^4 p}{D}$, which is only 2.3% in error.

In the case of transverse concentrated loads applied to a plate, the extrapolation formulas given by the author cannot be used, because the error is due not only to the substitution of finite differences for differentials, but also to the spreading of the load over a certain area.[45] In investigating skew plates under a central concentrated load,[46] the writer had to use higher order differences to obtain satisfactory approximations. It might be worthwhile to investigate the possibility of developing extrapolation formulas to problems of this type.

Mario G. Salvadori,[47] Assoc. M. ASCE.—Mr. Silverman points out that the Hencky "bar chain" method is equivalent to the finite differences approach and he emphasizes the advantages of a physical picture in solving engineering problems. The choice between a physical or a mathematical approach rests essentially on the mental attitude of the worker and on the relative efficiency of the two types of approach. In this connection it will suffice to note that the physical approach of Mr. Hencky would never lead to the investigation of the error and to the discovery of the extrapolations, without which his method is excessively laborious and inefficient. The problems dealt with in the paper illustrate the decided superiority of the mathematical approach in the present case. In other cases, a combination of physical intuition and mathematical knowledge usually gives the best results.

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[43] "Structural Skew Plates," by F. L. Ehasz, *Transactions*, ASCE, Vol. 111, 1946, p. 1011.

[44] "Die elastischen Platten," by Arpád Nádai, J. Springer, Berlin, 1925, p. 220.

[45] "Die Theorie elastischer Gewebe und ihre Anwendung auf die Berechnung biegsamer Platten," by H. Marcus, J. Springer, Berlin, Vol. 1, 2d Ed., 1932, p. 61.

[46] "Theoretische Untersuchungen über die Durchbiegung parallelogrammförmiger Platten unter zentrischer Einzellast," by G. Herrmann, *Publication No. 4*, Laboratoire de Photoélasticité de l'École Polytechnique Féderale, Verlag Leemann, Zürich, 1950.

[47] Associate Prof. of Civ. Eng., Columbia Univ., New York, N. Y.

The graphical "extrapolation of the extrapolations," suggested by Mr. Boley, although without sound mathematical justification, may prove useful. It can be applied to any converging series of approximate values and has the advantage of bringing the asymptotic value of the approximations to a point on a 45° line. It is doubtful whether the graphical extrapolation would give accurate results in the case of oscillating approximations.

Mr. Ramaswamy's example, which checks an unpublished result obtained by the writer, clearly indicates the powerfulness of the extrapolation procedure. It is interesting to add that the $(h^2 - h^4)$-extrapolation applied to the approximations $n = 2, 3, 4$, gives (in this case) $k = 20.166$, with an error of 0.03%.

The writer is grateful to Mr. Herrmann for showing the applicability of the extrapolations to problems other than those treated in the paper. The extrapolations can actually be used in connection with any method of successive approximations in which: (*a*) The successive values of the unknown depend on an increasing integer n and (*b*) the error is expandable into a series of negative even powers of n. Various procedures of this type, to which the extrapolations can be usefully applied, have been discussed elsewhere by the writer.[48]

The proof, given by Mr. Wang, that the error in the characteristic value of Euler's problem is of the h^2-type represents an important contribution to the theory of the finite differences method. It opens the path to a more general proof which is valid for all ordinary equations with constant coefficients.

The writer is unable to follow the reasoning in the last paragraph of Mr. Wang's discussion, however, and he does not see that it proves Eq. 98 to be correct for all equations with constant coefficients. In fact, Eq. 98 was derived for the particular case of Eq. 84 and this equation does not hold even for all beams with constant moment of inertia. The following fundamental theorem may instead be proved, by the error-formula (Eq. 34) given by the writer:

> *Theorem.* The error in the characteristic value of linear ordinary differential equations with constant coefficients evaluated by finite differences is of the h^2-type.

For this purpose notice that, according to the theorem of L. F. Richardson, e_1 and e_2, and hence $e_1 + k\, e_2$, are of the h^2-type. Therefore, the condition:

$$L_1\, y + K\, L_2\, y = 0 \tag{100}$$

is a sufficient condition (although not a necessary condition) for e to be of the h^2-type; but the solution y of an ordinary difference problem with constant coefficients is identical with solution Y of the corresponding differential problem[49] and, hence, by Eq. 30 the condition of Eq. 100, is satisfied when L_1 and L_2 have constant coefficients. Thus, for equations with constant coefficients the error e reduces to the form:

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[48] "Approximations and Extrapolations," by M. G. Salvadori, *Columbia Engineering Quarterly*, January, 1949.

$$e = -\frac{e_1 + K\, e_2}{L_2\, Y - e_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (101)$$

and is of the h^2-type.

It is now obvious that the extrapolations must give good results in connection with constant coefficient equations and that the accuracy of the extrapolations must improve by taking into account a larger number of terms in the series for the error. For example, the writer has found that, on a large number of cases, the $(h^2 - h^4)$-extrapolation gives better results than the h^2-extrapolation in connection with constant coefficient equations.

It is also interesting to note that the behavior of the successive approximations is monotonic in all the constant coefficient examples solved by the writer, whereas the behavior is seldom monotonic for equations with variable coefficients. For this last type of equation the sufficient condition of Eq. 100, in general, is not satisfied; and this fact may be proved by checking the values of the approximate solution at the pivotal points, obtained by finite differences, to note that they differ from the values of the integral of the differential equation at the same points. For example, the differential problem for the buckling of a strut, built in at one end and free at the other, under the action of its own weight q, may be written in the form:

$$\frac{d^2u}{dz^2} + k^2\, z\, u = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (102)$$

and

$$u'\,(0) = u\,(L) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (103)$$

and has a series-solution:

$$u\,(z) = C\left(1 - \frac{k^2\, z^3}{2\cdot 3} + \frac{k^4\, z^6}{2\cdot 3\cdot 5\cdot 6} - \frac{k^6\, z^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \cdots\right) \ldots\ldots (104)$$

(S. Timoshenko[2] develops Eq. 102; here $u = dy/dx$, $z = L - x$, and $k^2 = q/(E\,I)$.) By dividing the strut into two equal parts, the ratio of the middle point, slope $u_1 = u\,(L/2)$, to the end slope, $u_2 = u\,(0)$, becomes (with $k^2\, L^3 = 7.81$ and, hence, $k^2\,(L/2)^3 = 0.977$): $u_1/u_2 = 0.842/1 = 0.842$. The corresponding difference problem is given by the equations:

$$u_l + \left(\frac{k^2\, L^2}{n^2}\, z_i - 2\right) u_i + u_r = 0 \ldots\ldots\ldots\ldots\ldots (105)$$

and

$$u'\,(0) = u(L) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (106)$$

For $n = 2$, Eq. 105 and Eq. 106 give rise to the system of two equations:

$$2\, u_1 - 2\, u_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (107)$$

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[49] "Die gewöhnlichen und partiellen Differenzengleichungen der Baustatik," by Fr. Bleich and E. Melan, Springer, Berlin, 1927.

and

$$\left(\frac{k^2 L^3}{8} - 2\right)u_1 + u_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (108)$$

The root of the corresponding determinantal equation is $k^2 L^3 = 8$ ($e = 2.4\%$); and, with this value for k, the ratio u_1/u_2 becomes

$$\frac{u_1}{u_2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots (109)$$

Hence, the approximate u differs from the true u and Eq. 100 is not necessarily satisfied.

Finally, it must be noted that, for equations with variable coefficients, an increase in the number of terms of the error series taken into account usually reduces the accuracy of the extrapolations obtained by means of all the successive approximations. Therefore, these results seem to indicate that the error in the characteristic values of equations with variable coefficients is not of the h^2-type.

The reason why the extrapolations work so well in connection with any monotonic series of approximations remains to be investigated, although this result could well be of a purely empirical character.

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THE GEOCHEMISTRY OF EARTHWORK

By Hyde Forbes,[1] M. ASCE

With Discussion by Messrs. Warren D. Smith, D. P. Krynine, S. S. Gorman, Charles H. Lee, George S. Harman, and Hyde Forbes.

Synopsis

The geochemistry of earthwork involves the geologic and chemical processes that account for the kinds and quantities of mineral substances found in earth that has been "worked" in the presence of air and water. It should be advantageous for engineers and technicians concerned with earthwork to follow these geochemical processes and products, as well as the mechanical properties assumed by earth, in order that a complete and effective technique may be developed in the entire field of earthwork. This paper presents observational and test data relative to the geochemical processes and the mineralogical changes set up in working with earth, in excavations and in engineering structures.

A comparison is drawn between the natural processes of rock weathering, erosion, and deposition as observed by all, and those encountered in engineering construction. It is common knowledge that the rock-forming minerals making up the earth's crust are compounds of the elements and, although it is not necessary that the engineer know the infinite number of possible chemical combinations of the elements that have been effected, he should recognize the fact that compounds are being formed continuously (under natural conditions and agencies) which are important in connection with unyielding foundations and stability of slopes. Compounds may be affected by the union of elements during the mechanical processes involved in earthwork, the character of the product of such unions being important in the final production of stable, dense structures.

Those who select construction material and control its use in construction should consider its mineral content in order to obtain material which, when worked with air and water, will result in dense, relatively impervious, stable masses. Material that will react unfavorably with water and atmospheric gasses should be avoided. Similarly, excavated slopes and faces in natural material should be considered in the light of the following: (1) The mineral

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[1] Engr. and Geologist, Palo Alto, Calif.

character and structure of the excavated material; (2) its future action in contact with the atmosphere and moisture; and (3) the effects produced upon structures built in or adjacent to the excavated areas as the foundation soil becomes modified by geochemical change.

INTRODUCTION

Since the late 1920's the application of scientific investigation to natural materials and to construction design and control in earthwork has produced much literature relative to physical and mechanical properties. However, there are fundamental geologic phenomena involved which would appear to be of primary interest to the geologist and which have been generally neglected. These include the definition of the mineralogical characteristics of the material, the geochemical reactions they undergo upon excavation, and the changes that occur with handling, moistening, and compacting to produce general stability of foundations and slopes, and consolidation in earth structures. The vagaries of nature are legion and often unpredictable except from a rather broad point of view. The products of geochemical action in earthwork may vary greatly even under similar conditions of moisture and working. What little is known positively, conclusively, and completely is that the earth's crust and its constituents are subject to constant change. The accumulation of observational and test data during the working of earth, and after the completion of that work, should prove extremely valuable.

Materials encountered in earthwork are all products of rock. The reactions that occurred during the formation of rock resulted in a chemical equilibrium. The chemical system is one in which changes occur by the interaction of various natural agencies, resulting in the formation of a new set of combinations or mineral species recognized as the products of rock weathering and decomposition. The latter (weathered and decomposed rock), in turn, become stable for existing conditions or are subject to further change or decay. These species are the result of the chemistry of nature; but in earthwork construction the process may be continued and other compounds generated artificially in the reworking of these species as secondary units of geochemistry.

The formation of the secondary units is the result of relatively simple reactions. The principal agencies are atmospheric gases and water, both of which are intensely active in construction processes. Rock (or a rock-forming mineral with its associations) is the result of chemical change. It is subject to further change or transformation into other forms of mineral matter (under certain conditions) during a period of time dependent on those conditions and on the rock-forming minerals involved. The products of these changes (in the case of soil, for example), under proper conditions, are subject to reconsolidation to a degree dependent upon time, pressure, and association with other elements or compounds of elements. To summarize: This cycle of change is the result of the physical, mineralogical, and chemical constitution of matter; and the physical, mineralogical, and chemical changes possible within that constitution, under certain conditions, compose the geochemistry of earthwork.

WEATHERING, A GEOCHEMICAL CHANGE THROUGH GEOLOGIC PROCESSES

The results of weathering as a geological process are important to the engineer engaged in foundation excavation; and the construction procedure parallels, facilitaties, and accelerates the geologic process. Igneous rock originated as molten fluid in which the elements occur mutually dissolved under intense heat and pressure, the reduction of which allowed the elements to combine to form cohesive crystalline aggregates of distinct rock-forming minerals in chemical equilibrium. Crystalline metamorphic rock resulted from the formation of chemical combinations of elements present or injected under conditions of pressure or heat or both. Sedimentary rock is caused by the reconsolidation of the products of disintegration and decomposition of other rock through pressure, or the injection of other mineral matter (or both). Rock undergoes a gradual disintegration because each constituent mineral crystal has a distinctive coefficient of expansion and contraction, and the rock fabric is broken physically by temperature change between night and day, and the seasons. The partly disintegrated mass is then subjected to partial decomposition when attacked by atmospheric gases and dissolved by rain or percolating water. Solubles are formed and carried away and others may be reabsorbed or deposited; and the remaining material, some in hydrated form, contains products subject to transportation by moving water or to further development into soil. This process is the well-understood rock weathering—disintegration, decomposition, solution, and hydration—resulting in residual soils, in place, over parent rock and quarriable soft rock. Through erosion, transportation, and deposition, the product of rock weathering forms the alluvial accumulations of valley slopes and fills.

The readiness and rapidity with which geochemical change takes place in fresh rock surfaces were brought to the attention of the writer in 1915 in observing the foundation rock for the Gibraltar Dam on the Santa Ynez River in California. The dam site consisted of a thoroughly cemented, massive sandstone containing abundant fossil shells. The shells had provided the lime which percolating water (having absorbed carbon dioxide) dissolved and redeposited as the mineral calcite in all interstices and joint openings. The site was stripped to fresh, sound rock and hosed off before pouring concrete; but overnight it slaked and the surface became covered with rock flakes, sand, and white, chalk-like dust. This loose matter was cleared with wire brushes, and again the detritus formed before concrete was poured. The question arose as to the effect which such action would have on the bond between the rock and concrete, and a chemical analysis was made. The reaction was simple—calcium carbonate, cementing material, exposed to the air, separated into its component parts, gaseous carbon dioxide and earthy lime. Some of the detritus was then molded with a small quantity of water lying in a pot hole in the bedrock, and allowed to dry in free air. The result was "rock hardness," which is said to be a reversal of the first action—that is, lime water plus carbonated water caused the carbonate to become visible to the eye as the water evaporated.

The weathering of fresh cut slopes and the rate at which the slopes deteriorated to result in landslides has been described by the writer elsewhere.[2] The same reaction has been observed in tunnels and other excavation varying in rapidity according to the mineral constituents of the rock. Blasting was required in driving a tunnel in solid igneous rock under Parker Avenue in San Francisco, Calif., in 1939. However, the rock was attacked by the water vapor in the air, causing it to hydrate so rapidly that it was possible to scrape "clay" from the tunnel roof and walls. The tunnel required lining. The variation in the extent to which natural agencies attacked rock of varying mineral composition was evident in the geological surveys and investigations made in connection with nearly one hundred proposed dam sites in California and Arizona. Each dam site presented distinctive geologic conditions and mineral compositions in relation to the activity of geochemical change taking place, as evidenced by the quantity or scarcity of alteration products in the detrital cover, and the mineralogical character of the secondary products in the residuum. The end result of the geochemical change in the weathering process was the development of soil. Rock disintegrated to fragments; fragments were broken up to particle size; and the crystalloid particles were changed to amorphous substances and colloidal particles. The process was one of continuing decrease in grain size.

GEOCHEMICAL CHANGE ACCOMPANYING THE HANDLING OF EARTH

The geochemical processes of nature—oxidation and reduction, hydration and dehydration, and solution and consolidation—are duplicated or continued by the methods of engineering construction. All such action and reaction begin when a deposit of natural material is first disturbed and continue throughout the construction period, the degree of change and the resultant product being dependent upon the mineralogical character of the material at the site. Oxidation is most effective on surfaces exposed to the air, but both oxidation and

TABLE 1.—AVERAGE RESULTS OF GRADATION TESTS UNDER VARYING CONDITIONS; FELT LAKE DAM, STANFORD UNIVERSITY, CALIF.
(Percentage of Grain Distribution by Weight)

Pit No.	Material	LARGER THAN 1.0 MM			1.00 MM TO 0.074 MM			SILT AND CLAY		
		A	B	C	A	B	C	A	B	C
(1)	(2)	(3)	(4)	(5)	(3)	(4)	(5)	(3)	(4)	(5)
TYPE (*a*) HILL OF CLAYEY SANDSTONE (SOIL, SUBSOIL, AND DISINTEGRATED ROCK)										
1	Soil and rock fragments over clayey sandstone	16	15	13	63	61	55	21	24	32
7		42	39	34	40	41	40	18	20	26
9		41	38	32	42	41	36	17	22	32
TYPE (*b*) HILL CAPPED BY A FRESH-WATER DEPOSIT, INDICATING A LAKE BED										
4	Soil and gravel of lake beds	48	46	40	39	39	36	13	15	24
8		62	62	53	26	24	20	12	14	27

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[2] *Transactions*, ASCE Vol. 112, 1947, p. 393.

carbonization (as in the kaolinization of feldspars when carbon dioxide is absorbed) may extend beneath the surface. All processes are intensely active in, and with, material encountered in construction. Solution and hydration (through absorption of moisture as vapor or water) are accelerated, and the cycle of mineralogical and chemical change within the system continues until a final geochemical equilibrium is achieved under new conditions.

Felt Lake Dam (in California) *Construction Material.*—Reduction of the grain size, and change in the character of the borrow-pit material with handling, were noted during a check study on available construction material for Felt Lake Dam at Stanford University, in California, in 1929. Two types of materials were available (see Table 1): (*a*) Soil, subsoil, and disintegrated rock from a hill of clayey sandstone and (*b*) a hill capped by a fresh-water deposit presumed to be a lake bed. Pit 1 was shallow, principally soil, with few rock fragments. Pits 7 and 9 included the top of the disintegrated sandstone. Pits 4 and 8 (type (*b*), Table 1) contained gravel and boulders of the lake bed deposit. In Table 1 the results of the pit sampling are averaged to eliminate local differences. Cols. 4 contain the original screen analysis. As each pit was being dug, an average of three samples was taken and washed—the fines thus being separated from the coarse material, and dried in an electric oven. The fines were broken up with a wooden roller, mixed with the coarse material, and put through a set of standard screens. Samples A represent a later check with fresh samples cut from the wall of each pit through its entire depth (3 ft to 8 ft), broken by hand on the screens in an air-dried condition, and with shaking. The con-

FIG. 1.—SWANZY DAM SITE, NORTH OF CARQUINEZ STRAITS, NEAR VALLEJO, CALIF., SHOWING THE EXPLORATION TEST PITS OF 1929

sistently larger size found led to another check (samples C, Table 1) with representative samples quartered from the dump at each pit after being stacked in loose piles for some time. The increase in percentages of the fine particles was found to occur with the increase in handling and exposure to air and moisture.

Swanzy Earth Dam (in California) *Construction Material.*—Tests were made in 1929 and 1930 in connection with the design for Swanzy Dam, later incorporated into the water supply system of Vallejo, Calif. The limitations set by the capacity of the reservoir required on a pipe line from a well field and by the topography (see Fig. 1) made it necessary to construct the embankment of material excavated from within the site, using jointed shale. Surveys and test pits showed that approximately one half the required quantity could be obtained from wash or alluvial soil, one quarter from sandy clay subsoil, and one quarter from weathered shale. Gradation tests were made on each of these materials taken from the pits in the proposed borrow area, a typical example being shown in Cols. 2, 3, and 4, Table 2. Preliminary compaction tests were made with materials mixed in the $\frac{1}{2}$ to $\frac{1}{4}$ to $\frac{1}{4}$ proportions in a 4-in. cylinder. The shale content made it difficult to obtain complete compaction in the small cylinder. To approximate field conditions more closely, larger quantities of material were mixed and sprinkled on canvas and spread in layers in a steel box, 4 ft by 4 ft by 2 ft. A steel plate, 9 in. by 9 in., was fitted to a 4-ft handle, the whole weighing 25 lb; and this tamper was given an average drop of 18 in., until a thickness of about 9 in. of material was built up. Samples were cut from the center of the compacted material by a steel pipe with a cutting edge, 12 in. inside diameter, having a capacity of 0.5 cu ft. These cut samples were used for density determinations and permeability tests, and smaller samples were taken for moisture determinations and sizing tests. The result of one such

TABLE 2.—ORIGINAL AND COMPACTED MATERIAL USED IN CONSTRUCTION

Description (1)	Wash soil (2)	Sub-soil (3)	Shale (4)	Compacted mix (5)
(*a*) GRADATION; PERCENTAGES BY WEIGHT, PASSING EACH SIEVE				
Size of Mesh:				
1.954	97.7	98.4	85.0	96.5
0.864	84.1	90.8	55.0	81.8
0.503	73.0	84.3	45.1	75.0
0.381	65.5	78.5	40.6	72.2
0.279	58.7	75.4	39.0	71.0
0.173	45.5	63.0	33.2	66.6
0.140	41.7	59.0	32.3	65.7
0.074	35.0	50.6	28.3	63.3
(*b*) PHYSICAL CHARACTERISTICS				
Solubles in water, percentages by weight	0.3	0.4	0.2	0.2
Solubles in water with 5% HCl, percentages by weight	4.0	5.1	6.5	8.5
Dry density, in pounds per cubic foot	98.9	100.0	132.0	127.5
Moisture, percentages by weight	15.9	19.4	15.5	20.4
Approximate voids, percentages by volume	48.0	45.0	35.0	...

test on the compacted material (at an optimum moisture-density relation), in comparison with those on the original materials as obtained from the pits, is shown in Col. 5, Table 2.

Conn Valley Earth Dam in California.—Determinations made in connection with construction (that is, a comparison of the results of sizing tests made on borrow-pit material with those made on samples cut from the compacted fill) have not been entirely successful. Because borrow-pit test holes covered only a small proportion of the actual borrow area their value as representative examples of that area was to be questioned. Table 3 gives the results of an effort to determine the physical change due to geochemical development during the construction of the Conn Valley Dam[3] by the City of Napa, Calif., in 1945. The construction material was taken from the valley fill and alluvium of Conn Creek in four borrow areas, designated A, B, C, and D. Gravel consisted of all rock material larger than the ¼-in. mesh; sand was material, large enough to pass the ¼-in. sieve and held on the 0.074-mm screen; and silt and clay was that material passing the 0.074-mm screen.

TABLE 3.—COMPARISON OF MATERIALS SAMPLED AND MATERIALS IN THE DAM, CONN VALLEY BORROW PIT, NAPA, CALIF.

Column Headings

Col. 1.—Average grading of the composite samples, taken from 6 test boreholes in area A;
Col. 2.—Average grading of 58 samples cut from the fill rolled material from area A;
Col. 3.—Boring samples, area B;
Col. 4.—Average of 115 samples from area B after compaction in the fill;
Col. 5.—Boring samples, area C;
Col. 6.—Average of 92 samples from area C after compaction;
Col. 7.—Boring samples, area D; and
Col. 8.—Average of 16 samples from area D, after compaction in the pervious section of the dam.

Description	AREA A		AREA B		AREA C		AREA D	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
(*a*) GRADATION; PERCENTAGES BY WEIGHT								
Gravel	21.6	16.9	0.5	14.2[a]	4.0	2.2	61.9	66.9[a]
Sand	42.2	29.8	49.0	22.2	47.1	13.8	30.0	24.7
Silt and clay	32.2	53.3	50.5	61.6	48.9	82.9	8.1	8.3
(*b*) PHYSICAL CHARACTERISTICS								
Specific gravity	2.574	2.686	2.462	2.621	2.446	2.613	2.707	2.671
Approximate voids[b]	....	35.8	....	36.8	....	38.5	....	18.4
Moisture content[c]	....	13.9	....	17.2	....	19.3	....	6.4
Density, in pounds per cubic foot	....	120.3	....	121.0	....	119.4	....	144.8

[a] Material from the borrow pits was excavated deeper into the underlying gravel beds than the test boring in three cases. [b] Percentages by volume. [c] Percentages by weight.

Many earth structures in California have utilized (wholly or in part) soft rock broken to fragmental form during excavation, and the fragments decompose with further working. The St. Mary's Playground fill in San Francisco was made by excavating jointed and weathered sandstone by pick and shovel, dumping it loosely, and allowing it to settle. About ten years later more than 100 boreholes of 30 in. in diameter were drilled through the fill[4] allowing visual observation. All holes were found to penetrate a uniformly compact, relatively impervious, sandy, clay fill in which all semblance of the original fragmental character of the rock fill had disappeared, because of the kaolinization of the feldspar of the rock and the formation of amorphous, hydrous compounds from the original iron oxides with rainfall penetration. These observations and tests are not conclusive in themselves, but they substantiate geologic reasoning. It has been found that, in general, the physical composition or grain size of the material is of less significance than the mineral constituents.

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[3] *Western Construction News*, January, 1945, p. 98.

Fragments, grains, and particles are minerals or groups of minerals, subject to physical disintegration and decrease in grain size in the process of working or shortly thereafter. Alteration of the mineral compounds to other forms of mineral matter through oxidation and reduction, hydration, and dehydration, and loss of gaseous components follows disturbance due to handling or exposure to the atmosphere. Reaction occurs when the same material is worked with water, producing new mineral compounds, apparently of still finer particle component. It is found that, with the decrease or reduction of the average grain size, all tested physical properties of a soil are changed. The size ratio of one particle to another, and the composition of the mineral compounds of silicon, iron, magnesium, and aluminum in a soil, have been found related to the cohesion of the soil and to its plasticity. The character of the mineral colloids present is the controlling factor in the capacity of the soil to absorb water physically. Hydration and solution are effected. Base exchange takes place when the soil is rendered plastic and is kneaded in compaction, as ions are adsorbed from the water, and thus produces new mineral forms. In a number of tests the geochemical change was demonstrated by the alteration in the specific gravity of the particles, as a result of handling. With the attack of atmospheric gases on freshly excavated material the reduction of particle size is explained, largely, in the production of oxides, hydroxides, and carbonates which are important in the geochemical changes effected through water.

WATER

The most widespread evidence of geochemical change in natural materials is the effect of water in producing erosion and deposition. The transportation and accumulation of material by, and in, water facilitates the action. The principal agent that is active is water, through its oxidizing and solvent power. Table 4 is an example of the alteration which occurs when the mineral constituents of average types of matter are changing from crystalline rock to sedimentary rock. (The United States Geological Survey issues data on geochemistry.)

Similar results are caused by water in the transition of rock to residuum or soil over hillsides and, therefore, over excavated surfaces. Field observation and analysis show that water dissolves the more soluble minerals of the rock with the liberation of colloidal silica and with the formation of carbonates of lime, iron, magnesium, and the alkalies; oxygen reduces the ferrous compounds to ferric compounds, sulfides change to sulfates, and carbonates of the bases are reduced to oxides. Carbonates are reformed with the absorption of carbon dioxide and they are generally abundant in residuum. The iron carbonate is oxidized to the rusty coating of silty ferric hydroxide. The alkaline salts remaining in solution in the soil water dissolve silica and cause the undissolved residues to hydrate, transforming feldspar to kaolin and magnesian minerals into talc; iron becomes essentially limonite. These hydrates make up clay, and the geochemcial change is accompanied by an increase in volume. Residuum supplies the greater quantity of material to be transported by rainfall runoff and streams, until deposited as alluvium. Alluvial deposits resemble sedimentary rock in composition, being a combination of sandstone and shale with a larger percentage of organic carbon and colloidal hydrates. The material of valley fills and slopes consists of the original rock eroded from the surrounding watershed, less a part removed through solution, plus oxygen, carbon dioxide, and water absorbed in the formation of new mineral matter, and the organic matter introduced through growths. The gains exceed the losses.

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[4] *Transactions*, ASCE, Vol. 112, 1947, p. 399.

TABLE 4.—GEOCHEMICAL CHANGE FROM CRYSTALLINE ROCK TO SEDIMENTARY ROCK

Material	(a) PRIMARY MINERALS					(b) ALTERATION PRODUCTS			
	Quartz[a]	Feldspars[b]	Ferromagnesian[b]	Mica	Accessory	Hydrates[c]	Limonite[d]	Carbonates[e]	Oxides
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Crystalline rock	12.0	59.5	16.8	3.8	7.9	0.0	0.0	0.0	0.0
Sandstone	66.8	11.5	0.0	0.0	0.0	6.6	1.8	11.1	2.3
Shale	22.3	30.0	0.0	0.0	0.0	25.0	5.6	5.7	11.4

[a] In the form of free silica. [b] Unchanged. [c] Siliceous and aluminum hydrates. [d] Iron hydrate. [e] By the absorption of carbon dioxide.

Effect of Water in Earth Fill Construction, Swanzy Dam.—That water had an important geochemical function in the construction procedure was also evident in the tests made on the material available for the Swanzy Dam in the fall of 1929. The bedrock consists of a series of stratified shale and subordinate sandstone beds. The latter contained fossil shells and had been cemented to an average specific gravity of 2.5 (156 lb per cu ft). An adequate water supply was not readily available at the dam site, and the ready access to sea water in near-by Carquinez Straits led to the consideration of its use for construction purposes. It was believed that, if the only function of moisture in compaction is to lubricate the material so that the grains would slide into firmer bearing under the compacting load, sea water would serve the purpose as well as fresh water. The materials (see Table 2) were sprinkled with sea water to make them somewhat plastic, but could only be compacted to a maximum density of 108 lb per cu ft. The compacted material proved to lack cohesion and, when air dried, it was loose and crumbly. When soil water from one of the pits was used in the compaction tests it was possible to obtain a density of 127 lb per cu ft and the air-dried samples were well consolidated. Chemical analysis showed that the unstable bicarbonate in solution in the soil water formed normal carbonate when mixed with the calcareous material and worked. The compact material became cemented as the excess water evaporated. Sea water contains about 27.2 mg per liter of sodium chloride and but little carbon dioxide. The content of fairly unstable alkalies in the soil water aided in obtaining a hard, relatively impermeable material, whereas the use of sea water defeated this tendency through the reaction of the neutral and acid salts on the alkalies. Sea water made the material soft, as the absorption of calcium was limited and an insoluble binder was not precipitated in the interstices.

Water is the most active agent in the chemistry of nature and, apparently, in the geochemistry of earthwork. All minerals are attacked by water—some are attacked more intensely by water in which the alkalies are carried in alkaline or carbonated solution, and some react more strongly to acid solutions. All natural materials contain minerals which supply reagents in contact with water. The air transmits to water some of the most active reagents, such as gases and acids which are the products of organic decay, as well as carbon dioxide. The surface of soils and the soil cover contain humus and carbonaceous substances which are soluble in alkaline solutions. The solutions form humic acids, which are an important reagent in the production of siliceous hydrates and other colloidal substances. Any salt in solution will affect the solubility of a mineral in some manner, and soils (especially clayey soils) will absorb salts from water mixed with them or percolating through them.

TABLE 5.—AVERAGE RESULTS OF SOIL TESTS, TEST PIT AT FORT ORD RESERVOIR SITE IN CALIFORNIA

Depth below ground surface (ft) (1)	Classification of sand (2)	Weight, in place (lb per cu ft) (3)	Moisture content[a] (4)	Dry weight (lb) (5)	Colloidal content[b] (% of weight) (6)	Specific gravity of particles (7)	Solubility in 5% HCl (8)
0– 2	Top soil not tested[c]	....	....	....	....	....	....
2– 4	Noncohesive sand	108.3	9.9	98.3	6.4	2.641	3.12
4– 6	Slightly cohesive sand	107.5	8.2	98.7	11.0	2.683	1.40
6– 8	Cohesive sand with silt lenses	108.5	7.8	100.7	14.5	2.700	1.41
8–10	Reddish, noncohesive sand	110.0	8.4	101.5	10.6	2.657	1.68
10–14	Noncohesive sand	108.0	9.2	98.7	7.0	2.623	2.36

[a] Percentage of dry weight. [b] By elutriation. [c] Roots and vegetation in loose sand.

Fort Ord Reservoir in California, Proposed As Balanced Cut and Fill.—The function of water in connection with construction and construction material is that it, first, produces substances from rock and soil with which it reacts to produce other substances. This behavior was apparent in testing dune sand in two instances. The first instance occurred in 1942 at the site of the Fort Ord Reservoir, less than 2 miles from the shores of Monterey Bay, California. A soil profile of the proposed cut area was made through borings, and a test pit was dug at a location that promised average characteristics of the material to be rolled into a fill and embankment. Equipment such as described in connection with the Swanzy Dam tests was used and samples, in place, of 0.5 cu ft were cut from the bottom of the pit as it was being dug. The results appear in Table 5.

The silt was light colored and the sand contained only a small percentage of the rusty-colored iron hydroxide. Solubility tests showed an average loss of 0.2% in weight when a sample was agitated in water, and 2.20% was found to be soluble in a 5% solution of hydrochloric acid. The percentage of material that would react to form a "binder" when worked with water (thus increasing cohesion and stability) was small. The material from the test pit was given compaction tests following the procedure used on the Swanzy Dam. The maximum compacted density was 111 lb per cu ft (dry weight) achieved with a moisture content of 12.8%, by weight. Tests showed that the resultant material contained a slightly greater amount of soluble compounds (0.45% in water and 2.54% in a 5% solution of hydrochloric acid), with but little change in the average specific gravity of the particles. Compaction tests provided samples for quick shear tests made to determine the degree of cementation and consolidation. Cohesion was found to equal 575 lb per sq ft. Under given normal loads, the samples sheared with a horizontal pull (in pounds per square inch) of:

Normal load	Horizontal shear pull
11.1	12.8
27.8	26.1
76.0	55.5
117.0	95.9

The angle of friction of the compacted material was 38°.

A practical application of knowledge of the unsatisfactory constitution and behavoir of soils, obtained through observation and test, was made on this job. The dune sand at Fort Ord was constantly accumulating and had a sparse cover of brush and grass. The part to be excavated for the basin was largely unchanged through rainfall penetration. The induration of the material through the deposition or development of cohesive material was poor. The foundation fill and embankment to be located on a slope were subject to a fluctuating live load (maximum 1,500 lb per sq ft), and compaction was unsatisfactory. The compacted samples became unstable with moisture content in excess of 13% of their dry weight. There was little likelihood that the moisture could be controlled within the required limit during the winter months of February and March, 1942, when construction was required to meet the emergency program. There was no opportunity to select more favorable material without excessive haul. The writer reported that the sand to be excavated from the basin could not be consolidated sufficiently to provide the cohesion and resistance to shear required by the completed structure. Accordingly, the balanced cut-and-fill reservoir that had been planned was abandoned in favor of a pre-stressed concrete tank.

Sutro Forest Reservoir in California.—Material of the same origin and method of deposition on the west slope of Twin Peaks in San Francisco, however, had undergone such change as to be satisfactory—when used with other material to be excavated from within the site—for Sutro Forest Reservoir constructed by the Water Department in 1945. A eucalyptus grove had been planted closely over the site many years previously. Rainfall on, and through, the forest litter dissolved that matter and became charged with humic and carbonic acids. The action of these solutions on the feldspars, quartz, and iron compounds making up the sand (producing clayey substances throughout the upper 10 ft to 20 ft of the deposit) could be seen in the walls of the pits. Borings (maximum depth 145 ft) to bedrock gave core samples that showed the effect of the change caused by the percolation of acid and ferruginous waters. A soil profile was developed from observations of auger borings and sampling on 100-ft centers to a depth dictated by the bottom of the cut, or by the occurrence of bedrock within the cut. Selectivity could be practiced because cut quantities exceeded those required for the fill and embankment.

Samples were classified as "good," "fair," and "poor," according to the percentage of secondary products contained, as evidenced by color. Materials of each classification, to certain depths below ground surface, were mixed with an excess of water, patted into molds without any attempt at compaction, and allowed to dry slowly in the moisture-laden air of a screen house. Portions of the air-dried samples were agitated in distilled water, filtered, air dried, and weighed; then they were agitated with a 5% solution of hydrochloric acid, filtered, dried, and weighed. Clay capped each sample in the filter after each water washing, increasing in thickness from "poor" to "good." The test for solubility in weak acid showed the fresher dune sand to have lost an average of 4% by weight; and lens examination showed that most of the black sand (ferromagnetite) grains were dissolved. The hydrous hematite and limonite of the red-brown clayey sand was less soluble. The average results of these preliminary tests are given in Table 6.

TABLE 6.—PRELIMINARY TESTS ON CONSTRUCTION MATERIAL, SUTRO FOREST RESERVOIR, SAN FRANCISCO, CALIF.

Average specific gravity of sand particles	2.593
Average weight of particles (pounds per cubic foot) in solid dry mass	161.7

Classification	Weight, air dried (lb per cu ft)	Density (%)	Voids (%)	SOLUBLE IN:	
				Water	5% solution HCl
Poor	101.6	62.3	37.7	0.11	4.00
Fair	111.9	69.2	30.8	0.30	2.80
Good	117.8	72.8	27.2	0.61	2.30

In the normal weathering process, the usual cycle of geochemical action caused by water is as follows:

(1) Partial solution of soluble matter with the liberation of colloidal silica;

(2) The formation of the carbonates of iron, lime, magnesium, and the alkalies;

(3) The solution of the lime, magnesia, and alkaline carbonates;

(4) The hydration of the iron carbonate forming ferric hydroxide; and

(5) The hydration of the undissolved residues to hydrous silicates.

Clayey or hydrated material is created by the chemical combination of water and minerals; dehydrated, it is physically capable of absorbing and retaining water and solutions. Not only is it desirable that geochemical action shall have taken place in the material to be used in construction, but also that further reaction take place when the water is mixed with the materials in the compaction process. Most water used in construction is from normal water supplies low in salinity or alkalinity. It usually contains bicarbonates of bases in solution; but with the absorption of carbon dioxide from the air the solvent power is increased. The bicarbonates are held in solution until evaporation produces concentration, when the normal salts are precipitated; or the alkaline solution forms soluble silicates and these have an affinity for water, hydrating and changing to amorphous substances, which are effective "binder" material. Nearly all natural material has some strong base present, giving the mixture with water an alkaline reaction. The result is an exchange of bases between the silicates of the material and the dissolved salts.

Compaction and Consolidation

The Sutro Forest dune sand had become more hydrous than that of Fort Ord because the constituent minerals, in place, had undergone a geochemical change. The materials had a lower average specific gravity, were more cohesive, and became plastic at a lower moisture content. Compacted in a somewhat plastic state a further geochemical change took place, the end result being a stable mineral material having a higher specific gravity, which becomes more or less consolidated, depending upon the character of the mineral "binder" developed. This action was found to take place in working the materials from the borrow pits into the Conn Valley Dam. (Compare the specific gravities noted in Table 3 except for the gravel of the pervious section.) It was also found to occur when tests were made in 1948 with borrow-pit material to be used for the impervious clay core of an embankment proposed for the Alameda County Water District near Niles, Calif.

Impervious Clay Core Material, Niles Embankment.—The clay used was that found existing as wash from rainfall erosion of hills, consisting of Cretaceous shale, a blue-gray, carbonaceous, fine-grained to sandy-textured rock. The disintegration and decomposition effected through weathering, transportation, and deposition produced a gritty clay, rusty brown in color (due to iron compounds). This clay contained silt and colloidal particles of several oxides and hydrates. It was friable when dry, crumbly and lumpy when damp, and plastic when moist to wet. The tested physical characteristics of 4 representative borrow-pit core samples (Nos. 1 to 4, inclusive) are given in Table 7 (see samples 1 to 4). Sample 5 is cut from compacted mixtures of the borrow-pit material, taken at a maximum compacted density or optimum moisture-density relation. The most striking change that occurred when the mixture was worked was in the color, the resultant being a dark, gray-brown, compact, impervious mass.

The fact that soils enter into chemical and mineralogical exchange with solutions, giving off some elements and taking up others, has long been utilized by irrigation engineers in soil treatment. The function of water or moisture in embankment construction, and its effect in other engineering work, are of primary concern to civil engineers who are utilizing the geochemical and mineralogical changes which it produces in adapting natural materials to construction purposes. Engineers are faced with such changes in excavating, in tunneling, and in foundations for structures. The geochemical action that accompanies disintegration of material with excavation, and the decomposition that takes place when the fragmental material is exposed to atmospheric gases and solutions, has received scant, if any, attention in the literature. The passage of the roller (especially the sheep's-foot roller) mixes and entraps moisture and air with the material so that it can do its work. The amount of water used and the fineness of particle resulting during construction are such that complete reactions never occur in the share that they do in the laboratory, where materials and water in chemical equivalents remain after excess water has evaporated. The moisture-density relation is important in that a degree of plasticity is required to facilitate and accelerate geochemical activity and achieve maximum density. It is apparent that such relation will change, within limits, because of the mineralogical and physical make-ups of the materials, possibly in the same borrow area.

TABLE 7.—Comparative Tests on Clay Borrow-Pit Material, in Place, and As Compacted; Clay Core for the Alameda County Water District Embankment, Niles, Calif.

Sample No.	Density, as sampled (lb per cu ft) (1)	Moisture content (percentage by weight) (2)	Dry density (lb per cu ft) (3)	Colloidal fraction by elutriation (%) (4)	Specific gravity of particles (5)	Lower plastic moisture content (percentage by weight) (6)	Percentage Solubility in	
							Water (7)	5% solution of HCl (8)
1	101	14.3	88	32.0	2.780	15.4	6.0	3.0
2	106	9.8	96	32.0	2.820	16.5	4.0	4.0
3	112	13.8	99	30.0	2.840	17.1	5.0	3.5
4	110	11.6	98	29.0	2.800	15.5	6.0	4.0
5	144	16.1	124	20.0	2.930	16.5	3.0	20.0

The necessity of reducing the size of voids between particles, so that they can be consolidated in some degree through cementation, is obvious. The process of cementation begins as the chemical combination takes up moisture. In that process, the internal pressure that accompanies expansion holds the particles in place until induration is achieved. The induration may be entirely due to the action of the clay fraction. As the clay becomes less hydrous it tends to adhere to crystalline material; it becomes amorphous after the water taken up by the chemical system is "fixed" through evaporation, binding the coated particles together. The cementing may be caused by: (1) The solution of moist surfaces of minerals in contact with each other; (2) the base exchange in the presence of solution which forms amorphous film mineral binder; (3) the deposition of solubles from concentrated solutions; or (4) the combination of all these to increase the density of a given body of material. The hardpan and caliche lenses in alluvial deposits of the semiarid west are a common example of cementation. Carbonates are carried in solution as the water containing normal carbonic ions transmits carbonic acid to the material with which it comes in contact. The bicarbonate ion loses its carbon dioxide through evaporation and the normal carbonate is precipitated in the interstices of the soil. The degree of mineralogical change with consolidation of worked material will be evidenced by the increase in the specific gravity of its constituents. The specific gravity or density of a solid depends upon the nature of the chemical substances of which it is composed and the state of its molecular aggregation. The density of hydrous minerals is low (2.3 to 2.5), but as the change takes place it will be nearer that of unchanged minerals (2.6 to 2.9). The specific gravity is an index of fundamental importance in identifying change in the hydrous character of construction material.

Gordon Valley Dam, in California.—Shale is a sedimentary rock made up of closely packed, fine particles, having a low permeability. Colloidal matter formed in conjunction with water from minerals of particle size is capable of binding granular or fragmental material under the influence of pressure alone, and such process in nature results in compaction shale. The same process under earth fill construction procedure produces a cohesive, dense mass. Such action was evidenced during the construction of the Gordon Valley Dam[5] for the City of Vallejo in 1925. Most of the structure was composed of the valley fill upstream from the dam site; but, at a certain stage, a dark, thin bedded, carbonaceous shale, excavated from the spillway and outlet tunnel, was rolled into the dam. The fresh shale was little altered in place; it slaked and oxidized quickly with handling; it broke down to a powdery, dry, light-colored dust on the construction road and formed a blue-black clay when sprinkled and rolled into the fill. That there was organic material in the shale was evident because the release of carbon dioxide gas in the short outlet tunnel made it difficult to keep a match lighted.

Notes made at the time show that, generally, the volume of the measured fill in place exceeded the borrow-pit measurement, but the swell of the shale portion was greater than that of the alluvium. The swell of the shale portion was probably more pronounced because the organic materials release great quantities of carbon dioxide which, with water, form acid reagents that dissolve the alkalies readily. There is a constant exchange of bases in solution, with a regeneration (precipitation) of carbonates in the compaction, and a consolidation of the structure. The similarity between the natural geologic processes responsible for the formation of the shale and those taking place during construction was impressive. The maximum degree of consolidation accompanies the following essential construction requirements:

(*a*) Adequate compaction with heavy unit loads to break fragments and to force grains into closer contact;

(*b*) Sufficient moisture to give all the material an initial plasticity, to encourage base exchange, and to assure an adequate absorption of solutions; and

(*c*) Thorough kneading by the repeated passage of the roller, in order to hold water and air in the soil long enough to do their work, and to expel the excess progressively as the consolidation process continues.

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[5] *Western Construction News*, January 25, 1926, p. 15.

The writer has observed that, as shale or clayey material loses its carbon dioxide, it becomes lighter in color, the oxides of the alkaline bases are reformed through oxidation, and the carbonates are regenerated in compaction. The compounds of iron and aluminum and siliceous hydrates will increase tremendously in volume, exerting pressure if confined, because of the absorption of alkaline solutions. For example, the oxides of iron and aluminum, which are present in most material, can take up as much as 20 mols of water to 1 mol of oxide. Such action is frequently the cause of the failure of excavated slopes, retaining or basement walls, and foundations. During embankment or fill construction the geochemical action, with its resultant swelling in the process of compaction, is necessarily accompanied by the development of internal pressures. At any given level, these pressures probably exceed the pressure imposed by the load, although there will be a gradual reduction as the material consolidates. The water applied in sprinkling the material for Gordon Valley Dam, carrying its original impurities and absorbing others, was taken up by the mass partly to become the hydration water of clayey material and partly to be evaporated. The completed structure presents an exceptionally dense, stable mass with no noticeable settlement after twenty-five years.

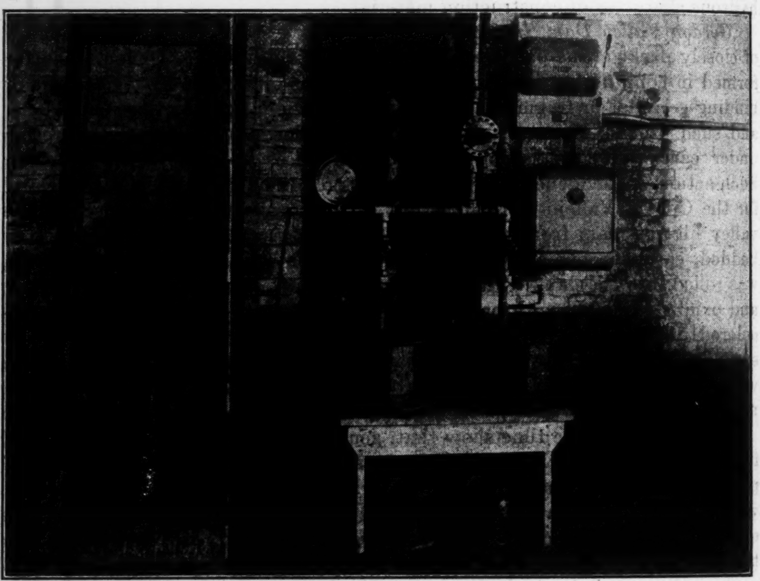

FIG. 2.—EQUIPMENT SETUP FOR PERMEABILITY TESTS, SWANZY RESERVOIR, IN CALIFORNIA, 1930

PERMEABILITY

Evidence of natural cementation of rock and rock material is common, resulting from induration due to atmospheric weathering, as well as from

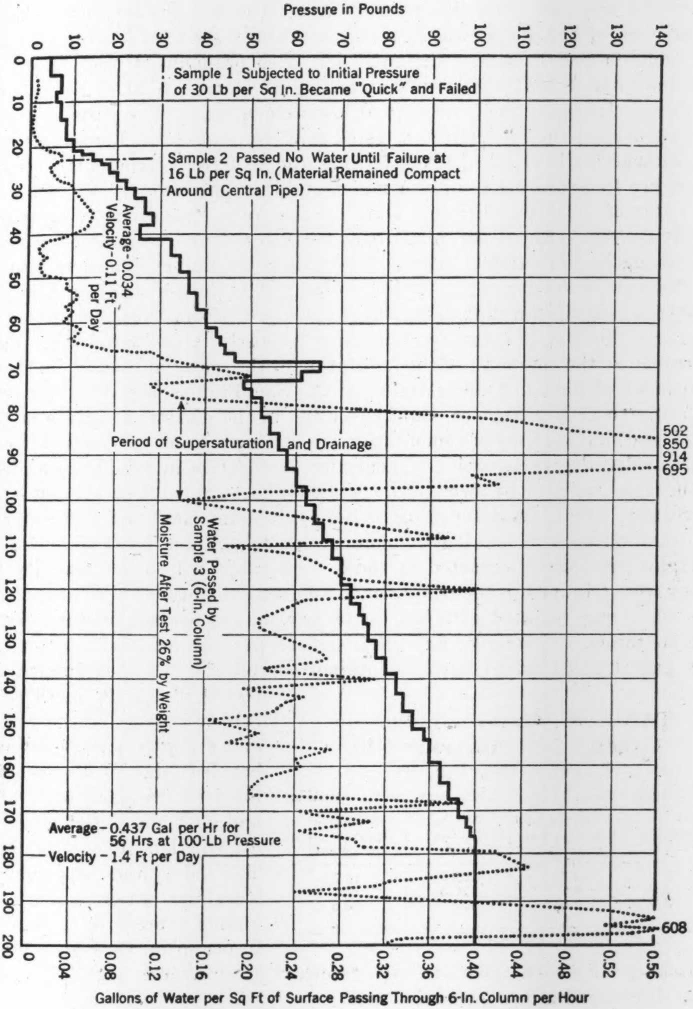

FIG. 3.—PRELIMINARY PERMEABILITY TESTS ON COMPACT SOIL SAMPLES (4-IN. LAYERS) SWANZY DAM, IN CALIFORNIA, 1930

injection of cementing material. Inspection of dam sites in sedimentary formations has revealed outcrops subjected to shell hardening[6] as a natural consequence of the oxidation, hydration, and secondary deposition of iron minerals cementing the grains and filling the interstices of the rock surfaces.

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[6] *Bulletin No. 26*, California Dept. of Public Works, Div. of Water Resources, Sacramento, Calif. 1930, p. 511.

The same material and action have indurated the face of road cuts in the dune sand areas of San Francisco, so that they readily shed rain water. At many locations in San Francisco the condition is so pronounced that drillers class the material as soft sandstone, extending as much as 20 ft below ground surface and capping the loose sand at lower elevations. The purplish to rusty-red staining of the pit walls and tested sand cores at Sutro Forest was due to the change in the sand of black ferrous iron to ferric compounds, eventually changing to hydrate limonite, a noncrystalline concretionary substance. At the sites of the Lake Pleasant Dam on the Aqua Fria River, and at Camp Verde on the Verde River in Arizona, faces in volcanic ash exhibited marked hardening and crystalline change, resembling crystalline limestone that rang to the blow of the hammer. The consolidation of the alkaline ash through the absorption of carbon dioxide was more or less uniform over the surface, producing a thick, solid crust over unchanged, loosely consolidated ash. Failure to recognize the character of the induration of the material making up the west abutment of the St. Francis Dam (on the San Francisquito Creek, about 40 miles north of Los Angeles, Calif.) resulted in the disastrous failure in 1928.[7]

These simple forms of cementation are duplicated and accelerated in earthwork. The degree to which cementation is effective in producing an impervious mass was evident through the permeability tests made on the test samples at Swanzy Dam. A description of the construction material and water, and the method used in making the compaction tests, have been presented. All samples cut from compacted material were subjected to permeability tests. They were trimmed in the container to an even thickness of 6 in. at both ends; the edges were painted with hot tar to prevent the passage of water between the container and sample; fine mesh screens were placed over each end; and pea gravel was used to fill the container completely. The equipment, illustrated in Fig. 2, consisted of a rubber gasket for the top edge and a copper gasket fitting the beveled edge of the container. Steel plates fitted with 8 bolts, and inlet and outlet pipes, were drawn tight over the gaskets. The inlet pipe was connected with the water supply main, having a pressure of about 140 lb per sq in. through a pressure regulator, and each inlet line was tapped for a pressure gage. The outlet pipe discharged into a measuring can.

TABLE 8.—CHARACTERISTICS OF COMPACT SOIL SAMPLES IN 4-IN. LAYERS (SEE FIG. 3)

Sample	1-ft depth compacted to (ft):	Weight (lb per cu ft)	Moisture (percentage by weight)
1	0.735	108.0	15.7
2	0.675	120.5	17.6
3	0.638	127.5	20.5

The tests were made simultaneously on 2 samples. The test results are stated in Table 8 and are shown graphically in Fig. 3. Sample 1 was compacted with sea water. The water was turned in at 30 lb per sq in. and all the fine material was washed from the container into the can immediately, there being no effective bond between grains. Sample 2 was the first compacted with soil water from the test pit. It remained consolidated until subjected to a pressure of 16 lb when a "pipe" developed between the inlet and outlet in the center of the sample. A solid plate was inserted below the inlet on sample 3; and that sample, as well as subsequent samples, was allowed to stand with water at a 4-lb pressure for 24 hours before pressures were built up. The permeability of the sample is expressed in rate of flow in gallons per hour through a 6-in. column over 1 sq ft of its cross section, with a vertical path of percolation under increasing heads. The temperature of the water was constant at 55° F. The results obtained with sample 3 were typical, and represented an average of those tested subsequently.

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[7] *Western Construction News*, April 10, 1928, p. 234.

The tests indicated that, for the heads that would prevail behind a structure completed to heights of from 60 ft to 70 ft, a soil mass would be consolidated by the cementation that accompanies working, and the compact structure of the soil and its impermeability would thus be maintained. The recommendation, based on the tests and investigations, was that the dam be designed without a paved facing or other provision against seepage through the structure. That recommendation was found to be warranted when, after the reservoir had been placed in operation, test borings were made through the embankment and no water was encountered above foundation rock. Similar tests for the permeability of the compacted clay core material in a proposed gravel embankment (Table 7, sample 5) showed that practical impermeability was achieved through the regeneration of abundant carbonates in the material, 20% soluble in a 5% solution of hydrochloric acid. Under these conditions, a thickness of 2 ft, pneumatically tamped between forms, would be entirely adequate to serve the purpose at a considerable saving in cost.

Conclusion

These observations and comments are not presented as criticism of test and construction procedure in general use, but rather as suggesting a broader base for the interpretation of results. Personal, practical experience has proved that, with a knowledge of the mineralogical and chemical characteristics of natural soils, and of their behavior when disturbed during construction, it is possible to utilize those materials with safety and economy. An understanding of the geochemical reactions resulted in a saving in cost of the Swanzy Dam by the elimination of the impervious core or facing planned in the original design. It is believed that economies can be effected in many ways, and failures can be prevented through a consideration of the geochemical action and reaction that are certain to occur in all engineering use of natural materials. From a geological point of view the processes (reconsolidation, for example) cannot be dealt with as invariable laws, but as tendencies. Local variations of mineral constituents of both earth and water create variation in the product, which is the union of elements from several sources—the earth, the air, and the water. It is neither possible nor necessary, to determine mathematical proportions readily, but it is important to recognize the facts and to be guided accordingly in construction. The observations and tests cited in reference to earthwork were the result of the application of elementary principles of geologic processes.

DISCUSSION

WARREN D. SMITH.[8]—A subject of paramount importance to engineers, and one very inadequately treated in textbooks on engineering geology, has been opened for discussion by Mr. Forbes.

Many engineers have studied little or no geology in their academic curriculum. This lack, however, does not belittle their competence as engineers, for only within recent years has the importance of geology been adequately recognized. The writer knows of one recently published engineering school curriculum in which no course in either geology or mineralogy is provided. A civil engineer should have some knowledge of the earth on which he is going to erect his structures. It is not enough to hire a geologist and leave the problem entirely to him. The engineer should know enough geology to enable him to cooperate with the geologist and appreciate the significance of geological findings.

Many of the reactions and principles brought out in Mr. Forbes' valuable paper have long been known to geologists,[9] but they have not been applied very widely to concrete practical engineering problems. According to Charles R. Van Hise[9] the crust of the earth is not static—there are changes continually taking place, either those that are anamorphic, like cementation, or those that are katamorphic, in which breakdown and disintegration are the rule. Through his activities, man often promotes these reactions and, if he does not realize this, he can get into serious difficulties.

Some of Mr. Forbes' examples from actual observations in practical work illustrate extremely well these metamorphic processes, as, for instance, the effect of water in earth fill construction on the Swanzy Dam in Vallejo, Calif. Again he cites the failure to understand the nature of the materials in the abutments of the St. Francis Dam[10] in California which led to that regrettable disaster.

The writer has observed numerous examples of profound geochemical changes which, if overlooked, lead to serious engineering troubles with landslides and rockfall. In western Oregon and Washington, for example, where there are certain types of rocks, tertiary volcanics, and sediments derived from them, in a region of copious rainfall, one can expect an unusual degree of weathering and, consequently, frequent landslides and rockfalls. This means that the slopes bordering all highways and railroads should be much lower than in other regions where these conditions do not prevail. Failure to realize these conditions has cost the State of Oregon, and railroad companies, great sums of money.

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[8] Prof. Emeritus, Dept. of Geology, Univ. of Oregon, Eugene, Ore.

[9] "A Treatise on Metamorphism," by Charles R. Van Hise, U. S. Govt. Printing Office, Washington, D. C., 1904.

[10] "Essential Facts Concerning the Failure of the St. Francis Dam: Report of Committee of Board of Direction," *Proceedings*, ASCE, October, 1929, p. 2147.

Nature makes no allowance for ignorance. Engineers must know more about geochemical processes—"or else!" This need resolves itself into four recommendations as follows:

1. Civil engineers should include more geology in their academic curriculum.
2. On all important civil engineering projects geologists with some knowledge of geochemistry should be consulted.
3. Researches in this field should be liberally supported. (Excellent work is now being done by several organizations, including that of the Bureau of Reclamation on the alkali problem in connection with concrete construction.)
4. Finally, there should be more frequent symposium discussions of the many aspects of this general subject of geochemistry, in which engineers, physical chemists, geologists, and contractors could take part.

Mr. Forbes has made a constructive contribution to a subject of first importance to all who are concerned with practical problems connected with the crust of the earth.

D. P. KRYNINE,[11] M. ASCE.—The constituents of the earth crust are never in a state of static equilibrium. Obeying the well-known rule of boundary conditions they tend to attain equilibrium with the environment. In turn, the latter is also variable because of the changes in temperature, humidity, pressure, and other factors. When extracted from the earth crust and placed in an engineering structure, materials such as rock and soils tend to attain the state of equilibrium with the new conditions of work. Hence, their behavior may substantially differ from that under natural conditions. It is decidedly advantageous for engineers to understand the geochemical processes that may develop in this connection, in order to avoid possible surprises.

Without presenting a comprehensive discussion of this interesting paper, the writer wishes to emphasize only two important topics—field decomposition of calcium carbonate and changes in specific gravity.

Field Decomposition of Calcium Carbonate.—The author depicts the decomposition of calcite in the sandstone foundation rock of the Gibraltar Dam on the Santa Ynez River, in California, as breaking of the calcium carbonate into lime and carbon dioxide. As the author states (under the heading, "Weathering, a Geochemical Change Through Geologic Processes"):

> "The site was stripped to fresh, sound rock and hosed off before pouring concrete; but overnight it slaked and the surface became covered with rock flakes, sand, and white, chalk-like dust. This loose matter was cleared with wire brushes, and again the detritus formed before concrete was poured."

The constituents of a sedimentary rock such as sandstone are: (*a*) Grains, (*b*) matrix or a groundmass of smaller grains in which the larger size grains are embedded, and (*c*) cement that infiltrates around grains and matrix. In the present case the cementing material was (at least partly) the product of decomposition of fossil shells consisting of the grains of calcite bound together with amorphous calcium carbonate (or more accurately microcrystals) as is the case of limestone. Chemically, the so-called high calcium limestone and calcite consist almost entirely of calcium carbonate. What was reduced to dust in the case described by the author was probably the apparently amorphous mass (microcrystals) bounding the crystals of calcite.

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[11] Cons. Engr., San Francisco, Calif.

The deterioration of limestone and the decomposition of the calcite are very common troubles in the building stone industry; and besides breaking into lime and carbon dioxide (as, according to the author, was the case of the Gibraltar Dam); the given material may undergo other deteriorations with the same general picture of failure as that given by the author. More specifically, the writer wishes to refer to the views of some French geologists and scientists[12] on the process of the field decomposition of calcium carbonate, which is defined in that case as a transformation of a part of the calcium carbonate mass into calcium sulfate (gypsum). Contrary to popular belief, such transformations are not limited to localities affected by smoke but may occur anywhere—as, for instance, examples were found in some old French country churches. The transformation itself is due to a base exchange process; and this reaction, being ionic, requires water for its completion. Hence, humidity is an essential factor for such transformations. On the other hand, sulfur salts are required for that reaction. In country districts, sulfur salts are practically absent in the meteoric water, such as rain water; but this water may acquire the necessary salts during its passage through soil, from which water is afterward pulled up by capillarity. In industrial cities, of course, rain water contains enough sulfates. Without entering into details, it also may be stated that the process of deterioration is accompanied and activated by bacteria.

In the final analysis the rotten rock possesses a thin, impervious, hard crust followed by a zone of deterioration that, being exposed, resolves into yellowish friable powder. A stripping of the outer surface of such a rock would cause pulverization of the exposed surface.

The writer ventures to believe that, in the case of the Gibraltar Dam, the rock was already in an advanced stage of deterioration (at least next to the surface) and that pulverization was caused by the stripping of the outer thin crust.

Change in Specific Gravity of the Material During the Construction Period.—Table 3 suggests that, in the case of the Conn Valley Dam, in Napa, Calif., in the three borrow pit areas, the material became finer after compaction and its specific gravity increased. The decrease in the coarseness of the material that undergoes compaction is easily understood, especially if the material contains breakable rock fragments.

The fact that the specific gravity of an earth material may change as a result of hydration or dehydration is clear to anyone who remembers how, from two metals of different density, an alloy of intermediate density may be obtained.

In the case of hydrated clay and similar materials, water broken into hydroxyl and hydrogen ions joins the atoms of the clay minerals located at the crystal lattices and "dilutes" their solid substance. The latter, however, still remains solid after hydration. The specific gravity of the hydrated product in many cases may be simply computed in the same way as the specific gravity of an alloy is computed from that of the two original metals. The tamping effect perhaps produces dehydration by throwing the components of constitutional water off the lattices, especially if that water is loosely accommodated at the lattices. It remains to be determined, more precisely, under what circumstances dehydration may take place during compaction.

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[12] "Etude des détériorations des pierres des monuments historiques," *Annales*, Institut Technique du Bâtiment et des Travaux Publics, Paris, December, 1949.

Again, is it possible that compaction may produce the base exchange and thus replace lighter atoms by the heavier ones, or vice versa? These and similar questions should be clearly answered for the benefit of engineers who specialize in compaction problems; and, to obtain such answers, careful additional field investigations, and possibly extensive laboratory research, are needed. Mr. Forbes' paper is a valuable beginning of the scientific study of a complex array of interesting and important geotechnical problems, compaction being only one of them; and the author is commended for having brought these problems to the attention of engineers.

S. S. GORMAN,[13] M. ASCE.—The subject of geochemistry applied to soils as foundations and in earthwork is not new to the Foundations and Soil Mechanics Committee of the San Francisco (Calif.) Section of the Society or to the San Francisco Section as a whole. The author (whose membership in the Foundation Committee dates from April, 1929) has frequently discussed the subject, and at the regular meeting of the section, April 21, 1931, presented an abstract in his section of the report of the committee on the question of subsidence and the foundation problem in San Francisco. This report was published by the section in September, 1932.

The writer has worked with the committee since 1939 in the study of related problems. The work of the committee was again presented at the regular meeting of the section on October 18, 1938. The author's paper was entitled "Geological Background of Soil Behavior" and was made available only to section members in the *Proceedings* of the section. In that paper, it was emphasized that soil is not a static thing but, when accumulated in its various land forms, changes of a physical and chemical character continue until all rock fragments are completely broken down to clay and grit.

Rock originated in the molten state, crystallizing out in the form of complex silicates. Soil in addition contains oxides, carbonates, sulfates, and chlorides derived through rock weathering and geologic processes. The mineral constituents of the soil are continually being attacked by chemical reagents. Oxidation and carbonization take place, forming new oxides and soluble carbonates. Hydrolysis of the oxides silica, alumina, and ferrous oxide, as well as of the sulfates and chlorides of the bases continues, reducing the soil particles to colloids.

Furthermore, rock is not always stable but, due to the forces of nature, breaks down under general geologic processes. The processes involve both mechanical disintegration and chemical decomposition which are frequently the cause of landslides and rockfalls in cuts. It has been found that instability results because of profound geochemical changes occurring in rock which are not attributable to surface weathering.

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[13] Asst. Chf. Engr., Board of State Harbor Commrs., San Francisco, Calif.

As regards the determination of the effect of such geochemical change, a practical application was made by the author in his investigation of and recommendation for the correction of an area of unstable hill in the San Francisco Naval Shipyard in 1947 as a basis for work contracted for by the writer's department. It was found that the original peridotite underwent serpentinization (a process of hydration) which caused a pronounced internal shearing movement resulting in a thoroughly slickensided mass of cohesionless, dark colored to black serpentine rock particles, in which sheared diabase sills occurred and over which a massive sandstone and chert formation had been completely shattered and crushed through the displacement caused by the swell in serpentinization.

These conditions were not revealed until partial excavation was undertaken, slides were developed, and test borings were made. Correction consisted in resloping the hill area to meet the rock conditions involving the removal of many thousands of yards of material.

Charles H. Lee,[14] M. ASCE.—A most intriguing field for research and one which may have great potentiality for the engineer interested in earthwork has been touched upon by Mr. Forbes. Through this field an engineer may ultimately attain a technical command of earth material which in some instances may approach that of the metallurgist in the field of alloys, or of the chemist in the production of synthetic materials. The scope of scientific knowledge on which such command will be based, however, is far broader than geochemistry. It includes geochemistry and mineralogy as well as a specific knowledge of all phases of the clay minerals and of the mixtures of clay and nonclay minerals. It involves the physical chemistry of colloids, the flocculation and dispersion in clay suspensions (when ions are present), and the chemistry of adsorption in pasty masses where the ratio of solids to liquids is relatively high. A knowledge of the structure and behavior of soil crumbs or flocs built up by the attachment of ions adsorbed out of solution onto a nucleus of inert particles is also involved. As stated by the author, it also requires a knowledge of the various chemical processes met with in geochemistry, such as hydration and dehydration, base exchange, oxidation and reduction, and loss or addition of gaseous components. In short, it includes not only geochemistry but also the fields of soil physics, soil chemistry, ceramics, the physical chemistry of natural cements, and many other fields of scientific knowledge.

To be of value, the results of research in any new field of human endeavor should be supported by evidence of a convincing character and must also be related to the systematized knowledge of those branches of science to which it pertains. The results of research must also be expressed in generalized form and must be provided with technical procedure such that practical workers can take it and apply it to the varied problems that confront them. This is particularly true in a new field of activity, such as the application of chemistry to earthwork, where the entire subject is still in an immature state. The paper illustrates the current awakening to a new phase in the handling of earth in engineering construction.

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[14] Cons. Engr., San Francisco, Calif.

Essential to any engineering study of clays or of altered rocks is identification. Often it is necessary to identify the mineral constituency of the rock both in its original condition and as modified by weathering or hydrothermal processes. Such identification should extend to internal texture, structure, and mineral composition. If the slow chemical changes that take place in nature can duplicate, continue, or reverse themselves with rapidity in manipulated earth (as suggested by the author), then identification of the beginning and end products should furnish conclusive proof. Modern petrographic methods as applied to engineering problems provide a valuable tool for this purpose. Petrography as now practiced includes, in addition to petrographic analysis, microscopical examination, microchemical analysis, X-ray diffraction analysis, differential thermal analysis, and various physical tests. Undoubtedly the paper could be strengthened by submission of petrographic or other proof that the "geochemical processes of nature * * * are duplicated or continued by the methods of engineering construction" (see under the heading, "Geochemical Change Accompanying the Handling of Earth").

Considering the paper more in detail, the author points out that earth is a mixture of mineral and organic solids with water and gases. He indicates the various changes which he believes take place in these components during the handling and compacting of earth in construction operations. These changes can be classified as follows: (1) Reduction in particle size; (2) alteration of the chemical constituency of the solid particles, and hence of their specific gravity; (3) effect of the chemical elements dissolved in the soil water; and (4) cementation through chemical action produced by compaction.

The author attempts to identify these changes with the more or less well-recognized changes that occur in rock materials during long periods of geological time, as the result of weathering and hydrothermal activity. He intimates that in manipulated earth these changes take place in a matter of hours, days, or years, instead of millenniums or eons.

With regard to the reduction in particle size, in the examples presented by Mr. Forbes, it is to be noted that the mechanical analyses to determine percentages of various particle sizes were made by sieving. The only pretreatment mentioned is that of washing out the fines, drying them in an electric oven, and breaking them up with a wooden roller, or breaking the soil by hand on the screens in an air-dried condition. Apparently no attempt was made to break up the soil particle aggregates by chemical pretreatment and vigorous stirring or long, continual shaking of the test sample in suspension. In making mechanical analyses of material similar to those described by the author the writer has found that, without such pretreatment, appreciable percentages of the particles held on the finest mesh are composed of soil particles aggregated into crumbs (flocculated) which, if subjected to the standard treatment followed in hydrometer analysis, would separate into their pre-existing smallest particle sizes (deflocculate). It is believed that the reduction in particle size indicated in Tables 1, 2, and 3 would not have appeared if the mechanical analyses had been performed by present-day methods as standardized by the American Society for Testing Materials (ASTM) and other agencies. Excavating from borrow pits, air drying in stock piles, and mechanical handling accomplished a measurable degree of breakdown which sieving alone failed to accomplish.

Change in the structure of a soil particle through chemical reaction, with the removal or addition of an element or radical comprising an integral part of the structure, constitutes alteration of the chemical constituency of the solid. Such change does not include a change in the adsorbed film of water around or within the cellular structure of soil particles, or a change in the ions or radical in solution in such water. Change in the chemical constituency of solid particles of rock or soil may take place through various processes, such as the addition or removal of constitutional water (including water of crystallization), oxidation or reduction, chemical breakdown of complex unstable compounds, solution, or the action of organic acids or strong alkalies. Most of these reactions proceed very slowly and for their occurrence require conditions which are not present in earth construction. Most important of these conditions is free circulation of soil water through the voids between solid particles which is necessary for solution, oxidation, and reduction, and the continued action of weak acids or alkalies. Circulation is required to maintain the reaction and to carry away its products. Long continued circulation of water through the soil, or through the joints and cracks of the rock, is an essential accompaniment of weathering, hydrothermal activity, the formation of caliche or hardpan, and other geochemical processes. Circulation of water does not take place in clayey materials compacted to the degree of impermeability required for storage dams of earth. In fact, one purpose of compaction is to prevent such circulation.

Change in specific gravity accompanying alteration in chemical constituency is probably infrequent in earth construction and is attested by the fact that standard ASTM and other procedures for making soil compaction tests do not provide for such a contingency in establishing the zero voids curve. The necessity for such provision is apparent, if changes of from 4% to 6% in specific gravity can take place in handling and compaction, as is indicated in Tables 3 and 7. Even greater percentage increases in specific gravity are represented by changes of from 2.3 or 2.5 to 2.6 or 2.9 stated by the author to accompany dehydration. Moisture differences between the actual and zero voids moisture curves vary greatly—from 2% to 5%. Discrepancies of from 4% to 6% or more in specific gravity, if of frequent occurrence, would have given trouble in analyzing the test results of compaction and would have become generally recognized by soil technicians. This has not been the case. The author states that all materials compared in Table 3 are valley fill and alluvium from the vicinity of the dam site. It has been the writer's experience that such materials from a limited area do not vary in specific gravity to the extent indicated for the borrow-pit samples—namely, 2.574, 2.462, 2.446, and 2.707. Light might be thrown on the matter if the author's method of determining specific gravity was detailed, and if information was furnished regarding the predominant chemical constituency of the various samples.

The author presents various examples of the effect of the chemical elements dissolved in the soil water. Experience with sea water in compaction at Swanzy Dam (Vallejo, Calif.) recalls the writer's use of sea water in sealing the pervious clay lining of the Exposition Lagoon at Treasure Island (San Francisco, Calif). Base exchange of sodium and calcium ions existing as electrolytes in the soil water was involved in both instances. The author states (under the heading, "Water: Effect of Water in Earth Fill Construction, Swanzy Dam"): "The compacted material proved to lack cohesion and when air dried, it was loose and crumbly." This flocculated condition was overcome by the writer by leaching with fresh water. As stated elsewhere,[15]

> "After leaching with fresh water to remove excess sodium ions from the soil-water solution, the colloidal aggregates disperse, changing the physical properties of the soil and filling the voids with a sticky colloidal gel which is impervious to water."

An essential feature of the employment of base exchange to alter the physical properties of a soil is the removal of excess electrolytes. This was discovered before 1915 by soil physicists in experimenting with alkali soils in California.

The author believes that cementation through chemical action takes place in the process of compaction and he cites, as comparable examples, the formation of hardpan and caliche in alluvial deposits of the West. This explanation of the cause of the stiffening and hardening that occurs in clayey materials during the compacting process is quite at variance with the current concept—namely, that it is the result of closer contact and interlocking of soil particles, bringing a greater proportion within the range of molecular attraction as well as increasing the viscosity of the colloidal gel surrounding soil particles by attachment to them of increasing numbers of free ions from the soil water. The paper needs to be strengthened by submission of petrographic or other specific proof in support of this novel thesis.

GEORGE S. HARMAN,[16] Assoc. M. ASCE.—The attention of the writer was first called to geochemical change taking place in rock in 1935, in connection with the stabilization of the slope in the James D. Phelan State Park Beach in San Francisco, Calif. The serpentine rock extended up the slope beneath dune sand, above the high-tide line. Surface water draining through the dune sand to serpentine rock caused the rock to hydrate rapidly. In so doing, it swelled and absorbed more water until what had been rock a short time before flowed over the beach sand as a green slime. At that time, intercepting drains were installed over the most unstable part of this area to keep surface water from accumulating above the rock. Continued observation since that time indicates that the slope has been stabilized in the area drained. However, another part of the site which seemed to be stabilized in a natural way (there being a free flowing spring) became unstabilized and started to slide. It became apparent that the entire area should be thoroughly investigated, and an extensive drainage system, both surface and subsurface, has been designed by the Department of Works, Bureau of Engineering, at the request of the Recreation and Park Commission of the City and County of San Francisco.

The author mentions briefly the effects of the geochemical change accompanied by the surface settlement of the rock fill placed from 1930 to 1933 to provide level surfaces at St. Mary's Playground in San Francisco.[4] Although accurate, precise level records were not kept to measure the settlements in this fill, the magnitude and rate at which it settled was far more than originally expected, disrupting walls and rock gutter drains. The original fill was made of sandstone fragments varying in size to blocks of approximately 12 in. in diameter. The sandstone, characteristic of a considerable area in San Francisco, contained calcium carbonate as veinlets and as filling in the pores, binding the grains together. There was a large percentage of voids in the fill, as placed; 36-in. holes were readily bored through the fill in 1941–1942 and they did not have to be cased. The walls of the boreholes soon became covered with a white efflorescence, explained by the author to be lime (CaO) left by the breakdown of the calcium carbonate of the rock, releasing carbon dioxide. The carbon dioxide was absorbed by the circulating water to form carbonic acid, which further attacked the minerals making up the rock, achieving in time the present compact sandy clay.

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[15] "Sealing the Lagoon Lining at Treasure Island with Salt," by Charles H. Lee, *Transactions*, ASCE, Vol. 106, 1941, p. 579.

[16] Supervising Engr., Recreation and Park Dept., San Francisco, Calif.

In San Francisco, bedrock slopes are steep. The overlying material consists, usually, of interlayered clay and sand due to intermittent accumulations of wind-blown or dune sand over weathered soil surfaces; and ground-water conditions are an important factor in strength and stability. The level areas are principally made by cutting and placing fill. Much of the fill consists of sand derived from leveling the dunes, but the upper thicknesses in many filled-in locations are the rubble left by the 1906 fire, and a miscellany of material. Most of the level areas of the city playground sites involve both cut and fill.

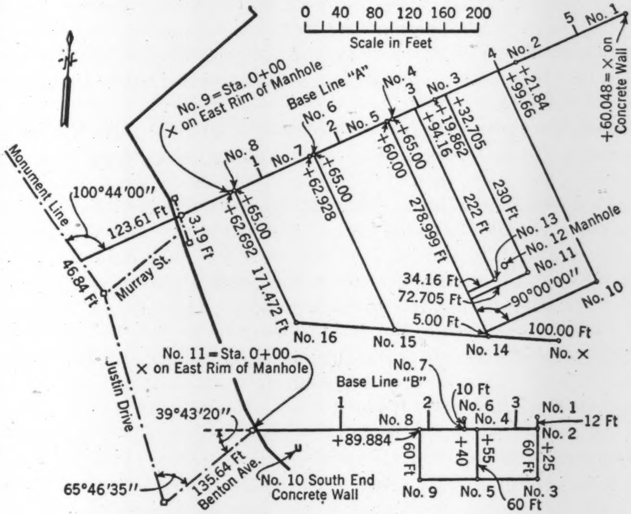
A bond issue of $12,000,000 was voted in November, 1947, to cover improvements on fifty-three separate recreational areas, involving concrete structures on some of these sites costing up to $350,000. On eighteen sites—because of either the proposed structure or the geological and ground conditions—it was decided that a consultant should be employed to investigate and advise the writer and the architect for the commission as to the allowable bearing capacity of the sites.

The St. Mary's site was one of the first investigated, and it was found that compactive settlement, and the change in the fill, had practically ceased (see Table 9). The material had achieved an average density of 108 lb per cu ft, dry, and developed enough shearing strength to support the structures planned. It was apparent therefrom that geochemical change occurring in fills is an important factor in their supporting value as well as in their stability.

The Potrero Hill building site, also in San Francisco, consisted, in part, of a rock fill blasted from the remainder of the site to provide a level area; this was done in 1931. The measured settlement has been slight. Test-pit excavation revealed the fill to be composed of highly siliceous rock of igneous origin, which suffered little, if any, decomposition but some crushing. The voids between the component rocks were still large, and the safe bearing capacity was judged to be 1,750 lb per sq ft, which required the spreading of column footings to cover 21 sq ft of bearing area.

The effects of age, character of material, penetration, and occurrence of water therein were investigated at the remaining sixteen sites. A number of the sites consisted of fills comprised largely of dune sand. Where the fills had been surfaced by impervious materials for tennis and basketball courts, the

TABLE 9.—Subsidence Record, St. Mary's Playground, San Francisco, Calif.


0 40 80 120 160 200
Scale in Feet
No. 9 = Sta. 0+00
× on East Rim of Manhole
Base Line "A"
No. 1
5
No. 2
4
+21.84
+99.66
+60.048 = × on Concrete Wall
No. 4
3
No. 3
+32.705
+19.862
+94.16
No. 6
2
No. 5
+65.00
+60.00
No. 8
1
No. 7
+65.00
+62.928
Monument Line
100°44′00″
123.61 Ft
3.19 Ft
+65.00
+62.692
171.472 Ft
278.999 Ft
222 Ft
230 Ft
No. 13
No. 12 Manhole
No. 11
34.16 Ft
72.705 Ft
90°00′00″
No. 10
5.00 Ft
100.00 Ft
46.84 Ft
Murray St.
No. 16
No. 15
No. 14
No. ×
No. 11 = Sta. 0+00
× on East Rim of Manhole
Justin Drive
No. 7
Base Line "B"
10 Ft
No. 6
No. 4
3
No. 1
12 Ft
1
2
No. 8
No. 2
39°43′20″
+89.884
60 Ft
+40
+55
60 Ft
+25
65°46′35″
135.64 Ft
Benton Ave.
No. 10 South End
Concrete Wall
No. 9
No. 5
No. 3
60 Ft

Point No.	Elevations, in Feet, Observed on:							Movement	
	Aug. 31, 1944	Nov. 21, 1944	Feb. 7, 1945	April 9, 1945	May 15, 1945	June 13, 1945	July 27, 1945	Feet	Direction
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
				(*a*) Base Line A					
1	151.644	151.636	151.642	151.648	151.649	151.644	151.645	0	....
2	150.274	150.293	150.294	150.300	150.300	150.295	150.296	0	....
3	150.538	150.551	150.558	150.558	Stake lost	....	....	....	....
4	150.370	150.377	150.379	150.382	150.384	150.375	150.375	0	....
5	150.230	150.233	150.236	150.244	150.248	150.238	150.235	0.010	North
6	150.554	150.566	150.568	150.571	150.573	150.559	150.555	0	....
7	150.594	150.619	150.622	150.623	150.621	150.605	150.604	0	....
8	150.297	150.302	150.300	150.303	150.307	140.300	150.301	0.005	East
9	154.931	154.931	154.934	154.933	154.933	154.931	154.931	0	....
10	144.866	144.856	144.891	144.902	144.895	144.880	144.872	0.03	South
11	145.479	145.466	145.461	145.470	145.460	145.446	145.441	0.02	South
12	146.344	146.341	146.339	146.347	146.345	146.342	146.342	0	....
13	146.166	146.166	146.160	146.172	146.168	146.162	146.158	0.02 0.01	South East
14	145.229	145.228	145.240	145.248	145.247	145.238	145.227	0.006	South
15	146.189	146.180	146.192	146.199	146.195	146.187	146.181	0.02	South
16	147.430	147.430	147.433	147.439	147.437	147.422	147.419	0.05	South
×	115.958	115.937	115.956	115.962	115.949	115.948	115.936	0.02	South
				(*b*) Base Line B					
1	118.608	118.617	118.609	118.607	118.594	118.587	118.585	0	....
2	118.740	118.752	118.748	118.752	118.737	118.729	118.724	0	....
3	117.824	117.817	117.836	117.837	117.827	117.815	117.797	0.04	South
4	120.817	120.824	120.823	120.821	120.812	120.809	120.798	0	....
5	119.508	119.497	119.503	119.504	119.499	119.491	119.472	0.01	South
6	120.744	120.814	120.824	120.819	120.818	120.743	120.728	0	....
7	121.164	121.179	129.177	121.175	121.166	121.163	121.152	0	....
8	120.936	120.946	120.946	120.948	120.940	120.936	120.928	0	....
9	119.848	119.841	119.879	119.881	119.871	119.855	119.829	0.02	South
10	123.330	123.328	123.324	123.322	123.323	123.328	123.330	0	....
11	128.913	128.917	128.917	128.918	128.918	128.915	128.912	0	....

underlying sand was found to be unchanged and to have characteristics similar to those described by the author (under the heading, "Water: Fort Ord Reservoir in California, Proposed As Balanced Cut and Fill"). The latest sand fill, in point of age, consisted of fresh dune sand whose dry density ranged well under 100 lb per cu ft. Cohesion was slight, the maximum cohesion of the core samples tested being 175 lb per sq ft, and the low value obtained for shearing strength under the existing surcharge and other considerations limited the bearing value to 1,500 lb per sq ft.

Hamilton Playground, however, occupies a sand fill since the sand dunes were leveled and the depressions were filled along Geary Street in this part of San Francisco before the turn of the century. It has long been landscaped and subject to rainfall penetration. The sand is classed as cohesionless, the average value for the core samples tested being 240 lb per sq ft. Streaks of sand in the mass were red and yellow-brown in color, because of the production of minerals through the oxidation and hydration of the original iron minerals of the dune sand, in association with water through rainfall penetration. These minerals produced some cementation, but it did not provide the sand the tenacity of "soft sand rock," described by the author in connection with the Sutro Forest Reservoir. That iron was carried in solution through the sand was shown by the fact that, from 10 ft to 11 ft below the surface of the fill, a piece of coal was brought up in the core sample which had been entirely coated with crystals of iron carbonate—the result of chemical reaction by water-carried minerals in solution with the carbon of the coal. The resulting change in the mineralogical character of the sand was reflected in the bearing capacity of this fill which, based on laboratory tests, was judged at 4,000 lb per sq ft.

HYDE FORBES,[17] M. ASCE.—A broad subject is involved in the consideration of geochemical changes taking place in natural material, with soil development, and during the working and placing of that material. The writer, constrained to brevity, realizes that he has barely "scratched the surface"; he generalized on the basis of his experience and hoped that discussion would add breadth to the present consideration. However, it was not contemplated that the broad fields noted by Mr. Lee, at the close of his opening paragraph, be considered. The writer has relied on his basic knowledge of geology, mineralogy, and chemistry in interpreting the geologic origin and the geochemical development of natural materials, and has used simple, well-known tests as a measure or "yardstick" in applying that interpretation.

In his discussion, Mr. Smith, a geologist, suggests that civil engineers should have more geology in their curriculum. It is also suggested that: "It is not enough to hire a geologist and leave the problem entirely to him"—inferring that the geologist lacks sufficient knowledge to apply geology to engineering problems. In a letter once received from William P. Creager, Hon. M. ASCE, it was stated: "Too many engineers know too little about geology and too many geologists know too little about engineering." That is true, and the field of specialization in engineering geology requires that a student major in civil engineering long enough to obtain a working knowledge of the fundamentals, then he can study geology objectively, not only as a pure science. The writer has previously submitted his proposed curriculum.[18]

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[17] Cons. Engr. and Geologist, Palo Alto, Calif.

Mr. Harman's discussion "points up" the importance of understanding equilibrium relationships in soil technology in interpreting the results of core sampling and the tests made on such samples in foundation investigations. The effect of water penetration, forming solutions, through material placed in fills or natural soils, is to provide an opportunity for the exchange of mineral ions at the suface of each soil particle and ions held in solution by water. This fact is illustrated by Mr. Harman in the reaction between water and coal in a fill. Such exchange leads to a change in the geochemical character and the physical behavior of the material. The soil particles tend to reach chemical equilibrium with the water so that an equilibrium distribution of the ions between the soil and the solution is approached. Secondary concretionary minerals, hydrous hematite and limonite streaking the Hamilton Playground sand mentioned by Mr. Harman, for example, are dehydrated; carbonates of the alkalies are formed, binding the particles of well-drained or dried soil together, thus increasing its resistance to shear, and increasing its supporting value.

Under the heading, "Field Decomposition of Calcium Carbonate," Mr. Krynine presents interesting examples of geochemical change involving calcium compounds. The foundation rock beneath the sand and gravel of the stream bed at the Gibraltar Dam was fresh as a result of scour. It is quite possible that it could have been affected by leaching below an apparently solid crust. However, the wire brushing and the immediate placement of concrete over the bedrock were successful in producing a watertight bond that proved satisfactory for thirty-five years. Both Messrs. Krynine and Lee discuss the changes in specific gravity of particles with working. Soil contains a large number of minerals among other matter so that an exact analysis and computation of specific gravity mathematically would appear impossible. The concept of the writer is a combination of those presented by Mr. Krynine—namely, dehydration of some minerals and base exchange effected through kneading action in the presence of water during the working of the soil. Mr. Lee asks that the method of determining the specific gravity be stated. The fine granular soil particles are made up of many crystalloid minerals; mineral and organic matter, either amorphous or colloidal; and gravel or rock fragments of larger size which are usually combinations of several minerals. Therefore, the average-particle, specific gravity of a soil is all that can be determined. This is most readily done in a specific gravity flask. The Le Chatelier apparatus, used by the writer in his tests, is available on most jobs where concrete batching or testing is being done.

The soils over a large area in Oregon are derived from volcanic sources and it would have added to current knowledge of their behavior had Mr. Smith presented the data obtained as the result of his investigations, which were presumably of a "katamorphic" nature according to his description. Since the preparation of the paper, the writer has had opportunity to investigate kaolinitic, ferruginous, and bentonitic clays derived from volcanic material, as well as pumice, in Oregon. With proper selectivity and treatment, the origin and development of those clays proved to be subject to a development classed as "anamorphic," in Mr. Smith's description. The origin and development may be of interest and the tests thereon answer questions raised by Mr. Lee's discussion. The "borrow" for the impervious clay core of the Toketee Diversion Dam of the California Oregon Power Company, on the North Umpqua River, consisted of a meadow in an oxbow of the river upstream from the site, and subject to overflow. The original overflow deposits consisted of pumice and andesitic sand, silt, and rock flour, with some basaltic sands and boulders. The humic and carbonic acid, picked up by waters passing through the vegetation and top soil, effected carbonization of the alkaline aluminum silicates, thus producing a kaolinitic clay (hydrous aluminum silicate), and released the alkaline minerals, which now occur as carbonates. The moisture content of the clay ranged from 50% to 80% of its dry weight; and it had to be processed through plowing and disk harrowing to dry it out for optimum compaction. When first stripped and plowed, the clay appeared as black clods which, when broken up and dried by the harrows, turned white in color (oxides of the alkalies) and broke into dust-size particles even with a moisture content of 25% by weight. When the material was compacted in field tests, and when it was compacted into the dam, the color changed to a dark gray, the carbonates reforming.

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[18] Discussion by Hyde Forbes of "Geology in Highway Engineering," by Marshall T. Hunting, *Transactions*, ASCE, Vol. 110, 1943, p. 337.

Mr. Lee found that variation of specific gravity within the same borrow area does not reach the extent cited for the Conn Valley Dam. The writer has found that such determinations vary widely from borrow area to borrow area and between test boreholes within the same borrow area as well as between samples from the same test boring; 54 test borings were made at 100-ft centers over the Toketee clay borrow area, and as many as 4 samples from each hole were selected for test. The content of unchanged pumice was relatively high in most holes and the variation in the specific gravity ranged from 2.169 to 2.45 in all samples. In a single hole, the specific gravity of the samples varied from 2.245 to 2.45. The average specific gravity of the borrow-pit samples was 2.37. The samples cut from the dam varied in specific gravity from 2.30 to 2.70, the average being 2.49. In the field compaction tests, and with rolling into the dam, the carbonates were regenerated and had the effect of cementing and hardening the clay. Tested at 20% moisture content, the compacted dry weight was 95 lb per cu ft. Cohesion was found to be 2,750 lb per sq ft, and the shearing strength under a surcharge of 5,000 lb per sq ft was 8,000 lb per sq ft. Practical considerations in working made it impossible to achieve optimum density which occurred when tested at 105 lb per cu ft (dry weight) compacted with 16% moisture content. The gray clay was found to be impervious and able to stand in a vertical face when the clay core was cut through to its foundation for installation of permanent closure.

The bright red clay at the site of Fish Creek Diversion Dam and Forebay (on a tributary of the North Umpqua River) consisted principally of hydrous hematite derived from the decomposition of basalt. The hematite blankets the basalt to depths of as much as 35 ft and in turn is blanketed by pumice ash. Similar variation in specific gravity determinations between borrow-area samples was found here, averaging about 2.46. The determinations made on samples cut after rolling showed an increase varying from 2.50 to 2.85 in specific gravity. Both the kaolinitic and ferruginous clays made excellent bonds with concrete and abutment rock. The Willow Creek Dam of the City of Medford, Ore., lies within a volcanic watershed and the reservoir site is the borrow area. The dam is to be built in 1951 but the preliminary tests on the available construction material show results consistent with the previous experience of the writer—namely, that there is an increase in specific gravity, with working, as minerals of higher atomic weight are formed under treatment and under a construction control designed to meet the mineralogical character of the material used.

Mr. Lee suggests more refinement in the determination of grain distribution than the sieve sizing used by the writer. The acid and other treatment suggested by Mr. Lee seems to defeat the purpose because it gives ample opportunity for the breakdown of individual grains and fragments. The writer has learned to keep test procedure as simple as possible, but always to conduct the tests in the same manner so that the results will be comparable. The material in question (natural material including soil) is extremely variable, and there is a large human equation for all test procedure devised in soil mechanics. Therefore, absolute determinations are not warranted and comparative results will prove to be ample. In the matter of compaction all test results are comparative. The maximum dry density which it is possible to achieve under optimum moisture varies greatly. The writer terms the absolute density as the percentage of the weight of materials in solid dry mass. That definition necessitates the determination of the specific gravity of the particles making up the mass and gives the percentage voids by volume. For example, the compacted dry density of some samples obtained at Toketee Dam was as low as 61% of absolute, which gives 39% voids. Such a voids content in granular material of larger size would mean permeability, but the grain-size distribution curves of the dam samples showed an average of more than 50% smaller than 1.0 mm and a clay content determined by elutriation of more than 30%. These tests are made quickly and at little cost; they are most informative during, as well as before, the construction period. The writer had adapted the same type of simple tests to foundation and stability investigations, not as a substitute for an understanding of the mineralogical and geological nature of the material and the processes through which it has emerged, but as an added tool in exercising geological judgment.

The profound geochemical change in the rock mass making up the hill area at the San Francisco Naval Shipyard, described by Mr. Gorman, not only had a considerable economic effect on the works constructed at that location, but has added greatly to the costs of other public works in San Francisco. The peridotite intruded originally into massive sandstone and chert beds, to be uncovered through erosion from Hunter's Point northerly across the city to Fort Point, the south end of the Golden Gate Bridge. The swell of the peridotite in serpentinization exerted enormous pressures so that the sandstone and chert bordering the mass are found completely crushed, with fracturing extending outward from that area. Such crushed zones become clay with the penetration of water, and are unstable, subject to slide, when disturbed through excavation (as at the

Parker Avenue Landslide at Lone Mountain).[19] In addition, the overlying sandstone formation has been deformed and subjected to overthrust. The overthrust extends many miles in a north-south direction through the built-up area of the city. The entire mass involved in the overthrust is cylindrical in section, one border making up the top and east face of Telegraph Hill and the other border making up the top and west slope of Russian and Nob Hills. Examples of the horizontal shear planes along which movement occurred at many levels in the mass can be seen near the top of the massive sandstone making up the east face of Telegraph Hill. Those having an opposite dip can be seen in cuts on the ridge of Russian and Nob hills. The excavation for the North Point Sewage Treatment Plant at the toe of the north slope of Telegraph Hill resulted in moving ground as the sheared, thoroughly slickensided rock of the zones of shearing raveled out and the horizontal sandstone masses were not able to cantilever over, unsupported. The results of the diamond drilling along Broadway indicate that the vehicle tunnel under that street will encounter horizontal shear zones of the westerly limit of the overthrust.

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[19] "Landslide Investigation and Correction," by Hyde Forbes, *Transactions*, ASCE, Vol. 112, 1947, p. 381.

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Paper No. 2443

FLOOD-CONTROL OPERATION OF TENNESSEE VALLEY AUTHORITY RESERVOIRS

By Edward J. Rutter,[1] M. ASCE

With Discussion by Messrs. Ray K. Linsley, Jr., and Edward J. Rutter.

Synopsis

The reservoir system of the Tennessee Valley Authority (TVA), in Kentucky, Tennessee, North Carolina, Alabama, and Georgia, has become of increasing effectiveness in reducing flood stages since Norris Dam was closed in March, 1936. With the closure of Fontana Dam on the Little Tennessee River in 1944 and Kentucky Dam near the mouth of the Tennessee River in 1945, the reservoir system for flood control on the Tennessee River was practically completed. The first real tests of operation for flood reduction occurred in January, 1946, January, 1947, and February, 1948, when severe storms passed over the basin. These storms would have produced the fifth, sixth, and seventh highest natural floods, respectively, at Chattanooga, Tenn., if the reservoir system had not regulated the crest stages; and the floods of February, 1948, and January, 1946, would have been the fifth and the eleventh highest, respectively, on the lower Tennessee River in the vicinity of Kentucky Dam.

This paper describes the actual flood conditions and actual and alternative hypothetical operations of the tributary and main-river reservoirs during these floods. Reductions in peak stage and discharge at each dam and at critical points are tabulated, and comparisons are made between the actual crest reductions and possible reductions if other methods of operation had been followed.

Another flood on the Ohio and Mississippi rivers occurred in April, 1948, after the major flood season in the Tennessee Basin and when the TVA reservoirs were being filled to maximum normal levels. The effect of the reservoirs on this flood is also discussed.

Introduction

As a result of heavy rainfall occurring in the 3-day period from January 6 to January 8, 1946, serious flood conditions were produced along the Tennessee

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[1] Hydr. Engr., TVA, Knoxville, Tenn.

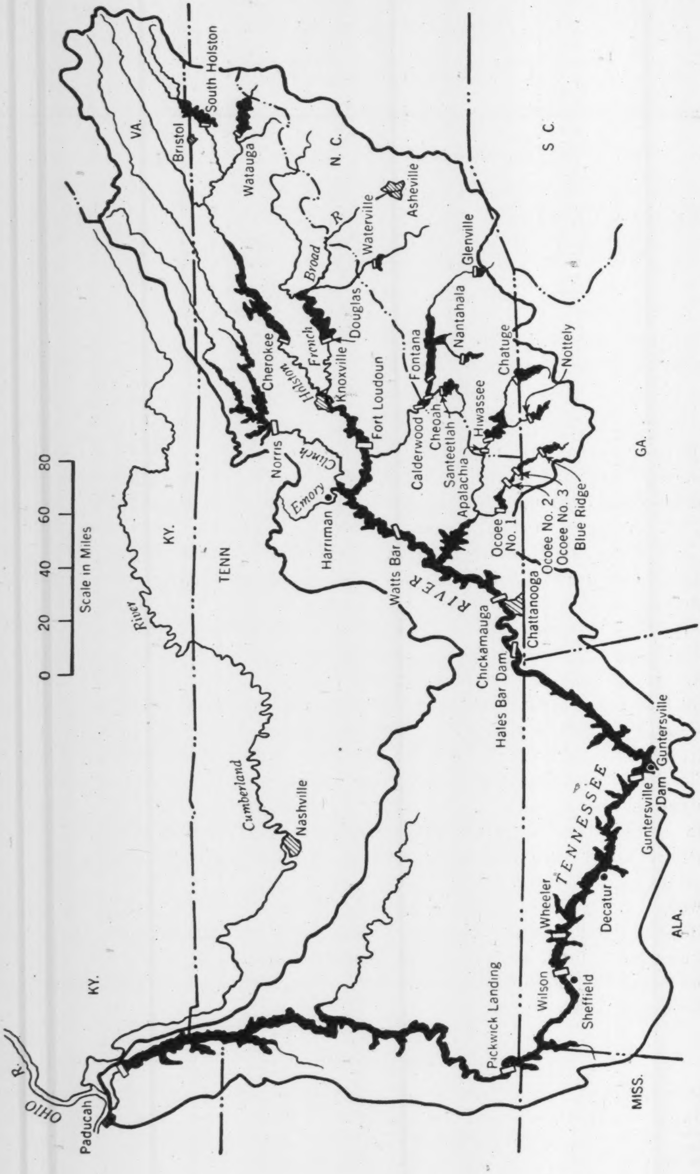
Scale in Miles
0
20
40
60
80
VA.
KY.
TENN
N. C.
S C.
GA.
ALA.
MISS.
KY.
Bristol
South Holston
Watauga
Waterville
Asheville
Broad R
French
Holston
Clinch
Emory
Douglas
Knoxville
Cherokee
Norris
Harriman
Fort Loudoun
Fontana
Nantahala
Glenville
Chatuge
Nottely
Calderwood
Cheoah
Santeetlah
Hiwassee
Apalachia
Ocoee No. 1
Ocoee No. 2
Ocoee No. 3
Blue Ridge
Watts Bar
RIVER
Chattanooga
Chickamauga
Hales Bar Dam
Guntersville
Guntersville Dam
TENNESSEE
Decatur
Wheeler
Wilson
Sheffield
Pickwick Landing
River
Cumberland
Nashville
OHIO
R.
Paducah

FIG. 1.—THE TENNESSEE RIVER SYSTEM

River and on the lower Ohio River at Cairo, Ill., about 30 miles west of Paducah, Ky. (see Fig. 1). In the following year (1947) another storm occurred in the 8-day period from January 14 to January 21, also producing serious flood conditions along the Tennessee River. Because the 1947 storm was relatively lighter on the lower Tennessee River Basin, that flood did not reach the same proportions on the lower Tennessee and Ohio rivers as did the 1946 flood. In February, 1948, for the third consecutive year, a storm occurred which would have caused damaging stages at Chattanooga and which would have produced record or near-record floods on some tributaries. Studies of the operation of the TVA reservoir system during these three floods show that peak stages were affected beneficially and that substantial savings in flood damages were obtained.

RAINFALL

1946 Storm.—The rainfall (see Fig. 2) producing the 1946 flood occurred in two periods, the first lasting for about 3 days, and the second for about 1 day. In the first storm the average rain falling on the area below Chattanooga amounted to 5.62 in. and on the area above Chattanooga it amounted to 3.93 in. The highest average amount in one day was 2.61 in. below Chattanooga and 2.18 in. above Chattanooga. Minor amounts of rain fell later during the flood but these had little effect on the peak rates of flow. Between the two storms, additional rainfall was predicted which did not develop, particularly over the eastern section, and the predictions influenced the operation of the main-river reservoirs by requiring that a greater part of the available storage should be reserved.

1947 Storm.—The rainfall producing the 1947 flood was not as intense as that in 1946, the highest average amount falling over the area above Chattanooga in one day being 1.93 in. Below Chattanooga the greatest rainfall in one day was 1.09 in. The 1947 storm, however, extended over a longer time, and the total rainfall above Chattanooga was greater than it was in 1946. Rainfall was reported for 8 days continuously from January 14 to January 21, 1947, but the amounts reported on the first and last days were insignificant. The total average amount above Chattanooga was 6.07 in., and below Chattanooga it was 4.13 in. Daily rainfall in the 1947 storm also is shown in Fig. 2.

1948 Storm.—From February 5 to February 10, 1948, precipitation averaging less than 1 in. per day occurred over the Tennessee River Basin. On February 11, additional rainfall was predicted of generally less than 1 in. On the morning of February 12 greater amounts occurred than had been predicted, ranging to as much as 3 in. at some points. Succeeding forecasts indicated more rain and the actual rainfall was closer to the forecasts than on February 11. On the morning of February 13, as much as 4 in. or 5 in. was reported for the preceding 24 hours, and on the morning of February 14 amounts up to 2 in. or 2.5 in. were reported. The rainfall was generally heavier over the area below Chattanooga than on the area above Chattanooga, and also was much heavier below the tributary dams than above them. The Emory River Basin, with an area of 865 sq miles on which there is no reservoir control, received more than 7 in. in 3 days at some points and was a large contributor to the crest at Chattanooga.

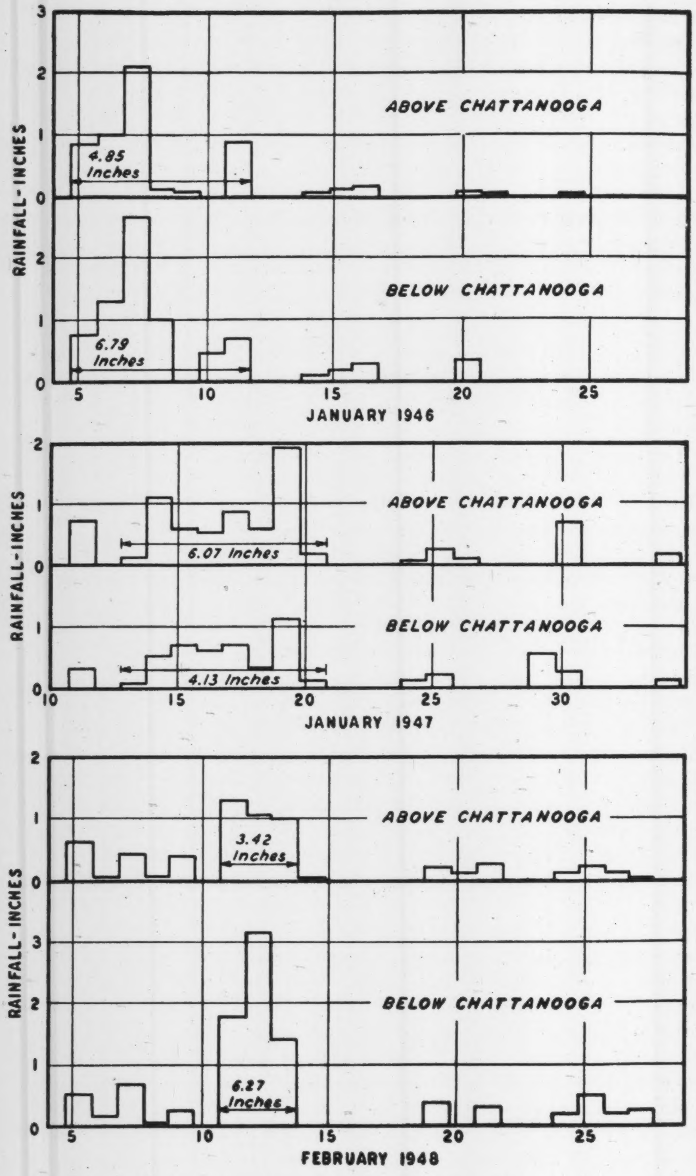

FIG. 2.—AVERAGE RAINFALL, TENNESSEE RIVER BASIN

The maximum daily rainfall above Chattanooga was 1.32 in. on February 12, and below Chattanooga it was 3.11 in. on February 13. The maximum point rainfall for a 24-hour period was 7.13 in. at Sugar Hill in the western part of the basin on February 12–February 13. Total rainfall for the 3-day period was 3.42 in. above Chattanooga and 6.27 in. below Chattanooga. Although the average above Chattanooga was only 3.42 in. in 3 days, some areas—notably the Emory Basin—received amounts comparable to those received on the lower basin.

Differences in the areal distribution of the storms above Chattanooga are shown by the amount of inflow stored in the tributary reservoirs (see subsequently in Table 3). The operation of the reservoirs was about the same during each flood—that is, small releases for power generation were made as required, but the relative amount stored in Cherokee and Fontana reservoirs was substantially more in the 1947 flood than in the 1946 flood, and substantially less was stored in Norris and Douglas reservoirs. The 1946 storm, however, was more intense over the area between Chickamauga and Watts Bar dams, more than 5 in. being reported for 1 day at several stations.

TRIBUTARY RESERVOIR OPERATION

The function of the tributary reservoirs in a flood-control operation is to retain that part of their inflow which would contribute to the flood at Chattanooga, and to release the stored water as soon as practicable after the flood. In great floods it will not be possible to retain all inflow. Consequently, some discharge will be made in the early part of each flood so that, in case the flood should develop into an extreme one, all storage capacity will not have been filled before the peak inflow occurs. The amount of this early discharge will vary with the reservoir stage. Flood-storage capacity reserved on March 15 at seven tributary reservoirs is given in Table 1.

TABLE 1.—FLOOD-STORAGE CAPACITY ON SOME TENNESSEE VALLEY AUTHORITY RESERVOIRS AS OF MARCH 15

Reservoir (see Fig. 1)	Drainage area, in square miles	FLOOD-STORAGE RESERVATION	
		In acre-feet	In inches
Cherokee	3,429	807,200	4.41
Douglas	4,541	1,019,800	4.21
Fontana	1,571	581,800	6.94
Norris	2,912	1,377,000	8.87
Hiwassee	968	245,100	8.13
Chatuge	189	75,100	7.45
Nottely	214	83,500	7.32

1946 Flood.—During the month of December, 1945, tributary reservoirs were drawn to normal flood-season levels for January 1, by substantial releases through the sluice gates. At the beginning of the 1946 storm on January 5, all tributary reservoirs were at or below the normal level for that date. During the flood, releases of water were limited to power production until after the peak stage was reached at Chattanooga (about January 10) when additional water was released through the sluices to reduce the reservoirs to normal levels. These levels were reached about 15 days or 20 days after the beginning of the flood.

Normal level is defined in this paper as the maximum level to which the reservoir may rise except during a flood-control operation. This level varies with the season of the year. For the tributary reservoirs it increases from January 1 to March 15 and then rises more rapidly to April 1. After April 1 the reservoirs may be filled to within a few feet of the top of spillway gates if there is sufficient stream flow, but they must be lowered during the summer and fall to the level for January 1.

1947 Flood.—In the fall of 1946 stream flow was below normal and the demand for power production was high. The tributary reservoirs, therefore, were reduced to levels substantially below normal for January 1, except in the case of Douglas Reservoir which was drawn to normal level by sluice gate releases. A small amount of runoff resulted from a storm on January 1, 1947, and the reservoirs were partly filled to their normal levels. At the beginning of the flood on January 14, 1947, the levels were still below normal, except that of Douglas Reservoir which was 2.7 ft above normal. Releases were limited to requirements for power production until January 23, flood danger at Chattanooga then being over, when additional releases were made through the sluices to return the reservoirs to normal levels. As in the 1946 flood, these levels were reached from 15 days to 20 days after the beginning of the flood.

1948 Flood.—At the beginning of February, 1948, the tributary reservoirs were substantially below their normal flood-season levels, having been drawn down in the preceding dry season for power generation. Some recovery in these reservoirs was possible from the runoff due to the rainfall on February 5 to February 10; but even on February 11, at the beginning of the heavy rainfall, Cherokee Reservoir was about 28 ft below normal; Norris Reservoir, about 30 ft; Fontana Reservoir, about 66 ft; Douglas Reservoir, 3 ft; Hiwassee Reservoir, 29 ft; Chatuge Reservoir, 22 ft; and Nottely Reservoir, 38 ft.

During the flood the entire inflow into the tributary reservoirs was stored except for a small amount used for power generation. The amount so released was only about 6% of the water passing Chattanooga at the flood crest. Much greater inflows could have been stored in the tributary reservoirs because, with the exception of Douglas Reservoir, by the end of the flood the water level had not risen to the normal flood-season level. Douglas Reservoir rose about 14 ft above its normal level but was lowered after the flood at discharge rates up to 19,500 cu ft per sec and by March 4 was returned to its normal level. The other tributary reservoirs continued to rise slowly after the flood. Table 2 shows the normal and actual headwater elevations for the three floods, the maximum level reached during each flood, the amount stored, and the remaining available storage. Storages are given in acre-feet and in inches over the tributary drainage area. The flood storage available at the beginning of each flood is the sum of the maximum amount stored and the remaining available storage shown in Table 2. Fig. 3 shows the actual elevations and discharges of the tributary reservoirs in the 1946 flood. Operations in the floods of 1947 and 1948 followed a similar pattern and are not shown.

These tributary water-control operations eliminated damage to the areas immediately below the dams, and at the end of the flood there was from two times to four times as much storage space remaining as had been filled during the flood.

The quantities of water actually stored, shown in Table 2, are maximum amounts but these amounts were not completely effective in every case in reducing the flood peak at Chattanooga. From January 8 to January 14, 1946, inclusive, and from January 17 to January 26, 1947, the actual discharges at Chattanooga were lower than the computed natural discharges. During the flood period from February 13 to February 19, 1948, the actual Chattanooga discharge was less than the natural discharge. It was also less than the natural discharge before and after these dates, but this 7-day period represents the time when tributary inflow should have been retained for the benefit of Chattanooga. The amounts stored in the tributary reservoirs during these periods, allowing for a time of travel to Chattanooga, represent more correctly the relative control afforded by each of the reservoirs.

TABLE 2.—TRIBUTARY STAGES FOR SOME TENNESSEE VALLEY AUTHORITY RESERVOIRS

Reservoir (Fig. 1)	Normal elevation[a]	Actual elevation[a]	Maximum elevation reached	MAXIMUM AMOUNT STORED		REMAINING STORAGE[b]	
				1,000 acre-ft	Inches	1,000 acre-ft	Inches
(a) 1946 FLOOD							
Cherokee	1,021.8	1,021.1	1,037.5	245	1.34	888	4.86
Douglas	936.9	935.3	964.9	408	1.69	901	3.72
Fontana	1,617.4	1,615.9	1,637.5	138	1.65	628	7.50
Norris	979.0	978.4	995.3	377	2.43	1,249	8.05
Chatuge	1,910.5	1,909.1	1,914.3	26	2.53	77	7.65
Nottely	1,744.0	1,749.4	1,760.5	28	2.48	64	5.61
Hiwassee	1,456.4	1,456.1	1,468.9	34	1.13	254	8.44
Total	....	....	....	1,256	1.75	4,061	5.69
(b) 1947 FLOOD							
Cherokee	1,024.0	998.6	1,036.4	451	2.47	907	4.95
Douglas	939.2	941.9	972.6	488	2.02	752	3.11
Fontana	1,620.3	1,589.2	1,639.4	296	3.54	614	7.34
Norris	980.2	969.5	993.2	491	3.17	1,300	8.38
Chatuge	1,911.1	1,904.0	1,911.8	35	3.48	97	9.63
Nottely	1,745.2	1,737.1	1,752.3	31	2.72	90	7.89
Hiwassee	1,458.1	1,439.8	1,479.2	102	3.38	222	7.37
Total	....	....	....	1,894	2.65	3,982	5.57
(c) 1948 FLOOD							
Cherokee	1,032.5	1,005.0	1,029.8	292	1.60	1,013	5.55
Douglas	948.0	945.0	964.6	296	1.23	907	3.76
Fontana	1,631.5	1,565.0	1,598.5	154	1.84	863	10.30
Norris	984.8	955.0	982.5	469	3.02	1,543	9.97
Chatuge	1,913.4	1,891.5	1,898.0	20	1.98	153	15.20
Nottely	1,749.8	1,711.5	1,728.2	20	1.76	135	11.82
Hiwassee	1,464.6	1,435.6	1,456.7	45	1.49	287	9.55
Total	....	....	....	1,296	1.81	4,901	6.85

[a] At noon on January 6, 1946, January 14, 1947, and February 11, 1948. [b] Between the maximum elevation reached during the flood and the maximum controlled elevation.

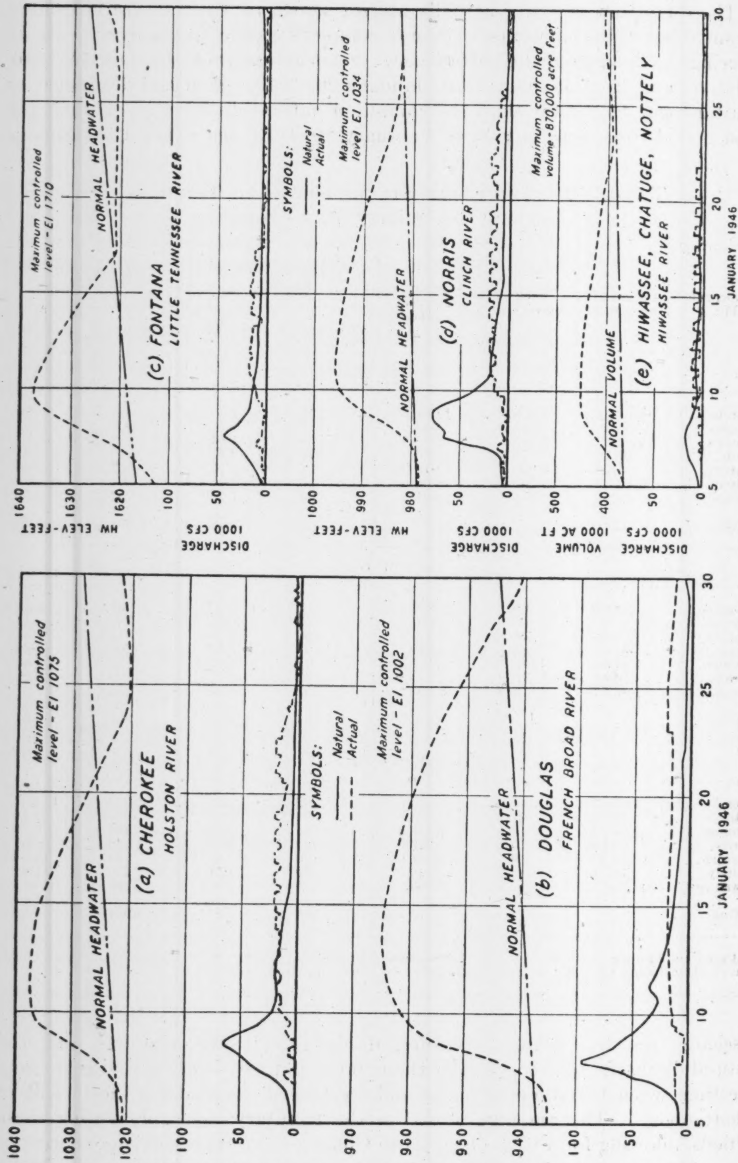

FIG. 3.—OPERATION OF TRIBUTARY RESERVOIRS; 1946 FLOOD

Table 3 shows the amount of inflow and outflow, storage in the reservoirs, the percentage of the inflow stored at each reservoir, the amount of water stored in each reservoir expressed as a percentage of the total amount stored in all reservoirs, and the area above each reservoir expressed as a percentage of the total controlled drainage area. These amounts are for the flood periods given previously; but, if a shorter period near the crest of the flood were assumed, a greater proportion of tributary inflow would be effective because at the end of the selected periods releases to return the reservoirs to their normal levels had been begun. For example, in the 1947 flood if a period corresponding to the 3-day period of January 20–January 22 at Chattanooga had been assumed, more than 83% of the inflow would have been stored, compared with 75.5% for the longer period.

TABLE 3.—EFFECTIVE TRIBUTARY STORAGE FOR SOME TENNESSEE VALLEY AUTHORITY RESERVOIRS

Reservoir	Assumed time of travel to Chattanooga, Tenn., in days	THOUSAND ACRE-FEET			STORAGE		Percentage of controlled drainage area
		Inflow	Outflow	Storage	Percentage of inflow	Percentage of total	
			(a) 1946 FLOOD				
Cherokee	3	354	100	254	72.0	21.3	25.6
Douglas	3	530	142	388	73.3	32.6	33.9
Fontana	2	250	136	114	45.5	9.6	11.7
Norris	2	510	134	376	73.7	31.5	21.6
Hiwassee	1	133	73	60[a]	45.1	5.0	7.2
Total		1,777	585	1,192	67.2	100.0	100.0
			(b) 1947 FLOOD				
Cherokee	3	557	106	451	81.0	25.0	25.6
Douglas	3	642	154	488	76.0	27.1	33.9
Fontana	2	358	76	282	78.8	15.6	11.7
Norris	2	637	169	468	73.4	25.9	21.6
Hiwassee	1	197	82	115[a]	58.4	6.4	7.2
Total		2,391	587	1,804	75.5	100.0	100.0
			(c) 1948 FLOOD				
Cherokee	3	348	47	301	86.5	22.8	25.6
Douglas	3	395	91	304	77.0	23.0	33.9
Fontana	2	186	32	154	82.8	11.6	11.7
Norris	2	502	25	477	95.0	36.1	21.6
Hiwassee	1	108	22	86[a]	79.6	6.5	7.2
Total		1,539	217	1,322	86.0	100.0	100.0

[a] Includes storage in Chatuge and Nottely reservoirs.

The storage of 86% of the inflow during the 1948 flood represents a high degree of control; and it compares with 67% stored in the January, 1946, flood and with 75% stored in the January, 1947, flood. Such a degree of control was possible because of the low reservoir stages at the beginning of the flood and will not necessarily be obtained in all future floods.

Table 3 also shows that in all three floods only Norris Reservoir stored water above its drainage area ratio. Fontana Reservoir in 1947 was the only other instance when more was stored with respect to its drainage area.

TENNESSEE MAIN-RIVER RESERVOIR OPERATION

The function of the main-river reservoirs in relation to a flood-control operation is to hold the headwater level at its minimum, or to lower it to that minimum in the early part of the flood and then to fill the reservoir during the period of peak inflow. The rate and degree of filling will depend on stream flow and rainfall predictions. As in the case of tributary reservoirs, the stored water is to be released as soon as practicable after the flood, considering flood conditions on the Tennessee, Ohio, and Mississippi rivers. The main-river reservoirs above Chattanooga are operated with that city as the primary consideration. Those below Chattanooga are operated primarily for reducing flood crests downstream from each dam, and for regulating the flow on the lower Ohio and Mississippi rivers.

The nominal flat-pool flood storage during the winter months in eight main-river reservoirs is as given in Table 4.

TABLE 4.—FLOOD-STORAGE RESERVATION ON SOME TENNESSEE VALLEY AUTHORITY RESERVOIRS DURING THE WINTER MONTHS

Reservoir	DRAINAGE AREA, IN SQUARE MILES		FLOOD-STORAGE RESERVATION	
	Each	Between dams	Acre-feet	Inches
Fort Loudoun	9,550	1,580	109,300	1.30
Watts Bar	17,310	3,277	377,600	2.16
Chickamauga	20,790	2,512	329,400	2.46
Guntersville	24,450	3,660	162,900	0.83
Wheeler	29,590	5,140	347,500	1.27
Wilson	30,750	1,160	52,500	0.85
Pickwick	32,820	2,070	418,400	3.79
Kentucky	40,200	7,380	4,010,800	10.19

In all studies of operation of these reservoirs, storage under a sloped profile was used. During a flood this may be substantially different from the flat-pool storage in Table 4.

1946 Flood.—At the beginning of the 1946 flood all main-river reservoirs were slightly above normal level, but within the range permitted. Normal level in the main-river reservoirs is constant from January 1 to March 15 or March 31, having been established by navigation requirements. After March 15 (or March 31) the level increases to within a few feet of the top of the gates, but later in the year it must be lowered to the level for January 1. During the 1946 flood no attempt was made to lower the three reservoirs above Chattanooga (Fort Loudoun, Watts Bar, and Chickamauga) to the minimum levels, and only part of the available storage space was filled after the flood commenced. Because of the continued prediction of additional rainfall, these reservoirs were held ready for storage of any additional inflow that might have developed. Operation of these reservoirs aided in reducing the peak at Chattanooga; but, because they were drawn down to initial levels soon after the peak, the reservoirs did not reduce the flood volume during the 7-day period from January 8 to January 14.

Guntersville headwater level at the dam fluctuated between El. 593 and El. 594 throughout the 1946 flood period. Because of lack of excess spillway capacity at low levels, Wheeler Reservoir rose about a foot before controlled storage was needed for peak reduction. The maximum headwater elevation reached was slightly above El. 554. At Pickwick Reservoir about 5 ft of storage space was used, and the discharge was increased to about 350,000 cu ft per sec. This reservoir was then lowered to discharge the previously stored water before the Ohio River crest arrived. When storage for Cairo peak reduction was required, Pickwick Reservoir was again filled to about El. 414.

On January 5, 1946, Kentucky Reservoir headwater was at its minimum navigation level for low flows. On January 6, when it was recognized that a substantial storm and a flood were developing, it was possible to lower Kentucky Reservoir at the dam without interfering with navigation requirements at upstream points. The release was increased from about 86,000 cu ft per sec to about 200,000 cu ft per sec. Beginning on January 8, when it became apparent that a 35-ft stage at Cairo would be exceeded and that much higher stages were possible, the discharge was further increased to about 400,000 cu ft per sec, in an attempt to draw the reservoir down to El. 346 at the rate of 1 ft per day. The rate of drawdown of 1 ft per day was not maintained, however, because of the rapid increase in inflow. The minimum elevation, El. 346, was not reached because of the necessity for storing to reduce the Cairo peak stage. Beginning on January 13, the discharge was decreased to about 264,000 cu ft per sec over a period of 3 days and then held uniform at that rate for about a week. The maximum reservoir level was El. 359.3 on January 19.

Actual discharges and reservoir elevations at the main-river dams are shown in Figs. 4 to 8.

1947 Flood.—Operation of the main-river reservoirs in the 1947 flood was generally similar to that in the 1946 flood. At the beginning of the 1947 flood (January 14) the reservoirs were at, or slightly above, their normal levels. The three reservoirs above Chattanooga were allowed to rise slightly above those levels until January 17 or January 18 when, in view of the continual rain and predictions of rain, drawdown to approximate normal levels was commenced. On January 20, the intense rain required the use of some of the available storage to prevent damaging stages at Chattanooga. However, a substantial volume of the available storage was held in reserve for possible additional runoff.

Guntersville Reservoir was allowed to rise slightly above normal level (El. 593) from January 14 to January 18, 1947; then it was drawn down to below El. 593 on January 19; and finally it was filled to the maximum level by January 22, during the period of high inflow. Wheeler Reservoir rose a small distance above normal level (El. 550) until January 20 and then, by holding a steady discharge, the reservoir rose to El. 554.2 on January 23. Pickwick Reservoir was drawn down from El. 410 on January 13 to El. 407.7 on January 20 and then filled to El. 410 on January 23. Complete use of the available storage was not required because of the low stages at Cairo.

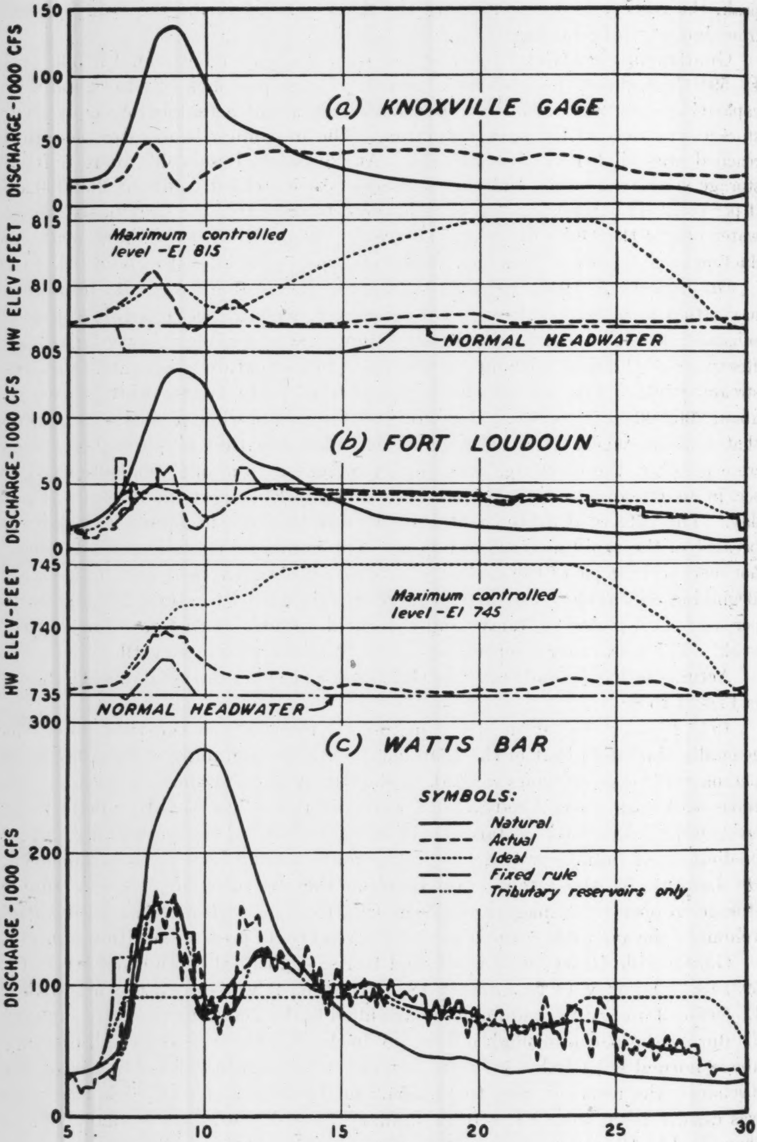

FIG. 4.—KNOXVILLE (TENN.) GAGE, AND OPERATION OF FORT LOUDOUN AND WATTS BAR RESERVOIRS IN TENNESSEE; 1946 FLOOD

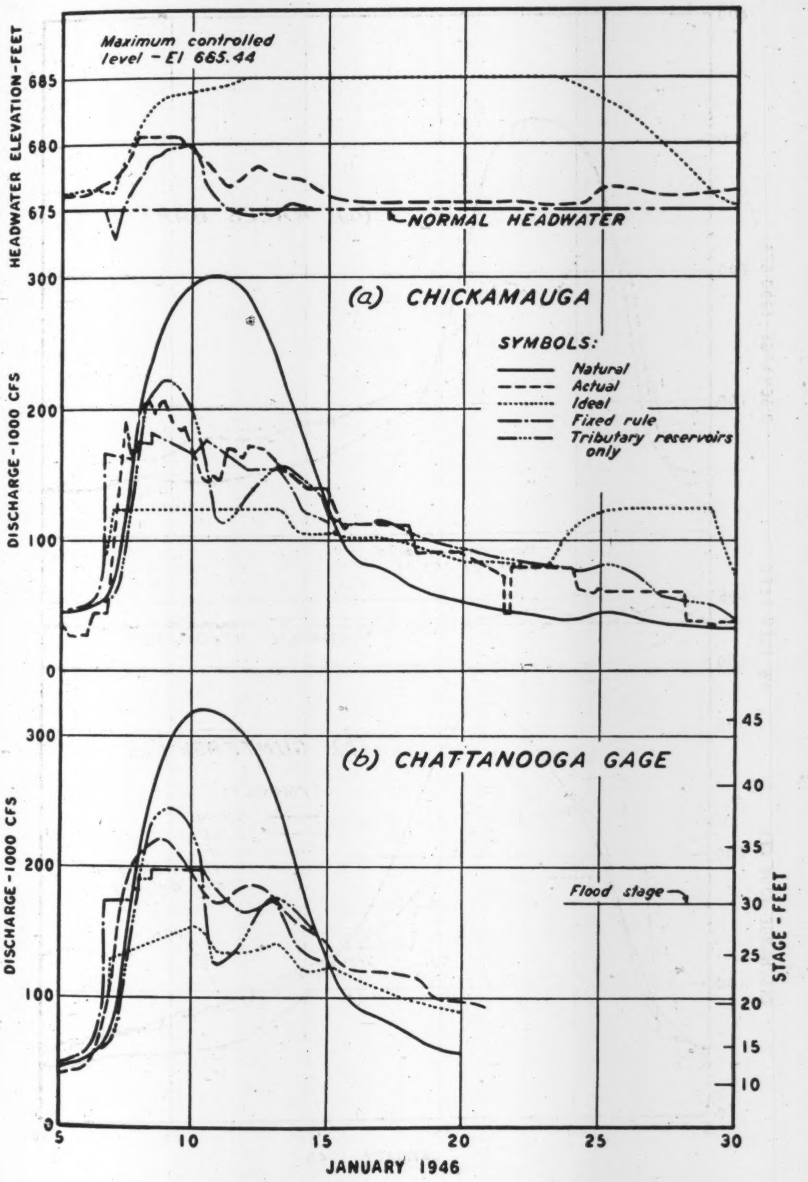

FIG. 5.—CHATTANOOGA (TENN.) GAGE, AND OPERATION OF CHICKAMAUGA RESERVOIR IN TENNESSEE; 1946 FLOOD

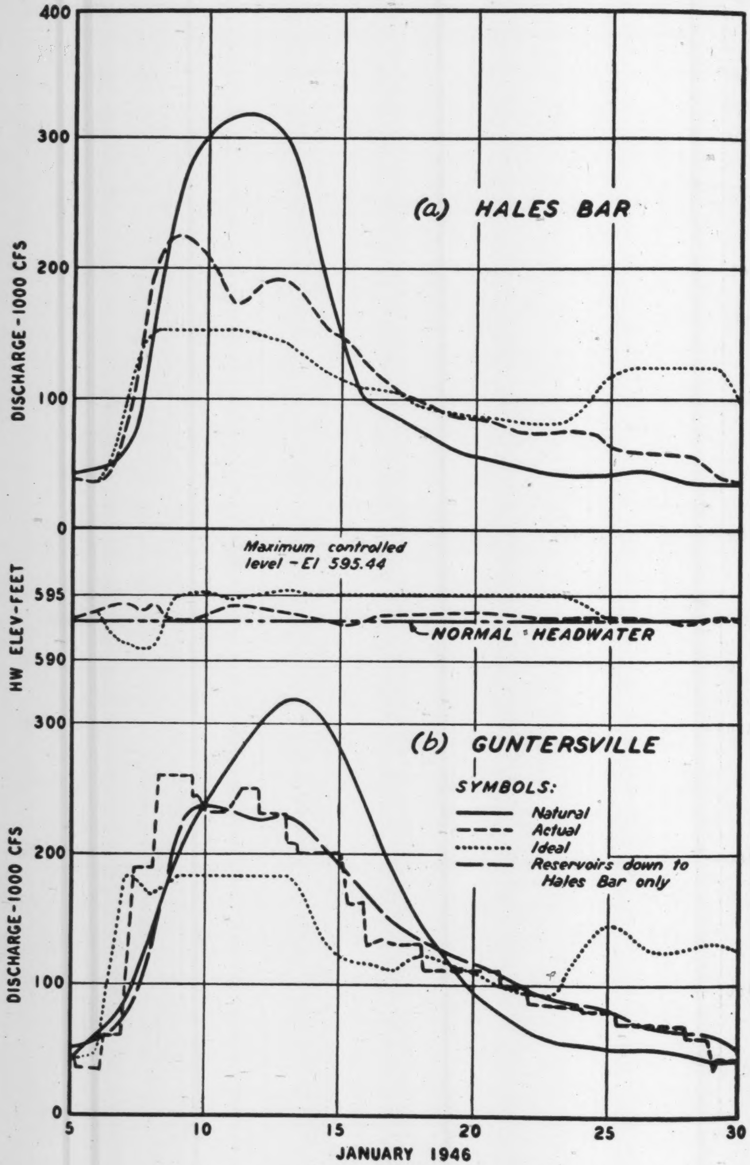

Fig. 6.—Operation of Hales Bar Reservoir in Tennessee and Guntersville Reservoir in Alabama; 1946 Flood

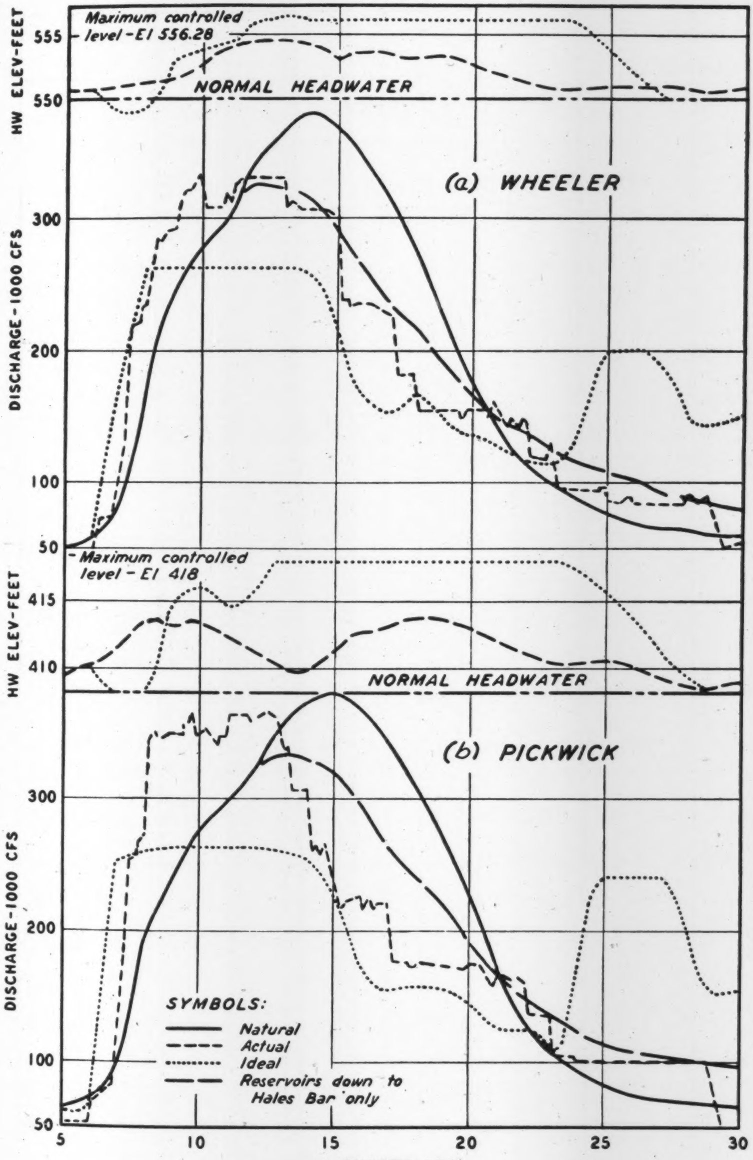

FIG. 7.—OPERATION OF WHEELER RESERVOIR IN ALABAMA AND PICKWICK RESERVOIR IN TENNESSEE; 1946 FLOOD

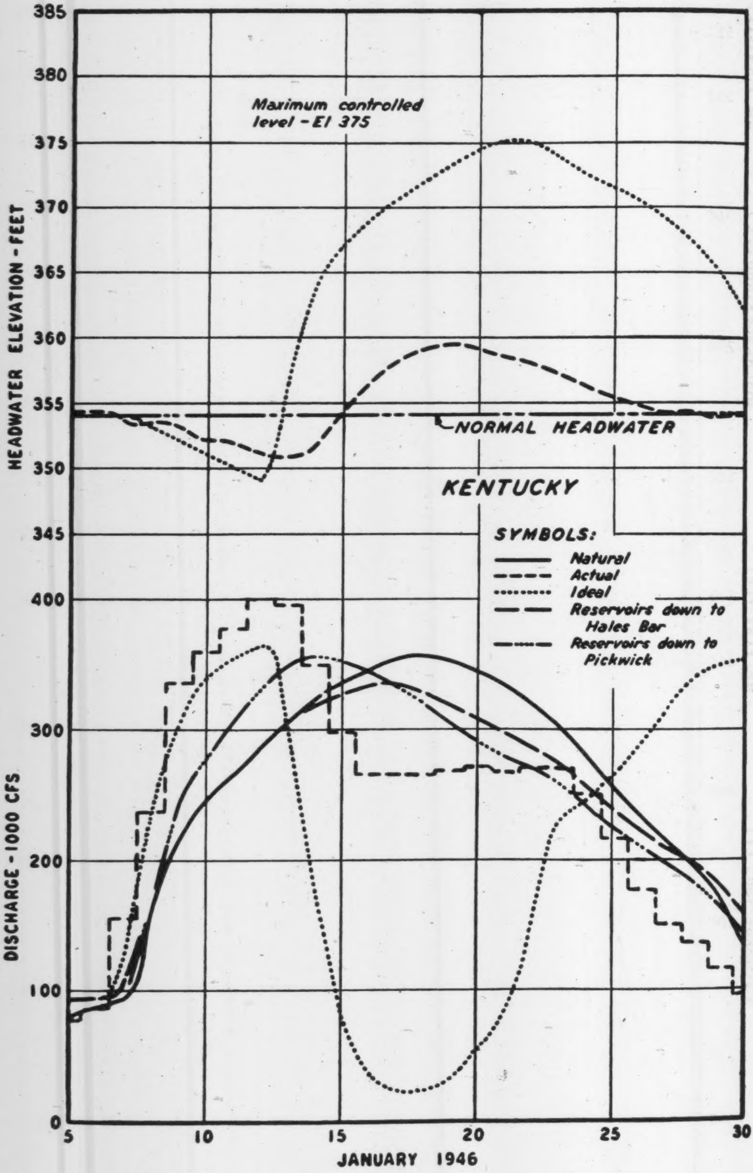

Fig. 8.—Operation of Kentucky Reservoir in Kentucky; 1946 Flood

Kentucky Reservoir was held at about normal elevation until January 18, 1947. On this date the Cairo stage stood at about 30 ft and drawdown of the reservoir was begun. In the 1946 flood, drawdown was started when the Cairo stage reached 35 ft, but it was found that drawdown should have been begun earlier to accomplish complete drawdown to El. 346. In the 1947 flood drawdown was begun sooner, therefore, in the attempt to reach the minimum level. On January 22, it was believed that the Cairo stage would not approach serious proportions and drawdown was stopped. Thereafter, Kentucky Reservoir was refilled to its normal level, El. 354, by February 3.

1948 Flood.—The main-river reservoirs were all at or close to their normal flood-season levels on February 5. Some slight filling then occurred because of the rainfall from February 5 to February 10. By February 11, however, these reservoirs had been returned to within 1 ft of their normal levels.

The three reservoirs above Chattanooga were utilized during the major flood period to control as much of the runoff from the area below the tributary dams as was prudent in view of the continued prediction of rainfall. Fort Loudoun was filled to about El. 810, Watts Bar to El. 742, and Chickamauga to El. 682.5. These reservoirs were promptly returned to their normal levels at gradually decreasing discharge rates after the crest had passed Chattanooga.

The main-river reservoirs below Chattanooga were operated to keep the discharge below each of the dams as low as possible, bearing in mind the possibility of a serious flood on the Ohio and Mississippi rivers. Guntersville and Wheeler reservoirs were almost completely filled, and the combined regulation of all reservoirs lowered the flood-crest discharges and stages below those two dams.

Wilson Reservoir was held at approximately the gate top level (El. 507.88); but, because of the excessive rainfall in that vicinity, it rose above that level and then at the height of the flood was drawn down to slightly below gate top level.

Pickwick Reservoir was practically filled to El. 418 by February 14 but was lowered immediately to be ready for a flood on the lower Ohio and Mississippi rivers.

The headwater at Kentucky Reservoir was lowered from El. 354 on February 13 to less than El. 350 on February 22, and then was raised in anticipation of the crest at Cairo. Stages at Cairo did not reach the point where a full flood-control operation was required at Kentucky Reservoir. Actual discharges and reservoir elevations at the main-river dams are shown in Figs. 4 to 8 for the 1946 flood.

NATURAL DISCHARGE

Natural discharge hydrographs for the three floods were computed at all multiple-purpose dams and at other critical points of flooding. "Natural discharge" means the discharge that would have occurred if these dams had not been built. Natural hydrographs at the smaller tributary reservoirs (Apalachia, Blue Ridge, Ocoee No. 1, Ocoee No. 2, Ocoee No. 3, the Alcoa reservoirs, and other small reservoirs) were not computed, and the effect of these reservoirs was disregarded in computing natural hydrographs at downstream points. The combined effect of these smaller reservoirs might be equivalent to as much as 1 ft on the Chattanooga gage. No attempt was made to determine the effect of Wilson Reservoir because of its small flood-storage capacity. At Hales Bar Dam there was no controlled flood storage and the effect of this reservoir is similar to that of a natural reach of the river.

Natural discharge, or reservoir inflow, on each major tributary was computed by combining the change in storage in each 6-hour period and the actual discharge during the same time, adding the two amounts if the reservoir was rising and subtracting the change in storage from the actual discharge if the reservoir was falling. The discharge thus computed is the total during the 6-hour period. However, for constructing a hydrograph it was assumed that this amount was a rate of flow occurring at the midpoint of the 6-hour period.

Check computations of the natural tributary hydrographs were made from rainfall data and upstream flow records by a method developed from pre-reservoir storms and floods. These check hydrographs agreed closely with those computed from reservoir-storage increments and actual discharge thus indicating, in the case of the large tributary reservoirs, that there was little difference between natural discharge and reservoir inflow.

Natural and actual discharges, in cubic feet per second, on the five major tributaries are shown in Fig. 3 for the 1946 flood. Actual headwater levels at the dams and normal maximum headwater levels are also shown. For Hiwassee River, the total volume in the three reservoirs (Hiwassee, Chatuge, and Nottely) is used because these were considered to act as a unit.

On the main Tennessee River the first step in the computation of natural discharge was the determination of the unmeasured flow into each reservoir by combining the reported daily discharges at the upstream dams and the tributary gaging stations, the discharge at the downstream dam, and the daily change in storage as represented by the change in headwater level and flow-storage curves. In this procedure a standard routing form was used, the reported discharges and headwater elevations were entered on the form, and the corresponding storage was read from the flow-storage curves. From these amounts the total inflow into the reach was computed; and, since the measured proportion of the total was known, the unmeasured flow was determined. Next, the natural tributary flows were combined with the unmeasured flow into the main-river reaches and their sum was routed downstream through the successive reaches with the aid of natural routing diagrams.[2] Natural discharges at the main-river dams and at Knoxville, Tenn., and Chattanooga are shown in Figs. 4 to 9 for the 1946 flood. Stages instead of discharges are shown at Paducah and Cairo, on the Ohio River.

The method used to compute inflow uses reservoir-volume curves which, of course, are mass curves. The difference between two points on the curve is the rate of increase in storage and an error made in reading a volume at a certain time would affect the inflow for two periods only, and would be compensating. Routing computations were checked by comparing total inflow with total outflow during the flood period, with consideration given to changes in storage. Computed natural discharge hydrographs were compared with hydrographs

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[2] "Flood Routing," by Edward J. Rutter, Quintin B. Graves, and Franklin F. Snyder, *Transactions*, ASCE, Vol. 104, 1939, p. 275.

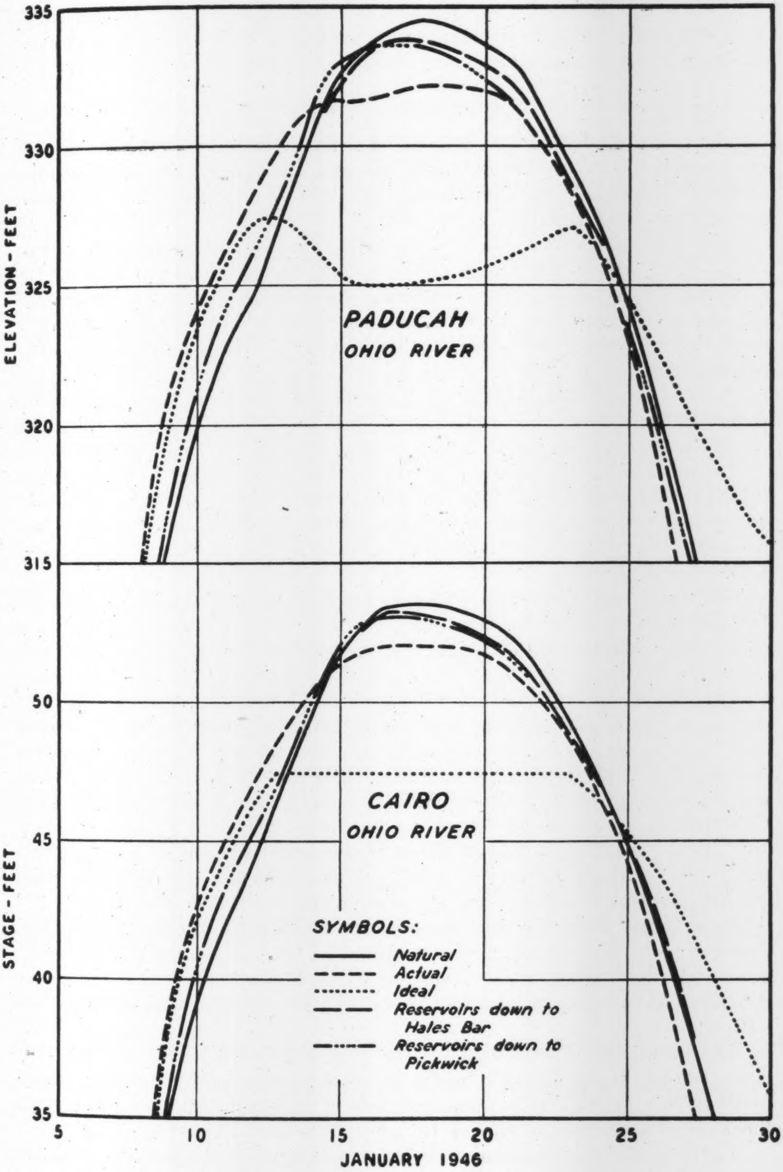

Fig. 9.—Effect of Operation of Tennessee Valley Authority Reservoirs on Stages at Paducah, Ky., and Cairo, Ill.; 1946 Flood

of floods occurring before the reservoirs were built to detect any inconsistency. A comparison of total flood volume with average rainfall also affords an overall check of the computations.

PEAK REDUCTIONS

Maximum discharges and stages and peak reductions for the three floods are given in Table 5. The difference between the actual and the natural hydrographs shown in Figs. 3 to 9 for the 1946 flood is the effect of the operation of the reservoir system.

The actual stages in Table 5 reflect the combined result of upstream storage, the backwater effect from a downstream dam, and the effect of lowering the water surface by channel improvement. As would be expected, the greatest reductions in stage occur immediately downstream from the large tributary reservoirs and, generally, reductions become smaller as the Mississippi River is approached.

Natural stages were obtained by applying the computed natural discharge to the rating curve applicable to each location before the construction of the dam.

In the 1946 flood the natural volume at Chattanooga for the 16-day period from January 5 to January 20, inclusive, would have amounted to 2,554,000 day-sec-ft. With an allowance for an average base flow of 34,000 cu ft per sec per day, this volume is equivalent to a runoff of 3.50 in. from the rainfall of 4.85 in., or 72%.

The contribution of the 600-sq-mile area between Chickamauga Dam and the Chattanooga gage was determined by subtracting the reported daily discharge at the dam from the discharge corresponding to the Chattanooga stage, consideration also being given to the stream gage record on South Chickamauga Creek. To establish the natural Chattanooga discharge, the local contribution determined from actual flows was added to the computed natural discharge at Chickamauga. This local contribution has become of great importance because it comes from the only part of the area above Chattanooga over which there is no control. The 600-sq-mile area is only 2.8% of the total area above Chattanooga; but in the 1946 flood it contributed 22,000 cu ft per sec, or about 10%, to the actual peak and 30,000 cu ft per sec (also about 10%) at the time the natural peak would have occurred. A discharge of about 10,000 cu ft per sec is equivalent to 1 ft in stage at the elevation reached in these floods.

In the 1947 flood the volume at Chattanooga in the 17-day period, from January 13 to January 29, inclusive, would have amounted to 2,767,000 day-sec-ft. With an allowance for an average base flow of 32,000 cu ft per sec per day this volume is equivalent to a runoff of 3.85 in., or 63% of the rainfall of 6.07 in.

The natural flood volume at Chattanooga for the 14-day period from February 12 to February 25, 1948, inclusive, would have been 2,089,000 day-sec-ft. With an allowance for an average base flow of 46,400 cu ft per sec per day, the flood runoff amounted to 2.50 in. from the rainfall of 3.42 in., or 73%.

In the 1946 flood a peak stage of 35.7 ft was reached at Chattanooga on the morning of January 9. Without the reservoirs this stage would have been

TABLE 5.—PEAK REDUCTIONS AT SOME TENNESSEE VALLEY AUTHORITY DAMS AND RIVER GAGES

Location	NATURAL PEAK		ACTUAL PEAK		REDUCTION	
	Discharge, in cubic feet per second	Stage, in feet	Discharge, in cubic feet per second	Stage, in feet	Discharge, in cubic feet per second	Stage, in feet
(*a*) 1946 FLOOD						
Cherokee	64,800	943.2	21,900	931.3	42,900	11.9
Douglas	95,900	892.5	21,200	877.3	74,700	15.2
Knoxville	136,000	824.3	50,000	814.5[a]	86,000	9.8
Fort Loudoun	138,000	765.0	65,000	752.3	73,000	12.7
Fontana	44,800	1,287.0	18,700	1,282.5	26,100	4.5
Norris	77,500	844.8	21,200	831.5	56,300	13.3
Watts Bar	281,000	705.6	163,000	694.0	118,000	11.6
Hiwassee	19,600	1,279.5	13,800	1,280.4[a]	5,800	−0.9
Chickamauga	301,000	669.4	208,000	659.4	93,000	10.0
Chattanooga	320,000	45.8	222,000	35.7	98,000	10.1
Hales Bar	319,000	626.3	226,000	618.3	93,000	8.0
Guntersville	320,000	581.9	260,000	577.6	60,000	4.3
Wheeler	380,000	508.8	332,000	508.7[a]	48,000	0.1
Wilson	380,000	431.0	348,000	428.7	32,000	2.3
Florence	380,000	429.2	348,000	426.2	32,000	3.0
Pickwick	380,000	397.5	365,000	395.9	15,000	1.6
Savannah	....	394.3	....	391.6	....	2.7
Kentucky	357,000	338.6	400,000	337.3	−43,000	1.3
Paducah	....	334.5	....	332.2	....	2.3
Cairo	....	53.5	....	52.1	....	1.4
(*b*) 1947 FLOOD						
Cherokee	63,000	942.9	18,800	930.2	44,200	12.7
Douglas	86,900	891.2	22,900	877.8	64,000	13.4
Knoxville	130,000	823.4	48,600	813.6[a]	81,400	9.8
Fort Loudoun	125,000	766.5	52,800	751.4	72,200	15.1
Fontana	55,500	1,288.7	16,700	1,282.0	38,800	6.7
Norris	52,000	839.0	21,000	831.5	31,000	7.5
Watts Bar	254,000	703.2	127,500	690.9	126,500	12.3
Hiwassee	35,800	1,284.5	14,300	1,280.2[a]	21,500	4.3
Chickamauga	285,000	667.9	160,000	655.3	125,000	12.6
Chattanooga	307,000	44.5	188,000	31.8	119,000	12.7
Hales Bar	309,000	625.7	199,000	615.7	110,000	10.0
Guntersville	301,000	580.7	227,000	575.7	74,000	5.0
Wheeler	344,000	508.7	295,000	509.1[a]	49,000	−0.4
Wilson	345,000	429.5	315,000	428.1	30,000	1.4
Florence	345,000	427.6	315,000	424.8	30,000	2.8
Pickwick	343,000	395.4	318,000	390.9	25,000	4.5
Savannah	....	391.4	....	383.7	....	7.7
Kentucky	307,000	332.8	303,000	330.0	4,000	2.8
Paducah	....	327.4	....	325.6	....	1.8
Cairo	....	42.7	....	41.1	....	1.6
(*c*) 1948 FLOOD						
Cherokee	47,500	939.0	7,000	925.9	40,500	13.1
Douglas	48,800	884.3	19,500	876.8	29,300	7.5
Knoxville	104,000	819.1	41,000	812.6[a]	63,000	6.5
Fort Loudoun	109,000	760.0	49,500	749.9	59,500	10.1
Fontana	27,200	1,284.2	4,900	1,279.0	22,300	5.2
Norris	66,700	842.4	6,000	826.8	60,700	15.6
Watts Bar	260,000	703.9	193,000	696.0	67,000	7.9
Hiwassee	22,800	1,280.6	3,600	1,278.9[a]	19,200	1.7
Chickamauga	287,000	668.2	196,000	657.6	91,000	10.6
Chattanooga	305,000	44.3	205,000	33.8	100,000	10.5
Hales Bar	297,000	624.8	194,000	617.6	103,000	7.2
Guntersville	293,000	580.1	244,000	577.2	49,000	2.9
Wheeler	349,000	508.8	326,000	509.0[a]	23,000	−0.2
Wilson	359,000	430.1	380,000	432.0	−21,000	−1.9
Florence	359,000	428.2	380,000	429.2	−21,000	−1.0
Pickwick	394,000	397.6	411,000	396.1	−17,000	1.5
Savannah	....	394.4	....	392.3	....	2.1
Kentucky	378,000	339.7	444,000	338.9	−66,000	0.8
Paducah	....	334.0	....	332.0	....	2.0
Cairo	....	48.7	....	46.75	....	1.95

[a] Tailwater elevations of these projects are greatly affected by normal pool level of downstream projects.

45.8 ft on January 10, or 10.1 ft higher than the actual. On the basis of a survey[3] made in 1938, damage of $6,500,000 would have been suffered, but this was almost entirely eliminated by the operation of the reservoirs. This sum was $11,800,000 under conditions and values existing at the date of the flood as determined by a house-to-house survey, made in 1948, of new construction since 1938 and by an estimated increase in the dollar value of development as of 1938.

In the 1947 flood the actual peak stage was 31.8 ft on January 20, and the natural peak would have been 44.5 ft 2 days later. This reduction of 12.7 ft prevented damages of $11,500,000 under 1947 conditions.

In 1948 the flood reached a stage at Chattanooga of 33.8 ft on February 14. Without the reservoirs the stage would have been 44.3 ft on February 15. This flood would have been the seventh highest at Chattanooga, and would have established the unique occurrence of three major floods in three consecutive years. The reduction in stage of 10.5 ft was worth $12,900,000 in averted damages under 1948 conditions, and made the total reduction in damage at Chattanooga from all floods including several smaller ones since 1936 about $42,000,000.

On the Tennessee River below Kentucky Dam the February, 1948, flood crest would have been 378,000 cu ft per sec without the reservoirs, and would have been exceeded in crest discharge only by the floods of 1897, 1882, 1875, and 1867.

At Cairo a peak stage of 52.1 ft was reached on January 17, January 18, and January 19, 1946. This flood was the highest that had occurred at Cairo so early in the year. Without the reservoirs it has been computed that the peak would have been 53.5 ft. The reduction of 1.4 ft represents a saving in damages downstream from Kentucky Dam of about $500,000 which is a relatively small amount because of the time that the flood occurred. Had the same flood occurred in April or May the saving would have been as much as five times greater for the same reduction.

TABLE 6.—COMPARISON OF FLOODS AT CHATTANOOGA, TENN.

Flood	Rainfall, in inches	Duration, in days	Flood runoff,[a] in inches	Crest stage, in feet
March, 1875...	11.40	7	6.94	53.8
April, 1886.....	9.47	7	6.63	52.2
March, 1917...	5.20	5	3.54	47.7
April, 1920.....	4.90	5	3.30	43.6
January, 1946..	4.85	4	3.50	45.8[b]
January, 1947..	6.07	6	3.85	44.5[b]
February, 1948.	3.42	3	2.50	44.3[b]

[a] After subtracting a base flow. [b] Computed natural stage.

At Cairo a flood crest of 46.75 ft was reported on February 23, 1948, which under natural conditions would have been 48.7 ft on February 24. Such a flood stage is not serious on the lower Ohio and Mississippi rivers at this time of the year and no crop damage was caused.

At many other points on the Tennessee Basin serious damages were prevented during these floods. Below each of the major tributary dams natural crest stages were greatly reduced, the reduction amounting to as much as 15.6 ft below Norris Dam in the 1948 flood. At Knoxville peak stages were reduced by 9.8 ft in both 1946 and 1947. No estimate of the damages prevented at these points has been made.

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[3] "The Chattanooga Flood Control Problem," *House Document No. 91*, 76th Cong., 1st Session, 1939.

COMPARISON WITH OTHER FLOODS

Table 6 shows the rainfall, runoff, and crest stage of seven floods at Chattanooga; and hydrographs of all except the floods of 1875 and 1886 are shown in Fig. 10. The storms producing these floods, however, were much different in their intensities and areal distribution; and only a general comparison can be made between average rainfall and crest stage. The 1917 storm was more uniform with respect to time and area than any of the others. The 1920 storm was greater over the southern part of the basin than over other parts, and most of the rain occurred in 2 of the 5 days. In January, 1946, there was a period of 3 days with only minor rainfall between two separate storms and the flood crest would have been produced by the first 3 days of rain. The 1947

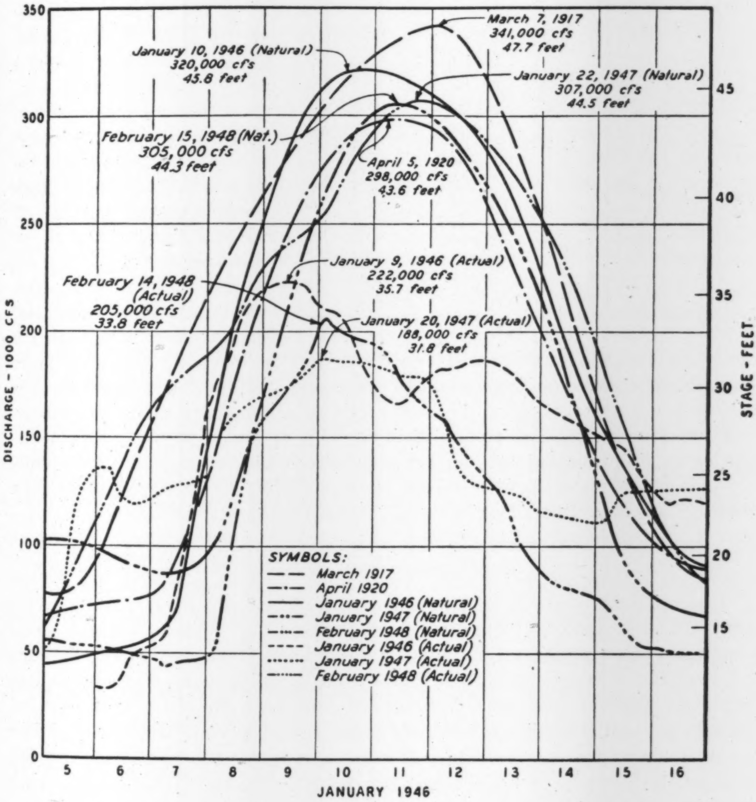

FIG. 10.—COMPARISON OF 1917, 1920, 1946, 1947, AND 1948 FLOODS AT CHATTANOOGA, TENN.

storm was similar to the 1917 storm but the rate of rainfall per day was less than in 1917. The 1948 storm, although having the lowest total rainfall of any of the five storms, produced a crest stage comparable with the others because of the high concentration of rain on the areas close to Chattanooga, particularly on the Emory River.

TABLE 7.—HIGHEST ELEVEN FLOODS ON LOWER TENNESSEE RIVER

Order	Date	Stage at Johnsonville, Tenn., in feet	Discharge, in cubic feet per second
1	March 24, 1897	48.5	475,000
2	January 31 to February 2, 1882	47.9	447,000
3	March 7, 1875	46.0	447,000
4	March 22, 1867	45.3	416,000
5	February 21, 1948[a]	....	378,000
6	February 21, 1884	43.4	377,000
7	March 20, 1884	41.4	373,000
8	April 15–16, 1886	42.1	372,000
9	January 3–4, 1927	41.0	367,000
10	March 31, 1899	40.2	360,000
11	January 18, 1946[a]	....	357,000

[a] Computed natural flow at Kentucky Dam.

As stated, the flood of February, 1948, would have been the seventh highest flood since 1867, being exceeded by the floods of 1867, 1875, 1886, 1917, 1946, and 1947. Never before have three large floods occurred in three consecutive years; but in 1917, 1918, and 1920 floods occurred which reached crest stages of 47.7 ft, 42.7 ft, and 43.6 ft, respectively.

Under natural conditions the February, 1948, flood would have been the fifth highest on the lower Tennessee River. It is difficult to make an accurate comparison of floods on this section of the river before and after the reservoirs because present stages at the Johnsonville (Tenn.) gage are not comparable with previous stages. A comparison may be made, however, between discharges at Johnsonville for pre-reservoir floods and computed natural discharges at Kentucky Dam for recent floods. Table 7 lists the highest eleven floods arranged in descending order of crest discharge.

ACCELERATION OF FLOODS

A comparison of the natural and actual hydrographs given in Figs. 4 to 9 shows the acceleration of the 1946 flood produced by operation of the main-river reservoirs. At Fort Loudoun Dam the large storage effect of Cherokee and Douglas reservoirs offsets any acceleration. Beginning with Watts Bar Reservoir, however, the actual hydrograph rises more rapidly than the natural hydrograph, and the difference between the two becomes progressively greater as the flood passes downstream. The peak reduction in discharge afforded by the tributary dams also becomes less until at Pickwick Dam in the 1946 flood the reduction is only 15,000 cu ft per sec as compared with 98,000 cu ft per sec at Chattanooga. At Kentucky Dam the acceleration was further increased by drawdown and in the 1946 flood the actual discharge exceeded the computed natural discharge by a substantial amount. From January 6 to January 14, 1946, a volume of 1,400,000 acre-ft was discharged which, under natural conditions, would still have been in the Tennessee River and discharged later.

A similar acceleration occurred in the 1947 and 1948 floods as shown by the difference between the rising limbs of the natural and actual hydrographs. In these floods, because of the relatively large volume stored in the tributary reservoirs, the acceleration was not substantial down to Chattanooga. Below this point, however, the acceleration became more pronounced. At Kentucky Dam a volume of 1,110,000 acre-ft was discharged between January 16 and January 25, 1947, which under natural conditions would still have been in the Tennessee River and discharged later. This acceleration was beneficial to stages on the Ohio and Mississippi rivers; and, when followed by filling to normal level at the time of the Ohio River crest stage, a substantial reduction was produced.

A computation was made to determine whether the 1946 flood could have been accelerated a greater amount. The only method by which this could have been accomplished was by drawing each reservoir to its minimum allowable headwater level and holding that level throughout the flood. It was found that some further acceleration would have been possible by this method, particularly on January 6 and January 7; but it would have required a knowledge of the size of the flood on January 5, when the storm was only beginning. Some of the additional acceleration would have been lost as the floodwater passed through Hales Bar and Wheeler reservoirs where the spillway capacity is limited at the lower levels, and also as it passed through the long (184 miles) Kentucky Reservoir reach. At Kentucky Dam the additional acceleration would have diminished to only a small amount and it can be stated, therefore, that the 1946 flood could not have been accelerated more than it actually was.

This accelerating effect of main-river reservoirs has been known for some time. In the early studies of the Tennessee Basin made by the Corps of Engineers and in all operation studies made by TVA it became apparent that this acceleration would occur and that it was an extremely desirable condition for the regulation of most floods on the Ohio and Mississippi rivers. The ability to discharge large volumes of floodwater from the Tennessee River earlier than under natural conditions and ahead of the crest on the Ohio River is a method of preserving storage capacity for use at the time of the Ohio River crest. The acceleration is caused, of course, by holding the headwater level constant, or by lowering it, at the time of high inflows. The effect of this operation is to decrease the amount of inflow going into storage and, as a result, to increase the amount appearing as discharge.

OPERATION ACCORDING TO A FIXED RULE

A study of operation of the main-river reservoirs above Chattanooga according to a fixed rule was made to discover whether an improvement could have been made over the actual operation. Under the assumed fixed rule, releases at Chickamauga, and also at Watts Bar and Fort Loudoun, depend on the Chattanooga stage and the Chickamauga headwater level, no knowledge of the rainfall or prediction of inflow being required. The rule was planned so that complete use would be made of the available storage in Chickamauga, Watts Bar, and Fort Loudoun reservoirs during a flood equal to the maximum of record, which was about 12 ft higher at Chattanooga than the flood of

January, 1946. When it is recognized that a flood is developing, the operation would be as follows:

1. Hold the normal headwater level at Chickamauga (El. 675) until a 20-ft stage is reached on the Chattanooga gage.
2. Increase the stage, not exceeding 30 ft, attempting to lower Chickamauga headwater to the minimum level, El. 673.
3. Continue the 30-ft stage until the Chickamauga headwater elevation, El. 677, is reached.
4. Increase the stage to 31 ft and hold it until the headwater elevation, El. 678, is reached.
5. Make successive 1-ft increases in the Chattanooga stage for each 1-ft rise in the Chickamauga headwater elevation until, with a 38-ft stage, the headwater elevation, El. 685, is reached.
6. Hold El. 685 until the peak of the flood is past and then promptly return the reservoir to normal level with a decreasing rate of release from Chickamauga Reservoir.
7. Hold the normal headwater level at Watts Bar and Fort Loudoun reservoirs until a stage of 20 ft is reached at Chattanooga. Thereafter, up to the time of the Chickamauga peak, hold discharges at 0.7 and 0.4, respectively, of the Chattanooga allowable discharge, unless they are limited because of minimum headwater or full reservoir. (By "allowable discharge" is meant the discharge indicated for the Chattanooga stage and the Chickamauga headwater level.)
8. When the headwater elevation begins to recede, return the three reservoirs to normal levels at rates consistent with flood conditions on the lower Tennessee River.

The foregoing rules were applied to the three floods in 1946, 1947, and 1948 as far as possible, assuming that the actual tributary operation would be repeated and that the normal headwater elevation would be maintained prior to the flood. The rule for release, at Fort Loudoun, of four tenths of the Chattanooga discharge could not be used because of the relatively low flows at Fort Loudoun. That reservoir was drawn to the minimum level and held there until the flood danger was past.

A comparison (see Figs. 4 and 5 and Table 8) of the fixed-rule operation with the actual operation shows that the maximum headwater levels in the three reservoirs were lower with fixed-rule operation. In the 1946 flood the peak stage at Chattanooga would have been 33 ft instead of an actual stage of 35.7 ft. The greater reduction would be obtained by increasing the release earlier and thereby holding the available storage for later filling. It is realized, of course, that there would be objections to the sudden increase from 20 ft to 30 ft even if it were physically possible to open the gates (actually the stage increased from 20 ft to 30 ft in about 18 hours), but it should always be borne in mind that the sooner the 30-ft stage is reached, the lower the peak may be made later.

For the 1947 and 1948 floods the fixed-rule operation would give only a small additional reduction (0.8 ft) at Chattanooga below the actual crest stage. The three main-river reservoirs, however, would have been in better condition for retaining additional flood flows if they had occurred, because of the lower headwater levels.

TABLE 8.—COMPARISON OF OPERATIONS FOR CHATTANOOGA, TENN.

RESERVOIR OPERATION		1946 FLOOD[a]			1947 FLOOD[a]			1948 FLOOD[a]		
Tributary (1)	Main river (2)	Chickamauga (3)	Chattanooga Flow (4)	Chattanooga Stage (5)	Chickamauga (3)	Chattanooga Flow (4)	Chattanooga Stage (5)	Chickamauga (3)	Chattanooga Flow (4)	Chattanooga Stage (5)
Natural	Natural	301	320	45.8	285	307	44.5	287	305	44.3
Actual	Natural	222	244	38.1	176	202	33.4	216	239	37.5
Actual	Actual	208	222	35.7	160	188	31.8	196	205	33.8
Actual	Fixed rule	181	198	33.0	162	181	31.0	171	198	33.0
Actual	Ideal	123	153	27.5	97	123	23.7	80	111	22.1

[a] Cols. 3 and 4 contain the maximum discharge, in thousands of cubic feet per second, and Col. 5 contains stage heights, in feet.

Operation of the tributary reservoirs according to a fixed rule also has been given considerable study. A tentative guide curve has been developed by which tributary reservoir discharges would be determined by reservoir stages. In the floods of 1946, 1947, and 1948, operation according to the tentative fixed rule would vary only slightly from the actual operation.

OPERATION OF MAIN-RIVER RESERVOIRS FOR MAXIMUM POSSIBLE REDUCTION

In addition to the study of the main-river reservoirs above Chattanooga according to a fixed rule, an investigation was made of an operation in which it was assumed that complete knowledge of the inflow would be available and that full use would be made of flood-control storage. As in the fixed-rule study, it was also assumed that the tributary reservoirs would have been operated as they actually were. This operation shows the results which could be approached if the "ideal" were realized. However, it seems almost impossible that such complete knowledge will ever be available in advance for a flood of major proportions, and this method merely indicates the possible results of "hindsight" operation.

This ideal operation consisted of drawing each main-river reservoir to its minimum level, provided the discharge did not exceed the peak rate produced later, and then of holding that level if possible until the discharge increased to a rate which, if maintained, would cause the reservoir level to rise to the maximum permitted. Such an operation requires a knowledge of the inflow and produces a flat-peak discharge below each dam. At Kentucky Reservoir a modification of the operation was made. Instead of producing a flat discharge below the dam, the reservoir was filled by releasing water at rates that would produce a flat peak at Cairo. The reservoirs were returned to their normal levels at rates consistent with downstream flood conditions. In the case of the 1946 flood the reservoirs were held full until there was no possibility that the releases would affect Cairo stages. In the 1947 flood they were returned to normal levels immediately by making the discharges equivalent to a 30-ft stage at Chattanooga.

Figs. 4 to 9 for the 1946 flood show, at each dam and at Chattanooga, Paducah, and Cairo, the results of the ideal reservoir operation. Hydrographs for the natural condition and for other operations are also given for comparison. The maximum stage in the 1946 flood at Chattanooga with ideal operation of main-river reservoirs was 27.5 ft; and, at Cairo, 47.4 ft. In the 1947 flood the stage at Chattanooga would have been 23.7 ft with the ideal operation; and, in the 1948 flood, it would have been 22.1 ft. Because of the relatively small magnitude of the 1947 flood at Cairo, studies of hypothetical reservoir operation were not made below Chattanooga.

Effect of Tributary Reservoirs on Chattanooga Stage

The effect, at Chattanooga, of the tributary reservoirs alone was determined by assuming that the main-river reservoirs were not built—that is, that natural conditions would exist in the Tennessee River. It was also assumed that tributary reservoir operation would be the same as the actual operation. The difference between the hydrograph thus computed and the natural hydrograph indicates the effect of the tributary reservoirs, and a comparison with the actual hydrograph shows the effect of the three main-river reservoirs.

Peak discharges and stages for the three floods computed on the foregoing assumptions are shown in Table 8 together with peaks for other conditions for comparison. Hydrographs of the 1946 flood are shown in Figs. 4 and 5.

It will be noted from Table 8 that in the 1946 flood the tributary reservoirs reduced the flood at Chattanooga from stage 45.8 to stage 38.1, or 7.7 ft, and that the three main-river reservoirs provided additional reduction to stage 35.7, or 2.4 ft. However, if greater use of the main-river storage had been made, greater reduction would have been possible, as indicated by the ideal operation. Operation according to the fixed rule also would have given a greater reduction because of the more efficient use of storage.

In the 1947 flood the tributary reservoirs reduced the flood from a stage of 44.5 ft to one of 33.4 ft, or 11.1 ft. The main-river reservoirs provided additional reduction to 31.8 ft or to 1.6 ft. In the 1948 flood the tributary reservoirs were responsible for a reduction of 6.8 ft and the main-river reservoirs for a reduction of 3.7 ft.

Effect of Reservoirs on Cairo Stage in the 1946 and 1948 Floods

Operation of all reservoirs above Chattanooga was assumed in this study as being primarily for the control of the flood at that point. It is desirable to know the effect of this operation at Cairo, and also the effect of operating the lower Tennessee River reservoirs. To compute these effects, two additional routing studies were made. In one the effect at Cairo of all reservoirs above Chattanooga was determined by assuming that there were no reservoirs below Hales Bar—that is, that actual discharge at Hales Bar combined with inflows below that point was carried downstream to Cairo under natural conditions. The difference between this hydrograph and the natural hydrograph indicates the effect of the reservoirs on the upper river.

In the second routing study, Guntersville, Wheeler, and Pickwick reservoirs were added, only Kentucky Reservoir being considered nonexistent. When the Cairo stage, computed for this condition, is compared with the hydrograph for reservoirs above Hales Bar, it shows the effect of Guntersville, Wheeler, and Pickwick reservoirs. When the Cairo stage is compared with the actual hydrograph, the effect of Kentucky Reservoir is obtained.

TABLE 9.—COMPARISON OF OPERATIONS FOR CAIRO, ILL.

Operation of reservoir system	1946 FLOOD				1948 FLOOD			
	KENTUCKY		PADUCAH	CAIRO	KENTUCKY		PADUCAH	CAIRO
	Discharge[a]	Tailwater elevation	Elevation	Stage, in feet	Discharge[a]	Tailwater elevation	Elevation	Stage, in feet
(1)	(2)	(3)	(4)	(5)	(2)	(3)	(4)	(5)
Natural	357	338.6	334.5	53.5	378	339.7	334.0	48.7
Reservoirs Down to—								
Hales Bar Dam	333	337.5	333.8	53.2	357	338.0	332.8	47.8
Pickwick Dam	356	337.6	333.6	53.1	372	337.6	332.4	47.7
All Reservoirs—								
Actual operation	400	337.3	332.2	52.1	444	338.9	332.0	46.7
Ideal operation of main-river reservoirs	364	335.0	327.4	47.4	439	338.1	330.6	43.5

[a] Discharges, in thousands of cubic feet per second.

Table 9 shows peak discharge at Kentucky Dam and peak stages at Paducah and Cairo for the various conditions. Hydrographs are shown in Figs. 6 to 9 for the 1946 flood. In the 1946 flood all reservoirs above Hales Bar Dam caused a peak reduction at Cairo of 0.3 ft; Guntersville, Wheeler, and Pickwick produced an additional reduction of 0.1 ft; and Kentucky, a reduction of 1.1 ft. The reduction of 0.1 ft attributed to Pickwick, Wheeler, and Guntersville reservoirs is small—almost negligible—but available storage was not completely filled because the flood did not reach critical stages at Cairo. It should also be noted in Table 9 that the regulated peak discharge from Kentucky Dam was greater than the natural peak discharge, but that the peak stage downstream was less. This result is possible, of course, only if the increased flows do not coincide with the Ohio River peak. Hydrographs in Fig. 9 also show that the major effect of upstream regulation appears after the Cairo peak and that Kentucky Reservoir is most effective on the peak, as would be expected.

FLOOD OF APRIL, 1948, AT CAIRO

The aforementioned three floods of 1946, 1947, and 1948 are examples of floods that occur during the winter or spring flood season on the Tennessee Basin, and which have relatively high flows in the Tennessee River and relatively low flows in the Ohio and Mississippi rivers. In such cases the tributary reservoirs store a large proportion of their inflow; and, if the water levels rise above normal elevations, they are returned to normal within a short time, usually in about 2 weeks. Moreover, moderately high stages may occur in the Mississippi River at this time of the year without causing any crop damage.

Floods of a different type, in which flows are relatively low in the Tennessee River and relatively high in the Ohio and Mississippi rivers and which occur later in the season when agricultural operations have begun, present different problems in reservoir operation. During these floods storage probably will be retained in the tributary reservoirs and not released until late in the year. The retention of tributary inflow over a period of months, combined with storage in the main-river reservoirs and the positive regulation afforded by Kentucky Reservoir, will reduce stages at Cairo over a long period. In addition to preventing the flooding of some agricultural lands, other lands may be entered and worked sooner than under natural flood conditions.

An example of this type of flood was that of April, 1948. Stages at Cairo began to rise above 30 ft about the middle of March and reached a crest stage of 51.6 ft on April 3 (see Fig. 11). Without the reservoirs the crest would have been 53.4 on April 4. Stages then fell until April 12 when, following heavy rainfall on the upper Ohio River, a second crest stage of 47.9 ft was reached on April 23, April 24, and April 25. Without the reservoirs this crest would have been 49.0 ft on April 21. Drawdown of Kentucky Reservoir was commenced on March 23 when it was certain that the Cairo stage would continue to rise above 40 ft, but the headwater level was returned to El. 354 when predictions did not indicate serious crest stages. On April 2, Kentucky Reservoir discharge was reduced to turbine capacity (60,000 cu ft per sec) and this rate was maintained for 7 days throughout the crest period at Cairo.

TABLE 10.—Inflow, Outflow, and Storage in Tributary Reservoirs of the Tennessee Valley Authority (in Thousands of Acre-Feet) During the 43-Day Period Corresponding to March 25–May 6 at Cairo, Ill.

Reservoir	Inflow	Outflow	Storage
Cherokee	725	304	421
Douglas	860	518	342
Fontana	548	176	372
Norris	785	319	466
Hiwassee	268	87	181[a]
Total	3,186	1,404	1,782

[a] Includes storage in Chatuge and Nottely reservoirs.

Kentucky Reservoir discharge was increased when Cairo stages fell below 50 ft to lower the headwater level. To regulate the second crest on April 23, April 24, and April 25 the discharge was reduced from 145,000 cu ft per sec on April 16 to 60,000 cu ft per sec on April 19, and this was not exceeded for the remainder of the crest period at Cairo. The natural Cairo crest would have been 49.0 ft on April 21. The reduction of 1.8 ft in the first crest represents about $1,000,000 in averted damages, and in the second crest the reduction of 1.1 ft represents about $600,000 in averted damages.

Actual stages at Cairo were lower than natural stages from March 25 to May 6. Except for power generation releases the entire tributary inflows were stored during a corresponding period, as shown in Table 10.

The tributary reservoirs stored 1,782,000 acre-ft or about 56% of their inflow during the 43-day period corresponding to March 25–May 6 at Cairo. This storage is equivalent to an average discharge of about 21,000 cu ft per sec.

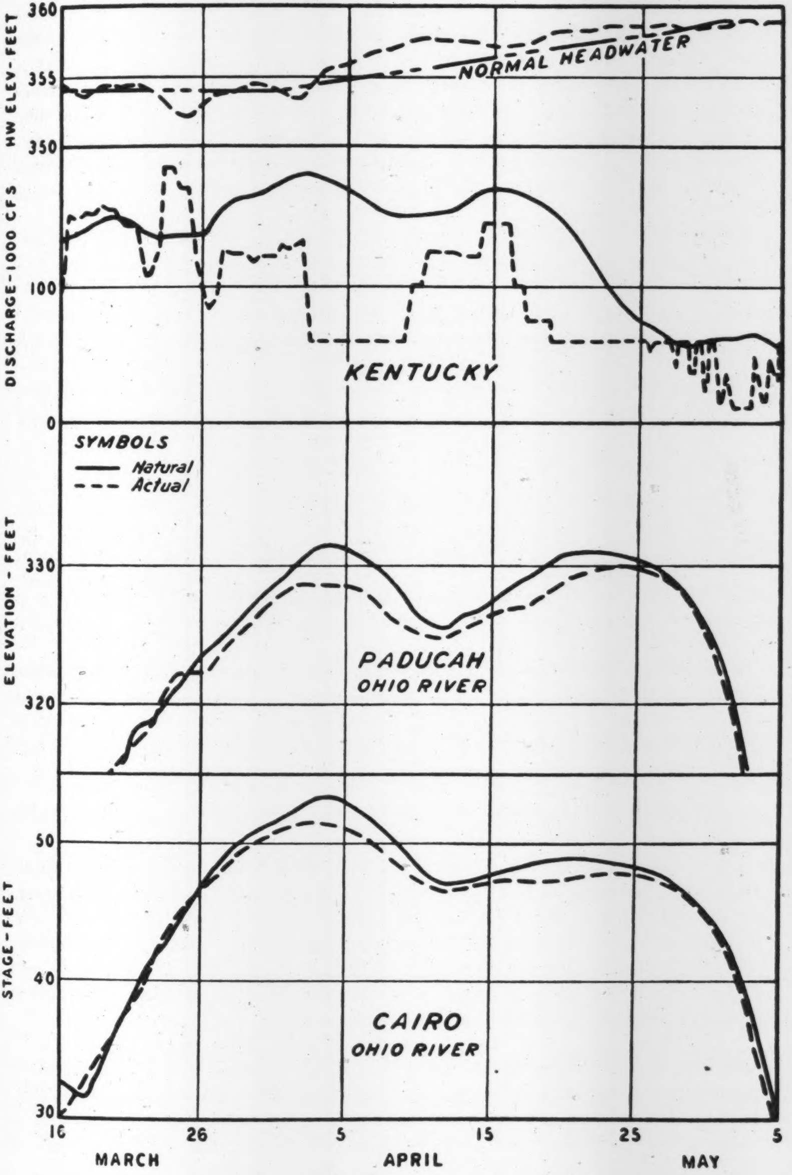

FIG. 11—EFFECT OF OPERATION OF TENNESSEE VALLEY AUTHORITY RESERVOIRS ON STAGES AT PADUCAH, KY., AND CAIRO, ILL.; FLOODS OF APRIL, 1948

The main Tennessee River reservoirs also were being filled from their flood-season levels to their maximum normal levels within this period and the total amount stored was 2,041,000 acre-ft, of which 898,000 acre-ft, or 44%, occurred in Kentucky Reservoir. This storage is equivalent to an average discharge of 24,000 cu ft per sec during the 43-day period, making the total average discharge retained by the reservoir system equal to 45,000 cu ft per sec, which is equivalent to about 1 ft at Cairo. Regulation by Kentucky Reservoir produced greater crest reductions than the average indicated by the storage over a 43-day period.

TABLE 11.—CREST STAGES AND CREST REDUCTIONS (IN FEET); FLOOD OF APRIL, 1948 (DISCHARGES ARE IN THOUSAND CUBIC FEET PER SECOND)

Location	April 3, 1948						April 23, 1948					
	Natural Crest		Actual Crest		Reduction		Natural Crest		Actual Crest		Reduction	
	Flow	Stage	Flow	Stage	Flow	Stage	Flow	Stage	Flow	Stage	Flow	Stage
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(2)	(3)	(4)	(5)	(6)	(7)
Kentucky...	180	332.9	131[a]	329.8	49	3.1	170	331.9	146[a]	330.2	24	1.7
Paducah, Ky.	....	331.4	....	328.7	....	2.7	....	331.0	....	330.0	....	1.0
Cairo, Ill.....	....	53.4	....	51.6	....	1.8	....	49.0	....	47.9	....	1.1

[a] Prior to the reduction of the outflow to turbine capacity.

Table 11 gives the natural and actual flood crests and crest reductions at Kentucky Reservoir and at Paducah and Cairo. No computations of hypothetical operations were made for this flood because of the relatively low flows on the Tennessee River.

CONCLUSIONS

The results of the studies reported in this paper lead to five conclusions, as follows:

1. The crest stage of floods of January, 1946, January, 1947, and February, 1948, were reduced 10.1 ft, 12.7 ft, and 10.5 ft, respectively, at Chattanooga by the operation of the TVA reservoir system and, in the 1946 flood, to 1.5 ft at Cairo. These reductions prevented damages of $37,100,000 at Chattanooga and of about $500,000 below Kentucky Dam. Reductions of two flood crests at Cairo in April, 1948, amounting to 1.8 ft and 1.1 ft prevented damages of about $1,600,000 below Kentucky Dam.

2. The reductions below the natural crest at Chattanooga in the 1946, 1947, and 1948 floods were about 55%, 61%, and 47%, respectively, of the greatest possible reduction, and at Cairo in the 1946 flood the reduction was about 25%. It should be recognized, however, that reductions as large as indicated to be possible by the "ideal" operation neither were needed nor were practicable in the 1946 flood in view of continued predictions of rainfall.

3. If a fixed rule of operation had been followed with the three main-river reservoirs above Chattanooga, an additional reduction of 2.7 ft would have been possible at that point in the 1946 flood, and the reservoirs would have been in as good condition for storing inflow from possible additional rainfall as they actually were when the crest had passed. In the 1947 flood and the 1948 flood the fixed-rule operation would have given only a slightly greater reduction than was actually obtained.

4. The acceleration of the flood by main-river reservoirs is indicated to have occurred as was revealed by the early planning studies. Natural storage capacity of the Tennessee River normally had an adverse effect on Ohio River flood crests because of the relative timing of the flood waves. The elimination of part of this storage by the operation of the main-river dams has contributed substantially to the reduction of Ohio River peaks. The use of reservoir storage at the time of the Ohio River crest has reduced those peaks still further.

5. Reductions in Cairo stages will occur over long periods when the TVA reservoirs are being filled to maximum normal levels after the flood season in the Tennessee Basin. If this filling is coincident with floods on the Ohio and Mississippi rivers, there will be a substantial saving in damages, as in the flood of April, 1948. Even with the reservoirs filled to maximum normal level a large amount of storage would be available (particularly in Kentucky Reservoir) to regulate a large flood at Cairo.

ACKNOWLEDGMENTS

The material presented herein is the result of studies conducted in the Flood Control and Drainage Branch of the Division of Water Control Planning of TVA. Nearly all members of the Flood Control Section participated in these studies at some time. Comments and suggestions by members of the Section and by Charles W. Okey and James S. Bowman, Members, ASCE, and by S. M. Woodward, Hon. M. ASCE, were greatly appreciated.

DISCUSSION

Ray K. Linsley, Jr.,[4] **Assoc. M. ASCE.**—Well organized and factual, Mr. Rutter's summary of the operation of an extensive reservoir system for flood-control purposes is a definite contribution to the field of hydraulic engineering. The literature contains many papers dealing with the construction of dams, but very few papers on the equally important subject of their operation.

Under the heading, "Operation of Main-River Reservoirs for Maximum Possible Reduction," the author very properly calls attention to the large difference between the actual reservoir operation and the "ideal" operation which would effect the maximum possible stage reduction at the point or points to be protected. He also emphasizes the fact that the "ideal" operation is almost impossible to accomplish because it presupposes a perfect knowledge of future stream flow. However, many projects for flood control, power, irrigation, and other purposes have been justified on the basis of benefits computed on the assumption of perfect operations. This overestimate of the benefits could well turn the balance between justification and rejection for projects with a low cost-to-benefit ratio.

Errors in river forecasting occur because the forecaster must work with data which cannot be measured with precision and with relationships which almost never quite attain perfect correlation. These sources of error will not frequently introduce serious error into forecasts prepared by competent forecasters if precipitation does not occur subsequent to the forecast. Despite extensive studies by meteorologists, high accuracy in weather forecasting has not been attained. The ordinary newspaper weather forecasts for 24 hours in advance verify with an accuracy of about 80%. Considering only forecasts of rain or forecasts made during rainy periods, the percentage of accuracy would probably be found to be somewhat lower. However, these forecasts, even if perfect, would be far from adequate. The river forecaster must know how much rain will fall, when it will occur, and how it will be distributed over the basin. For areas such as the Tennessee Valley the forecast must extend from 3 to 10 days into the future, depending on the current situation.

If a competent meteorologist can study a specific basin carefully and then discuss his forecast directly with the water control engineer, his advice will be found to have great value. Nevertheless, on many occasions reserve storage must be set aside as a safeguard against the unexpected and the ideal operation becomes impossible.

The eighth paragraph under the heading, "Peak Reductions," raises an interesting point. It is indicated here that potential damage to property built subsequent to the program of the Tennessee Valley Authority (TVA) is included in the estimated benefits of the several operations. If it can be assumed that the builders of these properties had possible alternative locations that would have been safe from flood, then the sites must have been selected with the assurance that they would be protected by TVA operations. Can such properties be properly included in the estimates of benefits from flood-control operations? If they are included, should not any future flood damage to these properties be charged as a cost of TVA operations rather than as the cost of unavoidable natural losses?

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[4] Associate Prof. of Hydr. Eng., Stanford Univ., Stanford University, Calif.

It would be interesting to have an indication of the probable error in the computed natural flows. Although the reduction in stage at Chattanooga (Tenn.) resulting from the operations is so large that any reasonable error would not materially affect the picture, the reduction claimed at Cairo (Ill.) is not far greater than the errors that might be expected in such involved computations. The determination of stage at Cairo from flows on the Ohio, Tennessee, and Mississippi rivers is probably subject to an error of several tenths of a foot. The error to be expected in the determination of the natural outflow from the Tennessee River would be difficult to estimate without the detailed computations. It would not be impossible for this error also to amount to several tenths of a foot in terms of the Cairo stage.

There are two sources of error in the computations which would tend to make the computed natural flows too high. First, the computed local inflows include the flow derived from rainfall on the surfaces of the reservoirs and, hence, are higher than would have occurred under natural conditions. Second, the accelerating effect of the reservoirs would tend to yield local inflows somewhat higher than under natural conditions. It is possible, of course, that these effects are negligible, but if they have not been investigated they should be.

EDWARD J. RUTTER,[5] M. ASCE.—Several questions pertinent to the flood-control operation of a large reservoir system have been raised by Mr. Linsley's discussion. Two points are of particular interest, one related to flood benefits and the other to accuracy of natural flow computations.

At Chattanooga, in floods occurring since the first reservoir came into operation, estimated benefits include damage to property in existence on the date of the flood, rather than only damage to property in existence before the first reservoir came into operation. Mr. Linsley suggests the possibility that development subsequent to the TVA program was built on land selected over possible alternative locations that would have been safe from flood. It is doubtful whether such alternative locations were available, because of topography, zoning restrictions, poor highway and railway connections, and the unavailability of municipal services. Moreover, complete protection against the maximum probable flood, or even the maximum known fllood, is not provided by the reservoirs, and a flood hazard still exists at Chattanooga. The city has been repeatedly warned of this situation, as have individuals and industries seeking flood-height information from TVA. It cannot be stated, therefore, either that alternative building sites safe from flood were available, or that sites were selected with the assurance that they would be protected by TVA operations. The writer believes it is correct to determine benefits on the basis of the status of the city at the date of the flood.

Mr. Linsley also questions the accuracy of natural outflows determined from inflows which, in turn, were computed from changes in reservoir levels.

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[5] Hydr. Engr., TVA, Knoxville, Tenn.

As stated by Mr. Linsley, there are two possible sources of error in this method: (1) The increased runoff from the greater water surface area of the reservoirs compared with that from natural land area; and (2) the difference in timing of the runoff attributable to the use of flat-pool storage in the tributary reservoirs. The determination of inflows by flow-storage curves eliminates this source of error in the main-river reservoirs.

The effect of greater runoff from a larger water surface area is believed by the writer to be negligible. In the case of Norris Reservoir, which has a relatively large water area with respect to its total drainage area, the water area at the level when the rain occurred in January, 1946 (approximate El. 980), is 20,000 acres or 31 sq miles. This is only slightly more than 1% of the total drainage area of 2,912 sq miles. If it is assumed that there would be 100% runoff on the increased water area instead of 70%, the increased runoff from the entire area would be 0.3%. Such an error could be ignored, of course, considering the accuracy of flow measurements in general.

To determine whether the second possible source of error was significant, computations of natural discharge at Chattanooga were made by a method which was completely independent of daily changes in reservoir volumes. This check method was based on average daily rainfall over the zones into which the total area was divided. Average daily rainfall, in inches of depth over each zone, was computed and converted to units of flow. These flows were then reduced so that the total from all zones was equal to the flood volume at Chattanooga. The sum of the reduced flows for each time interval, allowing for each zone's time lag, was assumed to be the valley inflow for that interval. This valley inflow was then routed to Chattanooga with routing diagrams constructed from the data on pre-reservoir floods. A comparison of crest flows (in cubic feet per second) at Chattanooga in the floods of 1946, 1947, and 1948, as computed from rainfall and by consideration of daily changes in reservoir volumes, is as follows:

Flood	Rainfall	Reservoir volume changes
1946	318,000	320,000
1947	298,000	307,000
1948	307,000	305,000

At least with respect to peak discharge at Chattanooga, this tabulation shows that there is no significant error in using reservoir-volume increments to determine inflows.

A somewhat different check method was used for computing natural discharge at the major tributary dams. Here, the amount of natural valley storage neglected when computing natural discharge from reservoir-volume increments is small compared with the volume of the flood; and, therefore, a more precise method was necessary. In addition to using a time period shorter than one day, the check method consisted of computing valley inflow from: (*a*) The measured discharge at stream gaging stations located above the influence of the dam; and (*b*) from the rainfall over the unmeasured area between the gaging station and the dam site. The total inflow thus computed was then routed to the dam site with diagrams constructed from data on pre-reservoir floods. A comparison of the two methods showed that there was no significant difference between them, except possibly at Norris Dam, where the peak discharge by the check method was lower and occurred later than did that determined by the reservoir-volume increment method. However, when transferred to Chattanooga, this difference had only a minor effect.

Although a check computation for natural discharge below Chattanooga has not yet been made, the writer feels that natural discharges as now computed for the lower Tennessee River are as accurate as those at Chattanooga. The accuracy of the natural stage at Cairo, of course, is dependent on the accuracy of the discharge computations for the Tennessee River and on the accuracy of the method used to transfer these discharges to Cairo. The writer does not believe, however, that errors would accumulate below the Kentucky dam site, suggested as a possibility by Mr. Linsley, any more than they would on the Tennessee River itself.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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TRANSACTIONS

Paper No. 2444

FLOATING TUNNEL FOR LONG WATER CROSSINGS

By Charles E. Andrew,[1] M. ASCE

With Discussion by Robert S. Mayo, Homer M. Hadley, and Charles E. Andrew.

Synopsis

The rapidly advancing requirements of vehicular transportation, caused by increased population and industry, occasionally necessitate an engineering attempt to span water barriers which formerly were considered either impossible to cross or economically unsound. The purpose of this paper is to present a preliminary study of such a problem. The solutions suggested and presented are by no means to be considered final in detail. It seems to the writer, however, that they do present the only possible method of constructing a vehicular bridge between the points indicated. Briefly stated, the problem presented to the writer, by the Director of Highways of the State of Washington, was to make a study of possible methods, if any, of constructing a permanent highway structure across Puget Sound between Seattle, Wash., on the east and the mainland on the west shore.

Territory and Population to Be Served

The immediate territory in the State of Washington to be served by the proposed facility includes the rapidly growing City of Seattle on the east shore and the City of Bremerton and Kitsap County on the west shore. The metropolitan population of Seattle is more than 500,000 (1949) and that of Bremerton and Kitsap County approximately 90,000. In addition, the entire north end of the Olympic Peninsula, containing approximately 5,000 sq miles lying to the west and including the Olympic National Park, is tributary to such a crossing. The present total population, on the west shore, is approximately 120,000. However, this population cannot be considered to represent the number of motorists who would use such a crossing within a few years after its construction, as a very large area of some of the most desirable residential property in the state, with hundreds of miles of shore line, would be made

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Note.—Published in March, 1950, as *Proceedings-Separate No. 8*. Positions and titles given are those in effect when the paper or the discussion was received for publication.

[1] Cons. Engr., Washington Toll Bridge Authority, Tacoma, Wash.

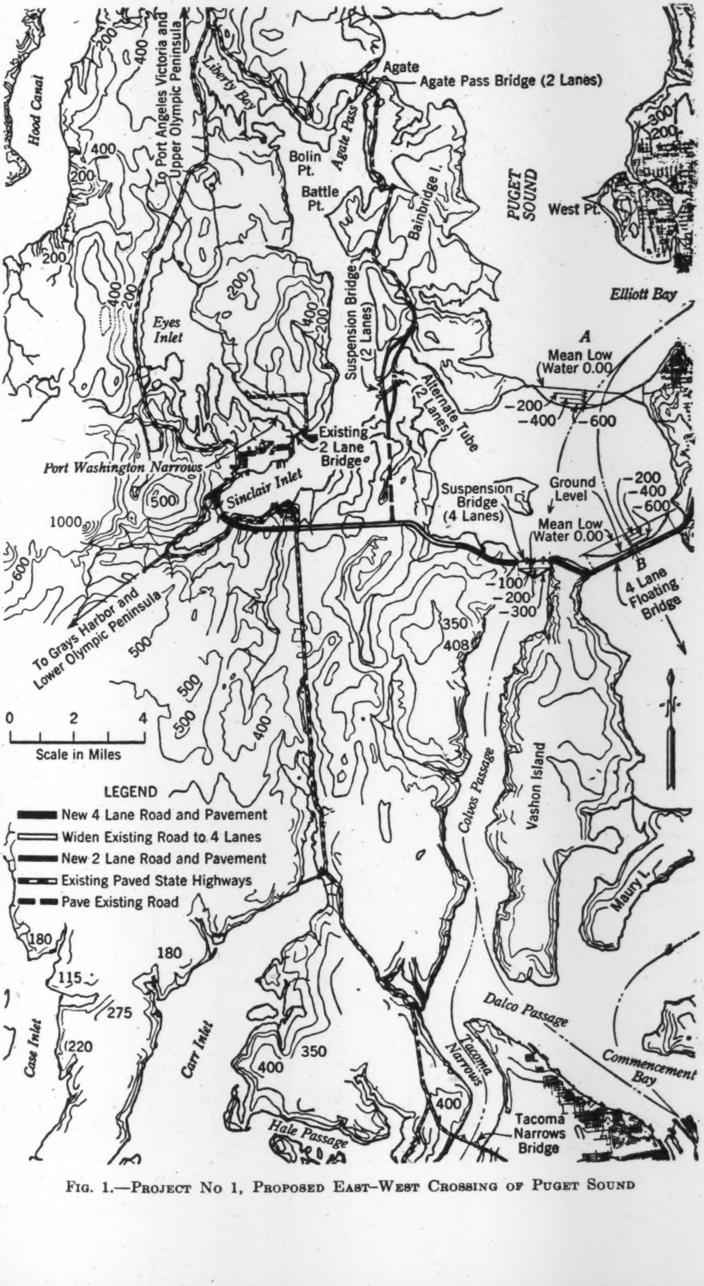
To Port Angeles Victoria and Upper Olympic Peninsula
Agate
Agate Pass Bridge (2 Lanes)
Liberty Bay
Agate Pass
Hood Canal
Bolin Pt.
Battle Pt.
Bainbridge I.
PUGET SOUND
West Pt.
Eyes Inlet
Suspension Bridge (2 Lanes)
Elliott Bay
A
Mean Low Water 0.00
Alternate Tube (2 Lanes)
−200
−400
−600
Existing 2 Lane Bridge
Port Washington Narrows
Sinclair Inlet
Suspension Bridge (4 Lanes)
Ground Level
Mean Low Water 0.00
B
4 Lane Floating Bridge
−100
−200
−300
To Grays Harbor and Lower Olympic Peninsula
350
408
500
1000
600
400
200
0 2 4
Scale in Miles
N
LEGEND
New 4 Lane Road and Pavement
Widen Existing Road to 4 Lanes
New 2 Lane Road and Pavement
Existing Paved State Highways
Pave Existing Road
Colvos Passage
Vashon Island
Maury I.
180
115
275
220
Case Inlet
Carr Inlet
Dalco Passage
Commencement Bay
Tacoma Narrows
Hale Passage
Tacoma Narrows Bridge

FIG. 1.—PROJECT NO 1, PROPOSED EAST–WEST CROSSING OF PUGET SOUND

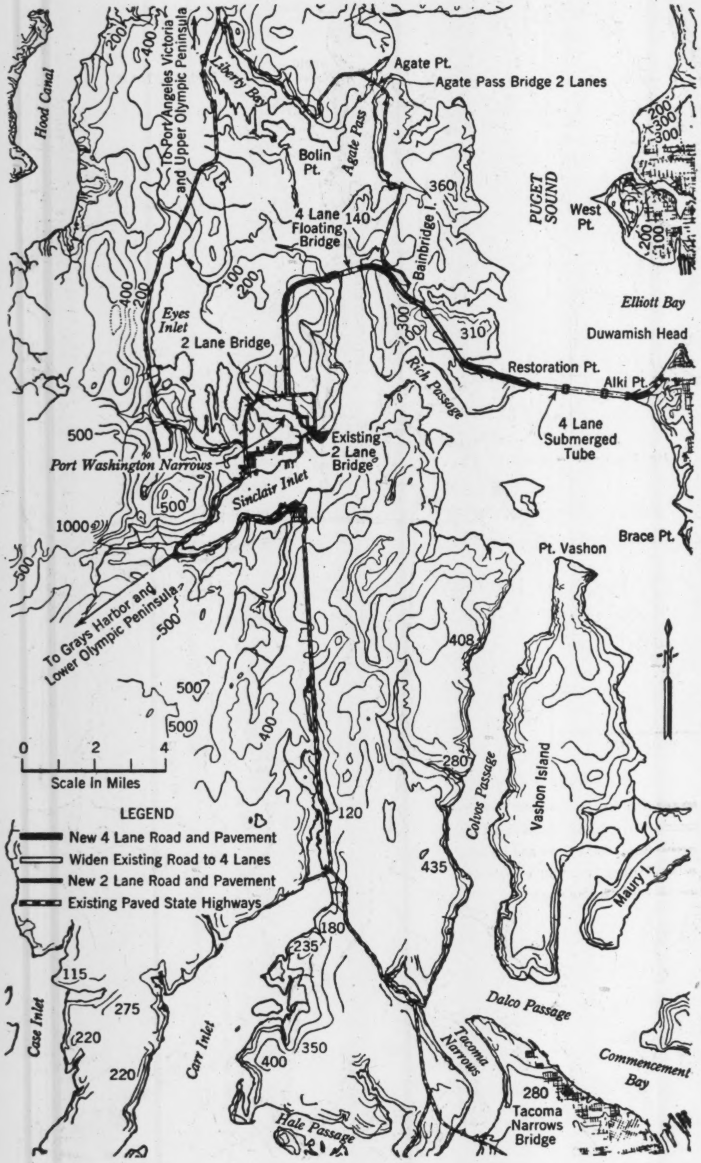
Hood Canal
200
400
To Port Angeles Victoria and Upper Olympic Peninsula
Liberty Bay
Agate Pt.
Agate Pass Bridge 2 Lanes
Agate Pass
Bolin Pt.
360
PUGET SOUND
200
300
300
West Pt.
200
100
4 Lane Floating Bridge
140
Bainbridge I.
100
200
400
200
Eyes Inlet
2 Lane Bridge
300
100
310
Elliott Bay
Duwamish Head
Rich Passage
Restoration Pt.
Alki Pt.
4 Lane Submerged Tube
500
Port Washington Narrows
Existing 2 Lane Bridge
Sinclair Inlet
500
1000
500
Brace Pt.
Pt. Vashon
To Grays Harbor and Lower Olympic Peninsula
500
408
N
500
500
400
0 2 4
Scale In Miles
280
Colvos Passage
Vashon Island
LEGEND
120
New 4 Lane Road and Pavement
Widen Existing Road to 4 Lanes
New 2 Lane Road and Pavement
Existing Paved State Highways
435
Maury I.
180
235
115
275
Case Inlet
220
220
Carr Inlet
350
400
Dalco Passage
Tacoma Narrows
Commencement Bay
280
Tacoma Narrows Bridge
Hale Passage

Fig. 2.—Project No. 2, Proposed East–West Crossing of Puget Sound

available to people who now refuse to be subjected to the inconvenience and delay of ferry traffic. With a permanent uninterrupted crossing available, it is reasonable to believe that the west shore would develop a tributary population of at least 200,000 within 3 years or 4 years after completion.

The City of Seattle, as well as the State of Washington, will experience a rapid growth due to the development of cheap power and large irrigation projects east of the mountains, and it is reasonable to believe that Seattle will have a metropolitan population of 650,000 within the next decade.

The present ferry system has been a constant subject of criticism for many years. Rates have been high and recently have become almost prohibitive. Thus, the west shore is facing a stalemate, as far as future development and the badly needed expansion from Seattle are concerned.

It was for this reason and furthermore in the interest of badly needed future development that such a study was undertaken. Other economic factors are discussed subsequently under the heading, "Economic Studies."

Physical Conditions

Puget Sound is an arm of the Pacific Ocean and therefore is salt water. The tidal range in the vicinity of Seattle is from −3.5 ft to approximately +14 ft, a range of 17.5 ft. Tidal velocities at the sites considered are relatively low due to the fact that the tidal prisms are greater than those in several other reaches of Puget Sound. A maximum velocity does not exceed 2 miles per hr.

Three sites have been considered. As shown in Figs. 1 and 2, water depths over distances of approximately 10,000 ft vary from 600 ft to 800 ft. As a consequence, all consideration of piers must be abandoned as impossible, and the water itself must be resorted to as a foundation.

Wind velocities of 80 miles per hr may be expected with normal maximum yearly velocities of from 55 miles per hr to 60 miles per hr. Wave action is choppy in character with maximum heights (trough to crest) of 8 ft or 9 ft. Ocean swells are practically dissipated before they reach Seattle. Depths of water for the study are taken from maps of the United States Coast and Geodetic Survey (USCGS); and the bottom condition, as indicated on these maps, is soft on the surface. The maps have been found to be very accurate but, of course, many of the foregoing conditions will require careful check before final determinations are made.

Results of Studies

Fundamental Considerations.—At the beginning of this study, seven fundamental considerations were self-evident:

(1) The water depths were so great that all thought of a structure supported by piers was useless, except in Colvos Passage, site A (see Fig. 1).

(2) Since the waters of Puget Sound constitute shipping lanes to Seattle, Tacoma, Olympia, and Bremerton, any structure must provide uninterrupted navigational channels of sufficient width and clearance for ocean-going vessels of any size.

(3) At all sites, some part or all of the structure must be supported by flotation.

(4) All anchorage members must be of such construction and material as would resist corrosion in sea water indefinitely.

(5) The added weight of the marine growths, present in Puget Sound waters, must be accounted for.

(6) Construction must insure against any considerable degree of leakage.

(7) A prospective future annual traffic of 10,000,000 vehicles requires four traffic lanes.

Possible Sites.—Examination of maps indicate that three possible sites are available. Two sites A and B are shown in Fig. 1. Site C to the north is almost identical with site A.

The center line of site A runs between Alki Point on the east shore to Restoration Point on Bainbridge Island, a distance of approximately 14,800 ft from shore to shore. A structure at this location would cross the only sea lane to Tacoma, Bremerton, and the southern end of Puget Sound. Therefore, it must permit unobstructed passage of ocean shipping. The depth of water, over 10,000 ft of the total distance, averages 700 ft with a maximum depth of 800 ft.

Sites B_1 and B_2 involve two main channels of Puget Sound. Each site leaves the easterly shore at Brace Point crossing over East Passage to Dolphin Point on Vashon Island, a total distance between shore lines of approximately 14,000 ft (see Fig. 1). For 11,000 ft of this total distance, the water depth averages 600 ft. From Vashon Island the line crosses Colvos Passage from Point Vashon to Point Southworth, a distance of approximately 4,600 ft. Maximum water depth at the center of the channel is 340 ft.

The center line of site C runs from West Point on the east shore to Skiff Point on Bainbridge Island, a distance of approximately 15,600 ft. This site (which is not shown in Figs. 1 and 2) is north of Elliott Bay and consequently crosses the only sea lane to Seattle, Tacoma, Bremerton, and the southern end of Puget Sound. Water depths are practically the same as those for site A. In any case, Bainbridge Island will be connected to the west mainland by bridges over either Port Orchard Bite or Rich Passage. From a traffic utility point of view, sites A and C are approximately equal and have a slight mileage advantage over site B. The latter, however, serves a larger potential population in that it affords a direct outlet from Vashon Island to the east and west.

Type of Proposed Structures at Site B.—Two types of structure are proposed over the East Passage (see Fig. 1). In one case, it is assumed that a permit can be obtained to close the east channel to ocean shipping with the exception of one 400-ft movable span in the main channel and that unobstructed passage of small boats near either shore line can be provided. The structure would consist of a floating type similar to, but of deeper draft and heavier wall construction than, the Lake Washington Bridge.[2] Main shipping lanes would be diverted to Colvos Passage where unobstructed passage would be provided under a 3,300-ft central span suspension bridge with the same vertical clearance as provided at the Tacoma Narrows Bridge.[3] The travel distance by boat to

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[2] "Building the World's Largest Floating Bridge," by Charles E. Andrew, *Civil Engineering*, January, 1940, p. 17.

[3] "Unusual Design Problems—Second Tacoma Narrows Bridge," by Charles E. Andrew, *Transactions*, ASCE, Vol. 114, 1949, p. 955.

Tacoma and Olympia is shorter by Colvos Passage than by the east channel and tidal velocities are comparable.

In the second case, if the aforementioned permit cannot be obtained, the east channel would be spanned by a twin tube structure with 50 ft of submerged clearance under mean lower low water and Colvos Passage would be spanned by a surface floating structure similar to the Lake Washington Bridge.

Type of Proposed Structures at Either Sites A or C.—Inasmuch as ocean shipping must be maintained at both sites A and C, the only type of structure possible is a floating tube providing 50 ft of clearance over its top. Obviously such a tube must receive its support through buoyancy as it would be impossible to rest it on the bottom of Puget Sound in water as deep as 800 ft. A filled base would be "out of reason" and would restrict the tidal prism to such an extent that tidal velocities would be prohibitive.

The three sites thus briefly described are the only ones that can be considered, because of the much greater length of crossings at any other points within a reasonable distance from the center of population of Seattle.

A surface floating structure over the east channel, combined with a long suspension span with high clearance over Colvos Passage, is without doubt the most conventional solution as both types of structures are well established by precedent (see Fig. 2). Such a structure requires an expenditure of funds considerably smaller than that required by other structures and, because of its location, is only slightly inferior as a traffic artery. It does have a bearing on water traffic, however, in that the main shipping lane south of Seattle must be shifted from the east passage to the west passage. The latter is not nearly so wide as the east passage although it is a more direct and slightly shorter route to Tacoma and a much more direct and shorter route to Olympia.

Whether or not a permit could be obtained for such a crossing has not yet been determined. No doubt a permit to build the submerged tube would be easier to obtain because it does not interfere with presently used navigation routes.

The Tube.—The writer recognized at the outset that the type of structure proposed may be considered rather bold; it presents many new problems in design, yet the general principles are relatively simple. Water is probably the most dependable of all foundation materials provided its laws of action (all of which are definitely known) are adhered to. The general principle that governs the design is restrained flotation.

Tube sections are designed as shown tentatively in Figs. 3 and 4 with such weight and dimension that, when bulkheaded at the ends, they will float well above water. Sufficient excess buoyancy of the tube section is provided so that it is equal to two and one-half times to three times the maximum amount of live load—which, in this case, includes vehicular traffic, weight of marine growths, vertical components of tidal anchors, etc.

To accomplish this purpose safely, careful computation of weight and displacement is necessary. An interesting side light is the maximum submerged weight that may be expected from marine growth.

A dense concrete, made of cement resistant to the harmful effect of sea water with deeply embedded reinforcement, has been selected. A minimum

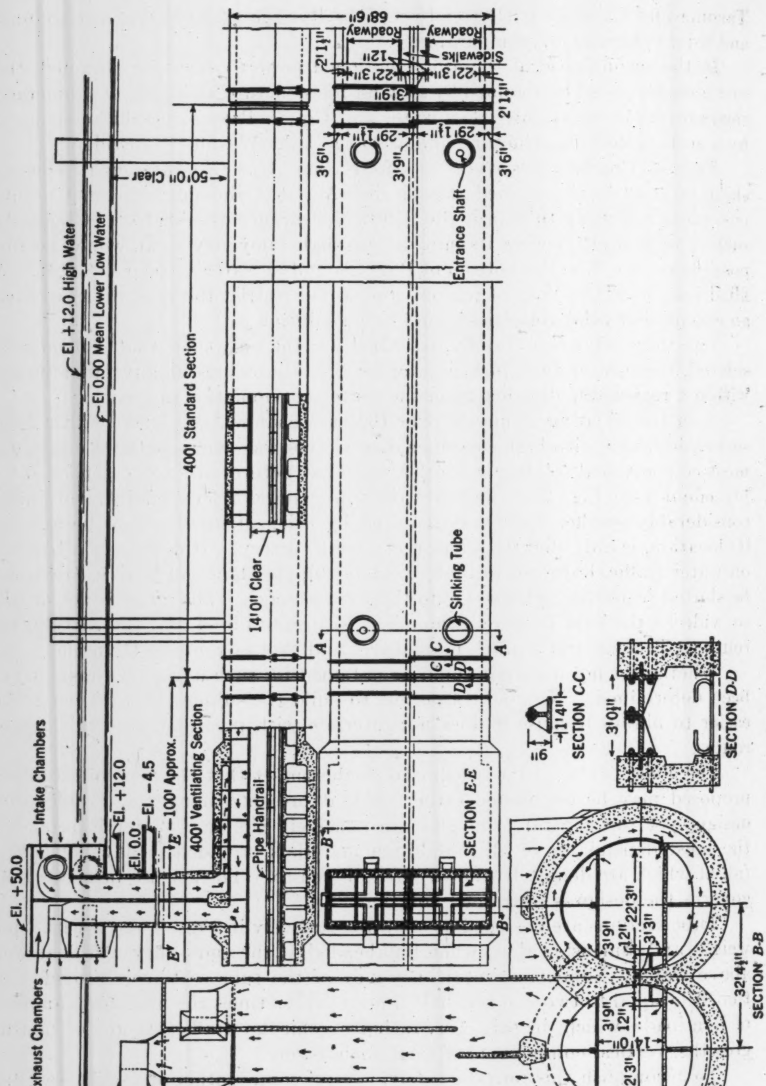
El. +50.0
Exhaust Chambers
Intake Chambers
El. +12.0
El. 0.0
El. −4.5
E
−100′ Approx.
400′ Ventilating Section
Pipe Handrail
B
SECTION E-E
400′ Standard Section
El +12.0 High Water
El 0.00 Mean Lower Low Water
50′0″ Clear
14′0″ Clear
Sinking Tube
Entrance Shaft
A
C
D
68′6″
Roadway
Roadway
Sidewalks
12″
22′3″
3′9″
29′1½″
2′1½″
3′6″
9″
1′4″
SECTION C-C
3′0″
SECTION D-D
3′9″
12″
3′3″
22′3″
32′4½″
SECTION B-B
14″
Exhaust Fans
El. +12.0
Mean Lower Low Water El. 0.00
El. −4.5

FIG. 3.—PRELIMINARY PLAN, FLOATING TWIN TUBES

thickness of tube shell equal to 3 ft is necessary to obtain the required weight that will keep the live-load anchors within a reasonable size (Fig. 4). In general, the shell thickness is determined by the weight and the accessibility for concrete placement rather than by stress considerations. Moments and shears, of course, determine the steel reinforcement. The required length of tube section (for example) at site A is 13,600 ft, with 2,300 ft of open shore approaches. This extreme length necessitates forced ventilation. A minimum of four ventilator towers is required—one at each end of the tube section forming the junction between floating and open shore approaches, and two in the main channel which must be floating with the tube section. These two floating ventilating towers offer an interesting and somewhat difficult problem. Fig. 3 shows a proposed solution.

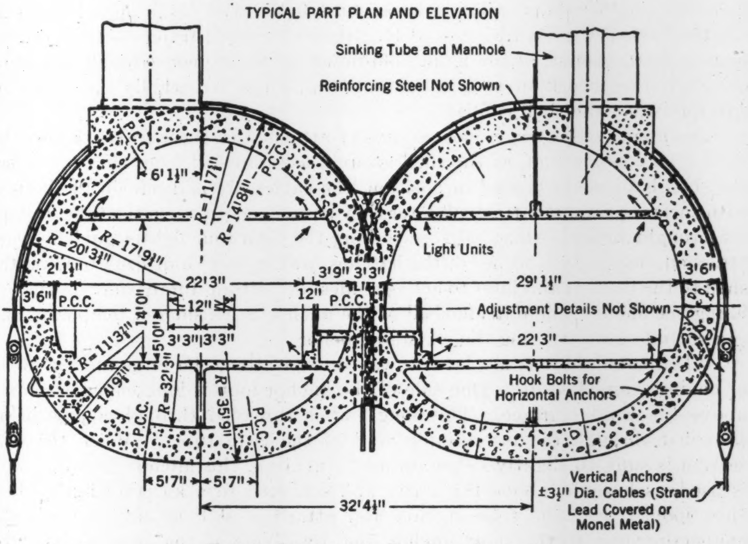
TYPICAL PART PLAN AND ELEVATION
Sinking Tube and Manhole
Reinforcing Steel Not Shown
P.C.C.
6'1½"
R=11'7½"
R=14'8½"
P.C.C.
R=20'3½"
R=17'9½"
2'1½"
Light Units
3'9" 3'3"
3'6"
22'3"
12"
29'1½"
3'6"
P.C.C.
14'0"
12"
P.C.C.
Adjustment Details Not Shown
5'0"
3'3" 3'3"
22'3"
R=11'3½"
R=14'9½"
R=32'3"
R=35'9"
P.C.C.
P.C.C.
Hook Bolts for Horizontal Anchors
5'7"
5'7"
32'4½"
Vertical Anchors ±3½" Dia. Cables (Strand Lead Covered or Monel Metal)

Fig. 4.—Section A-A, Enlarged Scale

Flotation and weight problems are fully susceptible to calculation. Cross sections of the towers below the powerhouses are made as small in area as possible within the area of tidal range to effect a minimum change in flotation due to the rise and fall of tide; and lightweight aggregate is used in the towers above the tube to decrease weight. The danger of ships colliding with these towers is of extreme importance; and, to guard against such a contingency, massive floating fenders of deep draft, firmly anchored at a considerable distance away from the towers, and completely surrounding them, are provided.

These fenders will weigh approximately 30,000 tons each. They not only will resist the impact of ship collision but will form a calm lagoon around the towers, impeding tidal velocity and eliminating wave pressures.

Live-Load Anchors.—Live-load anchors are required directly below the tube, on the bottom, to resist the tube uplift. For the most part, these anchors rest on a rather soft bottom material and are in water depths as great as 800 ft. The tentative design contemplates a flat shallow anchor spread over a comparatively large area. In principle, this anchor simulates the sole fish which lays on the bottom for long periods in swift tides without causing appreciable scour. It will have an underwater weight of from 500 tons to 600 tons and will exert a bottom pressure of approximately 500 lb per sq ft when attached to the submerged tube.

To construct and locate this type of anchor with reasonable accuracy, it is first necessary to design and construct a steel shell on shore which can be floated to the proper position and lowered to the bottom by derrick after water is admitted. The weight of such a shell will approximate from 60 tons to 70 tons under water. The shell is constructed with large open hatches in the top. Heavy cross frames and longitudinal trussed frames intersecting at points of attachment of the main hold-down cables are constructed as part of the shell framework in such a manner that they are wholly embedded in concrete when the shell is filled.

The main hold-down cables are finally attached to these frames before the shell is submerged and, as the shell is sunk, are unreeled from barges. When the shell is properly landed in position, the cables are laid out on the bottom with surface pendants attached; later they are picked up when the tube sections are brought into position. Before sinking the shell four lighter guidelines are attached, one at each corner of the hatches, which may be utilized to lower the shell to the bottom and later to act as guides for bottom-dump concrete buckets filling the shell. The outside shell of the anchor is considered temporary and can corrode away leaving the concrete anchor.

When tube sections are later towed into position, the large anchor cables or chains are picked up. One side of the anchor loop is long enough to reach above the water surface. This long end is threaded through a pipe insert provided through the intersection wall of the twin tubes. When the tube section is sunk to slightly below its final elevation, this anchor line (if a cable is used) is brought above the water and socketed to a known length. It is then looped over the tube section and attached with an adjustable-length connecting bar, to the short anchor line, by divers at the approximate level shown in Fig. 3. When all lines to a given tube section are connected, water is pumped from the tube and the section rises by flotation into the loops formed by the anchor lines. Tube levels are then checked and adjusted to the proper elevation at adjusting bars.

Lateral Anchors.—To hold the tube in alinement laterally against flood and neap tides, heavy bottom anchors, constructed in the same general manner as the live anchors, are provided.

A serious difficulty is the determination of what material should be used in both the live-load anchor lines and the lateral anchor lines since the water is salt and exceedingly corrosive. Maintenance of the vertical live-load anchor lines presents the greatest problem as it would be exceedingly difficult, if not impossible, to renew the lines. Two types of material have been considered—

(*a*) monel metal chains or cables and (*b*) lead shielded cables. Recent developments with titanium should be considered, although the cost would be excessive even if the metal were further developed.

It would be impracticable to use chains for lateral anchor lines on account of the weight and length; however, it is possible to develop a detail by which these lines can be renewed when necessary and the use of lead shielded steel cables or monel cables is proposed. In any case, extensive research will be necessary when, and if, funds and the opportunity to build this structure become available. Monel metal has been exposed successfully in salt water for many years and is probably the most promising material. It is felt that modern manufacturing, ingenuity, and metallurgy can satisfactorily solve the problem of corrosion.

General Plan and Method of Tube Construction.—The plan contemplates a twin tube of the general cross section shown in Fig. 4, providing four traffic lanes. The over-all length of the structure (site A) is approximately 15,900 ft, consisting of 1,150 ft of open approach at each end extending out to a point where the water is approximately 50 ft deep at low tide, and to the shore ventilating towers which are founded either on piling or on rock. The tube section, consisting of thirty-four sections, 400 ft long, extends across the sound between the shore ventilating towers for a distance of 13,600 ft.

The descending grade lines through the open approach structures are approximately 4%. From the shore ventilating towers, the grade lines continue on a 2.6% grade to points in the channel where a 50-ft depth below mean low water is reached over the crown of the tubes. A minimum of two additional ventilating towers is necessary in the main channel to divide the full length of the tubes into four sections of 3,400 ft each for the purpose of ventilation.

From a construction point of view, the proposed tube involves heavy special equipment and the solution and careful handling of details; yet, when analyzed, these features do not appear particularly difficult. The completed structure requires approximately 550,000 cu yd of concrete, of which 400,000 cu yd would be poured at a fixed mixing plant situated at the graving docks where the 400-ft sections of the twin tubes would be fabricated. As each 400-ft section is completed and set, bulkheads would be placed at each end. Water would then be admitted to the graving dock and the section floated into the sound.

Vertical sinking tubes are erected near each end of each tube (four in all). These tubes are attached to a connection framed into the tops of the tube sections and are of sufficient length to extend well above the surface of the water when the tube section is submerged to a depth of 50 ft under mean low water.

Before moving the section into deep water, tests for leakage are made by pumping a sufficient amount of water into the section to submerge it. Each section is then lowered by admitting additional water into the sinking tubes until the top of the tube section is 50 ft below water. This preliminary test is made at a point where the water is not deeper than from 90 ft to 100 ft. The section is permitted to stay in the submerged position for several days and then is raised to the surface by pumping water from the sinking tubes. If no leakage has taken place, the section is ready to tow into its final position in the structure. Prior to placing any section, both the live-load anchors and the lateral anchors with lines attached are set in place. The open approach structures, including the shore ventilating towers, will also be completed while the graving dock is being constructed and while the first tube sections are being fabricated and tested.

Tube placement will begin with the first sections from each shore and will advance toward the center of the crossing. The first shore section will be attached to an especially designed expansion detail framed into the outer walls of the shore ventilating towers. This joint will not only permit longitudinal expansion but also slight lateral and vertical movement. The constant temperature of the sound waters will keep the longitudinal expansion small, the greatest possibility of change being near the ventilating towers. In this particular location, however, seasonal variation in air temperature is not great.

In principle, the final placement of tube sections is not difficult. The sequence would be as follows: A tested bulkheaded section is towed into position and the long side of the vertical anchor lines is threaded through the hawse pipes at each end of the section. Water is then pumped into the tube section until it is submerged, the lateral anchor lines having been first attached. The tubes are then sunk slowly by admitting water into them until a depth of slightly more than 50 ft over the top of the section is reached. The vertical live-load anchor line is then looped over the top and connected to the short end by divers at a carefully predetermined length.

The tube is lifted into the anchor line loops by pumping water from the sinking tubes, and preliminary levels are taken. If the tube is found to be either high or low in relation to its proper elevation, it is again lowered to slack the anchor lines and the proper adjustment is made at the connecting bars in successive operations until the proper elevation is reached.

The first shore sections are attached through the expansion detail to the wall of the ventilating tower which takes the vertical uplift of the section at the shore end while anchor lines resist the uplift at the outer or seaward end. Succeeding sections are sunk in a similar manner and attached by heavy cast-steel rings to the section already in place, proceeding from each end toward the center of the channel. A tentative connection is shown in Fig. 3. As each successive section is placed, bulkheads are removed via the shore ends and reused in other sections to follow. The connections are then completed by filling in the tube walls inside the cast rings, reinforcing steel being lapped to establish continuity.

To gain weight for sinking, and to reduce, as far as possible, the amount of water which would be required to overcome buoyancy in the sinking process, all the interior concrete (except that at the end connections) such as roadway slabs and curtain walls, would be poured at the graving dock.

On first thought, the matter of maintaining proper trim in sinking a freely floating section of this size and weight seems to involve a difficult problem. A tentative schedule for performing this task is as follows: First, removable steel tanks are installed while the section is in the dock. Tanks that would contain from 70% to 75% of the sinking water required would be placed in the fresh-air duct below the floor, and tanks with a capacity somewhat greater than the remainder required would be placed on the roadway deck level.

The tanks in the fresh-air duct would be completely filled before the section is launched. The roadway tanks would be constructed with closely spaced watertight bulkheads dividing them into a series of small tanks. All tanks would be connected separately to a main pump line and an air line through valves; and the main water line and the air line would extend above the surface of the water through an entrance tube concentric with, and inside of one of, the sinking tubes.

This concentric tube would afford entrance to the section from the water surface at all times until the section is finally connected and entrance is established from the shore. The openings through the tube shell would then be closed by pouring concrete and the sinking tube would be removed for reuse on the following sections. As the test sinking starts, water is admitted to the closely bulkheaded roadway tanks, each compartment being filled symmetrically on each side of the longitudinal center of the tube beginning at the ends. In this manner, with all necessary tanks full, no end shifting of water can occur—final submergence, sinking, and trimming being accomplished by injecting additional water in the sinking tubes.

When the tube is finally sunk and the live-load anchor lines are connected, water is removed by compressed air in the reverse order of filling. The end connection is then made to the previous section; the sinking tubes are removed; and the bulkheads at the connected end are taken out through the shore end for reuse. Tanks are removed in the same manner. The end connection is then completed by filling in the ring of concrete inside the cast-steel connecting ring.

It is not possible to discuss further details in this paper. In fact, final plans have not been developed, and cannot be developed, until such time as further funds are appropriated to make a full investigation. The paper is intended merely to outline the fundamental principles of designing and constructing a structure which, in the writer's opinion, is the only type of crossing that can be devised to meet the physical conditions presently existing across Puget Sound.

ECONOMIC STUDIES

Quite extensive studies have been made to determine the economic feasibility of bridging Puget Sound between Seattle and the west shore. Census data are discussed herein under the heading, "Territory and Population to Be Served." Geographically, Seattle is situated on a relatively long narrow strip of land running north and south between Puget Sound on the west, and Lake Washington on the east. Its room for expansion is extremely limited by this fact. The unprecedented development to the east of Lake Washington and on Mercer Island, since the Lake Washington Bridge was completed, is evidence of the demand for more room. The bonds for the construction of that bridge were fully amortized in less than 10 years.

A situation now exists to the west of Seattle across Puget Sound, similar and parallel to that which existed on Lake Washington before the bridge was built, except that there is a far greater amount of existing traffic west of Seattle. Hundreds of miles of the most desirable waterfront property is undeveloped with almost an unlimited amount of residential property available as far as urban development is concerned.

It is impossible to serve this area by ferries except at toll rates which are so high that further development of the area is practically impossible and expansion of Seattle in this direction is eliminated. With the prospect of free transportation within a reasonable period of years the picture is entirely changed. Property values alone will increase many times the cost of bridging Puget Sound, and this territory will be brought within reasonable driving distance of the center of Seattle.

Preliminary estimates of cost for bridging Puget Sound vary from $50,000,000 to $61,000,000 depending on the type of structure selected, the maximum cost being for the tube shown in Fig. 1.

For comparison of transportation costs between perpetual ferry operation and the proposed structures, a period of 35 years, including the period of design and construction, has been used. Costs include maintenance and operation. The average cost of transportation, including maintenance and operation, in dollars per traffic unit for continuous ferry service for a period of 35 years is found to be $0.4766. The comparable average cost of unit transportation for the same period for the proposed structure shown in Fig. 2 was found to be $0.1955. After 35 years, the cost would decrease to $0.015 per unit for maintenance. All bonds would be retired from 21 years to 22 years after issue. For the proposed structure shown in Fig. 1, unit costs for the first 35 years would average $0.2297. All bonds would be retired approximately 23 years after issue. After 35 years, the average cost of continuous ferry operation would decrease to approximately $0.35 per unit and would continue at that rate indefinitely. A unit of transportation is taken as a value representing one of the total number of passengers, automobiles, trucks, and buses crossing.

CONCLUSION

From the foregoing comparable estimates of average unit transportation costs which the writer considers as conservatively representing the economic picture, it is quite evident that bridging Puget Sound by either method is financially sound. The proposed structures reduce unit costs over the first 35 years, respectively, to 41% and 48% of ferry operation cost and after 35 years to 2.2% and 8.4%.

The type of crossing proposed may be regarded as rather daring and unusual. It involves many tricky problems and details. However, it is believed worthy of consideration and, in the end, entirely feasible.

The physical conditions existing at the available sites impose difficulties that would not be present in many other cases. For instance, if the water were fresh, problems of metal corrosion would be very much simplified. If water depths were less (say, from 150 ft deep to 250 ft deep), anchorage problems would be much easier. Crossings as long as 6,000 ft or 7,000 ft would not require floating ventilating towers, etc. In such cases the feasibility of the design is even more evident.

DISCUSSION

Robert S. Mayo,[4] M. ASCE.—From an engineering standpoint, the idea presented in this paper is feasible; but, from an economic standpoint, it appears questionable that the future traffic would ever repay the great investment involved.

Several criticisms suggest themselves: (1) The ventilation of the 3-mile vehicular tunnel would be quite a problem and would be a continuing expense; (2) the external inspection of the tube and of the anchor cables would be expensive and, should repairs ever become necessary, the cost would be prohibitive; and (3) the most serious, should a leak ever develop in this tube, as the result of accident or of sabotage, the entire structure, 3 miles long, would be sunk beyond recovery in water from 600 ft to 800 ft deep and any motorists in the tube at that time would be drowned like rats in a trap.

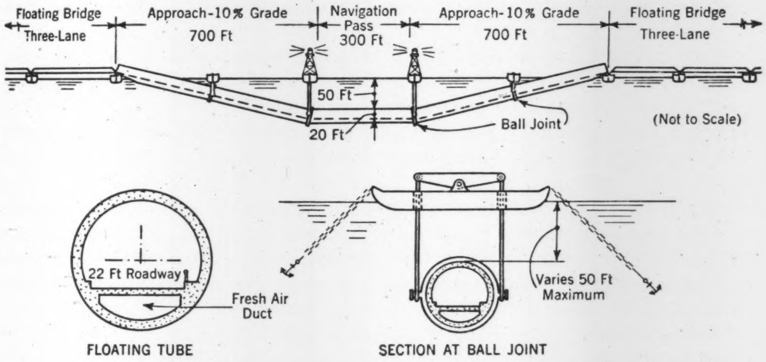
Floating Bridge
Three-Lane
Approach-10% Grade
700 Ft
Navigation
Pass
300 Ft
Approach-10% Grade
700 Ft
Floating Bridge
Three-Lane
50 Ft
20 Ft
Ball Joint
(Not to Scale)
22 Ft Roadway
Fresh Air
Duct
Varies 50 Ft
Maximum
FLOATING TUBE
SECTION AT BALL JOINT

Fig. 5

Mr. Andrew mentions his investigation of a floating bridge. Why not combine the ideas of the floating tube and the floating bridge as illustrated by Fig. 5? This crossing would have a total length of approximately 2.75 miles and it could be made up with large boatlike pontons, with a three-lane (or four-lane) bridge of economic span length. There would be no particular problem in the inspection and maintenance of such a floating bridge. Furthermore, the anchor and the anchor cables could be removed periodically for inspection.

For the navigation pass, the writer suggests the floating tube. The total length, including the approaches would be about one third of a mile. This tube would be comprised of five sections, each about 350 ft long, suspended from pontons. Each section is connected to the next section by a ball-and-socket joint so that there would be no internal strains due to twisting. These ball-and-socket joints would have a secondary advantage. They would allow the tube to be raised almost to the surface for inspection; and, should repairs ever be necessary, the entire floating tube could be taken ashore to a dry dock, service being maintained meanwhile by a floating drawspan.

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[4] Civ. Engr., Lancaster, Pa.

Mr. Andrew may object to funneling this traffic through one two-lane tube for the water crossing. The Pennsylvania Turnpike probably now carries more traffic than this proposed crossing will ever handle in its foreseeable life. The Pennsylvania Turnpike is four lanes throughout, except when passing through the seven tunnels, where it is reduced to two lanes. These two-lane tunnels handle all the traffic (about 4,000,000 vehicles per year) without noticeable delay.

The author is to be congratulated on his bold and novel approach to a very difficult problem.

Homer M. Hadley,[5] M. ASCE.—The proposal for crossing Puget Sound at Seattle (Wash.) by a floating tunnel is not to be criticized so much in terms of abstract thinking and conception as it is for the practical aspects of execution and financing. Assuredly a floating tunnel, anchored and restrained from vertical or horizontal movement and possessed of buoyancy in excess of all possible downward loads to come upon it, is a warranted and justifiable idea. However, if its construction involves such grave risks and hazards that contractors would hesitate to bid on the work, and bondsmen would hesitate to write surety bonds, and if neither would do so except at almost preposterously high figures; if continuous risks and hazards made bond buyers reluctant or unwilling to buy bonds, or made insurance men unwilling to write insurance on the structure—then, regardless of abstract merit, the project is impractical and, as such, may be dismissed from further consideration. There is little point in talking about a project on which bids, surety bonds, and sale of revenue bonds and insurance are not obtainable except at what would appear to be fantastic cost. This, to the writer, appears to be the case here.

Anyone familiar with Puget Sound geography, or examining United States Coast and Geodetic Survey chart No. 6446, knows (or will know) that there are three and only three narrow places in Puget Sound in the vicinity of Seattle. These are the sites which Mr. Andrew—and indeed almost anyone else—selects as possible sites for a crossing. It is unfortunate that Mr. Andrew's maps of Figs. 1 and 2 include only the westerly part of Seattle. They do not show the full city and its central business district, they do not show the full size of the waterway to the east of Vashon Island, and they do not designate the City of Bremerton, Wash., the principal center of population and industry on the west side of Pudget Sound. If the business center of Seattle were shown and the City of Bremerton were designated, it would be apparent to everyone looking at these maps that the Alki Point-Restoration Point site which Mr. Andrew designates as site A lies closest to a straight line between the centers of the two cities. Both of his other two routes depart quite widely, and by a considerable number of miles, from this desirable straight line. Moreover site A is the only location in solid rock anywhere along the entire length of Puget Sound, a fact of no mean advantage whatever is built there.

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[5] Cons. Engr., Seattle, Wash.

The proposal with respect to route B—to close off the main channel to the east of Vashon Island, now used by practically all large ships going to Tacoma, Wash., with a barrier such as a pontoon bridge of the Lake Washington type "but of deeper draft and heavier wall construction," and having only a 400-ft movable span—certainly deserves the characterizations Mr. Andrew bestows on his submerged tunnel (under the headings, "Results of Studies: The Tube" and "Conclusion"): "* * * rather bold * * *" and "* * * rather daring and unusual." To close the main, wide marine approach to Tacoma and compel shipping to use Colvos Passage on the west side of Vashon Island, one third to one quarter as wide, is certainly a novel thought.

It is true that Colvos Passage is a vastly wider and more spacious waterway than many a river port in the world enjoys. Nevertheless the fact remains that Tacoma now possesses both of these free and unobstructed approaches, that ocean shipping now uses the east passage, that the United States Navy base at Tacoma now possesses both approaches, that in foggy weather (of which there is considerable) a wide channel is much to be preferred to a narrow one, and, lastly, that approval for such a closure must be obtained from the United States War Department before it can be consummated. Bare sketches showing no more than the general features of the proposed construction, and its horizontal and vertical clearances, and requiring the expenditure of only a few dollars to produce, would be sufficient to accompany an application for War Department approval. A public hearing would follow, presently a decision would be forthcoming, and the question would be resolved. Since in all probability the decision would be adverse to the pontoon construction, it may be stated that all three of Mr. Andrew's sites involve the use of floating tunnels.

What is wrong with the floating tunnel idea? It is not original with Mr. Andrew. A Swedish patent was issued on the idea in 1890. In the 1939 report on the proposed Mackinac Straits bridge (Michigan) it is stated that a submerged tunnel was proposed for that site in 1920. Undoubtedly it has been considered at various times in the past, for sites much more favorable than that across Puget Sound at Seattle. Why has not one floating tunnel, short or long, that relies on "restrained buoyancy" for stability ever been built anywhere in the world? Undoubtedly this is because of the dangers and hazards involved in constructing a floating tunnel and because of the dangers and hazards inherent in it ever afterward even if it is built successfully.

What is the essential nature of a submerged tunnel of this type? It is a tunnel, a continuous, unbroken, uninterrupted hole from one end to the other surrounded on all sides by water, having no fixed and unyielding supports anywhere along its length and, of course, from its very use and service, devoid of transverse watertight bulkheads. At all times and everywhere along its length it is possessed of the urge to break loose from the anchor cables that hold it down and to lunge upward to the surface and to the atmosphere. Well, what can happen to it? It is anchored down, and nothing can happen to it. So likewise, nothing could happen to the Tacoma Narrows Bridge before its spectacular failure. This is certain, however: If anything does happen to rupture the shell of the floating tunnel—if anchor lines slip or fail, if anchors resembling the sole fish settle into the "soft on the surface" bottom of Puget Sound under their pressure of "approximately 500 lb per sq ft," if from any cause whatever water ever breaks through into the interior and floods the tunnel—then all buoyancy is lost and inevitably there follows complete, utter, 100% catastrophic loss of everything from shore to shore! Ruin is complete and overwhelming!

Who wants to write insurance on such a structure? What contractor wants to assemble, connect, and join the sections for such a tunnel? They are 400 ft long, 30 ft high; weigh thousands of tons out of water; and lose none of their mass, inertia, or size submerged. There is nothing to set them down upon and they must be brought into true alinement, both horizontal and vertical, for connection while tidal currents whose "maximum velocity does not exceed 2 miles per hr" press upon them and tend to throw them out of line. Then, once joined, all bulkheads except the last are removed and taken to shore through the completed portion of the tunnel which now possesses all the hazards plus that of a single removeable end bulkhead which the completed tunnel would have.

Who wants to enter into a binding and irrevocable contract to construct such a tunnel? Who wants to write the surety bond that, in the event the original contractor fails to fulfil the contract, his company will fulfil it? In Lewis Carroll's poem, "The Walrus and the Carpenter," the walrus invited the oysters to come and walk with them along the briny beach. Thereat

> "The eldest Oyster looked at him,
> But not a word he said.
> The eldest Oyster winked his eye
> And shook his heavy head—
> Meaning to say he did not choose
> To leave the Oyster-bed."

It is highly probable that contractors, bondsmen, and insurance men will react as did the eldest oyster. If they feel any impulse to do otherwise they had better get the poem and read what happened to the young oysters who did accept the invitation.

In this paper Mr. Andrew refers to the remarkable financial success of the Lake Washington Floating Bridge and the great developments that have occurred on Mercer Island and the east shore of Lake Washington. Reference can likewise be made to the success of the Transbay Bridge at San Francisco, Calif. What do these facts prove? Undoubtedly similar developments would follow the construction of a crossing of Puget Sound at Seattle, but the degree of similarity is conditioned by and dependent on the degree of similarity that the conditions on the west side of the Sound bear to those on Mercer Island and the east shore of Lake Washington, or to those on the east side of San Francisco Bay. Similarity of development likewise depends on the toll to be charged for passage. No toll as low as 25¢, reduced by commutation rates, will be charged for crossing Puget Sound with a $50,000,000 or $60,000,000 or $125,000,000 structure. This is simply because there is not the volume of traffic there now, or to be foreseen in the future, to justify such moderate transbay or floating bridge rates.

The sole industrial city in the entire area west of Puget Sound and east of Hood Canal is Bremerton (see Fig. 6) and its sole great industry is the naval shipyard there. This is the largest ship repair base north of San Francisco. During the World War II Bremerton grew tremendously, attained a population of 75,000. By 1950 it had shrunk to somewhat less than 30,000. Not until the war years was continuous rail connection established between Bremerton and the remainder of the state; and this connection extends southwesterly for 55 miles before it can swing back in an easterly direction toward the Puget Sound cities and the railroads across the Cascade Mountains. No continuous rail line extends northward either to the east or to the west of the great block of Olympic Mountains, which occupy most of the area between Hood Canal and the Pacific Ocean. The Chicago, Milwaukee, St. Paul and Pacific Railway has a road along the strip of benchland to the north of the Olympics and south of the Straits of Juan de Fuca, with car ferry conncetion to Seattle. Here in this north Olympic zone of Washington are three small cities: Port Angeles (about 12,000 population) with lumber mills and paper mills; Port Townsend (about 5,500 population) with an army post and a paper mill; Sequim (population about 1,000) with a shopping center for the surrounding agricultural and dairy district. By air Port Angeles is 60 miles distant from Seattle.

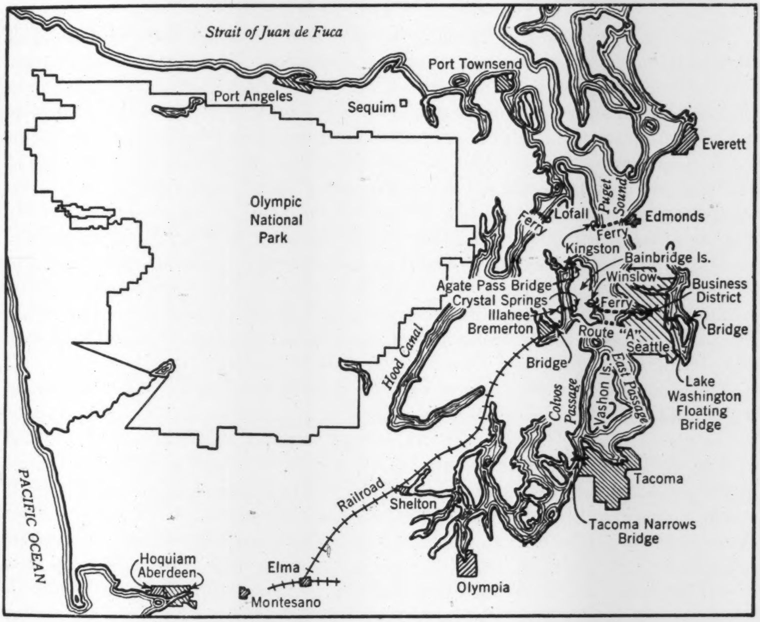
Strait of Juan de Fuca
Port Townsend
Port Angeles
Sequim
Everett
Olympic National Park
Ferry
Lofall
Puget Sound
Edmonds
Ferry
Kingston
Bainbridge Is.
Winslow
Business District
Agate Pass Bridge
Crystal Springs
Illahee
Ferry
Bremerton
Route "A"
Bridge
Seattle
Hood Canal
Bridge
Colvos Passage
Vashon Is.
East Passage
Lake Washington Floating Bridge
Tacoma
Railroad
Shelton
Tacoma Narrows Bridge
PACIFIC OCEAN
Hoquiam
Aberdeen
Elma
Olympia
Montesano

FIG. 6.—MAP SHOWING ENTIRE WESTERN AREA TO BE SERVED BY TRANS-SOUND CROSSING AT SEATTLE, WASH.

Most of the low lying areas between Puget Sound and the Pacific Ocean are covered with glacial deposits of clays, sands, and gravels. There are rich farms in certain areas, not so rich in others. There is magnificent scenery everywhere. There are hundreds of miles of beautiful waterfronts along Puget Sound, Hood Canal, and all the bays and inlets and passages that cut and indent the area. There are the ocean shores and those of the Straits of Juan de Fuca. Some day when the Columbia River is bridged at Astoria, Ore., and a few miles of road are built through easy country, a main north-south Pacific highway (coming up from San Francisco through the redwoods and along the wondrously beautiful Oregon Coast) can be brought into Bremerton and will contribute traffic to a trans-Sound to Seattle crossing.

The fact is, however, if anything is to be accomplished, both routes and all structures must be carefully studied and reviewed. Two examples of this fact are to be seen in Fig. 2. Since Bremerton is the main center of population to the west of Puget Sound; since Alki Point-Restoration Point route is the most direct route from the center of Seattle to Bremerton and for connection with all other points and areas of the entire Olympic Peninsula except Vashon Island and the territory immediately south of Rich Passage; and since neither of the other two routes offer any advantages to compensate for their considerably greater mileage—it may be accepted by everyone that the Alki Point route is the best one and is the one on which attention should be focussed and concentrated. Again, from Fig. 2, it is seen that Mr. Andrew's suggested highway swings northwesterly from Restoration Point almost halfway up Bainbridge Island to cross to the west mainland on a "4-lane pontoon bridge" and then completing the big loop by swinging back to Bremerton.

One would infer that there existed some strong and compelling reason for such a location. Actually there is none. About 2 or 2½ miles south of the pontoon bridge, the map shows a small creek emptying into the channel on the west side. Here is the shallowest water anywhere along the west side of Bainbridge Island, with 97 ft of extreme depth, most of it from 60 ft to 70 ft and a quarter mile on the west side not deeper than 18 ft. Since the water deepens both to the north and to the south of this line (which lies squarely athwart the north-south path of the glacier that once plowed through this area), it is a fair assumption that this submerged ridge is of firm, hard material.

At this Crystal Springs-Illahee location a structure on fixed unyielding foundations is possible. Why introduce 4 or 5 miles of useless distance between Seattle and Bremerton in order to build a pontoon bridge with a 500-ft required opening in it in tidal waters having a range of 17.5 ft? Fig. 2 is in error at the Illahee location in not showing the little creek forking and providing a southward extending branch up which the road could ascend to the top of the benchland on about a 5% grade.

Despite his negative attitude toward Mr. Andrew's proposals, the writer is sympathetic to the trans-Sound crossing project and would like to see it consummated. He believes it can be done, at not too great a cost, by construction free from great hazard, on which contractors would have no hesitancy in bidding; but it can only be effected by holding costs to a minimum and it will not be done with floating tunnels, regardless of Mr. Andrew's judgment that a floating tunnel is the "only possible method" of crossing Puget Sound.

Charles E. Andrew,[6] M. ASCE.—Answering Mr. Mayo's criticism, the writer believes that an expenditure as high as $75,000,000 is justifiable from an economic point of view. The total income from ferries in past years has been as high as $7,000,000, with a conservative average income of $5,500,000; 72% this sum is derived from the ferries that would be replaced by a sound crossing, or approximately $4,000,000. Induced traffic via the bridge would increase total traffic at least 100% and would result in an average income of about $5,400,000 at tolls approximately 35% lower than the 1948 tolls.

An income of $5,400,000 in the first year of operation will take care of debt service and will retire a bond issue of $90,000,000 in 20 years or less; also, of course, it will retire a $60,000,000 to $65,000,000 bond issue at a toll rate of 50% or more below 1948 rates.

It is of interest to note that the population of the State of Washington has increased almost 40% in the decade, 1940–1950; the ferry traffic at The Narrows has increased at the rate of 5.9% compounded since 1942; and automobile registration has increased at the rate of 8% a year in the 3 years since 1947.

Ventilation is a problem, of course, but has been solved successfully in other tubes. Its cost is taken care of in operating expense.

Metals which are practically everlasting in salt water are available for vertical anchor lines. Lateral anchors can be designed for renewal when necessary. External inspection by divers is quite practical in depths of 90 ft, which is the approximate depth of the bottom of the tubes. Deterioration of the heavy walls (which are 3 ft to 3.5 ft with steel imbedded 6 in.) should not be more rapid than that of walls of similar bottom tubes already constructed. Likewise, the danger of leaks is no greater than in existing tubes subject to greater hydrostatic head.

As to Mr. Mayo's suggestion for a tube in connection with a floating structure, a similar layout is under consideration, except for the use of twin tubes with longer opening and less approach grades. A single tube is insufficient to handle potential traffic. The tube suggested, however, does present practically the same hazards as those mentioned by Mr. Mayo. He indicated no method of ventilation, and ventilation would be necessary.

In answering Mr. Hadley's comments, it is necessary first to familiarize the reader with the conditions under which the investigation of bridging Puget Sound was made.

The writer was requested by the Director of Highways to make the investigation for the purpose of ascertaining whether or not it would be possible and feasible to cross Puget Sound by means of an uninterrupted highway, and in so doing, eliminate the present ferry system. The prime object of such an undertaking is to extend the present transcontinental highway (which now ends in Seattle) to the Pacific Ocean. Then it will serve not only the immediate territory on the west shore but also the entire vast area west of the sound, extending to the ocean. It will thus develop an area that is isolated by the handicap of ferry transportation.

It makes no practical difference whether the most northerly, central, or southerly location is selected. In constructing major highways across mountain ranges, the planner must use those passes which nature has provided whether they be north or south or east or west of the most direct line. In this study three possible sites are available and are being considered. In the writer's opinion, the population immediately adjacent to the west shore is not the whole picture by any means. However, if it actually were, the distance between the business center of Seattle and Bremerton by route A is only $1\frac{1}{2}$ miles less than by route B, whereas the distance between the center of population of Seattle and Bremerton is considerably less by route C than by either routes A or B. Thus, there is little choice as far as distances are concerned. It then becomes a question of cost of transportation (tolls). In this respect route B has the advantage because of its lesser cost.

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[6] Chf. Cons. Engr., Washington Toll Bridge Authority, Tacoma, Wash.

At the time this paper was written, the writer had indicated no preference between locations. On further study he has given preference to site B because it incorporates proved construction and less cost, while at the same time it serves more traffic. He does not condemn the use of floating tubes, however, if they should become necessary to solve the problem because of any possibility that they are dangerous, or because they cannot be financed, or because contractors will not undertake their construction. That is one man's opinion. Why was it that engineers experienced in major bridge construction condemned the Lake Washington Bridge and predicted dire consequences; yet it was successfully built and paid for? Why was it that many prominent men and engineers insisted that it was dangerous and impossible to build the San Francisco-Oakland Bay Bridge, in California, because of foundation depths far in excess of any built prior to that date? Why is a floating tube subject to hydrostatic pressures approximating 85 ft any more dangerous than the many bottom tubes now in existence, which are subject to greater heads and are held down by rock or earth instead of anchors? They require the same method of connection, the same alinement, and the same hazard of assembly.

As to the question of navigation, if site B is adopted, the writer contends that a floating bridge across the east channel will result in very little inconvenience to shipping to Tacoma or Olympia, Wash. Colvos Passage affords the shortest and most direct route to both cities. It is a mile wide and is 300 ft deep throughout its length, with maximum tidal velocities of less than 2 miles per hr. Any ship wishing to use the east channel may do so. It is a moot question as to why land transportation should always be subservient to water transportation even when the latter is slightly inconvenienced. Both are equally important to the development of the United States.

Mr. Hadley proposes a crossing of Port Orchard Channel at Illahee Crystal Springs instead of the one shown on route A. The length of his proposed crossing is nearly twice as long as the one proposed, and the United States Navy has refused to permit any bridge to be built south of the site proposed on route A. Mr. Hadley states that he is sympathetic to a sound crossing and infers that he has a cheap method of doing so. The writer is not informed as to his proposal. It is not a fact that the writer has said that the only method of crossing Puget Sound is by floating tunnels. The writer has said that, in his opinion, the floating tunnel is the only method of crossing by highway on routes A and C. There remains, of course, the continuance of consolidated ferries; but ferry transportation will never fully develop the vast area to the west of the sound and will always be more expensive in the long run.

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TRANSACTIONS

Paper No. 2445

INFLUENCE OF HEAVY LOADS ON PAVEMENT DESIGN TRENDS

By K. B. Woods,[1] M. ASCE

With Discussion by William S. Pollard, Jr., and K. B. Woods.

Synopsis

Highway pavement designs have been developed through extensive use of test roads, the use of theoretical or rational procedures—many of which have been checked by field tests and observations—and by the combined experiences of many highway engineers over a considerable period of time. These pavements, in general, proved to be adequate structurally for most sections of the United States for the period from 1920 to 1940.

Although information on the number and magnitude of loads which the highway systems carried through this period of time is somewhat meager, the available data, combined with general observations and experiences, point to the fact that the repetition of exceptionally heavy loads and the number of axle loads above 18,000 lb were relatively small in most places. In contrast, rigid pavements have failed, and flexible pavements have become rutted on a considerable mileage of the primary systems of many states since about 1940—particularly in highly industrial regions or in other areas where high concentrations of heavy loads occur. The occurrence of this widespread distress and the violation of existing truck weight laws (although apparently limited to a small percentage of total trucks) show that the failure can be attributed almost entirely to overloads, or in some instances, to exceptionally high concentrations of loads at or near the maximum legal axle-load limit.

The following primary conclusions have been drawn: (*a*) Strict enforcement of weight laws must be pursued; (*b*) the prevailing legal axle-load limits must not be increased lest the existing highway systems be destroyed by loads in excess of those for which the pavements were designed and constructed; and (*c*) large-scale research on a national basis must be initiated immediately to determine the most economical design of both truck and pavement.

Note.—Published in June, 1950, as *Proceedings-Separate No. 23.* Positions and titles given are those in effect when the paper or the discussion was received for publication.

[1] Associate Director, Joint Highway Research Project, and Prof. of Highway Eng., Purdue Univ., West Lafayette, Ind.

In connection with pavement design, available data indicate the need for better methods of compaction of base course materials for flexible and rigid pavements. Additional experimental work is needed to determine the economics and performance characteristics of rigid pavements constructed as thick slabs with heavy reinforcement or with various types and thicknesses of base courses.

INTRODUCTION

Early designs of flexible and rigid surfaces for the primary road systems within the United States have been developed by the use of test roads and other research procedures, by the accumulative experiences of design engineers and construction engineers, and by theoretical and rational analyses. The theoretical work and the data obtained from test roads established sound procedures for the design of rigid pavements in the early 1920's.

These designs, with some modifications throughout the years, proved to be reasonably adequate for primary roads throughout most sections of the United States. Because structural failures were not widespread, it is indicated that the designs were adequate for poor subgrade conditions and that these pavements were overdesigned for conditions of good subgrades. However, in the case of exceptionally heavy loads which occurred in fairly large concentrations in some sections of the East, these designs were not adequate and the use of base courses, thicker pavements, and modifications in joint design and reinforcing steel were required, notably in New Jersey.

During this same period of time (from 1920 through 1940) experiences were gradually accumulating for the design of flexible pavements. However, this situation is strikingly different from that of rigid pavements. Flexible pavements were developed during this period as a stage construction expedient rather than as a complete subgrade-pavement structure. As a result, data are somewhat meager in regard to the performance and thickness requirements of various types of bases and bituminous surfaces for the primary highway systems. This is particularly true in Indiana.

Two important conditions have altered the highway pavement design situation since 1940—namely, performance data accumulated in large-scale tests on runway pavements constructed to accommodate loads beyond those anticipated in highway use, and the destruction of rigid and flexible pavements on the primary highway systems of the United States which has been caused almost entirely by overloaded vehicles and extremely high concentrations of near-maximum legal loads. This destruction is evidenced in the pumping of rigid pavements and the rutting of flexible pavements.

DEVELOPMENT OF DESIGNS FOR RIGID AND FLEXIBLE PAVEMENTS

One of the earliest Portland cement concrete pavements built in the United States was constructed in 1891 in Bellefontaine, Ohio, and is still giving satisfactory service (1).[2] An extensive mileage of concrete was built in Wayne

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[2] Numerals in parentheses, thus: (1), refer to corresponding items in the Bibliography (see Appendix).

County, Michigan, in the early 1900's. In about 1920, Portland cement concrete pavements began to be used on an increasingly greater scale for heavily traveled roads in New York, California, Ohio, Indiana, Illinois, and Iowa. Some of the early slabs were of 4-in. and 5-in. uniform thicknesses (2); but, after 1920, most slabs were 6 in. thick or more. Some of the early slabs contained a thickened middle; but, after about 1920, the thickened edge became important as a design feature (3). As a result of the data collected from the Bates test road (4)(5)(6) and the Pittsburgh, Calif., and Arlington, Va., test roads (7)(8), the thickened edge design became common practice, although 7-in. and 8-in. uniform thicknesses continued to be used in many sections of the United States. Until about 1940, reinforcement was confined largely to the use of marginal bars and to wire mesh (9)(10). The use of a center-line joint became accepted practice after about 1926; however, contraction joints and particularly expansion joints did not become standard practice for primary roads until about 1935 (11)(12).

The United States Public Roads Administration (PRA) has reported (8) (*8a*)(*8b*)(*8c*) some important data relative to the Arlington tests. Data on warping, load transfer at joints, and many other details are included in these reports. At a later time, E. F. Kelley (13)(*13a*) summarized the status of the design of Portland cement concrete pavements for highways. Theoretical aspects of pavement design have been given consideration for many years by H. M. Westergaard, M. ASCE (14).

The evolution of rigid pavement design in Indiana follows rather closely that used in many other states. M. R. Keefe (7) summarized the Indiana background data in 1940. The early design, 1919–1921, was a 6–8–6-in. plain concrete section; in 1922, a 7–8–7-in. slab replaced the early type. An 8-in. uniform slab was adopted in 1923, and in 1926 the state adopted the 9–7–9-in. cross section, with a $\frac{3}{4}$-in. marginal bar along each edge; and longitudinal joints were installed for the first time. Expansion joints were adopted as standard practice in 1934 and these devices were used until about 1944. At present (1950) Indiana does not use expansion joints, but contraction joints are employed. The thickness of the slab remains at 9–7–9 in.

F. H. Eno (15), the PRA, the State of Michigan, and others initiated early studies on subgrades, but the wide-scale use of base courses under rigid pavements in areas of plastic soils did not occur until 1935. In 1934 H. F. Janda (16) showed that there existed good correlation between certain types of concrete cracking and the type of subgrade soil. C. A. Hogentogler (17) made a statistical analysis of the performance of pavements in 1928 and showed a definite relationship between performance and soil textures.

Since the early 1930's much attention has been devoted to a study of the structural characteristics of subgrade soils—as these characteristics influence the design of rigid and flexible pavements. Plate loading tests have been used for evaluating subgrade soils for both types of pavements by W. S. Housel, M. ASCE (18), the United States Navy (19)(20), and the Asphalt Institute (21). A symposium on methods of conducting load bearing tests was published by the American Society for Testing Materials (ASTM) (22) in 1947. The California bearing ratio (CBR) test has been used extensively by California in the design of flexible pavements for highways (23), the method being revised during World War II as a procedure for evaluating flexible pavements for military airfields (24)(25)(26). The triaxial compression test has been much used by many laboratories, and an effort has been made to relate the structural characteristics of subgrade materials, as indicated by this test, to pavement design requirements. The PRA has been active in this field (27), and the states of Kansas (28)(29) and Texas (30) have used this testing procedure extensively. W. Keith Boyd, M. ASCE (31), developed a cone test for evaluating subgrades and Fred Burggraf (32) developed a field-shear device. The California Department of Highways and other organizations (33) have conducted extensive experiments in the use of the Hveem stabilometer.

Macadam has been used for more than 150 years as a primary type of paving. The use of waterbound macadam and penetration macadam, with various types of bituminous-aggregate mixtures constructed as wearing courses, has prevailed in the United States for perhaps 75 years. However, flexible pavements have been employed primarily as resurfacing courses, and it has been only in recent years that any great amount of attention has been given to thickness design as it pertains to the subgrade-base-wearing-course combination. Sheet-asphalt and bituminous-concrete surfaces have been employed extensively, particularly in cities, but these types were often constructed over rigid bases or over some type of graded-aggregate mixture which had been under traffic for varying periods of time.

Just preceding and during World War II, many investigators were engaged in developing information on the design of flexible pavements. Prominent reports included those by A. C. Benkelman (34); W. H. Campen, M. ASCE, and J. R. Smith (35); A. T. Goldbeck, M. ASCE (36); Prevost Hubbard, Affiliate, ASCE, and F. C. Field (37); M. G. Spangler, M. ASCE (38); Roland Vokac (39); the committees of the Highway Research Board (HRB), National Research Council (40)(41)(42); A. T. Britton (43); F. N. Hveem, Assoc. M. ASCE (44); and G. E. Hawthorn (45). In addition, widespread interest has been shown in certain sections of the Great Plains, the West, and in the Middle West in the design of flexible pavements for use in secondary and even primary pavements. In addition to the reports from Texas and Kansas, mentioned previously, reports from New Mexico (46), Wyoming (47), Michigan (48), Minnesota (49), the western states (50), Missouri, and Kentucky, merit special mention. The reports on Missouri and Kentucky were received in letters from F. V. Reagel (February 21, 1949) and L. E. Gregg, Assoc. M. ASCE (March 9, 1949), respectively.

As a result of large-scale construction of civilian and military airports since about 1940, added emphasis has been given to the problem of pavement design—both flexible and rigid. Mr. Westergaard (51)(52) has extended his theoretical analysis of rigid pavement design to include loadings far beyond those which could reasonably be anticipated on highways. Robert R. Philippe, Assoc. M. ASCE (53)(54), has reported the results of extensive outdoor experiments with various types of rigid pavement slabs which were tested to failure. Both the United States Engineer Department (55)(56)(57)(58)(59)(60) and the United States Navy (61)(62) have released extensive reports covering the testing to failure of sections of airport runways at a variety of locations and situated on a large range of soil types. As a result of large-scale performance surveys of highways (63)(64) and of airports, there can be little doubt that the soil variable is a very important one and will become increasingly more important as conditions of loading become more severe.

TREND IN TRAFFIC LOADS

Limited data are available in regard to the number and magnitude of wheel and axle loads previous to 1935. At that time, PRA, in cooperation with most state highway departments, initiated programs of traffic counts and weight studies, and the collection of other types of factual data through the facilities of planning surveys. Robley Winfrey, M. ASCE (65), in 1935 reported data on actual service lives of pavements, but these data were not tied in with the most important condition of service—that is, loads. Rex M. Whitton (66) emphasizes the axle-load factor, stating (66*a*) that: "Most engineers agree that damage to road surfaces caused by overloading is due to wheel or axle loading and not to gross loading."

It is apparent that in the 1920's and 1930's primary highway systems carried but very few axle loads in excess of 18,000 lb. Furthermore, the ratio of the number of repetitions of the 16,000-lb axle loads to the 18,000-lb axle loads was much smaller in this period than was true in the years following 1940.

Some of the truck-weight information collected by the PRA and many of the state highway departments was summarized by John T. Lynch, Assoc. M. ASCE (67), in 1942. He concluded in part that, since the original loadometer operations in the period from 1936 to 1940, the frequency of heavy axle loads had increased materially in all regions of the United States. In this same paper, Mr. Lynch reported a large number of axle loads in the 20,000-lb to 22,000-lb range and a surprisingly great number more than 22,000 lb. The next year, Mr. Lynch (68) wrote (68*a*):

> "The average number of heavy axle loads passing each survey station in the 8 hours of operation was much greater in 1942 than in the 1936–1940 period, but was only slightly greater in 1943 than in 1942."

Thomas B. Dimmick in 1946 (69) supplied information on heavy axle-load frequency from data obtained in 1943 and 1944. This report grouped the data into frequency of axle loads between 14,000 lb and 16,000 lb, between 16,000 lb and 18,000 lb, and 18,000 lb and more. A large number of axle loads was reported in the 18,000-lb, and more, class (for the three eastern census regions, about 150 axles per 1,000 commercial vehicles daily being in the 18,000-lb, and more, class); unfortunately, information on the weight of the heaviest axle loads was not included.

In 1947, the writer, with F. H. Green, M. ASCE, and Harold S. Sweet, Jun. ASCE (70), made an analysis of the planning survey truck-weight data for Indiana, establishing a correlation between overloads and pavement destruction. Truck-weight data were available from twenty weighing stations for the years, 1936, 1942, 1943, 1944, and 1946. Survey procedure consisted of counting all vehicles and weighing trucks at each station for 8 hours, once each year.

The report showed a general relative decrease in the number of 10,000-lb loads and 14,000-lb loads since about 1943. In contrast, the number of axle loads of 18,000 lb, and more, showed a steady increase from 1936 with a material rise in 1946. Likewise, the number of axle loads greater than 20,000 lb increased significantly about 1944.

The maximum legal axle load in Indiana is 18,000 lb. A summary of the number of axles and the corresponding loads in excess of 18,000 lb, of one 8-hr period per station per year for all twenty Indiana stations in 1948 is shown in Table 1. It is to be noted that a few axle loads are in excess of 32,000 lb. The summary of the number of axle loads greater than 18,000 lb for the 8-hr survey at all twenty stations in Table 2 shows a large increase in the number of violations in comparing 1946 and 1947 with 1942, 1943, and 1944.

TABLE 1.—AXLE WEIGHTS IN EXCESS OF LEGAL LIMIT OF 18,000 LB IN 1948 SURVEY (TOTAL OF 369 AXLES ON 289 TRUNKS AND COMBINATIONS

Axle weight (lb) (1)	No. of axles (2)	Axle weight (lb) (1)	No. of axles (2)	Axle weight (lb) (1)	No. of axles (2)	Axle weight (lb) (1)	No. of axles (2)	Axle weight (lb) (1)	No. of axles (2)
18,200	34	20,000	17	21,800	5	23,800	1	26,400	1
18,400	33	20,200	8	22,000	12	24,000	3	26,600	1
18,600	30	20,400	10	22,200	2	24,200	2	26,800	1
18,800	28	20,600	9	22,400	4	24,400	2	27,400	1
19,000	27	20,800	11	22,600	5	24,600	1	27,800	1
19,200	22	21,000	11	22,800	2	25,000	1	28,400	1
19,400	23	21,200	7	23,000	2	25,200	1	28,800	1
19,600	21	21,400	4	23,200	1	25,400	1	30,000	1
19,800	13	21,600	4	23,600	2	26,000	1	32,000	1

TABLE 2.—SUMMARY OF LOADS GREATER THAN 18,000 LB

Description	NO. OF TRUCKS AND COMBINATIONS				
	95	98	71	230	217
Number of axles loaded more than 18,000 lb	116	118	75	289	274
Year recorded	1942	1943	1944	1946	1947

In testing airport runways, wheel loads of from 20,000 lb to 35,000 lb are common (55)(61), with extreme loads as high as 150,000 lb or more (53). These are much higher than most highway loads, although this variation may not be as great as indicated at first inspection, when consideration is given to some of the exceptionally high illegal highway loads as well as to the repetition-of-load factor.

STRUCTURAL FAILURES OF PAVEMENTS

Failures of primary pavements because of structural deficiencies were not widespread previous to about 1940. Failures were observed in some instances where pavements were too thin or in locations of frost heave, peat bog subsidences, or embankment settlement (particularly at culverts and bridge approaches), and in certain specific areas of high concentrations of heavy loads.

During certain seasons, adverse climatic influences contributed to the deterioration. These rather occasional failures probably had considerable influence in the development of better procedures for subgrade and embankment compaction, methods for handling peat bogs, subgrade treatments to control frost heave, and for posting roads during the spring breakup period (71).

In the years just preceding World War II, this condition changed significantly and structural failures became common on the primary system throughout many sections of the United States. These failures invariably occurred on roads that carried a heavy volume of truck traffic, covering areas where the subgrade materials were predominantly claylike or plastic in character (Fig. 1). Many rigid pavements were destroyed by pumping (72), and many flexible pavements showed a marked tendency toward rutting in the wheel tracks (Fig. 2).

Pumping, as a primary problem in Indiana, was first noted in 1940 (72). Short sections of several primary roads in the northwest corner of the state and in some sections near Fort Wayne were the first to be in distress. Extensive performance surveys in 1940–1941 failed to show structural failures to any great extent, as stated in unpublished reports of the Joint Highway Research Project at Purdue University in West Lafayette, Ind., in 1940 and 1941. However, this problem became severe during World War II, and in 1943 most of the rigid pavements in the state were surveyed to determine the extent and magnitude of the failures. Primary attention was given also to the spring breakup of flexible pavements. These data were reported at the 1943 meeting of the

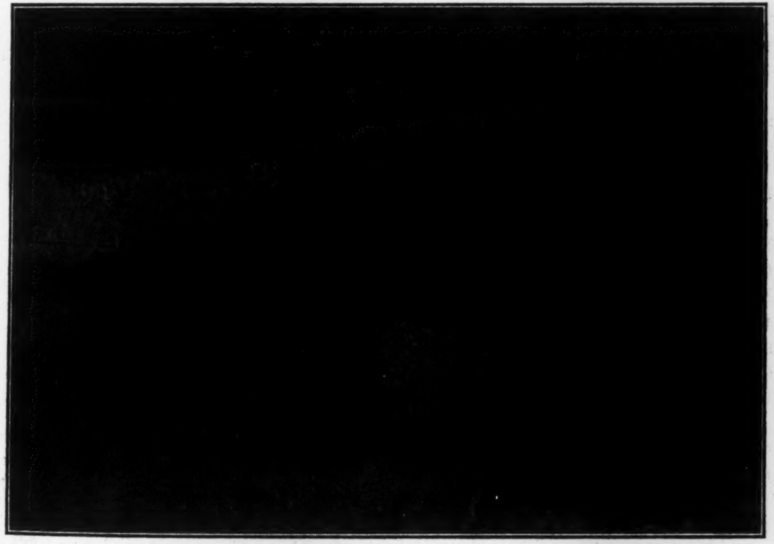

FIG. 1.—NEAR FAILURE ON OUTSIDE LANE OF DUAL LANE PAVEMENT CAUSED BY PUMPING

HRB (72). It is pertinent to note that 245.4 miles of pumping pavements was reported in the entire state. At that time, it was apparent that the soil variable was an important one because pumping was limited to sections of primary roads located in plastic soil areas. It was also obvious that the number and the magnitude of truck loads were important variables since pavement pumping was confined to the primary truck routes. Pavements on secondary roads with comparable designs, ages, and subgrade soils did not pump.

FIG. 2.—RUTTING OF A BITUMINOUS CONCRETE PAVEMENT

By 1947, the problem of pumping was widespread in the State of Indiana, and this fact led to a resurvey of the primary highway system, which showed (70) that the mileage of pavement pumping had almost doubled between 1943 and 1947. When the performance-survey data were correlated with the truck-weight data, it was indicated that the pumping in Indiana was directly related to the number and magnitude of overloads. The second report confirmed the data reported in 1943, pumping being confined to the heavily traveled truck routes located on plastic soils. However, it was noted that the problem had spread into areas of soils of lower plasticity than was true at the time of the 1943 survey. Furthermore, 47.8 miles of pavements constructed on deep sand subgrades were faulted, although they were not pumping. This circumstance was important in considering the use of these same materials as granular bases to eliminate pavement pumping in plastic soil areas.

Early mention of pavement pumping was made by J. W. Poulter (73) in connection with the use of mudjack materials to correct pumping pavements and faulted joints. The Illinois Division of Highways reported difficulty with pumping in 1939 (74), and the HRB appointed a committee to study the problem in 1942 (75). Detailed surveys have been made by various state highway departments in cooperation with the Portland Cement Association (76) (77). Field studies have been made in regard to the effect of the various design features including the thickened edge, expansion and contraction joints, the type of subgrade soil, and the use of various types and gradations of base courses. The work of the HRB committee confirms that pavement pumping is on the increase, that it occurs in areas of claylike materials, and that it is limited to roads which are subjected to a large number of heavy loads (78).

Information on the failures of flexible pavements as related to loads is meager. As mentioned previously, the use of flexible pavements has been widespread for secondary roads and as resurface materials for primary roads, at least in the Middle West—particularly in Indiana. As a result, the field-performance data on thickness requirements for the base-pavement combination are adequate for drawing only tentative conclusions. In two or three instances in Indiana, waterbound macadam bases with bituminous concrete surfaces have been constructed adjacent to Portland cement concrete pavements, which have been in distress from pavement pumping. In two instances, the bituminous surfaces have shown definite signs of rutting, thus indicating that the overload problem is of some consequence to this type of pavement also. In accelerated traffic tests on flexible airfield pavements Mr. Boyd (60) shows extensive rutting on pavements tested to failure. A similar type of failure is shown by F. A. Robeson (56).

DESIGN FEATURES RELATED TO STRUCTURAL BEHAVIOR

In considering the problem of pavement pumping as related to the design features of rigid pavements, it is significant that the problem has occurred on pavements with a wide range in design features and a wide range of pavement ages. In Indiana, pumping has occurred on 25-yr-old pavements with a thickened middle; it has been evident on pavements of uniform thicknesses ranging from 6 in. to 8 in.; and pumping has prevailed on the old Bates type of design, such as a 9–7–9-in. cross section with longitudinal bar or wire mesh reinforcement and without expansion or contraction joints. Some modern pavements with contraction and expansion joints, or with expansion joints alone, have pumped within 2 years or 3 years after construction. Many of these same roads, or roads of similar designs, carried highway traffic for years without encountering widespread structural failures preceding 1940. It must be concluded, therefore, that the entire range of designs was inadequate to carry the overloads to which they were abruptly subjected, and that minor variations in design had little to do with the problem of pumping. Certain features of design, however, are prominent and should be emphasized.

In regard to thickness of rigid pavements, available information indicates that pavement pumping is not eliminated on claylike subgrades by lightly reinforced slabs up to 10 in. thick. Pavements of thicknesses less than 10 in. have been giving good service for many years on primary roads in Indiana when subgrades are predominantly granular.

In analyzing the large amount of field information available on pumping, it is apparent that joints, particularly poorly maintained expansion joints, contribute appreciably to the problem. Since considerable data already indicate that expansion joints are not generally required on highway pavements (79)(80)(80*a*), it appears that the elimination of this design feature will greatly improve structural efficiency. Mr. Kelley (81) recognized this possibility in 1940 when he stated (81*a*) that some engineers conclude:

> "* * * when expansion joints are used it is not necessary to provide enough expansion space to effect complete relief of restraint but on the contrary, it is desirable from a structural standpoint to provide only enough space to keep compressive stresses within safe limits."

The State of New Jersey (82) requires the use of expansion joints, but an endeavor is made to keep the pavement in compression and to provide adequate load transfer between slabs at joints.

Contraction joints, as well as longitudinal joints, cause some difficulty from the standpoint of pumping, but their use can scarcely be considered a primary cause. There can be little argument in regard to the need for the longitudinal joint; and, since most design engineers feel the need for contraction joints as a method of controlling cracking, it is likely that these design features will be continued. More emphasis will be given to methods of maintaining joints.

Increased attention will doubtless be paid subgrade soil, and design will include the factor of subgrade soil type as an integral part of the subgrade-base-pavement combination. With pumping being almost entirely confined to plastic soils, and with the available evidence pointing to the potential severity of rutting of bituminous pavements on these same soils, the soil variable indeed appears to be the most important of all design factors.

Drainage and associated maintenance must continue to be considered important. Since water is one of the most important factors involved in the problem of pumping, extra precautions should be taken in the original design to keep the pavement and subgrade as dry as possible. Coupled with drainage should be maintenance policies that require a periodic sealing of all cracks and joints to minimize the seepage of surface water into the subgrade.

Design Trends

If a design procedure must be established to construct pavements that will perform reasonably well for high concentrations of legal loads as well as for illegal loads, it is appropriate to consider design trends. As far as rigid pavements are concerned, much study is being given to the use of heavily reinforced slabs as contrasted to the improvement of poor subgrades and the use of base courses. As to bituminous pavements, there is a trend toward the use of macadam bases, and much thought is being given to methods of compacting granular materials in base courses. The compaction of granular base courses for rigid pavements is equally important.

The PRA, in cooperation with the State Highway Commission of Indiana, constructed several miles in test sections of continuously reinforced concrete pavements in 1938. Reports of the performance of these sections have been made available at periodic intervals (83)(84)(85). As pavement pumping has increased through the past several years, more and more attention has been given to these experimental sections. Since about 1946, several additional experimental test sections have been constructed in other states. In 1947, H. W. Russell and J. D. Lindsay (86) reported on such a pavement constructed in Illinois, and in the same year William Van Breemen (87) made a similar report on a project in New Jersey. W. R. Woolley (88) has been interested in the use of this type of pavement design, particularly from the standpoint of the elimination of expansion joints and susceptibility to pumping. Additional data will be required and perhaps several years of service testing will be needed before this type of design can be evaluated fully. However, continuous reinforced concrete design remains as one of the outstanding potential solutions to the destructive forces of a large number of excessively heavy axle loads.

Experiments with base courses have been conducted for many years, particularly in connection with subgrade treatment to prevent frost heave, to improve drainage, and, more recently, to eliminate pavement pumping. From the standpoint of pumping, it appears that the base course serves most effectively as an insulation between pavement and subgrade rather than as an addition to improve the structural capacity.

E. A. Henderson and W. T. Spencer (89) reported important data in regard to 10-yr-old experimental base course sections on U. S. Highway No. 30 in Indiana. The treatments included eight different base courses, varying in type from bituminous soil mixtures to limestone screenings and dune sands. Thicknesses of treatment ranged from 3 in. to 6 in. Small amounts of bituminous materials mixed with plastic subgrade soils reduced pavement pumping markedly. The dune sand was particularly effective in eliminating pumping, but the faulting at joints was of measurable and noticeable amounts. This observation, coupled with other performance-survey data showing considerable amounts of faulting on heavily traveled pavements constructed on deep sand-soils, indicates that the sand is being further compacted under the pavement slab by the vibration resulting from repetitions of heavy loads. This being true, precautions need be taken against the use of a deep layer of poorly graded, granular base courses and, in addition, special emphasis must be given to compaction during construction (90).

As far as rigid pavements are concerned, it appears that heavy duty pavements for truck routes will require either 8-in. to 10-in. slabs with heavy reinforcement, or thinner slabs with less reinforcement, underlain by well-compacted base courses. As an economical expedient in some areas well-graded granular materials (91)(92) and treated bases (93)(94) should be used. The selection of bituminous and concrete overlays to strengthen pavements which are structurally inadequate will probably continue as good design practice (95)(96); however, experience indicates that moving slabs must be stabilized before overlays can be used satisfactorily.

In regard to flexible pavements, the trend appears to be toward the use of waterbound macadam bases of considerable thickness, or selected granular materials with base and surface courses constructed of bituminous concrete (95). With the tendency toward rutting, it appears that additional emphasis must be placed on methods and procedures that can be employed during construction to obtain greater compaction of the granular materials. The use of small amounts of liquid bituminous material or cement as binder may partly solve the problem; the development of some method of hardening a claylike soil may be a partial answer to obtaining higher densities in the compaction of base course materials on claylike subgrades.

It appears at the present time that the large amount of data collected on the design of thick rigid and flexible surfaces for exceptionally heavy airplane loads cannot be applied directly to the design of highway pavements because of two factors: (*a*) The repetition of load on a highway pavement is considerably greater than that which can be expected on a normal airport runway or taxiway, and (*b*) the wheel loads and impact factors for trucks and other heavy vehicles using the highway are entirely different from those of the airplane.

SUMMARY

On the basis of published information, field test sections, and performance records of pavements in Indiana and many additional states, data available on the effect of heavy loads on pavement design may be summarized as follows:

1. The subgrade soil is the most important variable and the one which has received insufficient attention in the design of rigid and flexible pavements for roads carrying heavy industrial traffic. The performance information available shows a correlation between performance and soil texture—with drainage properties and position of the road surface in respect to the surrounding terrain being only slightly less important.

2. The most perplexing unknowns in connection with the design of rigid and flexible pavements for roads carrying heavy truck traffic are those of the ultimate number and magnitude of the loads themselves. Designs for rigid pavements developed on the basis of theoretical analysis, test roads, and field experiences have proved to be reasonably adequate for the period from 1920 to 1940 in most sections of the United States when loads were rarely in excess of the legal limit. In contrast, widespread structural failures have occurred and are continuing to occur on the roads of the primary systems of several states, including Indiana, in which illegal loads of considerable magnitude have become frequent in occurrence. Available truck-weight data show that the frequency of illegal loads has increased steadily since about 1940, and that the magnitude of these loads, particularly axle loads, is still increasing.

3. Corrective procedures to meet the present emergency should include: (*a*) Strict enforcement of existing weight laws, particularly axle loads; (*b*) the adoption of a policy by the state highway departments whereby salvage operations—including the stabilization of pumping slabs and the use of overlays—will be attempted in the early stages of pumping; and (*c*) the posting of roads and the establishment of load limits during the spring breakup period in an endeavor to protect existing pavements which are structurally inadequate.

4. In considering the generally established design procedures for rigid pavement, available data indicate that these are inadequate—particularly from the standpoint of subgrade soil evaluation. Furthermore, it is indicated that the thickened edge, thickened middle, and other minor variations are relatively unimportant from the standpoint of pavement pumping; however, the elimination of expansion joints should improve the structural efficiency of rigid pavements, although this change alone will not necessarily eliminate the difficulty. It has not been proved thus far that pavement pumping can be eliminated by thick pavements, up to 10 in. In Indiana there are many hundreds of miles of pavements constructed with an old Bates type of design (9–7–9-in. cross section with longitudinal bars or wire mesh reinforcement) giving excellent performance after 20 years or more of life. These roads are located on "nontruck" routes and have been placed on a wide variety of soils with claylike materials predominating. In contrast, most of the pavements of this type of design located on primary routes have been resurfaced or destroyed. These data indicate that this type of design is adequate for carrying only a small number of trucks, but that it is inadequate for primary roads under present loadings. In regard to flexible pavements, there is evidence that rutting of the surface occurs on

pavements carrying heavy truck loads and constructed on plastic soils. This research should be directed toward compaction requirements for the subgrade and base. Also, research is needed to establish thickness requirements of the base-pavement combination for prevailing conditions of loading. The use of nonplastic bases or bases with low plasticity indexes, as well as cement—or bituminous-treated granular bases—is indicated for both secondary and primary roads particularly in the midwestern and western states.

5. In regard to design trends, from currently available information, it appears that rigid pavements will be designed by one of two procedures: (*a*) The use of soil treatments to correct inferior subgrades, the additional use of base course, and the employment of slabs of medium thickness and reinforcement of some standard type of design of the slab; and (*b*) the use of thicker slabs with continuous reinforcement and the elimination of all the transverse joints. In general, longitudinal joints and contraction joints are likely to continue in standard practice, but the trend is definitely away from expansion joints. In connection with flexible pavements, the available data show that new methods and procedures for compacting subgrades and bases to obtain a maximum of strength from the pavement-base-subgrade combination are needed, and research on base course materials and admixtures should be continued. As to soils, there is a basic need to evaluate soils and constructional materials on a regional basis and thus to standardize design by soil regions rather than by political boundaries.

6. The establishment of large-scale research programs is required, preferably on a national basis, to determine the most economical design of both the truck and the pavement. In this respect, the states can obtain excellent preliminary information by contracting for the construction of test sections. These sections should include various thicknesses of rigid and flexible pavements over various types and thicknesses of base courses on both the primary and secondary road systems.

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(91) **"Subbases for Rigid Pavements,"** *Technical Bulletin No. 122*, Am. Road Builders' Assn., Washington, D. C., 1947.

(92) **"A Study of Course-Graded Base-Course Materials,"** by Edward A. Willis, *Public Roads*, Vol. 24, No. 9, 1946.

(93) **"Progress Report on California Experience with Cement Treated Bases,"** by T. E. Stanton, F. N. Hveem, and J. L. Beatty, *Proceedings*, Highway Research Board, National Research Council, Vol. 23, 1943, pp. 279–295.

(94) **"Base Stabilization with Tar,"** by Ralph D. Witt, *Proceedings*, Am. Road Builders' Assn., Washington, D. C., 1940, p. 677.

(95) **"Traffic Testing of Overlays on Rigid Airfield Pavements,"** by Herman L. Gaines, *Proceedings*, Highway Research Board, National Research Council, Vol. 25, 1945, pp. 85–95.

(96) **"Some Observations on Resurfacing Portland Cement Concrete Pavements with Bituminous Mixtures,"** by Wayne R. Woolley, *ibid.*, Vol. 26, 1946, pp. 311–319.

(97) **"Axle-Load and Gross-Load Trends,"** by John T. Lynch and Thomas B. Dimmick, *Public Roads*, Vol. 25, No. 12, February, 1950, p. 279.

(98) **"Traffic Trends on Rural Roads in 1949,"** by Thomas B. Dimmick, *Public Roads*, Vol. 26, No. 5, December, 1950, p. 85.

DISCUSSION

William S. Pollard, Jr.[3] **Jun. ASCE.**—An excellent survey of highway load and design trends since about 1920 has been presented in this paper.

In support of the traffic load trends stated in the paper (under the heading, "Trend in Traffic Loads") and concentrating on axle loads greater than 18,000 lb, Table 3 is submitted as portraying the high axle-load trend in Illinois through 1948. The marked and progressive increase in violations is evident, looming exceptionally large when the increase from 1942 to 1948 is reported as a percentage of the 1942 values.

TABLE 3.—Axle Weights in Excess of Legal Limit of 18,000 Lb in Illinois[a]

Axle-Load Group, in Pounds		Percentage of Total Axle Loads Weighed								
		Single-truck units			Tractor truck semitrailers			Trailer combination[b]		
From	To	1948	1942	1936	1948	1942	1936	1948	1942	1936
18,000	18,999	0.25	0.34	0.00	3.41	1.41	0.13	10.31	3.51	0.00
19,000	19,999	0.10	0.00	0.00	1.88	0.40	0.07	5.93	1.75	0.00
20,000	21,999	0.04	0.00	0.00	1.39	0.28	0.00	3.87	0.00	0.00
22,000	23,999	0.08	0.00	0.00	0.30	0.00	0.00	0.00	1.75	0.00
24,000	25,999	0.00	0.00	0.00	0.10	0.00	0.00	0.00	1.75	0.00
26,000	27,999	0.00	0.00	0.00	0.04	0.00	0.00	0.00	0.00	0.00
>28,000		0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Total, overloaded axles		0.47	0.34	0.00	7.12	2.09	0.20	20.11	8.76	0.00
Percentage increase in violations, 1942 to 1948		38	....	....	240	....	....	129	....	....

[a] Adapted from tabulated data, from short surveys distributed over the state, presented in the "Thirty-first Annual Report of Division of Highways," Dept. of Public Works and Bldgs., Springfield, Ill., 1948, p. 179. [b] Tractor truck semitrailers with trailers and trucks with trailers.

Relative to corrective measures for the nation-wide highway distress caused by this trend toward overloads, the writer is in agreement with Mr. Woods' suggested procedures. However, since it is generally conceded that high overloads in even small numbers can inflict damage costing millions of dollars to repair, it is the opinion of the writer that the "strict enforcement of existing weight laws" suggested in the paper (item 3(*a*) under the heading, "Summary") should include the establishment by states of on-the-spot unloading and of mandatory, uniform, practically prohibitive fines for violators. Furthermore, fines so assessed should be restricted to highway use. Such a policy, rigidly enforced, would certainly decrease the number of overloads on a highway system in a short time, and would contribute in part to the repair of damage already effected by the violator group—a group representing only from 5% to 10% of the total trucks on the road. When he knows the penalty for homicide in advance and is aware that in most cases apprehension is inevitable, the average individual takes pause before taking a life, regardless of provocation. The parallel is evident.

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[3] Instr. of Civ. Eng., Univ. of Illinois, Urbana, Ill.

An expedient to the enactment of suitable legislation to cover the handling of violators would be to inform the public, more completely, of the existing situation—its cause and the steps necessary to correct this situation. The press and popular magazines are the logical avenues of approach for such an action. The average highway user, made aware of this unjustifiable drain on "his" tax money, is capable of contributing toward considerable group pressure—pressure of a desirable nature in this instance.

A direct reference to the fatigue properties of concrete would have been desirable in this paper. Extensive tests (4)[3a] have established the fact that the use of a minimum design safety factor of 2, on the basis of the predominating wheel load, provides protection against failure in fatigue, regardless of the number of load applications. Those interested in bringing about an increase in allowable axle loads might attempt to justify this increase on the basis of this safety factor, choosing to emphasize the safety factor itself and to minimize its reason for existing. Any effort in this direction should be vigorously opposed, not with the view of opposing progress as such but with a determination that any pseudo progress that would in actuality constitute a step backward will not be permitted. If it is deemed an over-all economic necessity that greater axle loads be permitted, then highways must be built to accomodate them and their travel must be restricted to highways so designed. These "musts" are fundamental requirements for maintaining the level at which the national highway system now exists.

K. B. Woods,[4] M. ASCE.—Lively interest continues in the subject of the destructive effects of heavy loads on highway pavements and the trends in pavement designs. During 1950, the Highway Research Board, in cooperation with several state highway departments, the Bureau of Public Roads, the Automobile Manufacturers Association, and the American Trucking Association, initiated large-scale controlled field tests on a section of road in Maryland. The purpose of the test road was to study the performance of a concrete pavement under the operation of commercial-type vehicles loaded to 18,000 lb and 22,400 lb on single axles and 32,000 lb and 44,800 lb on tandem axles.

In his discussion, Mr. Pollard submits some traffic-load trend data for Illinois covering the years 1936, 1942, and 1948. In February, 1950, John T. Lynch and Thomas B. Dimmick reported some nation-wide data on axle-load and gross-load trends. The report stated that

> "* * * the total number of heavy axle loads found on the highways in 1948 was eighteen times the number in the 1936–37 period, and the rate of increase appears to be accelerating rapidly (97)."

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[4] Associate Director, Joint Highway Research Project, and Prof. of Highway Eng., Purdue Univ., West Lafayette, Ind.

In December, 1950, in an article on traffic trends on rural roads in 1949, Mr. Dimmick reported that

> "In 1949 more than 5 percent of all trucks and combinations exceeded a state legal weight limit and 16 percent of the combinations were illegally overloaded in some particular (98)."

Comparing the pre-World War II period with the information available in 1949, he reported a 750% increase in the frequency of occurrence of axle loads in excess of 22,000 lb. Rather strict enforcement of truck-weight laws in the various states may account for the slight decrease in the number of over-loaded trucks per thousand in 1949 as compared with 1947 or 1948. Mr. Dimmick reported 75 instances of overloading per thousand cases in 1949, a rate of 66 per thousand cases in 1947, and 85 over-loaded vehicles per thousand cases in 1948.

Mr. Pollard's reference to fatigue properties of concrete is a worthy addition to the subject of rigid pavement design trends. The subject of fatigue action in rigid pavement slabs has been neglected a long time.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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TRANSACTIONS

Paper No. 2446

PAVEMENT BEARING CAPACITY COMPUTED BY THEORY OF LAYERED SYSTEMS

By Guthlac Wilson,[1] M. ASCE, and G. M. J. Williams,[2] Assoc. M. ASCE

With Discussion by Messrs. Robert Ruckli, Hugh Q. Golder, and Guthlac Wilson and G. M. J. Williams

Synopsis

This paper describes a method for calculating the bearing capacity of rigid and flexible pavements, which is based on the theory of layered elastic systems. Graphs are presented from which the required pavement thickness can be obtained directly, if the loading and the physical properties of the pavement and subgrade materials are known. It is shown how these stresses can be allowed for in cohesive and in cohesionless soils.

Commonly used methods for computing the bearing capacity of pavements are discussed, particularly with reference to the degree in which they permit a count to be taken of the various factors affecting the bearing capacity of pavements.

Pavement thicknesses are calculated for typical cases by these methods and also by the method described in the paper, and the results are compared.

1. Introduction

The main content of this paper is the application of the methods of the theory of elasticity to the calculation of the bearing capacity of pavements. No claim is made that this method is the best one for estimating the proper thickness of pavements. The writers believe, rather, that a semi-empirical method, such as the California bearing ratio (CBR) method,[3] is the soundest. They do believe, however, that it is advantageous to have a theoretical yardstick, applicable if the materials concerned were "perfect," with which to compare any empirical method. Such a yardstick should include all the factors that affect the bearing capacity of pavements. The method presented involves

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Note.—Published in May, 1950, as *Proceedings-Separate No. 16*. Positions and titles given are those in effect when the paper or the discussion was received for publication.

[1] Partner, Scott and Wilson, Cons. Engrs., London, England.

[2] Asst. Engr., Scott and Wilson, Cons. Engrs., London, England.

[3] "Development of CBR Flexible Pavement Design Method for Airfields: A Symposium," *Proceedings*, ASCE, January, 1949, p. 3.

the application of principles published by D. M. Burmister, Assoc. M. ASCE, in 1943[4] and again in 1945[5] under the descriptive name of "theory of layered systems." The writers believe that this method includes more relative factors than any other.

In 1946, discussing a paper by H. Q. Golder,[6] one of the writers (Mr. Wilson) made the suggestion that the bearing capacity of pavements should be calculated from stresses both in the pavement and in the soil as determined by the Burmister theory, rather than from the assumption of an arbitrary limiting deflection as proposed by Mr. Burmister. In 1946, L. Fox extended[7] Mr. Burmister's computations to the analysis of the stresses in the soil, thus opening the way for the determination of the bearing capacity of a pavement supported on a purely cohesive soil subjected to no initial shearing stresses.

In this paper the writers make allowance for the weight of the pavement and soil, and for the magnitude of the earth pressure at rest, and they extend the investigation to any elastic soil, the state of failure of which can be represented by the Coulomb equation:

$$\tau = c + \sigma \tan \phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which τ is the intensity of shear stress; c is a factor denoting cohesion per unit area; σ is the intensity of direct stress; and ϕ is the angle of internal friction.

2. Factors Affecting the Bearing Capacity of Pavements

The methods of design which are most widely used at present are the method of H. M. Westergaard,[8] M. ASCE, for rigid pavements and the CBR method for flexible pavements.[9,10] The methods proposed by R. Glossop and Mr. Golder in 1944[11] and by Mr. Golder in 1946[6] (which are said to be applicable to rigid pavements) have had some use as have the methods proposed by the American Civil Aeronautics Authority[12] (CAA) and by Norman W. McLeod.[13] The former[12] is stated to be applicable to both rigid and flexible pavements and the latter[13] to flexible pavements only.

The factors affecting the bearing capacity of pavements, and the extent to which the various methods mentioned take account of these factors, are indicated in Table 1.

In all methods which involve the strength or elasticity of the foundation the uncertain factor is the deterioration of the foundation, and the difficulty is the decision as to how samples should be conditioned to make a sufficient, yet not excessive, allowance for this deterioration.

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[4] "The Theory of Stresses and Displacements in Layered Systems and Applications to the Design of Airport Runways," by D. M. Burmister, *Proceedings*, Highway Research Board, National Research Council, Vol. 23, 1943.

[5] "General Theory of Stresses and Displacements in Layered Systems," by D. M. Burmister, *Journal of Applied Physics*, Vol. 16, 1945, p. 89.

[6] "The Relationship of Runway Thickness and Undercarriage Design to the Properties of the Subsoil," by H. Q. Golder, *Airport Paper No. 4*, Inst. C. E., 1946.

[7] "Computation of Traffic Stresses in a Simple Road Structure," by L. Fox, *Proceedings*, 2d International Conference on Soil Mechanics, Rotterdam, 1948, Vol. II, p. 236.

[8] "Stresses in Concrete Pavements Computed by Theoretical Analysis," by H. M. Westergaard, *Public Roads*, April, 1926, pp. 25–35.

[9] "Soil Tests for the Design of Runway Pavements," by T. A. Middlebrooks and G. E. Bertram, *Proceedings*, Highway Research Board, National Research Council, Vol. 22, 1942, p. 144.

[10] "Preparation of Subgrades," by O. J. Porter, *ibid.*, Vol. 18, 1938, p. 324.

[11] "Construction of Pavements on a Clay Foundation Soil," by R. Glossop and H. Q. Golder, *Road Paper No. 15*, Inst. C. E., 1944.

[12] "Classification of Soils for Airport Construction," by Henry Aaron, *Proceedings*, 2d International Conference on Soil Mechanics, Rotterdam, 1948, Vol. V, p. 190.

[13] "An Investigation of Airport Runways in Canada," by Norman W. McLeod, *ibid.*, Vol. IV, p. 167.

A justifiable criticism of the CBR method is that, in some cases, tests on soaked specimens lead to lack of economy, as the foundation never attains as weak a condition during the life of the pavement, and many engineers now base their designs on the CBR of specimens conditioned to the weakest state which, in their judgment, the foundation will ever attain. Valuable work has been published on suitable methods of conditioning specimens[14] or of modifying the CBR found from soaked specimens.[15]

TABLE 1.—FACTORS TAKEN INTO ACCOUNT, IN VARIOUS DEGREE, BY VARIOUS METHODS OF PAVEMENT DESIGN

Abbreviations: A = Civil Aeronautics Authority; B = Burmister; C = California Bearing Ratio; G = Glossop and Golder (1944) and Golder (1946); M = McLeod; and W = Westergaard.

Factor	Directly	Indirectly	Possible	Not at all
Strength of pavement material	W, B	C, A, M	....	G
Elasticity of pavement material	W, B	C, A, M	....	G
Stresses in pavement except applied load[a]	....	C, A, M	W,[b] B[b]	G
Strength of foundation	G, C, M, B	A	....	W
Elasticity of foundation	W, B	C, A, M	....	G
Pre-existing stresses in foundation	B	C, A, M	G	W
Deterioration of foundation	C	A	B, W, G, M	....
Frequency of loading	....	C, A, M	B	W, G
Tire pressure	W, G, B	C, M	....	A

[a] Stresses from causes other than applied load, such as temperature change and shrinkage. [b] Can be allowed for by appropriate reduction in permissible tensile stress.

The CBR method applies only to the design of flexible pavements, but J. K. M. Henry and H. Grace[16] have applied it to the design of rigid pavements by assuming that a rigid pavement of a given thickness is equivalent to a flexible pavement of twice that thickness. Their experiment has been successful. Evidence has been presented[17] to show that this suggestion is compatible with other design methods.

Mr. McLeod's method[13] appears to result in a range of thicknesses generally proportional to, but thinner than, those derived by the CBR method and thus to have a lower factor of safety. The method adopted by the CAA, in which the soil properties are deduced from index tests on the materials, appears to lead to a lower factor of safety than does the CBR method in the case of weak materials and to a greater factor of safety in the case of strong materials.

3. PROPOSED BASIS OF COMPUTING BEARING CAPACITY

In 1943, Mr. Burmister[4] published a mathematical solution of the problem shown in Fig. 1, comprising a total load P uniformly distributed over a circle of radius a on the surface of a layer of elastic material, which in turn rests on a semi-infinite elastic body. The superficial layer of thickness, h, has elastic properties as follows: E_1 is Young's modulus of elasticity; and μ_1 is Poisson's ratio. The elastic constants for the underlying material are E_2 and μ_2, respectively.

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[14] "A Procedure for Evaluating the Influence of the Moisture Content of the Subgrade on the Thickness Required for Flexible Pavements of Airfields," by H. Loxton, H. M. Beavis, and M. D. McNicholl, *Proceedings*, 2d International Conference on Soil Mechanics, Rotterdam, 1948, Vol. II, p. 218.

[15] "Methods of Flexible Pavement Design: Wyoming Method of Flexible Pavement Design," by I. E. Russell and D. J. Olinger, *ibid.*, Vol. V, p. 165.

[16] "The Incorporation of Decomposed Granite in the Design and Construction of Pavements in Hong Kong," by J. K. M. Henry and H. Grace, *ibid.*, Vol. IV, p. 190.

[17] *Ibid.*, Vol. VI, p. 130.

Mr. Burmister has published formulas giving the stresses set up in both materials in terms of the two parameters, E_1/E_2 and h/a, assuming μ_1 and μ_2 to have the same value.[4] He considered both the case in which the materials are fully continuous with no relative movement at the interface, and the case which arises when the interface is frictionless so that no shear stress acts across it. Subsequently, Mr.Fox[7] published values for the stresses on the axis of symmetry in the lower material. These constitute an evaluation of the stresses set up in a simple pavement if it is assumed that the pavement material and the subgrade behave elastically, and that the load acts uniformly over a circular area. The writers have also assumed that the pavement and the subgrade are fully continuous at the interface.

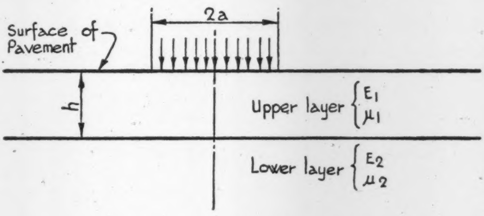

FIG. 1

Failure of such a pavement may begin in either of two ways: The pavement itself may fail initially, leading to overloading of the subgrade and to general failure; or the subgrade may fail first, causing the pavement to fail because it is inadequately supported. Consequently, the stresses in neither material can exceed the permissible stress for that material.

The pavement will nearly always begin to fail when the tensile stress at the base exceeds the permissible stress for the pavement material. This is particularly true of concrete pavements, and is the only method of pavement failure considered in this paper. The subgrade fails when the Coulomb equation (Eq. 1) is first fulfilled at any point on the axis of symmetry of the loaded area.

Mr. Burmister gives expressions for the tensile stress in the pavement on the axis of symmetry, from which Fig. 2 may be derived by numerical integration after the manner adopted by Mr. Fox. In Fig. 2, q represents the intensity of pressure on the loaded area; and σ_c, the resulting tensile stress in the base of the concrete pavement. Consequently, if σ_c is taken as the maximum tension allowable in the pavement, these curves will give the least thickness, h, necessary to prevent failure of the pavement in terms of a, the radius of the loaded area.

This allowance alone is not sufficient to insure a safe design because the strength of the subgrade has not been considered, and it may be necessary to increase the pavement thickness to prevent the subgrade from being overstressed. The stresses in the subgrade arise from two sources—the load applied on the surface and the dead weight of the soil and the pavement. The stresses due to the former are given by Mr. Fox; but, to estimate those set up through dead weight, a value must be assumed for the coefficient of subgrade earth pressure at rest, K_o. It will be shown subsequently in Section 5 that K_o tends to decrease with depth. The value to be used in the calculation is that which will apply at the critical depth. This depth is shown in the illustrative curves for the cases when $c = 0$, whereas in the cases when $\phi = 0$ the writers introduce a new parameter, χ, which allows for the variation of K_o with depth.

In this paper it is assumed that the subgrade and the pavement material have the same density, γ. If, then, any point on the axis of symmetry of the loaded area is considered, the stresses from these two causes can be added together to obtain the total resultant stresses at that point. Thus, the resultant vertical stress is given by

$$\sigma_{vr} = q\,\sigma_I + \sigma_v \ldots\ldots\ldots\ldots\ldots\ldots (2a)$$

and the resultant horizontal stress by

$$\sigma_{hr} = q\,\sigma_{III} + \sigma_h \ldots\ldots\ldots\ldots\ldots\ldots (2b)$$

in which σ_I and σ_{III} are, respectively, the vertical and horizontal stresses due to unit tire pressure, as derived by Mr. Fox; and σ_v and σ_h are the vertical and horizontal stresses pre-existing in the soil (see Section 5). Since the point in question is on the axis of symmetry, σ_{vr} and σ_{hr} will be the maximum and minimum principal stresses, so that failure will occur when Eq. 1 is satisfied:

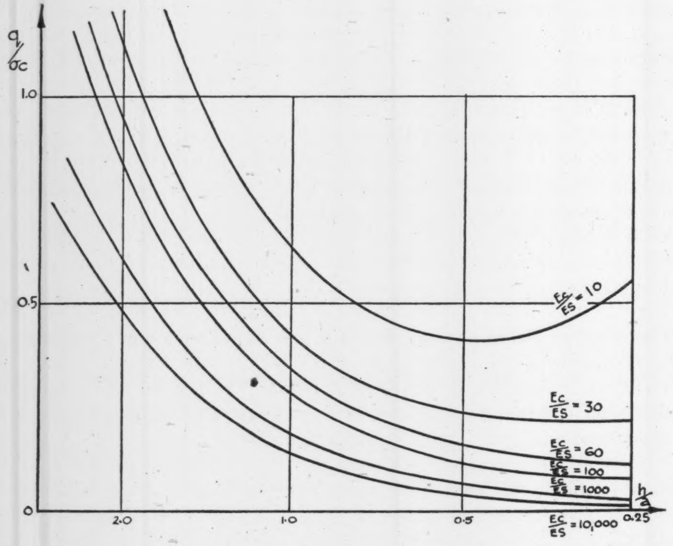

FIG. 2.—PERMISSIBLE VALUES OF q/σ_c IF CONCRETE FAILURE OCCURS BEFORE SOIL FAILURE

$$\frac{\sigma_{vr} - \sigma_{hr}}{2} = c + \frac{\sigma_{vr} + \sigma_{hr}}{2} \tan\phi \ldots\ldots\ldots\ldots (3a)$$

or

$$\frac{q\,(\sigma_I - \sigma_{III}) + \sigma_v - \sigma_h}{2} = c + \left[\frac{q\,(\sigma_I + \sigma_{III}) + \sigma_v + \sigma_h}{2}\right] \tan \phi \,..(3b)$$

Eq. 3*b* gives the value of *q* that causes Eq. 1 to be satisfied at that point—that is, the value which causes failure to occur.

By repeating the process at several points on the axis of symmetry, the least value of *q* that involves failure anywhere can be found. This value will give the maximum permissible tire pressure if failure is not to occur, and is shown by *h*/*a*-lines on a special set of design charts[17a] devised for various combinations of values for the parameters *c*, ϕ, E_1/E_2, and *h*/*a*. In this descriptive paper it is not feasible to reproduce a complete set of charts such as would be necessary in practice. The writers make use of fifteen such charts.[17a] In the first five a separate chart is drawn for E_1/E_2-ratios of 10, 100 (see Fig. 3), 300, 600, and 1,000, respectively. The values of other factors are: $\phi = 0$; $\chi = 0$ and 1; and $K_o = 1$ and maximum. Three sets of two each are for $K_o = 0.5$ (Fig. 4) and 0.75, respectively, with E_1/E_2-ratios at 10, 60, and 100.

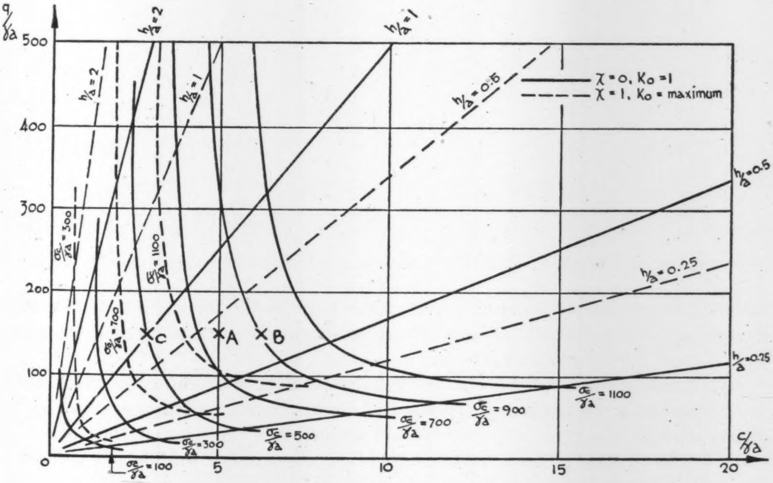

FIG. 3.—SAMPLE CHART $q/(\gamma\ a)$ VERSUS $c/(\gamma\ a)$; $\phi = 0$ AND $E_1/E_2 = 100$

Holding K_o constant at 0.5, two sets of two each are for $\phi = 30°$ ($E_1/E_2 = 10$ and 100) and $\phi = 37.5°$ ($E_1/E_2 = 10$ and 100). In these charts terms representing stresses or pressures have been divided by the term $\gamma\ a$ to obtain nondimensional results. Alternately for a fixed value of *q* these curves will give the minimum value of *h*/*a* if failure is not to occur in the subgrade, so that the least pavement thickness required to safeguard the subgrade can be found.

[17a] A complete set of full-scale drawings (Order No. 16C) will be mailed postpaid on receipt of $2.00 (50% discount to ASCE members).

Thus, the theory gives two values for the pavement thickness: That derived from Fig. 2 which insures that the pavement itself is not overstressed, and that derived from the design charts such as Figs. 3 and 4 insuring that the subgrade is safe. To obtain a completely safe design it will be necessary to take the greatest of the several thicknesses thus determined.

For convenience the data in Fig. 2 are represented on the design charts by the $\sigma_c/(\gamma\ a)$-lines. These lines give the tire pressures that would cause the maximum permissible tensile stress to be set up in the base of the pavement. Several curves are given on each chart to allow for variations in the tensile strength of the pavement material.

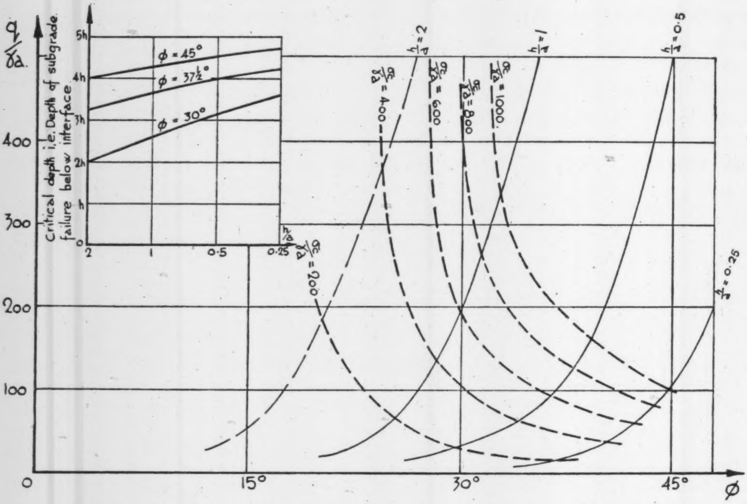

FIG. 4.—SAMPLE CHART $q/(\gamma\ a)$ VERSUS ϕ; $c = 0$, $E_1/E_2 = 60$, AND $K_o = 0.75$

An example of how the charts may be used to determine the pavement thickness necessary to meet a given set of circumstances is illustrated in Fig. 3. This particular chart is for a cohesive soil with no internal friction, for which the ratio $E_1/E_2 = 100$. There are two sets of curves relating to different values of K_o, which factor is fully discussed in Section 5. Consider the curves for the case when $K_o = 1$ and suppose $q/(\gamma\ a) = 150$ and $c/(\gamma\ a) = 5$ for a particular combination of subgrade and loading. These values, taken as ordinate and abscissa, respectively, fix point A in Fig. 3, corresponding approximately to $h = 0.70\ a$, which gives the least thickness required to safeguard the subgrade against failure. Suppose the pavement material is such that $\sigma_c/(\gamma\ a) = 900$. This value, in conjunction with $q/(\gamma\ a) = 150$ gives point B in Fig. 3, showing that the least thickness to prevent failure from occurring in the pavement is equal to $0.65\ a$.

The thickness that must be taken for design is the greater—namely, $0.70\ a$ corresponding to subgrade failure, which shows that the subgrade will fail before the pavement. On the other hand, if the pavement material were such that $c/(\gamma\ a) = 500$, point C, Fig. 3, would have been obtained, giving $h = 1.0\ a$ as the least thickness for safety. This thickness is larger than that necessary to safeguard the subgrade, so that now the pavement rules and the design thickness must be increased to 1.0 a.

4. THE PHYSICAL CHARACTERISTICS OF THE SOIL

It is usually assumed that the state of stress of a soil at failure can be represented by Eq. 1 and that most soils fall into one of two classes—purely cohesive or cohesionless. The Mohr envelopes of these two classes are shown in full lines in Fig. 5.

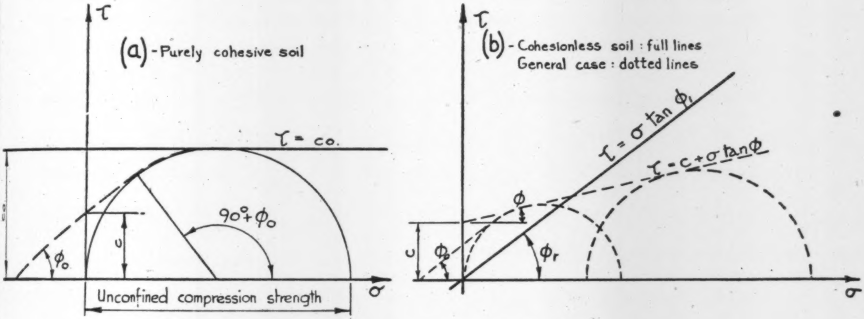
τ
(a) - Purely cohesive soil
τ = co.
co
c
90°+φ₀
φ₀
Unconfined compression strength
σ
τ
(b) - Cohesionless soil : full lines
General case : dotted lines
τ = σ tan φᵣ
τ = c + σ tan φ
φ
c
φ₀
φᵣ
σ

FIG. 5.—MOHR ENVELOPES OF TWO CLASSES OF SOILS

It does not seem probable that the tensile strength of a purely cohesive soil is equal to the compression strength, and one of the writers (Mr. Wilson) has suggested[18] that a closer approximation to the true envelope of failure in such a case is as shown in heavy dotted lines in Fig. 5(a), the angle ϕ_o being that determined from the direction of the shear planes in an unconfined compression test specimen. In the general case of a soil having both cohesion and friction the envelope would be as shown in the dotted lines in Fig. 5(b). This type of envelope has been assumed in the preparation of the design charts typified by Figs. 3 and 4. The angle ϕ_o has been assumed to be 30° in cases when $\phi \leqq 30°$ and to be equal to ϕ, when $\phi > 30°$.

The envelope of failure and the modulus of elasticity should be determined on the basis of quick triaxial tests in the case of taxiways and warm-up areas. In the case of runways and roads it is possible that high speed tests (such as those at Harvard University[19] (Cambridge, Mass.) on the investigation of the safety of the slopes of the Panama Canal) would be more suitable, in view of the very brief time that the load is being applied. The modulus of elasticity of soils is known to vary enormously with the minor principal stress.[20] Therefore, this stress should be suitably regulated during the tests, and an allowance should be made for the stresses existing in the soil before application of the wheel load. These stresses are considered in Section 5.

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[18] *Journal*, Inst. C. E., Vol. 24, 1945, p. 544.

[19] "Strength of Soils Under Dynamic Loads," by A. Casagrande and W. L. Shannon, in "Panama Canal—The Sea-Level Project: A Symposium," *Transactions*, ASCE, Vol. 114, 1949, p. 755.

[20] "A Contribution to the Study of the Elastic Properties of Sand," by G. Wilson and J. L. E. Sutton, *Proceedings*, 2d International Conference on Soil Mechanics, Rotterdam, 1948, Vol. I, p. 197.

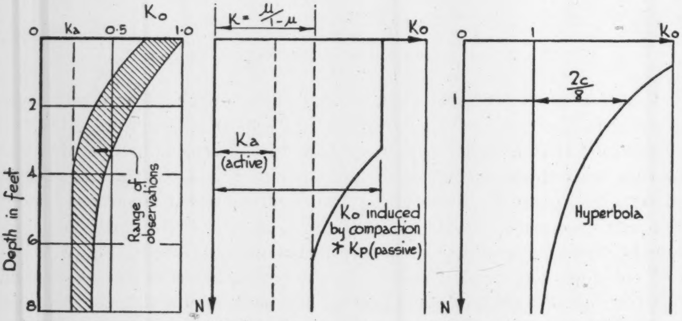
The strength of cohesive soils, both natural and remolded, depends on the moisture content, and it is as necessary in this method as in all others to test the soil at the greatest moisture content that is likely to exist during the lifetime of the pavement.

5. THE PRE-EXISTING STRESSES IN THE SOIL

Before any wheel load comes on to the pavement there exist, at any point in the soil, effective vertical stresses equal to the weight per unit of area of soil and pavement above the point in question, less any reduction due to buoyancy.

The effective horizontal stresses at any point can bear any relationship to the vertical stresses that is compatible with the envelope of failure, and the actual relationship in any case will depend on the history of the stratum.

In a cohesionless stratum the ratio of horizontal stress to vertical stress (K_o) can vary between $K_a = \frac{1 - \sin \phi}{1 + \sin \phi}$ and $K_p = \frac{1 + \sin \phi}{1 - \sin \phi}$, whereas in a frictionless stratum the horizontal stress can vary from the vertical stress by $\pm 2\,c$, and K_o can vary, therefore, between $1 \pm \frac{2\,c}{\gamma}\frac{1}{z}$, in which z is the depth below the surface of the stratum.



(a) Test on sand (b) Theoretical Ko Cohesionless soil compacted in layers (C) Theoretical maximum Ko Cohesive soil compacted in layers

FIG. 6.—PRE-EXISTING STRESSES IN THE SOIL

In the case of a frictionless stratum it might be useful to introduce a new term, the proportionate compaction, χ, defined by the equation:

$$\sigma_h = \sigma_v + \chi\, 2\,c \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

in which σ_h and σ_v are the vertical and horizontal stresses in the unloaded stratum. Parameter χ will vary between $+\,1$ and $-\,1$.

It can be shown[21] that the actual value of K_o is likely to be a maximum at the surface in the case of strata in the bottom of cuts or in banks that have been built up by compaction in layers, whether the soil is cohesive or cohesionless. This condition seems to be confirmed by a test. Fig. 6 shows the actual values of K_o found in a test in a sand stratum compacted in 6-in. layers and the nature of the variation of K_o indicated by theory.

The importance of the value of K_o is immediately apparent when it is realized that, if K_o had its minimum value, the soil could carry no load beyond its own weight. Therefore, effective compaction will increase the carrying capacity of the subsoil.

6. Basis of Comparison with Other Methods

In the attempt to emphasize any underlying common features in this and other methods, it is proposed to compare the results reported by Messrs. Westergaard and Glossop and Golder, and the results from CBR methods, with the proposed method. It will first be necessary to make an estimate of the linking magnitudes—that is, to establish a kind of "rate of exchange" between the measure of soil properties that form the basis of the various methods. These soil properties are:

(a) The strength of the foundation, represented by Eq. 1; and
(b) The elastic properties of the foundation material E_2 and μ_2, which are related to the CBR and the modulus of subgrade reaction, K_u.

In the case of an elastic half space the settlement of a rigid circular plate on the surface is given by the formula:

$$\delta = \frac{(1 - \mu^2{}_2)}{2\,E_2}\,p\,a \qquad (5)$$

in which p is the average pressure on a plate of radius a. Hence,

$$K_u = \frac{p}{\delta} = \frac{2\,E_2}{a(1 - \mu^2{}_2)} \qquad (6)$$

which is a relation between K_u and E.

Coefficient K_u is usually determined from the pressure required to cause a settlement of 0.05 in. on a plate of 15-in. radius. Consequently, it is found that

$$E_2 = 21\,K_u \qquad (7)$$

assuming $\mu_2 = \frac{1}{3}$.

Young's modulus E_2 can also be derived from the CBR by Eq. 7, if the appropriate deflection is chosen. A deflection of 0.00325 in. in the case of a disk with a 0.97-in. radius (such as is used in the CBR test) corresponds to the standard deflection of 0.05 in. found in the bearing test with a 30-in. plate. This deflection is very close to the origin of the load-deflection curve in the CBR test and the slope of the initial tangent may be taken as the appropriate value of p/δ. Assuming a parabola through the origin and the standard CBR pressures corresponding to deflections of 0.1 in. and 0.2 in. (Fig. 7), the slope of the tangent is 30,000 lb per in.[3] Hence,

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[21] "Bearing Capacity of Screw Piles and Screwcrete Cylinders," by G. Wilson, *Journal*, Inst. C. E., March, 1950, p. 4

$$E_2 = 409 \text{ CBR} \quad \ldots\ldots (8)$$

and

$$K_u = 19.5 \text{ CBR} \quad \ldots\ldots (9)$$

The CBR could also be related to the strength of the foundation material by the formula proposed in 1947 by Karl Terzaghi,[22] Hon. M. ASCE, for the bearing capacity of a circular foundation, provided that an assumption is made about the deflection at failure,

$$q_{dr} = 1.3\, c\, N_c + \gamma\, D_f\, N_q + 0.6\, \gamma\, a\, N_\phi \quad \ldots\ldots (10)$$

in which N_c, N_q, and N_ϕ are constants depending on ϕ, the angle of internal friction, and D_f is the depth of the footing below the surface of the ground. If failure occurs at a settlement of about 10% of the radius—that is, when an 100% CBR corresponds to a pressure of 1,000 lb per sq in., causing a settlement of 0.1 in.,

$$\text{CBR} = \frac{100 \times q_{dr}}{1{,}000} = \frac{q_d}{10} \quad \ldots\ldots (11)$$

The CBR test is usually performed on clay soils without a surcharge so that the second term disappears from the right-hand side of Eq. 10 and the results obtained by this mode of calculation are in fair agreement with experimental values for this type of soil. Sands, on the other hand, are normally tested with a surcharge that could be allowed for by making the term γ/D_f equal to the pressure exerted on the sample by the surcharge. However, even if this is done, Eq. 11 gives results which are much smaller than those commonly obtained by experiment, so that it appears that the sample obtains additional restraint from the sides of the container and from the friction on the base of the surcharge. Consequently, Eq. 10 holds for clay soils but not for sands, and Eq. 9 is no longer applicable. However, a number of empirical relationships have been given connecting the CBR with K_u, one of which[23] the writers have adopted for cases when the CBR exceeds 10. Below this value, Eq. 9 is used and the resulting relationship between the CBR and K_u is shown at

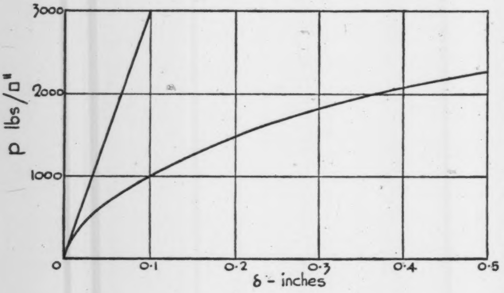

FIG. 7.—STANDARD TEST OF A 100% CBR MATERIAL; PLUNGER, 1.95 IN. IN DIAMETER

the top of Fig. 8. There seems to be no reason why Eq. 7 should not still be applicable.

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[22] "Theoretical Soil Mechanics," by Karl Terzaghi, John Wiley & Sons, Inc., New York, N. Y., 2d Ed., 1947.

[23] "Handbook of Soils, Concrete and Bituminous Materials," H. M. Stationery Office, London, 1944, p. 191, Fig. C18.

Table 2, Cols. 1, 2, and 3, gives soil properties for four of the typical soils cited by Mr. Golder in 1946.[24] The data for the clays were obtained from Eqs. 8 to 11. As Eq. 10 is not applicable to sands, CBR values were chosen arbitrarily for loose and dense sands (types 3 and 4). The value for K_u (Col. 4), in pounds per cubic inch, was then derived empirically as outlined in the previous paragraph, and Eq. 7 was used to evaluate E_2 (Col. 6), in pounds per square inch. In Col. 3, Table 2, c is in pounds per square foot.

It was thought advisable to devise, as far as possible, a graduated scale connecting the various soil properties, rather than to use the range of values that Mr. Golder arbitrarily assigned on the basis of experience (a range with which the writers do not disagree). However, these relationships between the soil properties have been derived solely for comparing pavement thicknesses obtained by various methods, and it is not suggested that they should be applied in practice. The relationship between the modulus of elasticity and the modulus of subgrade reaction as determined by the plate bearing tests may be fairly reliable; and that between the modulus of elasticity and the CBR may be very far from the truth as the load-settlement curve is rarely parabolic, as assumed. It may vary from a straight line to something like a fourth-power curve. The derivation of the CBR from the strength characteristics of the material is probably reasonably accurate in the case of clays.

TABLE 2.—LINKING OF SOIL PROPERTIES CONSIDERED IN VARIOUS METHODS

Soil type (1)	Description (2)	Shear strength (3)	K_u (4)	CBR (5)	E_2 (6)
1	Soft clay...........	$c = 400$	41	2.1	858
2	Stiff clay...........	$c = 2{,}000$	205	10.5	4,300
3	Loose sand.........	$\phi = 32°$	230	15	4,800
4	Dense sand........	$\phi = 38°$	500	50	10,500

7. COMPARISON OF METHODS FOR DESIGNING CONCRETE PAVEMENTS

Fig. 8 shows the thickness of concrete pavement as required by Messrs. Westergaard, Glossop and Golder,[11] and Golder[6] in 1946, and the present method. It also shows the thickness of flexible pavement required by the CBR method for comparison with the suggestion made by Messrs. Henry and Grace.[16]

Scrutiny of the illustrations shows that the proposed method gives results generally proportional to those given by Mr. Westergaard's method. In the cases when failure occurs in the soil, a slight benefit arises if the soil can be so compacted that the "natural" horizontal stress is a maximum—that is, $\chi = 1$.

The suggestion made by Messrs. Henry and Grace appears to be correct when the CBR is between 5 and 10. For smaller values a concrete pavement could be less than half the thickness of a flexible pavement; but, if the CBR exceeds 10, the thickness of a concrete pavement would need to be more than half that of a flexible pavement. The CBR of the soil considered by Messrs. Henry and Grace was 3.

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[24] "The Relationship of Runway Thickness and Undercarriage Design to the Properties of the Subsoil," by H. Q. Golder, *Airport Paper No. 4*, Inst. C. E., 1946, p. 5, Table II.

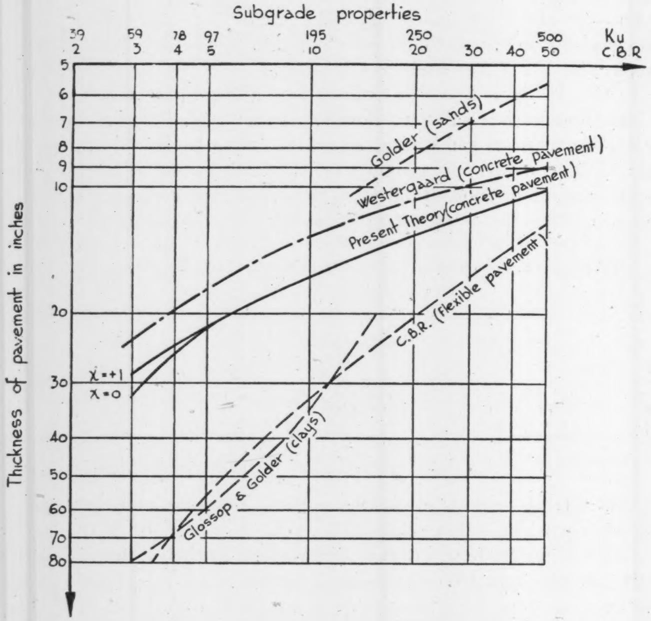

Fig. 8.—Comparison of Design Methods (Wheel Load, 100,000 Lb; and, in Pounds per Square Inch, the Tire Pressure Is 100 and the Tensile Strength of Concrete Is 450)

The method proposed by Messrs. Glossop and Golder agrees closely with the CBR and, therefore, it would appear to be uneconomical for the design of concrete pavements. Mr. Golder's method,[6] on the other hand, appears to be unsafe, for the concrete would crack.

8. Application to Flexible Pavements

The horizontal pressure at the base of an unloaded flexible pavement of compacted gravel or hard core is equal to $K'_o \gamma h$ (the coefficient K'_o being the ratio of horizontal to vertical stress in the unloaded pavement). Such a pavement may safely be loaded until the maximum tensile stress, $\overline{rr}$, at its base due to the surface load reaches this value. This stress may be calculated in the same way as the stress for concrete pavements, using suitable values for the ratio of the moduli of elasticity of the pavement and the subgrade materials. Since the CBR is proportional to the load required to produce a given compression in a sample of the soil, the ratio of the moduli may be taken approximately as the CBR of the subgrade material divided into 100 minus the CBR of a "perfect" pavement material. If this is done, loads known to be safe in practice would theoretically induce tensile stresses equal to many times the compressive stress $K'_o \gamma h$, even when very large values are taken for K'_o.

Apparently the pavement does not act elastically, and it is suggested that it behaves approximately as follows: Fig. 9 shows the effect of variations in the ratio of the elastic moduli E_1/E_2, on the stress $\overline{rr}$ and shows that a reduction in E_1/E_2 causes a reduction in rr. If E_1/E_2 is initially based on the CBR of the subgrade, $\overline{rr}$ increases rapidly with an increasing load and its safe value, $K'_o \gamma h$, is reached very quickly. The pavement must then yield, which is roughly equivalent to a decrease occurring in E_1, its elastic modulus; and this action results in a decrease in $\overline{rr}$, which may continue until $\overline{rr} = K'_o \gamma h$ when equilibrium is reached—that is,

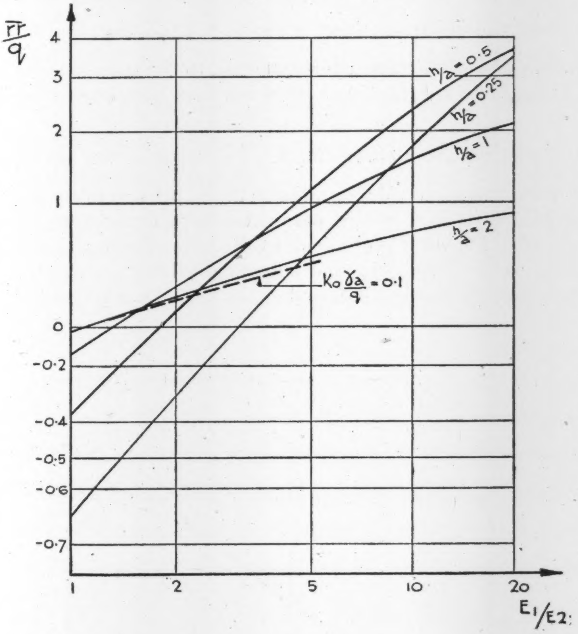

FIG. 9

$$\frac{\overline{rr}}{q} = K'_o \frac{\gamma a}{q} \frac{h}{a} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

For a given value of $K'_o \frac{\gamma a}{q}$, Eq. 12 defines a line in Fig. 9 intersecting the curves corresponding to various values of h/a so as to give the apparent value of E_1/E_2 when $\overline{rr}$ is equal to $K'_o \gamma h$ in any particular pavement. A typical intersecting curve is given for $K'_o \frac{\gamma a}{q} = 0.1$, showing that if, for instance, $h/a = \frac{1}{4}$ the pavement behaves as if $E_1/E_2 = 4.5$. It seems reasonable to assume that the stresses in the subgrade are approximately equal to those set up when E_1/E_2 is equal to 4.5, and they may be calculated from the Burmister theory, after the manner suggested by Mr. Fox. Consequently, if $K'_o \frac{\gamma a}{q}$ is given, the maximum shear stress in the subgrade may be calculated for each value of h/a. Equating this quantity to the cohesion c, a curve may be drawn giving safe values of a/h as a function of c/q for purely cohesive subgrades.

Fig. 10 shows two such curves for $K'_o \frac{\gamma a}{q} = 0$ and 0.1, assuming $K_o = 1$ in the subgrade. As these values represent the upper and lower limits likely to be encountered, variations in this parameter have little effect on the result. The third curve in Fig. 10 was obtained by direct application of the Boussinesq theory, assuming the pavement and subgrade, jointly, to constitute a semi-infinite, homogeneous elastic solid, and equating the maximum shear stress in the subgrade to its cohesion. The similarity between the curves suggests that flexible pavements behave very much according to the Boussinesq theory, an hypothesis stated in 1949 by R. de l'Hortet,[25] who showed the close relation between results obtained by the CBR method and those obtained by the Boussinesq theory.

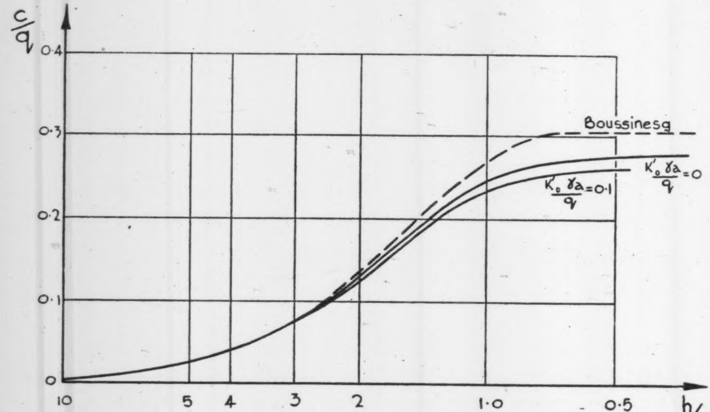

FIG. 10.—RELATION BETWEEN THE THICKNESS REQUIRED FOR A FLEXIBLE PAVEMENT ON A FRICTIONLESS SUBGRADE, AND THE COHESION (COEFFICIENT OF EARTH PRESSURE AT REST IN THE SUBGRADE, $K'_o = 1$)

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[25] "Runway Strength Correlation Between the Distribution of Stresses in the Soil Derived from: (1) The C.B.R. Method of the Determination of Runway Thickness; (2) the Theory of Boussinesq," by R. de l'Hortet, *Proceedings*, 2d International Conference on Soil Mechanics, Rotterdam, 1948, Vol. V, p. 185.

It is difficult to obtain a direct comparison between the thickness derived for a particular case by the CBR method and the present method. For exact equivalence the two methods should give the same thickness for any frictionless soil and loading. Thus, if for any soil the least thickness required by the CBR is used to compute c from Fig. 10, the results should be independent of the particular wheel load and tire pressure chosen. Taking thicknesses from a CBR chart,[26] the relationship in Fig. 11 was obtained. Three curves are shown working with wheel loads of 4,000 lb, 40,000 lb, and 70,000 lb, the discrepancy suggesting that the CBR has a slightly higher safety factor for the smaller wheel loads. The results for the cohesion are considerably affected by the tire pressure, which is not allowed for very closely by the CBR method. A pressure of 60 lb per sq in. was assumed, but a slight increase or decrease in the case of the lightly loaded or heavily loaded wheels, respectively, would bring the curves more closely together. The broken line in Fig. 11 shows the relationship between the CBR and the cohesion of the subgrade which was obtained by an independent method in Section 6. It will be seen that the two methods give very similar results.

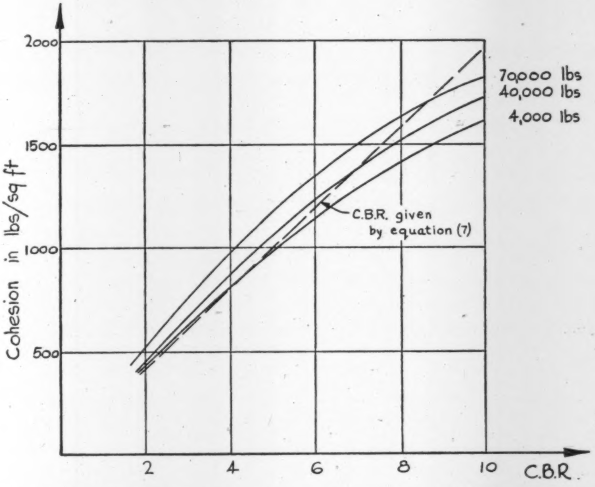

FIG. 11.—RELATION BETWEEN THE CBR AND THE COHESION OF SUBGRADE MATERIALS

The method may be applied to frictional subgrade materials by assuming the same values for E_1/E_2 and by taking Eq. 1 as the criterion for subgrade failure.

In the case of thick flexible pavements (of thickness h approximately greater than the radius a), the pavement material may also fail in a manner similar to the failure of material under a simple foundation, a possibility that can be allowed for by the usual methods.

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[26] "Handbook of Soils, Concrete and Bituminous Materials," H. M. Stationery Office, London, 1944, p. 191.

9. LIMITATIONS OF PROPOSED METHOD

The obvious limitations of the proposed method are that it necessarily relates only to a load on the surface of the upper layer of a distinct two-layered system. Edge and corner conditions that control the design of concrete surface pavements, and the gradual merging of flexible surfaces into subgrade, via base and subbase, cannot be provided for in the theory.

Edge and corner conditions, which are considered only in Mr. Westergaard's method, could be investigated by the methods of three-dimensional photoelasticity. Loads could be applied to large blocks of bakelite through the edge and corners of metal sheets of varying thickness and varying elastic moduli. Direct comparison with Mr. Westergaard's method for the "interior" of slabs will be useful, in any case.

However, the method is directly applicable to pre-stressed concrete pavements and to the "schipol sandwich" type of construction.[27] Both these types of pavement seem destined for wide adoption.

A limitation that is less immediately apparent, but which is, nonetheless, real, is that the modulus of elasticity of soils depends on the state of stress and that, consequently, the distribution of stress from a superimposed load depends on the pre-existing state of stress in the system. Against this limitation, as was so brilliantly stated by A. Caquot[28] in 1948, is the fact that it seems to be possible to "educate" soils, by repetition of loading, to behave more as elastic materials.

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[27] "Soil Investigations and Runway Construction at Schipol Airport," by J. W. Clerx and L. J. H. Weinberg, *Proceedings*, 2d International Conference on Soil Mechanics, Rotterdam, 1948, Vol. IV. p. 197.

[28] "Foundation Pressure and Settlements of Buildings on Footings and Rafts: General Report," by A. Caquot, *ibid.*, Vol. VI, Section VI, p. 112.

DISCUSSION

Robert Ruckli.[29]—Because they have practically no tensile strength, flexible pavements cannot be analyzed by applying the Burmister theory[4] to cases in which the ratio E_1/E_2 is small. One might expect that the difference between the moduli of elasticity E_1 and E_2 would diminish the factor of concentration ν, introduced by O. K. Fröhlich,[30] but the writer, with the help of the principle of minimum stress energy, found that the decrease is small. For a single load, a ratio $E_1/E_2 = 10$ and a depth $h = 30$ cm (1 ft), it was found $\nu = 2.82$, rather than $\nu = 3$ for the homogeneous half space.

Possibly the following procedure would lead to a more conclusive result: Instead of a discrete variation of the modulus of elasticity from E_1 to E_2 assume that E decreases, beginning at the top, in such a manner that no horizontal tensile stress occurs along the z-axis. A function $E = f(z)$ would be derived that would fulfil this condition. The pavement, base, and subbase would then be constructed so that the modulus of elasticity of the foundation would vary approximatively in accordance with the same law. In this manner, it would be possible to evaluate the degree to which the quality of the foundation (E) will influence the stress distribution in the subsoil and, therefore, to compute the thickness of the pavement.

By this procedure, also, one could find an answer to the question of whether multi-layer systems really behave approximately according to the Boussinesq theory,[25] as the authors suggest in their discussion of Fig. 10. At the very best, the Boussinesq theory is not very simple, and the writer doubts that a mathematician could be found who could generalize it to the solution of this complex problem.

Hugh Q. Golder.[31]—Another theory for the design of pavements is offered in this paper. The method of approach would seem to be unobjectionable but, as is the case in all theories, its applicability must rest on some assumptions. In the method described by the writer and R. Glossop[11] the same was true, and one assumption—namely, that the tensile strength of the concrete was neglected—was heavily criticized. However, this method had the almost unique advantage of having been devised after the occurrence of pavement failures, against which it was then checked. The design criterion was chosen as one out of three, in order to give good agreement with the case of actual failure, for the conditions existing. The same criterion was later checked against other failures and found to give reasonable agreement. It was later found that the method gave good agreement with the wholly empirical California Bearing Ratio (CBR) method.

The later method for frictional materials (including materials both frictional and cohesive), devised by the writer,[24] has not been checked against the failure of pavements on a sand foundation. The reason for this is simple: The writer has been unable to discover any case in which a pavement on sand has failed due to failure of the foundation, although he has consistently asked, on every possible occasion, for information concerning such a failure.

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[29] Rosenbergstrasse 42, Berne, Switzerland.

[30] "Druckverteilung im Baugrunde, mit besonderer berucksichtigung der plastischen eischeinungen." by O. K. Fröhlich, Springer-Verlag, Wien, 1934, p. 23.

[31] Director, Soil Mechanics, Ltd., London, England.

The foregoing evidence confirms the writer's belief that the variation in thickness of slab given by the Westergaard method[8] does not reflect, accurately, the variation in the properties of the foundation material when going from, say, soft clay to a dense sand; and it does not reflect the effect of this variation on pavement behavior. Incidently, this second method can be made to agree with the Glossop-Golder method for the case of clays, simply by the choice of a suitable factor of safety.

Messrs. Wilson and Williams have made many assumptions, of which they are fully aware, and which are concerned partly with their endeavor to obtain a comparison between the different methods. In Fig. 10, for example, the Boussinesq curve takes them straight back to the original assumption of the Glossop-Golder method.

In their comparisons of the different methods (Fig. 8), Messrs. Wilson and Williams state: "Mr. Golder's method,[6] on the other hand, appears to be unsafe for the concrete would crack." Properly, they should have added "according to the Westergaard theory or the Wilson-Williams theory." As stated, the writer has yet to find a case in which the concrete did crack on a sand subgrade. Although making no claim that the Golder method gives the correct answer, the writer feels that it gives the right degree of difference between the thickness of pavement required on a dense sand and a soft clay and, to the writer's knowledge, it is the only one that does so. In discussing the writer's 1946 paper,[6] Mr. Wilson referred to a possible method such as he has now produced, stating: "the matter would certainly require to be confirmed by practical tests on a large scale, or by experience."

Although applauding the initiative and energy of the authors, the writer feels that a new method such as this should be used with reserve until it has been checked against experience with pavements designed down to a failure condition.

In Section 5, the authors state that "* * * if K_o had its minimum value, the soil could carry no load beyond its own weight." The statement that "K_o had its minimum value" is equivalent to stating that the Rankine active state of stress exists in the ground. This means that, on all planes inclined at $45°-\phi/2$, incipient failure exists; but this is not a condition of foundation failure. As shown by W. J. M. Rankine, M. Ritter, and Karl Terzaghi, Hon. M. ASCE, in different ways, the power of a frictional material to support a load is considerable (particularly with a wide load), even when the active state is assumed below the load. In this connection it would be interesting to know how the value of K_o was measured in the test referred to, since this is a most difficult thing to do.

GUTHLAC WILSON,[32] M. ASCE, and G. M. J. WILLIAMS,[33] ASSOC. M. ASCE.—The interesting suggestion by Mr. Ruckli appears particularly applicable to flexible pavements founded on base and subbase courses of decreasing strengths on a weak subgrade.

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[32] Partner, Scott and Wilson, Cons. Engrs., London, England.

[33] Asst. Engr., Scott and Wilson, Cons. Engrs., London, England.

In answer to Mr. Golder, the writers refer to Fig. 8 as a convincing demonstration that the Westergaard method is an accurate reflection of the properties of the foundation material. The thickness of concrete pavement given by the Westergaard method is consistently between 80% and 85% of the thickness given by the method presented in the paper. Lack of recorded instances of cracking in concrete pavements on sand subgrades does not necessarily support Mr. Golder's view. Rather, it is likely that all such pavements were designed by the Westergaard theory, to which they form an eloquent testimonial.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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TRANSACTIONS

Paper No. 2447

LONG-TERM STORAGE CAPACITY OF RESERVOIRS

BY H. E. HURST[1]

WITH DISCUSSION BY VEN TE CHOW, HENRI MILLERET, LOUIS M. LAUSHEY, AND H. E. HURST.

SYNOPSIS

A solution of the problem of determining the reservoir storage required on a given stream, to guarantee a given draft, is presented in this paper. For example, if a long-time record of annual total discharges from the stream is available, the storage required to yield the average flow, each year, is obtained by computing the cumulative sums of the departures of the annual totals from the mean annual total discharge. The range from the maximum to the minimum of these cumulative totals is taken as the required storage.

The results of the investigations given here were applied in computing the storage required in the Great Lakes of the Nile River Basin, the length of time chosen as a basis of the estimates of storage being 100 years. These reservoirs, in combination with the other projects described, would enable the Nile to be developed for irrigation to the fullest possible extent. Finally, a relation has been derived between the storage capacity on the main stream and the amounts required on the tributaries to produce its equivalent.

It is thought that the general theory may have other applications than the design of reservoirs for the storage of water.

1. INTRODUCTION

A number of projects has been investigated which will enable Egypt and the Sudan to develop irrigation from the Nile River to its fullest extent.[2,3] The broad plan involves storage of water from good years for use in bad ones, and necessitates reservoirs of sufficient capacity to meet the shortages that might occur during a century. Much research had been done on the capacity

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NOTE.—Published in April, 1950, as *Proceedings-Separate No. 11*. Positions and titles given are those in effect when the paper or the discussion was received for publication.

[1] Scientific Consultant to the Ministry of Public Works, Egypt; formerly Director-Gen., Physical Dept., Cairo, Egypt.

[2] "The Nile Basin," Vol. VII, by H. E. Hurst, R. P. Black, and Y. M. Simaika, *Physical Department Paper No. 51*, Ministry of Public Works, Cairo Govt. Press, Cairo, Egypt, 1946.

[3] *Civil Engineering and Public Works Review*, September, 1948.

of such reservoirs and the results were used to determine the size of the reservoirs required in the Great Lakes of the Nile Basin for what was called "century storage."

Some preliminary results of investigations were released in 1938[4] which were amplified in 1946;[3] but in neither case was there an adequate statement of the theoretical investigation. The latter has been reserved for this paper.

Storage problems were treated first by the late Allen Hazen,[5] M. ASCE, in 1914, when he studied the discharge of thirteen American rivers, dealing with them mainly in a graphical manner, on probability paper. The longest record was one of 45 years, but longer sets of observations were made by combining records from several rivers. The objection to this procedure and the uncertainty in extending the normal Gaussian frequency distribution to the future were recognized by Mr. Hazen, but he thought it gave the best results then obtainable.

The work was extended by the late Charles E. Sudler,[6] M. ASCE, by similar graphical methods. Much of Mr. Sudler's work was based on artificial records in which information taken from a short record of a stream was extended to a period of 1,000 years by writing, say, 50 annual runoff values on cards and then shuffling and drawing a card from these 1,000 times. Again the actual data were necessarily confined to short periods since stream measurements in the United States had rarely been made for as long as 50 years. Thus, the results are mainly based on a few short-period records of natural river flow extended to long periods by random repetition. It is clear that the process can give very little information about what may happen over long periods and may be completely misleading since it excludes values higher or lower than those which have already occurred. This was recognized by some of those who contributed to the discussion on Mr. Sudler's paper. Moreover, the graphical methods employed do not lead to any concise and easily understandable presentation of results.

In the present investigation the methods of the theory of probability are applied, but the data are long-period records of natural events which are shown to have certain similarities, although they may be records of river discharge, rainfall, temperature, annual growth rings of trees, or annual deposits of clay in lakes. The average results can be expressed quite simply by two equations which apply to the storage of other substances as well as water. It is shown that, in regard to storage, records covering such short periods as 30 years or 40 years may be very misleading, but their usefulness can be extended by analyzing a large number of phenomena which have been recorded for periods as long as a century or more.

The problem of what storage is required on a stream to give a certain minimum discharge was first investigated in the case of Lake Albert (in the Belgian Congo and the Uganda Protectorate) to determine the size of an over-year storage reservoir which would be required in order to equalize the outflow over a number of years. It leads to a statistic R, which is the range from maximum to minimum of the curve obtained by plotting the cumulative totals of the departures from the mean of the annual discharges taken in order. Such a curve of cumulative totals is sometimes called a mass curve. Values of R have been computed for long series of observations of rainfall, temperature, and other natural phenomena, and these have been filed for reference in the Engineering Societies Library.[6a]

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[4] "The Nile Basin," Vol. V, by H. E. Hurst and P. Phillips, Cairo Govt. Press, Cairo, Egypt, 1938, p. 86.

[5] "Storage to Be Provided in Impounding Reservoirs for Municipal Water Supply," by Allen Hazen. *Transactions*, ASCE, Vol. LXXVII, December, 1914, p. 1539.

[6] "Storage Required for the Regulation of Stream Flow," by Charles E. Sudler, *ibid.*, Vol. 91, 1927, p. 622.

Nomenclature.—The letter symbols introduced in this paper are defined where they first appear, in the text, or by illustration, and are arranged alphabetically in the Appendix, for convenience of reference. A few specialized terms used in the paper are defined as follows:

Century Storage.—The computed storage capacity of a reservoir, as required in the Great Lakes of the Nile Basin, necessary to meet the shortages that may occur in 100 years (Section 1).

Continued Sum.—Successive cumulative totals (Section 1).

Cubic Meter.—English units, 0.0008107 acre-ft.

Equatorial Lakes.—Lake Victoria (in Uganda, Tanganyika, and Kenya), Lake Albert (in the Belgian Congo and Uganda), and other bodies of impounded water in the southern Nile Basin, at the equator; also referred to as the "Great African Lakes," or simply the "Great Lakes" (Section 2).

Gaussian Frequency Distribution.—The normal frequency curve first advocated by the German mathematician, Karl Freidrich Gauss, who maintained that all skew distributions tended to become normal as the number of events was increased and that skewness was an evidence of inadequate data.

Meter.—English units, 3.28083 ft.

Milliard.—One thousand million things—that is, 10^9.

Varves.—Annual layers of silt as deposited in a lake or other body of still water. As individual layers differ in thickness and character, a succession of such layers forms a characteristic group which can be identified as of contemporaneous deposition in whatever deposit it may be found. It is thus possible, by combining different sections, to measure the time involved in the deposition of the entire group of sediments and to construct a time scale in a manner similar to that employed in the study of annual rings in trees (*Webster's New International Dictionary of the English Language*).

2. Statement of the Problem

Given a reservoir of large capacity, how can its outflow be regulated economically to meet the deficiencies of low years, and what is the capacity required to guarantee a certain minimum discharge in these years? These are the basic questions.

As a concrete example to illustrate the methods, consider the case of Lake Albert. If the past annual discharges from the lake are given, the size of reservoir required to maintain the maximum possible steady discharge during the period can be determined. This is the mean discharge for the period, and the capacity of the reservoir is obtained by adding, algebraically, the departures from the mean to form a series of accumulated totals. The difference between the highest and the lowest of these accumulated totals—that is, the range *R*—is (*a*) the maximum accumulated storage when there is never a deficit, (*b*) the maximum accumulated deficit when there is never any storage, or (*c*) their sum when there is both storage and deficit. This range *R* is the storage required to maintain the average discharge.

[6a] 29 W. 39th Street, New York 18, N. Y.

Increased losses due to storage are disregarded because, unless they are small, the site is not suitable for over-year storage. On a reservoir like the Aswan Reservoir in Egypt there is heavy evaporation uncompensated by any rainfall, and the surface area of the reservoir is several times that of the natural river. Therefore, there is an appreciable extra loss as a result of storage, so that, even if there were the capacity at Aswan, it would be unsuitable for over-year storage. In the case of the Equatorial Lakes their size makes it possible to store a great volume of water with only a small proportional increase of surface area. Moreover, evaporation and rainfall are nearly equal; so increased evaporation loss due to increased surface is compensated by increased gains from rainfall.

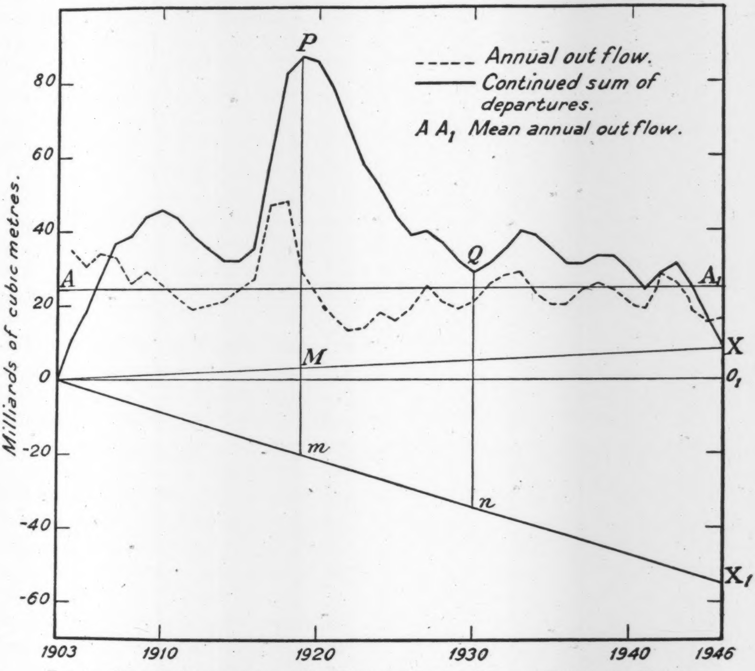

FIG. 1.—ANNUAL OUTFLOW FROM LAKE ALBERT IN THE BELGIAN CONGO AND UGANDA PROTECTORATE (1904 TO 1946) AND THE CURVE OF CONTINUED SUMS OF DEPARTURES (ONE MILLIARD CUBIC METERS = 810,700 ACRE-FT)

Fig. 1 shows the annual total discharge from Lake Albert based on recorded lake levels and gage-discharge curves. The curve of accumulated departures from a base near the mean is also shown. The actual values and the method of computation are shown by reference to Table 1.

TABLE 1.—DISCHARGE FROM LAKE ALBERT (BELGIAN CONGO—UGANDA PROTECTORATE) IN MILLIARD CUBIC METERS (1 MILLIARD CUBIC METERS = 810,700 ACRE-FT)

Year (1)	Q (2)	$+d$ (3)	$-d$ (4)	d^2 (5)	Σd_c (6)
1904	35	11		120	11
1905	31	7		50	18
1906	34	10		100	28
1907	33	9		80	37
1908	26	2		0	39
1909	29	5		20	44
1910	26	2		0	46
1911	22		2	0	44
1912	19		5	20	39
1913	20		4	20	35
1914	21		3	10	32
1915	24		0	0	32
1916	27	3		10	35
1917	47	23		530	58
1918	48	24		580	82
1919	29	5		20	87
1920	23		1	0	86
1921	17		7	50	79
1922	13		11	120	68
1923	14		10	100	58
1924	18		6	40	52
1925	16		8	60	44
1926	19		5	20	39
1927	25	1		0	40
1928	21		3	10	37
1929	19		5	20	32
1930	21		3	10	29
1931	26	2		0	31
1932	28	4		20	35
1933	29	5		20	40
1934	23		1	0	39
1935	20		4	20	35
1936	20		4	20	31
1937	24		0	0	31
1938	26	2		0	33
1939	24		0	0	33
1940	20		4	20	29
1941	19		5	20	24
1942	29	5		20	29
1943	26	2		0	31
1944	18		6	40	25
1945	15		9	80	16
1946	16		8	60	8
Σ	1,040	+122	−114	2,310	
N	43	(8)			
Mean	24.19	0.19			
Base	24				

The discharges, Q, in Col. 2, are usually given to one tenth of a milliard cubic meters but the last digit is not certain. For this computation the discharges are written to the nearest milliard and this reduces most of the departures, d, to a single digit, which is enough for purposes of computation. The omission of the last digit makes no significant difference to the mean or the standard deviation, σ, but lessens considerably the labor of computation. The computation is arranged to be self-checking as far as possible.

The mean of the series is first computed, and then a base is chosen near the mean. In this case the mean is 1,040/43, or 24.19, and 24 is selected as a base from which to compute departures, being the nearest whole number to the mean, and so keeping all the departures to one or two digits.

After writing the departures the positive and negative values of d (Cols. 3 and 4) are added separately. In this case the sum of the positive is 122 and of the negative, 114. The difference is 8 and $8/N$ should be the difference between the mean and the base. This latter difference gives a check on the correctness of the departures, and also on their continued sum Σd_c (Col. 6). If the departures were reckoned from the mean their final algebraic sum would be zero; otherwise, it is N (mean minus base), or 43 (24.19 − 24), and, of course, this is also the algebraic sum of the totals of the positive and negative departures, d, thus affording a check on the continued sums. With a long series of observations it is advisable to compute the totals of the columns, Σ on each page to check the work piece by piece.

In computing the standard deviation, σ, the mean square deviation must be reduced by (mean minus base)^2, or by $(24.19 - 24)^2 = 0.19^2 = 0.04$, to give $\sigma^2 = \frac{2{,}310}{43} - 0.04$; and $\sigma = 7.33$. This is the only part of the work which is not self-checking and must be checked by repetition. The last digits of the square are omitted, which means in this case that the squares are given to the nearest ten which is near enough provided there are 50 or more observations. The effect of the approximation on σ is about 1% in this instance.

The departures are reckoned from a base which is not the mean, and, if the continued sums of these are plotted, a simple graphical method will yield the continued sum of departures from the mean or any other base. In the present case the mean is 24.2 and the base, 24. Every departure from the base is therefore 0.2 greater than the corresponding departure from the mean, so that the sum of the first n departures is $n \times 0.2$ greater than the sum of the first n departures from the mean. The sum of all the departures from the base is $N \times 0.2$. Thus, the continued sum of any number of departures from the mean can be found by joining the point representing the last continued sum to the origin and taking this line as a new axis. This is a useful construction for finding the continued sums from any base, when the sums referring to some base have been plotted.

In Fig. 1 the continued sums of departures from base 24 are plotted and the new axis OX giving continued sums of departures from the mean is drawn. The range, R, of the summation curve of departures from the mean is $\overline{\text{PM}} = 84$ milliards.

Examples of other uses of the graphic method of obtaining continued sums are given, subsequently, in Figs. 3 and 6.

In the case of Lake Albert the accumulated sum of departures rose to a maximum in 1919 (point P, Fig. 1) and then fell to zero by 1946 (point X), so that the next period would begin with no stored water. Had the operation been begun in 1920 there would have been a deficit to start with. This fact shows the necessity of making a reservoir some time before it will be required so that it can be partly or wholly filled before its storage is needed. It is clear from Fig. 1 that the storage required to maintain the mean discharge over the period is the range from maximum to minimum of the accumulated departures. It is also clear that, if the capacity of the reservoir is just enough to make up the maximum accumulated deficit, when this has been done, the reservoir will be empty and there will be no margin of stored water for the future. Moreover, the engineer is regulating the flow after the event, with full knowledge of what has happened. The problem of the required capacity is not a simple one and cannot be solved by adding departures from the mean, considering only Lake Albert, but must be dealt with on a wider basis. The discharge of the main Nile River at Aswan may be taken as a further illustration of the difficulties.

Table 2, for example, is based on three widely differing assumptions: First, that a computation was made in 1909, using 38 years of record; second, that a computation was made in 1946, using the succeeding 36 years of record; and, third, that a computation was made in 1946, using the total period of 74 years of record. The difference between the results of these computations shows how misleading the simple process can be, both as to the size of the reservoir needed and as to the benefit to be expected from it. Thus, it is clear that more extended investigation is required, and this can only be made by the examination of many sets of river discharges extending over long periods, or of other data with similar characteristics. At the same time some promise of a solution is possible by exploring theory.

TABLE 2.—COMPARISON OF VARYING RESULTS, COMPUTATIONS FOR THE MAIN NILE RIVER

Description	1871–1908	1909–1945	1871–1945
Average annual discharge, Q, milliards m^3.............	103	83	93
Range of accumulated departures, R, milliards m^3......	201	83	476

The first part of this paper attempts to find a theoretical relation among the range, R, of the summation curve of departures from the arithmetic mean, the variability of the phenomenon expressed by its standard deviation, σ, and the number of years of observation, N, on the assumption that the observations are random events following the normal Gaussian frequency distribution. This course was adopted because many natural phenomena have frequency distributions which are approximately normal if the order of occurrence is disregarded, and it was thought that the examination of this case, which is a first approximation to those occurring in nature, would indicate the form of relation to be sought. Gauss' normal frequency curve (which is also the curve for the frequency distribution of errors of observation) gives the frequency of departures of a quantity from its arithmetic mean in relation to their magnitudes. If y is the number of departures lying between $x + \frac{1}{2}$ and $x - \frac{1}{2}$ when the total number of values of the quantity is N, the equation of the curve is

$$y = (N/\sigma \sqrt{2\,\pi})\, e^{-x^2/2\sigma^2} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Incidentally, the assumption that the discharge of a river is a random event is the basis of the works of Messrs. Hazen[5] and Sudler;[6] under this same assumption a theoretical solution has been found.

3. THE FREQUENCY DISTRIBUTIONS OF RANDOM EVENTS

It is well known that the Gaussian normal frequency curve (Eq. 1) fits many distributions of events in which success or failure are equally likely and in which each event is independent of the next—for example, the frequency of accidental errors in a set of observations. It is also well known that an approximation to this curve is given by the terms of a binomial expansion. When a set of m coins is tossed, the probability of the occurrence of r heads and $(m - r)$-tails at any throw is ${}_mC_r\,(\frac{1}{2})^r\,(\frac{1}{2})^{m-r}$ in which ${}_mC_r$ is the number of combinations of m different things taken r at a time.

If the set is tossed N times (the number N being fairly large), the average frequency of occurrence of 0 heads and m tails, 1 head and $(m - 1)$-tails, etc., is given by the terms of the binomial expansion:

$$N\,(\tfrac{1}{2} + \tfrac{1}{2})^m = N\,(1 + {}_mC_1 + {}_mC_2 + \cdots + {}_mC_{m-1} + 1)\,(\tfrac{1}{2})^m \ldots\ldots (2)$$

As m increases, Eq. 2 approximates the normal curve, and if $m = 10$ the approximation is fairly good; in fact, for many purposes it is good enough.

Table 3 shows how the number of heads would be expected to be distributed on the average when a set of 10 coins is tossed 1,024 times—that is, the binomial distribution. The distribution derived from the normal curve, and one actually obtained by trial are also given.

TABLE 3.—COMPARISON OF FREQUENCIES

Distribution	Number of Heads, r											
	0	1	2	3	4	5	6	7	8	9	10	Total
Binomial	1	10	45	120	210	252	210	120	45	10	1	1,024
Normal Gaussian	2	10	43	116	212	258	212	116	43	10	2	
Actual experiment	0	16	53	122	209	240	194	138	46	5	1	

In the normal distribution the numerals represent the nearest integer. The relation between the binomial and the normal distribution has been treated by G. Udny Yule,[7] who defines the normal distribution as the limit of the binomial distribution when m is indefinitely increased. For theoretical purposes the computer can use whichever form is the more convenient.

4. CALCULATION OF THE RANGE OF CONTINUED SUMS OF DEPARTURES OR STORAGE FOR A NORMAL FREQUENCY DISTRIBUTION

Assume a number of consecutive values of an element whose variation from its mean is distributed normally, as in the binomial expansion already discussed, but where there is no correlation between successive values or groups of values, as in tossing coins. For example, consider the definite case of tossing a set of $2\ m$ coins N times, calling a head a gain and a tail a loss. For present purposes each toss of the $2\ m$ coins is recorded as the number of heads minus the number of tails, and the departures of this quantity from its mean are added to find the range R between the maximum and the minimum of this summation curve. The range thus found, from N tosses of a set of $2\ m$ coins, may differ slightly from the range found by the summation of heads minus tails from $2\ N\ m$ tosses of a single coin. This can be easily seen by tossing a set of 10 coins one after the other, and recording individual results. Then the sums of heads minus tails are the same at the end of every set of ten whether individual tosses are considered or not. It is as if the continued sum were plotted using single tosses, every tenth point being marked. The curve obtained by taking the tenth points would be a close approximation to the curve of individual points, but the latter would usually have a slightly greater range. For example, suppose that the maximum of the curve was produced by a set of 6 heads and 4 tails which would put the end-point of the set two units above the end-point of the preceding set. If the 6 heads occurred and then the 4 tails, there would be a point on the curve recording individual sums of heads minus tails which would be six above the end-point of the previous set—that is, the range would be increased by four. For the present this difference is ignored, but it will be considered later.

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[7] "An Introduction to the Theory of Statistics," by G. Udny Yule, Charles Griffin and Co., London, 2d Ed., 1912, p. 317.

In the case selected for analysis $2\,m$ coins are tossed N times, heads and tails or gains and losses are equal at the end of the trial, and n is written for $N\,m$.

The number of orders or arrangements in which n gains and n losses can occur is

$$ {}_{2n}C_n = \frac{(2\,n)!}{(n\,!)^2} \qquad (3)$$

in which ${}_{2n}C_n$ is the number of combinations of $2\,n$ different things taken n at a time. Among these orders there are ${}_{2n}C_{n+h}$ instances in which, at some point in the process of tossing, losses exceed gains by h or more.[8]

There are three cases:

Case	Condition	No. of arrangements
(*a*)	Losses never exceed gains	$\frac{{}_{2n}C_n}{n+1}$
(*b*)	Gains never exceed losses	$\frac{{}_{2n}C_n}{n+1}$
(*c*)	Losses never exceed gains and gains never exceed losses	$(n-1)\,\frac{{}_{2n}C_n}{n+1}$

In case (*a*) the range is the maximum value of gains minus losses; in case (*b*) the range is the maximum value of losses minus gains; and in case (*c*) the range is the sum of the maximum of gains minus losses, and the maximum of losses minus gains.

Summing all values of the range to obtain a mean value, and writing G for number of gains and L for number of losses,

$$\text{Mean range} = \frac{\text{maximum }(G-L) + \text{maximum }(L-G)}{{}_{2n}C_n}$$

$$= \frac{2\text{ maximum }(G-L)}{{}_{2n}C_n} \qquad (4)$$

To find the mean range see Table 4. There are ${}_{2n}C_{n+1}$ arrangements in which, at some point, gains exceed losses by one or more; and there are ${}_{2n}C_{n+2}$

TABLE 4.—COMPUTATIONS TO DETERMINE THE MEAN RANGE

Gains minus losses	No. of arrangements	Products
1	${}_{2n}C_{n+1} - {}_{2n}C_{n+2}$	$1\,({}_{2n}C_{n+1} - {}_{2n}C_{n+2})$
2	${}_{2n}C_{n+2} - {}_{2n}C_{n+3}$	$2\,({}_{2n}C_{n+2} - {}_{2n}C_{n+3})$
3	${}_{2n}C_{n+3} - {}_{2n}C_{n+4}$	$3\,({}_{2n}C_{n+3} - {}_{2n}C_{n+4})$
....	....	....
....	....	....
$n-1$	${}_{2n}C_{2n-1} - {}_{2n}C_{2n}$	$(n-1)\,({}_{2n}C_{2n-1} - {}_{2n}C_{2n})$
n	${}_{2n}C_{2n}$	$n\,({}_{2n}C_{2n})$
Sum	${}_{2n}C_{n+1}$	${}_{2n}C_{n+1} + {}_{2n}C_{n+2} + \cdots + {}_{2n}C_{2n}$

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[8] "Choice and Chance," by W. A. Whitworth, Cambridge, England, 4th Ed., 1886, Chapter on Priority.

in which gains exceed losses by two or more. That is, there are ${}_{2n}C_{n+1} - {}_{2n}C_{n+2}$ arrangements in which gains exceed losses by one.

Referring to Table 4, $2^{2n} = 1 + {}_{2n}C_1 + {}_{2n}C_2 + \cdots + {}_{2n}C_{n-1} + {}_{2n}C_n + {}_{2n}C_{n+1} + \cdots + {}_{2n}C_{2n}$. Hence, the sum of the products is equal to $\frac{1}{2}(2^{2n} - {}_{2n}C_n)$. The arrangements in Table 4 include all except those in which gains never exceed losses, whose number is $\frac{{}_{2n}C_n}{n+1}$ (or ${}_{2n}C_n - {}_{2n}C_{n+1}$), and in which the maximum of gains minus losses is 0. Hence, the mean maximum value of gains minus losses is $\frac{2^{2n-1} - \frac{1}{2}\,{}_{2n}C_n}{{}_{2n}C_n}$ and this is also the mean maximum of losses minus gains. Therefore, the mean range is

$$R = \frac{2^{2n}}{{}_{2n}C_n} - 1 \qquad (5)$$

Since n is large, Eq. 5 can be simplified by the approximation for $n!$ of James Stirling which is as follows:

$$n! = \sqrt{2\,n\,\pi}\left(\frac{n}{e}\right)^n \qquad (6)$$

Hence,

$$ {}_{2n}C_n = \frac{(2\,n)!}{n!\,n!} = \frac{\sqrt{4\,n\,\pi}\,(2\,n/e)^{2n}}{[\sqrt{2\,n\,\pi}\,(n/e)^n]^2} = \frac{2^{2n}}{\sqrt{n\,\pi}} \qquad (7)$$

and

$$R = \sqrt{n\,\pi} - 1 = \sqrt{N\,m\,\pi} \qquad (8)$$

since the product $N\,m$ is large.

Thus the average range of the accumulated sums (number of heads minus number of tails), when $2\,m$ coins are tossed N times ($N\,m$ being large), increases as the square root of the number of tosses, exactly like the accumulated error on a line of leveling.

The standard deviation of the binomial distribution produced by tossing $2\,m$ coins is

$$\sigma_r = \sqrt{2\,m \times \tfrac{1}{2} \times \tfrac{1}{2}} = \sqrt{\frac{m}{2}} \qquad (9)$$

This is the standard deviation of the number of heads (r) or tails ($2\,m - r$). The standard deviation σ_d of the number of heads minus the number of tails is twice σ_r, or $\sqrt{2\,m}$. Substituting for $\sqrt{2\,m}$ in Eq. 8,

$$R = \sigma_d\,\sqrt{\tfrac{1}{2}\,N\,\pi} = 1.25\,\sigma_d\,\sqrt{N} \qquad (10)$$

In Eq. 10, R is the range of the continued sum of $2\,m\,N$ tosses of a single coin, and is a little larger than the range obtained from N tosses of a set of $2\,m$ coins. The average difference $\bar{d}$ between the two computations for range can be found as follows:

Toss a coin a large number of times and plot the continued sums of heads minus tails, marking a given number of them off in sets of $2\,m$. The set of

$2m$ that contains the maximum sum is selected for scrutiny. If there are r heads and t tails the end-point of the set is $r - t$ higher than the end-point of the previous set. The number of orders in which r heads and t tails can occur is ${}_{2m}C_r$.

The number of orders in which at some stage of the tossing the number of heads shall exceed the number of tails by h or more is ${}_{2m}C_{m+h}$. At the end of the set, heads exceed tails by $r - t = 2r - 2m$. The number of orders in which at some stage heads exceed tails by more than $2r - 2m$ is ${}_{2m}C_{2r-2m+1}$. In all these cases the maximum of the summation curve of individual tosses will exceed the maximum of the curve derived from sets of $2m$ tosses. The mean value of the excess $\bar{d}$ can be obtained by a procedure similar to that already employed in finding the mean range of the summation curve. The result of this is

$$\bar{d} = \frac{1}{2^{2m}}\left[\frac{(2m)!}{(m-1)!\,(m+1)!} + 3\,\frac{(2m)!}{(m-2)!\,(m+2)!} + 5\,\frac{(2m)!}{(m-3)!\,(m+3)!} + \cdots + (2m-3)\,2m + (2m-1)\,1\right] \quad \ldots(11)$$

There is a similar effect on the minimum of the summation curve, so that the average range of the summation curve of individual tosses is greater than the range of the curve from sets of tosses by $2\bar{d}$. Some values of $2\bar{d}$ are given in Table 5. The correction $2\bar{d}$ to the range is independent of N and so has a less proportionate effect as N increases.

TABLE 5.—TYPICAL VALUES OF $2\bar{d}$

Magnitude	VALUES OF m			
	4	5	6	10
$2\bar{d}$	1.5	1.7	2.0	2.7
Correction[a]	4.1	4.3	4.5	4.8

[a] Percentage correction to R when $N = 100$.

5. EXPERIMENTS WITH RANDOM EVENTS

Table 6(*a*) gives the results of an actual experiment in which sixpences were shaken in a box and thrown onto a table. They have been analyzed in some detail to give a practical example of the foregoing theory. Their frequency distribution has been given in Table 3, in which there were 1,024 throws of 10 coins ($m = 5$) which resulted in a distribution close to the theoretical.

The observations were recorded as they occurred and the continued sums were computed from 0 to 1,000 observations. In this case (because, at the finish, heads and tails are nearly equal) the actual values (heads minus tails) are taken as departures, and not the departures from the mean (Col. 1, Table 6(*a*)). At the finish there are 74 more heads than tails; so the mean value of $h - t$ per toss is 0.074. This is corrected graphically by the aforementioned method. Fig. 6, discussed subsequently, shows the result of one of the experiments.

In each case R (Cols. 3, Table 6) is corrected to what it would have been if the departure had been reckoned from the mean. The theoretical value of σ is 3.16 for a binomial distribution.

Table 6(*a*) shows that $R/(\sigma\sqrt{N})$ tends to be constant and the values from 1,000 trials (Cols. 4) agree closely with the theoretical values (Cols. 5). The trials illustrate a point not examined in the theoretical work, and that is the variability of σ, R, and $R/(\sigma \sqrt{N})$ from set to set of 100 throws.

The maximum variations, as percentage departures from the mean, are as follows: σ, 11%; R, 50%; and $R/(\sigma \sqrt{N})$, 39%. The standard deviation (or standard error) of σ for a set of 100 throws is about 7%.

Another device for obtaining a normal frequency distribution is what may be called a probability pack of cards (Table 6(*b*)). In this pack the cards are numbered +1, −1, +3, −3, +5, −5, +7, −7, +9, and −9, and the numbers of each kind are proportional to the corresponding ordinates of a normal frequency curve. They are: 13 one's; 10 three's; and 6, 3, and 1, respectively, of the others. The approximation of these numbers is fairly close. The cards are first well shuffled and then cut, and the number on the exposed card is recorded. The cards are reshuffled slightly and cut again, and so on. The numbers recorded may be taken as corresponding to observations of a quantity whose frequency distribution conforms to the normal Gaussian curve.

TABLE 6.—EXPERIMENTS WITH RANDOM EVENTS

DEFINITIONS OF COLUMN HEADINGS

Cols. 1. Primary Observations:
Table 6(*a*), Excess of heads minus tails = $h - t$.
Table 6(*b*), Sum of recorded numbers = Σ.
Table 6(*c*), Excess of even numbers over odd numbers = H.
Col. 2. Standard deviation of corresponding primary observations (Col. 1) = σ.
Col. 3. Range of accumulated deviations = R.
Col. 4. Computation of $\dfrac{R}{\sigma\sqrt{N}}$ from observed data.
Col. 5. Theoretical values of data in Cols. 4 corrected for the effect of tossing in sets.

No. of Trials	(*a*) TEN SIXPENCES TOSSED 1,000 TIMES					(*b*) PROBABILITY CARDS CUT 1,000 TIMES					(*c*) SERIAL NUMBERS OF BONDS				
	$h - t$	σ	R	$R/(\sigma\sqrt{N})$		Σ	σ	R	$R/(\sigma\sqrt{N})$		H	σ	R	$R/(\sigma\sqrt{N})$	
	(1)	(2)	(3)	(4)	(5)	(1)	(2)	(3)	(4)	(5)	(1)	(2)	(3)	(4)	(5)
1-100	26	3.08	39	1.28	1.20	−26	3.16	40	1.27	1.20	−10	3.42	39	1.14	1.20
101-200	24	3.10	25	0.81	....	+38	3.00	30	1.00	....	52	3.22	32	0.99	....
201-300	−26	3.36	44	1.31	....	− 2	3.99	35	0.88	....	− 4	2.30	24	1.04	....
301-400	−30	2.88	29	1.01	....	−44	4.06	36	0.89	....	−30	2.86	50	1.75	....
401-500	+24	3.59	40	1.11	....	+28	3.87	65	1.68	....	18	2.79	47	1.68	....
501-600	−32	3.06	31	1.01	....	− 2	3.58	48	1.34	....	6	3.13	56	1.79	....
601-700	+12	3.56	44.5	1.25	....	+10	3.57	33	0.92	....	8	2.97	46	1.55	....
701-800	+42	3.02	28.5	0.94	....	−22	3.11	44	1.42	....	30	2.78	44	1.58	....
801-900	+38	3.03	25.5	0.84	....	+ 4	3.82	78	2.04	....	−48	3.02	49	1.62	....
901-1,000	− 4	3.50	54	1.54	....	−14	4.25	48	1.13	....	8	3.44	40	1.16	....
Mean	....	3.22	36.1	1.11	1.20	....	3.64	45.7	1.26	1.20	....	2.99	42.7	1.43	1.20
1-500	+18	3.20	87	1.21	1.23	− 6	3.62	81	1.00	1.23	26	2.95	84	1.27	1.23
501-1,000	+56	3.23	94	1.30	1.23	−24	3.67	111	1.35	1.23	4	3.09	87	1.26	1.23
1-1,000	+74	3.22	121	1.19	1.23	−30	3.64	116	1.01	1.23	30	3.02	88	0.92	1.23
Mean	....	....	....	1.19	1.22	....	....	....	1.15	1.22	....	....	....	1.20	1.22

This is a quicker process than tossing and counting coins. To toss 10 coins 100 times required about 35 min, whereas shuffling and cutting for 100 cards required 20 min. Table 6(*b*) records the results of 1 trial of cutting 1,000 times. As in Col. 3, Table 6(*a*), in each case R is corrected to what it would have been if the departures had been reckoned from the mean; and, again, $R/(\sigma\sqrt{N})$ in Cols. 4 approximates the value obtained theoretically for the case when N is large (Col. 5).

A third set of chance events was examined (Table 6(c)). From time to time newspapers announce the numbers of bonds that have been drawn for redemption. A collection of unconnected random events was made by dividing the numbers of the bonds into groups of ten, as follows:

	(a)	(b)	(c)		(d)	(e)	(f)
3	5	7	0	3	6	0	3
3	8	1	0	3	8	7	6
3	9	8	9	3	9	9	1
4	1	0	5	4	1	1	7
4	2	4	5	4	2	5	9
4	4	0	4	4	4	1	6
4	5	3	4	4	5	3	6
4	6	6	8	4	6	7	1
4	7	7	3	4	8	0	0
4	9	2	2	4	9	2	3
	–	–	–		–	–	–

Number of Even Digits Minus Number of Odd Digits

	(a)	(b)	(c)		(d)	(e)	(f)
	−2	+0	+2		2	−4	−2

The left-hand column of a group was ignored as only a few different digits are likely to occur in it. The number of even digits less the number of odd digits in columns (a) (b) (c), (d) (e) (f), etc., are recorded. This difference yields a sequence of numbers exactly analogous to the sequence previously obtained by tossing coins, and the labor of producing it is much less.

By this method a sequence of 1,000 random numbers composed of the even numbers from 0 to ±10 was made and this was used as before. Table 6(c) gives the results of this trial.

In each case R (Col. 3) has been corrected to what it would have been if the departures had been reckoned from the mean. The theoretical value of $R/(\sigma\sqrt{N})$ (Col. 5) has been corrected for the effect of tossing in sets.

Although the mean value of $R/(\sigma\sqrt{N})$ found from the three sets of random events in Table 6 is very close to the value found by mathematical analysis (Cols. 5 compared to Cols. 4) yet individual values for sets of 100 observations vary considerably. The extreme ranges in the three cases are as follows:

Table	$\frac{R}{\sigma\sqrt{N}}$
6(a)	0.81 to 1.31
6(b)	0.88 to 2.04
6(c)	0.99 to 1.79

The mean value of $R/(\sigma\sqrt{N})$ from the thirty sets of 100 observations is 1.27 and the standard deviation is 0.32. The relation between R, σ, and N for random events having been established theoretically and having been found to agree with practice, it is now necessary to find in what respects natural phenomena are similar to random events and in what respects they differ.

6. Characteristics of Natural Phenomena

The phenomena that chiefly concern the subject of this paper are river discharges. Unfortunately records of these rarely cover periods of more than 50 years, although levels (which are usually an indication of the discharge) have been recorded over longer periods. The discharge of a river is closely related to the rainfall on its basin, and rainfall records for many places cover more than 100 years. The investigation began with these phenomena.

The main similarity of river discharges, levels and rainfall, and random events is that their frequency distributions conform approximately to the normal curve, although they are usually slightly skew. As a rule the observations are too few to give very regular distributions, but the flood levels (nearly 1,000 of them), of the Nile River at Cairo, Egypt, recorded on the Roda gage, give a very regular distribution which is well fitted by a normal curve. This distribution is shown in Fig. 2.

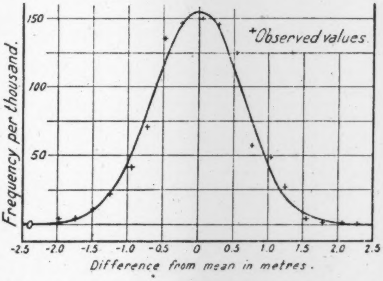

Fig. 2.—Maximum Annual Gage Reading at Roda Gage, Cairo, Egypt, for 1,080 Years Between 641 A.D. and 1946 A.D.

Although in random events groups of high or low values do occur, their tendency to occur in natural events is greater. This is the main difference between natural and random events, an example of which is the discharge of the Nile River at Aswan. The long series of records of flood levels at Cairo shows the same phenomenon.[9] There is no obvious periodicity, but there are long stretches when the floods are generally high, and others when they are generally low. These stretches occur without any regularity either in their time of occurrence or duration.

In the investigations of Messrs. Hazen and Sudler the frequency characteristics of river discharges are assumed to be like those of random events, and this is the common assumption in estimating probability of high or low floods. It will be shown that this assumption must be used with caution and, in cases in which storage over long periods is concerned, it is only an approximation.

7. The Range (R) of the Continued Sums of Departures for Natural Phenomena

The investigation began experimentally, before any attempts to find a mathematical theory were made, by finding R for any available long series of river discharges. Such series were scarce and so the work was extended to rainfall data, which are more plentiful. When it was found that rainfall data gave results similar to the river data the work was extended to other long series of natural events.

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[9] "Flood-Stage Records of the River Nile," by C. S. Jarvis, *Transactions*, ASCE, Vol. 101, 1936, p. 1012.

For example, Fig. 3 shows a summation curve of departures for the maximum levels of a lake on the Dalalven River in Sweden. There are 176 years of observations, divided into periods of 44 years, 88 years, and 132 years; and the ranges, R, of the summation curves for the departures from the mean in each of these periods are found.

The sums of departures from the base OX are plotted. For the sum of departures from the mean for the entire period the curve is referred to the axis OD. Similarly, the summation for the first 44 years is referred to line OA, for the second 44 years to line AB, for the first 88 years to line OB, and so on. The range, R, of the last period BD is $\overline{QN} + \overline{PM}$.

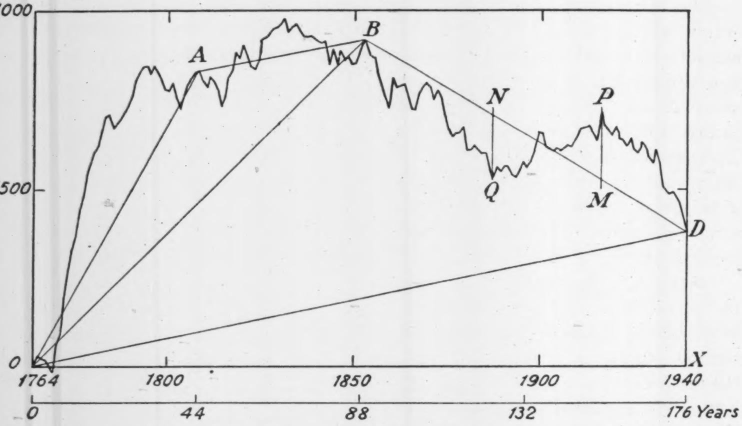

FIG. 3.—SUMMATION CURVE OF DEPARTURES, MAXIMUM LEVELS OF A SMALL LAKE ON THE DALALVEN RIVER IN SWEDEN

8. VARIATION OF R WITH N AND σ

The theoretical investigation shows that R is proportional to σ in the case of random events, and this must be approximately the case for any events that have normal or nearly normal frequency distributions. The device of expressing deviations from the mean in terms of σ is a common one, derived from the expression for the normal curve; and, since R is a function of these deviations, R/σ has been calculated. Complete tabular records of the computation including mean values of N and K $\left(= \dfrac{\log (R/\sigma)}{\log (N/2)}\right)$ are in the complete original manuscript, cataloged for reference in the Engineering Societies Library.[6a] Seventy-five different phenomena were used, and 690 values of R were computed for different values of N.

A summary of the results is given in Table 7 in which the phenomena are divided into groups according to their nature and the value of N, and log (R/σ) is plotted against log N in Fig. 4.

In general, R has not been computed for periods of less than 30 years because, for very short periods, it has little practical value although it may have some formal value. For $N = 2$, R/σ has the exact value, unity. The longest period used is 2,000 years and this is for clay layers in Lake Saki in the Crimea. Fig. 4, in which the means of small groups of observations of the same kind and length of period have been plotted, shows that log (R/σ) is a linear func-

TABLE 7.—RELATION OF R/σ AND N FOR GROUPS OF PHENOMENA

Phenomena	No. of cases	N years	$\frac{R}{\sigma}$	Log N	Log $\frac{R}{\sigma}$	K
(*a*) GROUP OF 99 CASES						
River levels, discharges, and runoff	8	35	7.5	1.54	0.85	0.68
	8	45	8.9	1.65	0.94	0.70
	8	62	13.1	1.79	1.08	0.72
	9	108	16.4	2.02	1.19	0.69
Roda nilometer	12	105	19.6	2.02	1.27	0.75
	10	208	36.5	2.32	1.54	0.77
	9	309	53.9	2.49	1.72	0.79
	8	420	60.3	2.56	1.77	0.78
	7	511	67.7	2.71	1.82	0.75
	6	613	81.3	2.79	1.89	0.77
	5	716	104	2.85	2.01	0.79
	4	820	122	2.91	2.08	0.80
	3	927	129	2.96	2.11	0.79
	2	1,040	130	3.02	2.12	0.78
Mean of 99 cases	....	....	....	2.28	1.48	0.75
(*b*) GROUP OF 168 CASES						
Rainfall stations with one value of R^a	7	71	14.0	1.82	1.12	0.74
Two groups of values of R^b	20	48	8.7	1.70	0.92	0.66
	12	98	13.7	1.94	1.11	0.68
Rainfall stations with three groups of values of R	66	38	8.2	1.57	0.91	0.72
	38	78	14.4	1.87	1.13	0.72
	25	121	22.2	2.08	1.31	0.74
Mean of 168 cases	....	....	....	1.79	1.08	0.70
(*c*) GROUP OF 109 CASES						
Temperature and pressure stations with 3 values of R	27	37	7.4	1.55	0.85	0.68
	18	73	12.0	1.86	1.06	0.68
	10	110	18.0	2.04	1.24	0.71
Temperature stations with 4 values of R	24	44	8.8	1.63	0.93	0.70
	12	88	15.1	1.94	1.17	0.71
	12	132	21.0	2.12	1.32	0.72
	6	175	27.1	2.24	1.43	0.74
Mean of 109 cases	....	....	....	1.81	1.05	0.70
(*d*) GROUP OF 85 CASES						
Annual growth of tree rings	41	50	13.6	1.69	1.11	0.80
	21	100	26.6	2.00	1.43	0.84
	14	200	45.0	2.30	1.64	0.82
	4	300	79.9	2.48	1.90	0.87
	4	462	76.6	2.66	1.87	0.79
	1	900	187.0	2.95	2.27	0.86
Mean of 85 cases	....	....	....	1.97	1.36	0.80

TABLE 7.—(*Continued*)

Phenomena	No. of cases	N years	$\frac{R}{\sigma}$	Log N	Log $\frac{R}{\sigma}$	K
(*e*) GROUP OF 90 CASES						
Thickness of annual layers of mud; Tamiskaming, Ont., Canada, and Moen, in the Sogne District, Norway	44	50	10.9	1.70	1.02	0.73
	22	100	21.3	2.00	1.31	0.77
	11	200	42.7	2.30	1.58	0.79
	4	300	82.5	2.48	1.90	0.87
	3	400	126	2.60	2.09	0.91
	4	550	115	2.74	2.00	0.82
	2	1,100	181	3.04	2.19	0.80
Mean of 90 cases	....	....	....	1.98	1.30	0.77
(*f*) GROUP OF 114 CASES						
Thickness of annual layers of mud, Lake Saki in the Crimea	40	50	9.7	1.70	0.98	0.70
	40	100	15.3	2.00	1.17	0.69
	20	200	25.0	2.30	1.39	0.70
	8	500	47.9	2.70	1.66	0.69
	4	1,000	84.0	3.00	1.91	0.71
	2	2,000	179.0	3.30	2.24	0.75
Mean of 114 cases	....	....	....	2.06	1.22	0.69
(*g*) GROUP OF 25 CASES						
Sunspot numbers and wheat prices	12	64	12.4	1.77	1.06	0.72
	6	124	22.1	2.09	1.34	0.75
	7	237	16.9	2.36	1.22	0.60
Mean of 25 cases	....	....	....	2.01	1.17	0.69
(*h*) GROUP OF 259 CASES						
Weight mean of 259 cases[c]	....	97	17.8	....	....	....

[a] Rainfall stations with one value of R; includes temperature at one station. [b] Rainfall stations with two groups of values of R; includes temperature and one pressure. [c] N ranges from 81 to 120.

tion of log N and can be expressed as

$$\log \frac{R}{\sigma} = K \log \frac{N}{2} \qquad (12)$$

In each main group of observations a line drawn through the point $\log (R/\sigma) = 0$, $\log N = 0.30$ ($N = 2$), and the center of gravity of all the observations is a good fit. The slopes of these lines (K) vary from 0.69 to 0.80.

Whether there is any theoretical significance in the fact that K is approximately $\frac{3}{4}$ is not known. The value of K has been calculated for each of the 690 individual values of R/σ and the frequency curve shown in Fig. 5 has been drawn. The data for this curve are as follows:

Description	K
Number of values	690
Mean value	0.729
Standard deviation	0.092
Range of individual values	0.46 to 0.96

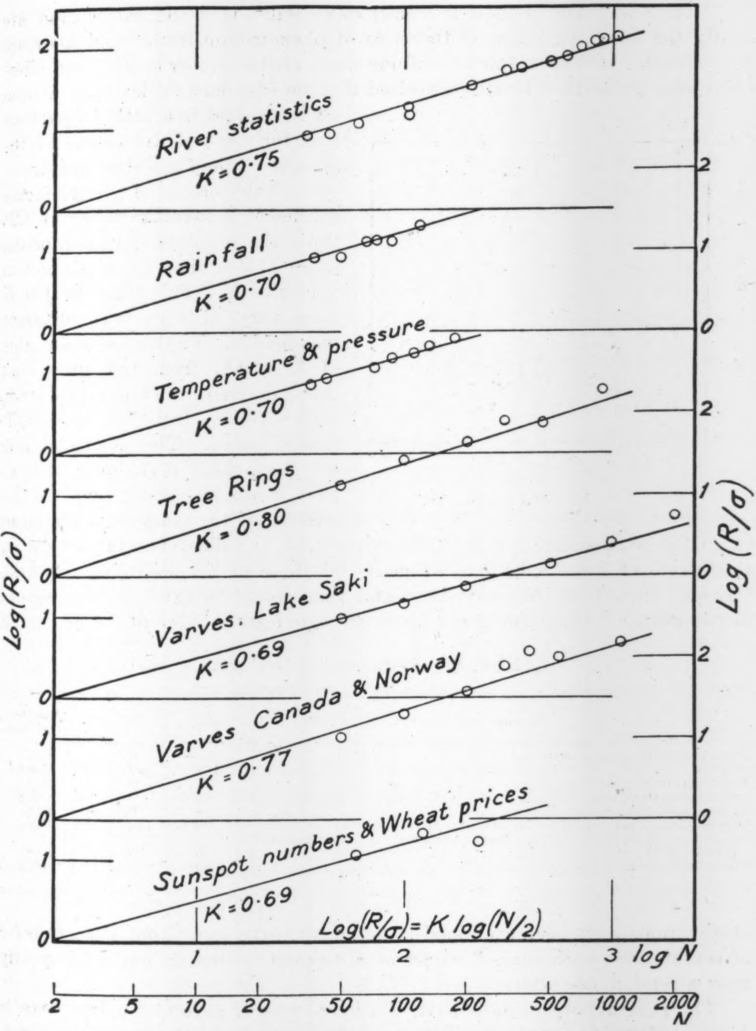

FIG. 4.—RELATION BETWEEN RANGE OF SUMMATION CURVE R, THE STANDARD DEVIATION σ, AND THE YEARS OF RECORD N

Fig. 5 shows that there is a slight skewness in the distribution of K, the mode tending to be slightly greater than the mean. However, it will be seen that the normal curve which has been fitted is a close approximation to the observed distribution. Referring to Table 7, a summary of K-values can be compiled as in Table 8.

Table 8 and Fig. 5 show a remarkable result—that the values of K are nearly the same regardless of the class of phenomenon from which they are determined. It is necessary to examine them a little further to see if the differences are significant. It will be noticed that the standard deviation of K from all the values is a little larger than that for most of the values for the groups. When the standard deviation of the means of groups is calculated it is found to be about 4.25 times the standard deviation of the mean of a similar group of random values of K. This suggests that K does vary slightly with different phenomena. The largest mean value of K is 0.81 from tree rings, and this is derived from four sets of trees, each set being from one of four different places. The means of sets are 0.84, 0.80, 0.81, and 0.81, so that it would appear that the departure of the trees from the mean of all the phenomena is significant. Similarly, the difference in K between the varves of Lake Saki (the Crimea) and those of Tamiskaming (Ontario, Canada) and Moen (Norway) is greater than would be expected on a chance distribution. Tamiskaming and Moen are significantly different. The rainfall observations have been examined in detail, with the result that the means for fifteen stations with some 7 values of K to each station do not differ greatly from a random distribution.

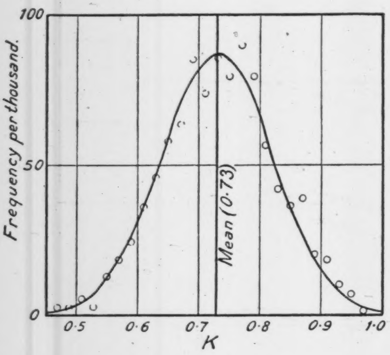

FIG. 5.—FREQUENCY OF INDEX K

TABLE 8.—SUMMARY OF K-VALUES

Phenomena	No.	Mean	Standard deviation
River levels, discharges, etc.	99	0.75	0.077
Rainfall	168	0.70	0.069
Temperature and pressure	115	0.70	0.085
Annual growth of tree rings	85	0.81	0.078
Varves (Lake Saki in the Crimea)	114	0.69	0.064
Varves (Tamiskaming, Ont., Canada, and Moen, Norway)	90	0.77	0.094
Sunspot numbers and wheat prices (combined as miscellaneous phenomena)	25	0.69	0.086
Means and totals	690	(0.729)	(0.092)

The general conclusion is that, if a long series of several hundred years is available for a phenomenon, it is probably best to take the mean value of K from these as an indication for the future. However, if only a short period of from 100 years to 200 years or less is available, the best plan is to take the mean value for K from all available material. The mean adopted for this purpose is 0.72 instead of 0.73, in order to give a little more weight to the more precise measurements relating to rivers, rainfall, and temperature. The mean value adopted, therefore, is $K = 0.72 \pm 0.006$. In using this result it is to be noted that it is the mean value of a quantity that has ranged from 0.46 to 0.96, and, therefore, some margin of safety must be allowed.

The probable error of a single value of R/σ can be derived by differentiating Eq. 12 with respect to K and using the data in Table 8, which yield a probable error of 24%. Confirmation of this value is obtained by taking the values of R/σ for rainfall, river discharge, temperature, and pressure for values of N between 80 years and 120 years, and correcting them to a period of 100 years. The probable error calculated from these corrected values is 23%. Fig. 10, introduced subsequently, shows how R/σ can be reduced by 23%, by reducing the draft by 0.04 σ.

In Fig. 5. the mean representing all the lines is represented by

$$\frac{R}{\sigma} = \left(\frac{N}{2}\right)^{0.72} = 0.61\,N^{0.72}\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

which means that R/σ increases more rapidly with N in the case of natural phenomena than in the case of chance events for which the power of $N = \frac{1}{2}$. The value of R/σ for $N = 100$ from Eq. 13 is 16.8. In the computations previously cited[2] the equation $R/\sigma = 1.65\sqrt{N}$ was used, which gives $R/\sigma = 16.5$ for $N = 100$.

If a phenomenon is exactly periodic, with constant amplitude, R will have a definite maximum which will be reached at the end of each complete period. In this case R will not vary in the manner already found for natural phenomena. The phenomena considered in this paper, however, are not of this type. In the case of sunspot numbers (which are the nearest approach) the periodic part does not have a constant amplitude, and has irregular variations superposed. These irregularities cause R to increase as N increases and the mean value of K is 0.75, with no indication that the phenomenon is different from the many others considered.

The very striking result from the investigation is the small variation of the index K over such a wide range of phenomena, and hence the possibility of its utility in many fields not mentioned in this paper.

9. The Variation of the Mean and The Standard Deviations Derived from Short-term Records

When records of a natural phenomenon extend over long periods there are considerable variations of both means and standard deviations from one period to another. The primary practical interest of this paper is in water supply and, hence, in rainfall. Unfortunately 200 years is about the longest series of observations obtainable for rainfall, and few stations reach 150 years. Some of these records are given in Table 9. They are divided into periods of 40 years or 50 years to illustrate the recorded variations. This division is chosen because quite frequently water supply or hydroelectric projects have to be based on such short records since nothing more is available. The reservoirs in the Great African Lakes are a case in point.

The average range of variation of the mean values of the rainfall in Table 9 is about 14%, whereas the standard deviations vary on the average by 23%.

This is more than would be expected from the ordinary theory for random distributions. The same is true in the case of temperature.

The uniformity in the values of R shown in the preceding section encourages the examination of some of the long-period records, a summary of which is compiled in Table 10. In Table 10(a) the variability of means for 50 years and 100 years is compared with what would be expected if the frequency were random, and not affected by any tendency for high or low values to be grouped together.

TABLE 9.—VARIATION OF MEANS AND STANDARD DEVIATIONS IN THE CASE OF RAINFALL

Station	Duration of records (years)	Length of period (years)	No. of periods	Mean		Standard Deviation	
				Maximum	Minimum	Maximum	Minimum
Philadelphia, Pa.	126	42	3	42.7	41.4	7.0	5.6
Boston, Mass.	128	43	3	44.2	38.2	7.4	5.0
Madras, India	133	44	3	51.3	48.4	15.6	14.3
Rome, Italy	151	50	3	886	777	17.1	15.7
Stockholm, Sweden	161	40	4	57.8	38.3	13.5	7.8
Padua, Italy	171	43	4	940	793	179	147
Milan, Italy	171	49	6	104	95	21.2	15.1
Zwanenbourg, the Netherlands	211	53	4	77	69	13.1	10.8

The computed value of the standard deviation (σ) of a mean for 50 years (Col. 4) is obtained by taking the standard deviation obtained from all the individual observations for the period given in Col. 3 and dividing it by $\sqrt{50}$. Cols. 5, Table 10, are the ratios Cols. 3/Cols. 4. It appears from Table 10(a) that the means are much more variable than they would be in the case of a random distribution, the standard deviation of a 50-year mean being 2.5 times

TABLE 10.—VARIATIONS OF MEANS AND STANDARD DEVIATIONS

Phenomenon (1)	No. of years, N (2)	(a) Standard Deviations of Means						(b) Standard Deviations of Standard Deviations					
		50-Year Means			100-Year Means			50-Year Periods			100-Year Periods		
		Actual (3)	Computed (4)	Ratio (5)	Actual (3)	Computed (4)	Ratio (5)	Actual (3)	Computed (4)	Ratio (5)	Actual (3)	Computed (4)	Ratio (5)
Varves:													
Lake Saki, Crimea	1,000	2.0	0.9	2.2	1.8	0.6	2.8	2.6	0.6	4.2	3.1	0.5	6.5
	1,000	1.5	0.9	1.7	1.0	0.6	1.5						
	1,000	2.2	1.3	1.7	1.6	0.9	1.7						
	950	2.2	1.2	1.8	2.0	0.9	2.3						
Moen, Norway	1,000	4.1	1.9	2.1	3.5	1.4	2.5	5.6	1.2	4.8	4.5	0.9	5.0
Tamiskaming, Ont., Canada	1,200	4.2	0.8	5.5	3.5	0.6	6.2	1.11	0.36	3.1	1.6	0.3	5.4
Ring Thickness:													
Sequoia trees	900	0.12	0.04	2.8	0.10	0.02	5.6	0.034	0.014	2.4	0.032	0.011	2.9
Flagstaff pines	500	0.09	0.04	2.1	....	....	....	0.040	0.029	1.4	....	....	....
Roda gage (free of trend)	1,050	0.24	0.10	2.4	0.21	0.07	3.0	....	....	....	0.145	0.042	3.4
Means	....	....	....	(2.5)	....	....	(3.2)	....	....	(3.2)	....	....	(4.6)

as great as it would be for a random distribution. Similarly, the standard deviation of a 100-year mean is 3.2 times as great as in a random distribution. This implies that, over a long period, the maximum value is likely to be higher and the minimum lower than would be predicted from the application of the ordinary theory of probability to the records from a short period.

Table 10(*b*) shows the variability of the standard deviation for periods of 50 years and 100 years in those cases where the records cover a long time. If a distribution is normal and randomly distributed as in chance events, the standard deviation of the standard deviation is $\sigma/\sqrt{2N}$. This is the value called "computed" (Cols. 4). The actual value (Cols. 3) is found in the usual way from the standard deviations for all the 50-year periods, etc., and Cols. 5 contain the ratios, Cols. 3/Cols. 4. Again it appears that the standard deviation for these natural phenomena is more variable than would be the case in a random distribution.

Summary.—Although many natural phenomena have a nearly normal frequency distribution, this is only the case when their order of occurrence is not considered. Consequently, their representation by a random distribution in the usual way is only a first approximation. The tendency to occur in groups makes both the mean and the standard deviation computed from short periods of years more variable than in the case of random distributions. Thus the odds against the occurrence of some particular extreme value (say, a high flood) are likely to be less than those calculated from a short period of records on normal probability theory.

It was thought that it might be possible to derive an expression for the effect of grouping, involving the correlation coefficient between successive values of the variable. Experiments with the natural number $1, 2, \cdots, N$ and $-1, -2, \cdots, -N$, however, showed that this was not the case. The greatest value of R is obtained by placing the positive numbers in one group and the negative in another and is independent of the order in the groups. The correlation between successive numbers depends on the order within the groups. In the foregoing example, the coefficient of correlation may vary from 1 to $\frac{1}{2}$ while R remains constant at $N(N + 1)$. The numbers can also be arranged to give small values of R with correlation coefficients varying from $\frac{1}{2}$ to -1.

10. STORAGE TO GUARANTEE A DRAFT LESS THAN THE MEAN DISCHARGE

If the discharge from a reservoir over a period is kept steady at the mean, the range of the curve of progressive sums of departures from the mean is taken as the storage required to keep this constant discharge. The final sum of the departures is zero. If departures are taken from a base less than the mean, this corresponds to the case in which the reservoir discharge is not allowed to fall below this base value.

The storage required in this case is the greatest accumulated deficit. This is illustrated in Fig. 1, of Lake Albert discharges, in which the summation curve, or curve of accumulated departures from base 24, is plotted, axis OO_1, Fig. 1, and the summation curve for departures from the mean is referred to axis OX.

If departures are computed from a new base 0.2 σ below the mean, as explained previously, the summation curve must be referred to the inclined axis OX_1, where the ordinate XX_1 is $43 \times 0.2\ \sigma$ or 63 milliards. It is obvious that, as the inclination of the axis OX_1 changes, different points on the summation curve may become maxima, or minima, and there is no simple connection between the storages required to guarantee the mean and to guarantee some lesser discharge. This is clearly shown in Fig. 6, in which the result of tossing 10 coins 100 times is shown in the form of a curve of the continued sums of the number of heads minus the number of tails.

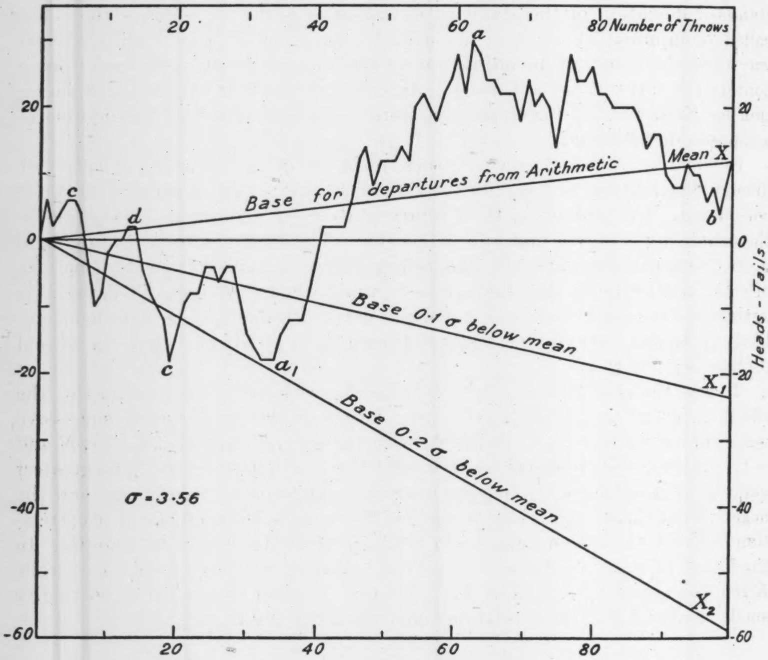

FIG. 6.—SUMMATION CURVE OF HEADS MINUS TAILS, IN TOSSING 10 COINS 100 TIMES

At the end of the 100 tosses, heads exceeded tails by twelve, so that the departures of heads minus tails are reckoned from a base 0.12 below the mean. This is corrected in Fig. 6 by drawing the axis OX which is the base for the summation curve of departures of heads minus tails from their mean. Similarly, the axes OX_1 and OX_2 can be drawn for the summation curves of departures from the bases 0.1 σ and 0.2 σ below the mean. The two points aa_1 are the maximum and minimum of the summation curve of heads minus tails and also of the summation curve of departures of heads minus tails from its mean. The range of the first curve is greater than the range of the second.

The maximum accumulated deficit with the mean as base is given by the points ab, but when a base 0.1 σ is selected below the mean the maximum accumulated deficit is given by cc_1; for a base 0.2 σ below the mean, it is dc_1. The storage required to maintain a draft below the mean can be computed arithmetically; but it is extremely laborious and by far the quicker and more reliable method is the graphic one.

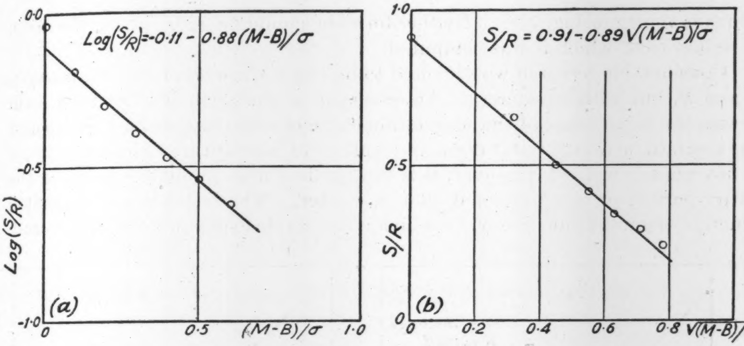

Fig. 7.—Analysis of Tossed Coins, Cards, and Drawing of Bonds to Determine a Relation Between Draft B and Maximum Deficit S (See Table 11)

TABLE 11.—Values of S/R for Given Ratios $(M - B)/\sigma$ (Means of 10 Experiments of 100 Trials Each)

Media	R	Values of $(M - B)/\sigma$					
		0.1	0.2	0.3	0.4	0.5	0.6
Coins	36	0.62	0.47	0.40	0.34	0.31	0.26
Cards	46	0.71	0.56	0.45	0.36	0.29	0.23
Bonds	43	0.62	0.48	0.39	0.32	0.27	0.23
Means	42	0.65	0.50	0.41	0.34	0.29	0.24
Logs		−0.19	−0.30	−0.39	−0.47	−0.54	−0.62

11. THE RELATION BETWEEN DRAFT AND STORAGE FOR CHANCE EVENTS

Let the storage required to maintain a draft B less than the mean (M) be S. Values of S (the maximum deficit) have been computed from the experiments with coins, probability cards, and drawing of bonds, as described. They have been calculated for every set of 100 experiments and for values of $M - B$ of 0.1 σ, 0.2 σ, etc. The ratio S/R has been tabulated and mean values are given in Table 11.

The results are also given in Fig. 7(a) in which log S/R is plotted against $(M - B)/\sigma$, the relation being

$$\log \frac{S}{R} = -0.11 - 0.88 \frac{M - B}{\sigma} \qquad (14a)$$

A little better fit is obtained from the equation—

$$\frac{S}{R} = 0.91 - 0.89 \sqrt{\frac{M - B}{\sigma}} \ldots\ldots\ldots\ldots\ldots\ldots (14b)$$

which is shown in Fig. 7(*b*). Neither formula should be used outside the range of values from which it was computed.

Considerable research was devoted to finding a theoretical relation between *S* and *R*, but without success. An extension of the method of Section 5 will enable the mean range of the summation curve of departures from a base below the mean to be found, but it does not result in a simple form. This mean range is not what is wanted, however; it is the maximum accumulated deficit in the entire period, or any part of it that is wanted. The latter is a complicated function of gains minus losses, for which no algebraic expression has been found.

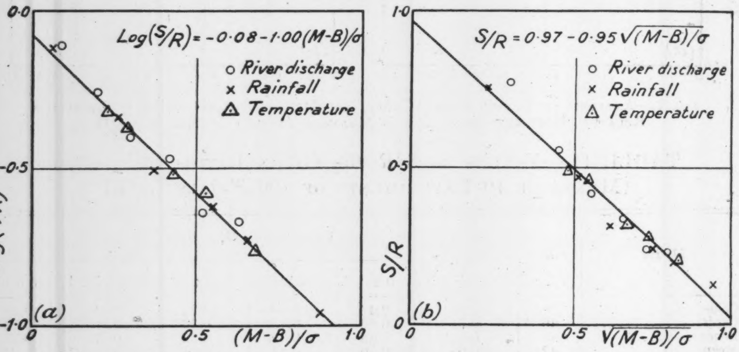

Fig. 8.—Analysis of River Discharge, Rainfall, and Temperature to Determine a Relation Between Draft *B* and Maximum Deficit *S*

An interesting point in the case of coins, cards, etc., is that, if the mean range is called R_1 when departures are measured from a base less than the mean, then the ratio R/R_1 is approximately equal to S/R.

TABLE 12.—Relation Between Draft *B* and

Magnitude	Values of									
	0.06	0.09	0.1	0.20	0.23	0.26	0.29	0.30	0.37	0.4
					(*a*) Rivers, Rainfall, and Temperature					
S/*R*	0.76	0.77	....	0.56	0.49	0.47	0.46	0.42	0.32	....
Log (*S*/*R*)	−0.12	−0.11	....	−0.26	−0.32	−0.34	−0.37	−0.40	−0.51	....
					(*b*) Tree Rings and Varves, Roda Gage					
S/*R*	....	....	0.64	0.48	....	....	....	0.35	....	0.27
Log (*S*/*R*)	....	....	−0.19	−0.32	....	....	....	−0.46	....	−0.57

12. THE RELATION BETWEEN DRAFT AND STORAGE FOR NATURAL PHENOMENA

The results of a number of computations of the storage (S) required to guarantee a minimum draft (B) less than the mean have been compiled. Table 12 gives the mean values of these detailed computations. Curves of the exponential and square root forms have been fitted to these mean values and the results are shown in Figs. 8 and 9. As far as closeness of fit is concerned, over the range of observations, there is no significant difference between one type of curve and the other. At some future time, it may perhaps be possible to decide that one or the other has some theoretical justification, but this has not so far been possible. The results of the curve fitting may be studied in Table 13. The curves that have been fitted to the means of all the observations are shown in Fig. 10.

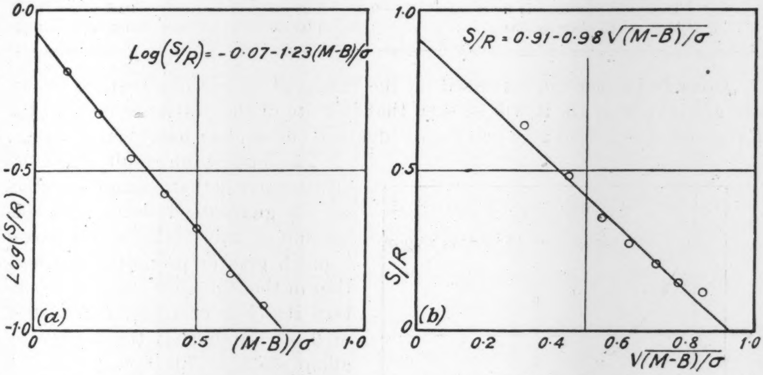

FIG. 9.—ANALYSIS OF TREE RINGS, VARVES, AND RODA GAGE TO DETERMINE A RELATION BETWEEN DRAFT B AND MAXIMUM DEFICIT S

STORAGE OR MAXIMUM DEFICIT S (MEAN VALUES)

$(M - B)/\sigma$												Magnitude
0.42	0.43	0.5	0.52	0.53	0.55	0.6	0.63	0.66	0.68	0.7	0.88	
(THIRTY-EIGHT PHENOMENA; $N = 96$ YEARS)												
0.34 −0.47	0.32 −0.52	....	0.24 −0.64	0.28 −0.58	0.24 −0.62	....	0.23 −0.67	0.20 −0.73	0.20 −0.76	....	0.09 −0.96	S/R log (S/R)
(SEVEN PHENOMENA; $N = 430$ YEARS)												
....	....	0.21 −0.68	....	....	....	0.15 −0.82	....	....	....	0.12 −0.92	....	S/R log (S/R)

The equations show very little variation among the three groups of phenomena. For example, if $(M - B)/\sigma$ is 0.25, S/R from the exponential curve for each group is 0.47, 0.47, and 0.42 and from the square root curve for each group is 0.46, 0.49, and 0.42. These results, again, illustrate a striking uniformity.

TABLE 13.—COMPARISON OF CURVES SHOWING RELATION BETWEEN DRAFT *B* AND STORAGE OR MAXIMUM DEFICIT *S*

Fig. No.	Media	Log (S/R) =	S/R =
7	Cards, coins, and bonds (Eqs. 14)....	$- 0.11 - 0.88\,(M - B)/\sigma$	$0.91 - 0.89\sqrt{(M - B)/\sigma}$
8	Rivers, rainfall, and temperatures....	$- 0.08 - 1.00\,(M - B)/\sigma$	$0.97 - 0.95\sqrt{(M - B)/\sigma}$
9	Tree rings, varves, and Roda gage....	$- 0.07 - 1.23\,(M - B)/\sigma$	$0.91 - 0.98\sqrt{(M - B)/\sigma}$
....	Mean of all observations............	$- 0.08 - 1.05\,(M - B)/\sigma$	$0.94 - 0.96\sqrt{(M - B)/\sigma}$

Considering the curves based on the means of all the observations, which are given in Fig. 10, it will be seen that in spite of the difference in form, the foregoing expressions give practically identical curves over nearly all the range.

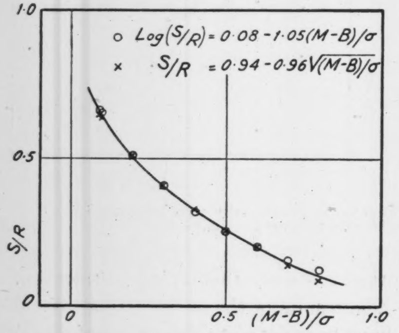

FIG. 10.—MEAN OF ALL OBSERVATIONS ANALYZED TO DETERMINE A RELATION BETWEEN DRAFT *B* AND MAXIMUM DEFICIT *S*

The important result shown by all the cases is that a small reduction in the guaranteed draft from the maximum value (the mean) makes a much greater proportional reduction in the storage required to maintain it. For example, a reduction of 0.1 σ diminishes the storage by about 35%. This fact provides a very practical method of applying a factor of safety to compensate for the natural variations of R.

13. COMBINATION OF STORAGE FROM DIFFERENT SOURCES

The previous theory has dealt with the storage of one variable supply, as for example, the water flowing out of Lake Albert. It was shown (see Eq. 13) that, on the average, the capacity required to produce a steady discharge equal to the mean over a period of N years, on a stream whose standard deviation of annual discharge is σ, is given by $R = \sigma \left(\frac{N}{2}\right)^{0.72}$. If two streams are involved, with variations whose discharges have a normal distribution, then if σ_1 and σ_2 are their standard deviations, and σ is the standard deviation of their combined flow, it can be shown that

$$\sigma^2 = \sigma^2_1 + \sigma^2_2 + 2\, r_{1,2}\, \sigma_1\, \sigma_2 \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

in which $r_{1,2}$ is the coefficient of correlation between the two discharges. It follows from Eq. 15 that, for a given value of N,

$$R^2 = R^2_1 + R^2_2 + 2\, r_{1,2}\, R_1\, R_2 \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

If the two tributaries vary exactly together, $r_{1,2} = 1$ and $R = R_1 + R_2$. Except in this case the necessary capacity to equalize the discharge of the main stream is less when it is all in one reservoir on the main stream than when it is in reservoirs on each of the tributaries. It is possible under some circumstances for the discharge of the main stream to be equalized by a reservoir on a tributary.

14. Practical Applications

As mentioned in Section 1 the investigation arose out of the need to control the flow of the White Nile by reservoirs in Lake Albert and Lake Victoria. The scheme favored by Egypt was to construct a single large reservoir in Lake Albert of such a capacity as to equalize the outflow over a century. Such a scheme requires the simplest type of regulation, but it was not acceptable to Uganda because of the large area of land which would be flooded for long periods. The Uganda Government proposed the combination of a large storage reservoir in Lake Victoria with a small regulating reservoir in Lake Albert. Later a regulator at the exit of Lake Kioga in Uganda was seen to be a necessary part of this combination to eliminate the very variable effect of Lake Kioga on the Victoria Nile. The regulation of the combination is more complicated and has been the subject of much study.

In the more simple plan for Lake Albert the capacity of the reservoir would be large compared with the natural fluctuations of the lake. The method of working would have been to build the reservoir 15 years or 20 years before it was required to be in complete action, and, during this time, to fill it partly full, leaving a sufficient reserve capacity to control unusual floods that might occur after the project was working to full capacity and which would then do damage to the Sudan. The simplest working of the reservoir would be to send down a constant annual discharge slightly less than the mean, distributing this discharge as required through the year. This operation would lead to occasional years in which a discharge of more than the average would be necessary to avoid overfilling the reservoir.

This procedure was adopted in the report[2] for making estimates of the value of the projects and of the size of the works required. In these estimates the guaranteed discharge was less than the mean discharge by twice the probable error of the mean, plus a further small reduction. The actual procedure would depend on circumstances, beginning, no doubt, with what was thought to be a safe draft. If the reservoirs showed signs of reaching high levels, the draft would be increased and, if the contents diminished considerably, the draft would be reduced. During the early years of regulation, before the full draft was required, experience would be gained which would enable a sound scheme of regulation to be drawn up based on more information than is now available.

A modification of this simple regulation would include what the writer has called "virtual storage," in which an excess on the Ethiopian tributaries can be virtually stored by reducing the outflow from Lake Albert.

Lake Victoria, as the main storage reservoir, has the advantage that its area is 12.5 times that of Lake Albert. Therefore, a much lower dam will give the required storage; but the regulation is complicated by the fact that the natural range of rise and fall of the lake is greater than the extra range required for regulation. This would not matter if there were plenty of room for regulation, but the interests of the inhabitants around the lake place a definite limit on the height to which the lake can be raised, and the natural range of the lake plus the extra effect of regulation must be fitted within the allowable limits. This requirement fixes the capacity needed in Lake Albert, where extra storage over that required for the annual regulation is necessary to provide against unusual floods. The regulation of the two lakes in combination, therefore, departs from the simple one of a steady annual discharge out of Lake Victoria and a steady annual discharge out of Lake Albert, which would be almost automatic. The more complicated regulation will depend for success upon a close study of rainfall, runoff, discharges of tributaries, and trends of lake levels. However, the time that will elapse before the project is working will provide additional data, and experience in regulation.

It is fortunate that the Nile supply comes from two sources—Ethiopia and the Great African Lakes—and that, as far as available information goes, the correlation between their contributions is small. It is unlikely, therefore, that both sources would suffer extended periods of drought together, and this can provide a factor of safety against an unusual sequence of low years on the Lake Plateau which includes Uganda and parts of Kenya, Tanganyika, and the Belgian Congo. To take advantage of this the scheme of projects reported[2] must be considered as a whole and worked as a combination. Should an unusual succession of low years on the Lake Plateau cause a serious decline in the stored water, it would be necessary to make full use of virtual storage. Then, by beginning to store a little earlier in the main Nile reservoirs, the quota from the lakes could be reduced and the possibility of a dangerous depletion of the storage would be removed.

15. CONCLUSION

This paper has been mainly confined to theoretical matters, and the practical outcome therefrom has been reported elsewhere.[2]

16. ACKNOWLEDGMENTS

W. Borgquist and H. Munding, of the Kunglige Vattenfalls Styrelsen, Stockholm, Sweden; and C. E. P. Brooks, of the British Meteorological Office, supplied the writer with useful series of data on the thickness of annual deposits of clay in lake beds and other data (see Section 7). Basic data on the annual growth of American trees was obtained from the researches of A. E. Douglass.[10] The source of much of the meteorological information is the "World Weather Records" in the Smithsonian Miscellaneous Collections.[11] Other information is from various sources.[12]

Thanks are due to R. P. Black, director of research, Inspectorate General for Nile Control, for continued assistance and advice, and to W. Lawrence Balls,

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[10] "Climatic Cycles and Tree Growth: A Study of the Annual Rings of Trees in Relation to Climate and Solar Activity," by A. E. Douglass, *Publication No. 2890*, Carnegie Inst., Washington, D. C., 1928.

[11] "World Weather Records," collected from official sources by F. Exner, G. C. Simpson, G. Walker, H. H. Clayton, and R. C. Mossman, *Smithsonian Miscellaneous Collections*, Vol. 79, Smithsonian Inst., Washington, D. C., 1927.

[12] "Geochronologia Suecica Principles," by Gerard de Geer, Almqvist & Wiksells, Stockholm, Sweden, 1940.

late of the Egyptian Ministry of Agriculture, for valuable suggestions. Nagib Boulos, Sayed Abdel Moneim Hassanein, and other members of the writer's staff rendered valuable assistance with a large amount of numerical work.

APPENDIX. NOTATION

The following letter symbols have been adopted for use in this paper and in the discussion:

B = annual draft from storage S on a stream, less than the mean annual flow of the stream;
${}_mC_r$ = number of combinations of m different things taken r at a time;
d = departure:
  $\bar{d}$ = average difference between the range of the continued sums of $2\,m\,N$ tosses of a single coin and N tosses of $2\,m$ coins;
  Σd_e = continued sums, or cumulative totals;
e = base of natural logarithms;
G = number of gains;
h = total number of instances in which the losses exceed the gains;
h = number of heads r in N throws; $h = r\,N$;
K = an index $= \dfrac{R/\sigma}{\log N/2}$;
L = number of losses;
M = mean annual draft of water from storage;
m = number of coins in any toss N; ${}_mC_r$ (see C);
N = number of annual discharges recorded; period of observation; also, number of tosses, of a set of m coins; number of departures from a mean;
$n = N\,m$;
Q = discharge, annual;
R = range, from maximum to minimum, of the curve obtained by plotting the cumulative totals of the departures from the mean Σd_e, of the annual discharges of a stream, taken in order;
r = number of heads in m tosses of a coin; also, number of heads; ${}_mC_r$ (see C); $r_{1.2}$ = coefficient of correlation between two variables;
S = storage required to maintain a draft B less than the mean M; maximum storage deficit;
t = number of tails, $m - r$ in m throws or tosses;
x = abscissas;
y = ordinates;
Σ = "sum of"; and
σ = standard deviation:
  $\sigma_r = \sigma$ of the number of heads or tails;
  $\sigma_d = \sigma$ of the number of heads minus the number of tails—that is, $2\,r - 2\,m$.

DISCUSSION

Ven Te Chow.[13]—The solution of many hydraulic engineering problems requiring the use of past hydrologic data is often handicapped by the short lengths of the periods of record. Mr. Hurst has contributed a desirable and new idea in proposing that the data of long-period records covering a large number of natural events can be adapted to supply the deficiency in hydrologic data, although great accuracy cannot be expected from the application of the very general results, thus obtained, to the individual cases. In order to perfect the idea many raw data should be assembled and processed. The amount of time and labor consumed in the preparation of the paper must have been truly tremendous.

The statement following Eq. 12—

> "In each main group of observations a line drawn through the point log $(R/\sigma) = 0$, log $N = 0.30$ $(N = 2)$, and the center of gravity of all the observations is a good fit."

—seems open to some question. In each main group of observations, the writer has tried to fit a straight line by the method of least squares, with the results shown in Table 14. There is no evidence that all the equations will be satisfied by the condition log $(R/\sigma) = 0$, log $N = 0.30$. The equation derived for sunspot numbers and wheat prices cannot be trusted because the number of points to be fitted (three points) is too small. The parameters in other equations can be found to vary within reasonable ranges—from $a = 0.30$ to $a = 0.43$ and from $b = 0.77$ to $b = 0.94$.

TABLE 14.—Equations of Best Fit

Phenomena	Number of groups	Equations; $R/\sigma =$
River level, discharges, etc.	14	$0.30\ N^{0.88}$
Rainfall	6	$0.43\ N^{0.79}$
Temperature and pressure	7	$0.37\ N^{0.83}$
Annual growth of tree rings	6	$0.36\ N^{0.88}$
Varves (Lake Saki in the Crimea)	7	$0.30\ N^{0.94}$
Varves (Tamiskaming, Ont., Canada, and Moen, Norway)	6	$0.43\ N^{0.77}$
Sunspot numbers and wheat prices	3	$4.04\ N^{0.29}$

A line designed to fit the main groups of all observations by the method of least squares was also found. It is expressed by

$$\frac{R}{\sigma} = 0.31\ N^{0.87} \qquad (17)$$

Eq. 17 is plotted to a log-log scale in Fig. 11, and appears to fit better than Mr. Hurst's line. The writer believes that a more satisfactory result would be obtained by fitting the line directly to all individual cases instead of to their main groups. In the latter method the weight of individual observations has been disregarded.

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[13] Hydr. Eng. Laboratory, Dept. of Civ. Eng., Univ. of Illinois, Urbana, Ill.

For $N = 2$, the ratio R/σ has the exact value $R/\sigma = 1$; and, for $N = 1$, $R = 0$ (or for $\log N = 0$, $\log (R/\sigma) = -\infty$). If the first condition ($N = 2$) is to be emphasized, the other condition ($N = 1$) must also be taken into consideration. Consequently, the true theoretical plot for $\log (R/\sigma)$ against $\log N$, if considered as a continuous function, is not a straight line, but a curve, as shown in Fig. 11. This curve is asymptotic to the $\log (R/\sigma)$-axis and tends (as N increases) to approximate the line defined by Eq. 10.

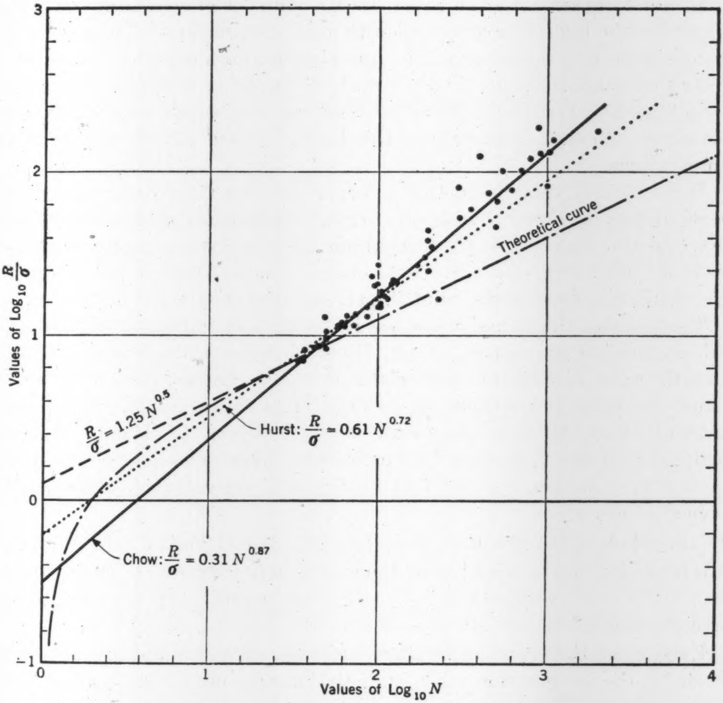

FIG. 11

Also, in Section 8, the author states:

> "In general, R has not been computed for periods of less than 30 years because, for very short periods, it has little practical value although it may have some formal value."

This statement is correct also because, for very short periods, the standard deviation computed from the observed data is subject to great variation, and, hence, a reliable value of R/σ cannot be obtained easily. Formulas, such as Eqs. 10, 13, or 17, may be accepted as valid only for large values of N, and, therefore, it is difficult to reason that the linear relationship between log (R/σ) and log N must hold, down to the point $N = 2$.

HENRI MILLERET.[14]—The paper by Mr. Hurst is of special interest to the writer since he himself is concerned with a similar problem. The general conclusions in Section 4 are reasonable, providing for both economy and safety.

As the author has stated, there are objections to using the Gaussian normal probability law; actual distributions of natural phenomena always have a certain skew. Nevertheless, he uses this law as a basis for his researches and computations.

It is now well established that a formula having three coefficients fits the observed data far better than one having two coefficients (such as the Gaussian law). In this connection, the logarithmic transformation, known as Gibrat's law is still used successfully.[15] It fits observed data of such different events as river discharge, flood peaks, rainfall, personal income, and population.[16]

The fact that the Gaussian law was found to check quite well with the many various phenomena reported by Mr. Hurst is understandable since, in all the cases, the range of observations is relatively small. For instance, in Table 1 the annual discharge, in milliards, ranges from 13 to 48 (ratio 3.7). It is possible that for such a small ratio, a Gaussian curve will always fit. It must be emphasized, for instance, that rivers in North Africa have much larger ratios. Mr. Hurst himself stresses the fact that his formulas are valuable only within the range of observations.

Nevertheless, the use of a "skew formula" would yield a safer theoretical basis for research, and would allow the results to be extended to more "erratic" cases. It should be added that hydraulic engineers still favor the formulas with three constants.

Even when the Gaussian formulas are used, an actual mean is not recommended in the computation. The statistical mean is not known in advance, but can be decided. The best values are available by using probability paper, and plotting observed events against Gaussian frequencies. The ordinary equation of the line of best fit then yields the statistical "true" value of the mean and, at the same time, of the standard deviation, which is not possible by ordinary arithmetic.

Technical literature on this subject is very scarce, and papers on researches such as Mr. Hurst has reported will be always welcomed by hydraulic engineers.

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[14] Engr., École Nationale des Ponts & Chaussées Société des Grands Equipements Marocains, Agadir, French Morocco.

[15] "Rainfall and Runoff," by Edgar Eugene Foster, Macmillan Co., New York, N. Y., 1948.

[16] "Aménagement hydroélectrique des cours d'eau: statistique mathématique et calcul des probabilités," by R. Gibrat, *Revue Générale de l'Électricité*, tome XXXII, Nos. 15 and 16, Paris, 1932.

Louis M. Laushey,[17] Jun. ASCE.—It has been shown that reservoir storage capacity can be determined by statistical relationships based on many observations of several different natural phenomena. The theory is based on a normal distribution from the mean and a random sequence of occurrence of values. Because of the long-term cyclic effects of nature, observed data do not check the theoretical equation, and corrections have been determined. The author must be admired for his thoroughness and sympathized with for the many computations involved. One simplification will be proposed.

The measure of dispersion, σ, can also be expressed by the mean absolute deviation, m. The variable, σ, could be replaced in Eq. 12 by a function of m to provide an alternate form that may have advantages. The choice of variable is naturally a matter of personal opinion, and the standard deviation has certain significance, but the following points are offered to support the possible change to the mean absolute deviation, m:

1. Computations are simplified considerably by the omission of the squares of deviations;
2. Work is completely self-checking;
3. Most books on statistics show the following relationship between σ and m for a normal curve:

$$\frac{m}{\sigma} = \sqrt{\frac{2}{\pi}} = 0.798 \qquad (18)$$

or, therefore,

$$\frac{R}{m} = 1.253 \left(\frac{N}{2}\right)^{K} \qquad (19)$$

4. The constant 0.798 is stable and varies little for curves deviaitng from a normal curve;
5. The value m is just as rational a choice since R is directly a function of the deviations, not of the squares; and
6. Computations of turbulence fluctuations to be presented indicate that m should be a satisfactory variable.

One more natural phenomenon can be added to the list already studied. Measurements were made of the variations in the squares of the velocity in a turbulent stream of water. The distribution of the variations from the average was symmetrical, but slightly flatter than a normal curve. It is an interesting comparison because 300 points were recorded in just one minute, not hundreds of years. All computations were made from the same 300 points to study the changes in variables within one representative group of points. Hence, the exact value of K was not expected for the turbulence variations since it may vary from point to point or with time.

Table 15 shows the results of ten groups of 30 points, five groups of 60 points two groups of 150 points, and one group of 300 points—all recorded consecutively at intervals of one fifth of a second. The longitudinal component is in the direction of motion, and the transverse component is at right angles.

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[17] Asst. Prof. of Civ. Eng., Carnegie Inst. of Technology, Pittsburgh, Pa.

The value of K' was computed from the mean absolute deviation by assuming $\sigma = 1.253\ m$. Values of K were computed from the standard deviation. Agreement is close, variations from the mean are about the same, and the assumed value of 0.798 checked. In Table 1, this ratio was calculated to be about 0.75 for the 43-year period. With more values, this ratio probably would approximate the theoretical within a small percentage.

The average value of K was about 0.66, which is less than 0.72 as determined by the author. Perhaps this is due to the small number of values used, but it is more likely that the tendency for cycles to occur during the short interval is not as great as found in long-term natural phenomena.

TABLE 15.—MEAN ABSOLUTE DEVIATIONS AND STANDARD DEVIATIONS OF TURBULENCE FLUCTUATIONS

m = mean absolute deviations of velocity-squared fluctuations;
K' = exponent of $\frac{N}{2}$ similar to K, but computed from $\sigma = \frac{m}{0.798}$; and
Time interval between each of 300 points is 0.2 sec.

Group	No. of points per group	No. of groups	GROUP AVERAGES					
			$\frac{R}{\sigma}$	K	K'	m	σ	$\frac{m}{\sigma}$
(*a*) LONGITUDINAL COMPONENT								
1	30	10	6.34	0.666 ±0.088	0.656 ±0.089	2.47	3.08	0.805
2	60	5	8.37	0.628 ±0.088	0.629 ±0.086	2.47	3.12	0.793
3	150	2	16.64	0.629 ±0.105	0.631 ±0.106	2.50	3.17	0.780
4	300	1	27.9	0.664	0.666	2.50	2.16	0.793
Average	....	....	....	0.647	0.646	....	....	0.793
(*b*) TRANSVERSE COMPONENT								
1	30	10	6.46	0.681 ±0.052	0.695 ±0.057	2.76	3.57	0.771
2	60	5	9.96	0.663 ±0.079	0.676 ±0.086	2.77	3.63	0.761
3	150	2	17.21	0.657 ±0.030	0.664 ±0.027	2.81	3.64	0.773
4	300	1	25.7	0.648	0.656	2.81	3.67	0.768
Average	....	....	....	0.662	0.673	....	....	0.768

There is no reason for the author to attempt to find a theoretical value of K for natural conditions, other than 0.5 for completely random occurrences, and no disturbing cyclic effects on the variables. Because of cyclic effects and different natural forces affecting each event, the values of K should be expected to vary slightly with each natural phenomenon.

H. E. HURST[18].—For their appreciative remarks and useful suggestions, the writer wishes to thank the discussers of this paper. Mr. Chow has raised the point that the lines in Fig. 4 are not the lines of best possible fit to the observed points; and he calculates other lines using the method of least squares to determine two parameters instead of the single parameter chosen by the writer. Of course, the larger the number of parameters used, the better the fit will be. The reasons that determine the choice of a formula are either theoretical ones, or they are matters of practical convenience. The latter usually lead to choosing the simplest formula that fits the facts. In the second case there is no justification for complicating a formula with additional parameters if the resulting improvement of fit is not significant.

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[18] Scientific Consultant to the Ministry of Public Works, Egypt; formerly Director-Gen., Physical Dept. Cairo, Egypt.

In the present case no theoretical formula has been discovered for natural phenomena, but it is a fact that all curves pass through the point corresponding to $N = 2$, $R/\sigma = 1$. Mr. Chow remarks that $N = 1$ must also be considered, but when $N = 1$, both R and σ are zero, and R/σ is indeterminate. Therefore, as the curve can only be determined for integral values of N, it stops at $N = 2$.

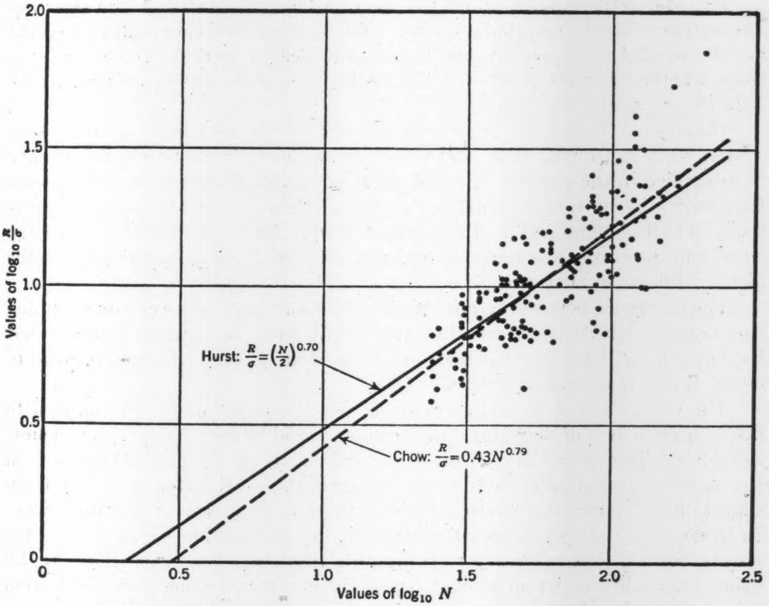

FIG. 12

The curves given by Eq. 12 clearly fit the observations very well. Therefore, Eq. 17 was compared with Eq. 12 to see if the difference was significant. Rainfall records (which are, perhaps, the most important group in practice) have been chosen, and the results of 168 calculations of R over periods ranging from 24 to 211 years have been plotted in Fig. 12. Lines representing Mr. Chow's Eq. 17 and the writer's Eq. 12 are also shown. There is a fairly large scatter among the individual points. In fact, compared with this scatter, the difference between the lines is not large enough to be significant. A rough confirmation of this fact was obtained by taking two groups of values of log (R/σ), one for log $N = 1.48$ to 1.52, and the other for log $N = 1.98$ to 2.02, and finding the probable errors of their means without allowing for the small differences in log N. The differences between the lines were less than their probable errors and so were not significant. This conclusion applies to the other groups since the range of individual values of K is much the same in all of them. For the practical purpose of prediction and of deciding what value of R to use in a particular case, predictions are not likely to be extended to periods of less than 30 years, or more than 100 years, and any differences in formulas would be, therefore, entirely negligible.

In view of the foregoing and the remarks in the paper on Fig. 5 and Table 8, there seems to be no reason for adopting a more complicated formula than the one used in the paper.

Mr. Milleret's remarks about the use of a Gaussian curve or of a skew frequency curve illustrate a similar point. The writer agrees thoroughly about the use of probability paper, a valuable means of dealing with frequency distributions whether normal or skew. Probability paper is constantly used in the writer's practice.

Theoretical frequency distributions can be derived for purely chance events, such as tossing coins, and these are the only events to which they have been applied in the paper. It is not usually possible, however, to arrive at the frequency distribution of a natural event on theoretical grounds except, perhaps, as to its general form. In the paper R and σ have been calculated in every case from the actual observations, and they depend on no assumptions as to the form of frequency distributions.

The difference between R/σ for natural and for pure chance events is not due to skewness but to the fact that the distribution of natural events is not haphazard. It happens that irregular successions of high or low years tend to occur in any long series of years.

The 690 values of R and σ computed for the paper might be analyzed to discover any effect of skewness, but the labor would be very heavy and of doubtful value. The distribution of many natural events, such as those dealt with in the paper, approximates to the normal, and there are often insufficient observations to determine whether the distribution is normal or definitely skew. In these cases, much useless labor is saved in computation by keeping to the normal form if it is necessary to compute a frequency curve, or still more, by using probability paper on which to draw the curve. If some general equation is to be used for skew frequency curves, the question arises as to which of the various forms that have been proposed is to be used, and this problem again is more likely to be decided on grounds of practical convenience than on theoretical considerations.

Mr. Laushey's suggestion to replace the standard deviation by the mean absolute deviation is a good one and would save labor, unless the standard deviation were needed for some other reason. The values of R/σ calculated by him for turbulence fluctuations form a useful addition to those collected in the paper and have the advantage, as he states, that a long series can be produced in a short time. It would be interesting to note what results would come from, say, two series of 1,000 each for comparison with the long-term records in the paper.

After discussion had been formally closed, a letter was received from S. P. Wing, M. ASCE, containing some valuable practical comments on the importance of fixing reservoir capacity by probability considerations, with which the writer is entirely in agreement. Mr. Wing refers to the economic difficulties that have arisen in the past through faulty determination of water supplies and of the required capacity to regulate them, and remarks "that the size of the reservoir to equalize the flow depends upon the risk one is willing to take of going dry; that it is a matter of probabilities." A method of assessing risks is given.

Mr. Wing believes that

> "* * * one conclusion of the author, if left without comment, might lead the reader to question the applicability of standard probability methods to these types of problems."

The conclusion referred to is that, although groups of high and low values do occur in purely chance events, their tendency to occur in natural events is greater. One example of this tendency is the records of the Nile River at Aswan and Roda in Egypt (Section 6 and Tables 2, 9, and 10). In commenting on this conclusion, Mr. Wing examines the long series of records of the Roda gage and, after removing the trend of the readings to rise as time goes on (which is usually thought to be due to silt deposition raising the bed), he finds that there are still discontinuities. He believes that

> "* * * a substantial part of the non-randomness found by the author in his numerical computations based on the Nile record arose from its non-homogeneous character."

It is quite true, as Mr. Wing states, that there are discontinuities in the Roda gage records, but they account for very little of the tendency for periods of high or low years to occur more frequently under those circumstances than in pure chance distributions. This can be seen by considering the record from 640 AD to 1440 AD which is more complete and homogeneous than the later part and is previous to the principal discontinuities. If this record is divided into periods of 100 years after removing trend, the mean value of K from the eight cases is 0.74, whereas if seven overlapping periods of 200 years are taken, the mean is 0.75. These shorter periods cannot be much affected by discontinuities.

Mr. Wing believes that thickness of tree rings and varves are also subject to criticism, although on different grounds. For example, tree rings may vary with changes of temperature or with disease, as well as with rainfall, and varves may vary with changes of availability of load or changes in culture or regime. This is true, but in the case of trees the causes, though complex, are natural. It is true also in the case of varves, since only in the case of Lake Saki (in the Crimea) can human agency have had any effect, and that only perhaps during the last tenth of the record.

Excluding Roda gage, tree rings, and varves and using river discharges, rainfall, temperature, and pressure, the mean value for K is 0.70, which is close to the mean for all the data (0.72). Assuming $N = 100$ years, the formula derived from natural events, with $K = 0.70$ yields $R/\sigma = 15.5$, whereas the formula derived from tossing coins would give 12.0 (Table 6). It cannot be doubted, therefore, that in calculating storage for practical purposes, an extension of the ordinary methods based on the probability theory is required. This does not mean that the methods should be discarded, but they must be modified to meet such cases as the foregoing where the order of events as well as their magnitudes is important.

In conclusion it may be stated that work is continuing on the methods to be adopted in regulating long-term storage reservoirs. The results emphasize—should it be necessary to do so—the importance of allowing a reasonable factor of safety on the value (R) calculated for the storage.

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THREE-DIMENSIONAL DISPLACEMENT DIAGRAMS FOR SPACE FRAME STRUCTURES

BY WALTER WORTHINGTON EWELL,[1] JUN. ASCE

WITH DISCUSSION BY MR. W. WATTERS PAGON

SYNOPSIS

Because the Williot-Mohr displacement diagram has proved to be an inestimable aid in planar structure analysis, it is conceivable that a similar diagram applicable to space structures might also have some merit. The development of such a diagram, suitable for determining the displacement of any space frame joint along the three coordinate axes of space, has been the purpose of this paper.

Three-dimensional displacement diagrams have been presented for a simple space pedestal and a second-degree indeterminate space truss. The space pedestal displacement diagram illustrates the graphical method in its simpler aspects. The displacement diagram for the space truss circumvents none of the problems inherent in the method and, furthermore, demonstrates the use of a rotational correction diagram. Both graphical examples presented in this paper have been developed progressively in three separate phases to meliorate any fears of an overly complex problem.

Elementary principles of descriptive geometry are involved in the orthographic representation of the displacement diagram. These geometric principles should be thoroughly understood, however, before reading this paper.

All displacement values obtained graphically have been confirmed by least work analyses of both structures. The complete research, including analytical verifications, has been filed for reference in the Engineering Societies Library.[1a]

INTRODUCTION

Since its inception, the Williot-Mohr displacement diagram has gradually been adapted to the solution of many planar structural problems previously

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[1] Asst. Prof., The Johns Hopkins Univ., Baltimore, Md.

[1a] 29 W. 39th St., New York 18, N. Y.

managed with the tedious, unwieldy, and sometimes abstruse classical methods. The most important use of the displacement diagram, aside from the determination of the intrinsically important truss deflection, has been in the analysis of statically indeterminate structures. With this significant graphical method, the calculation of end moments at truss panel points has been simplified and the construction of influence lines for redundant reactions and redundant internal members has been facilitated. Other important applications of this displacement diagram have been suggested.[2] Today the Williot-Mohr diagram exists as a vital part of the information required by the structural engineer working with planar structures of all types. The applications of the diagram, although manifold, probably have not been exhausted.

Multitudes of space structures are in existence in the United States today. Framed pedestals, water tank towers, pipe line bridges, electric power transmission towers, and domes are only a few examples of the space frames that abound. Many more structures—undeniably space frames—have been designed as planar structures to avoid the tedium of a three-dimensional analysis. With the ever-increasing use of welded joints manifestly facilitating the connections of members in space, and with the growing knowledge that many constructions can be more economically designed as space structures, the erection of true space frames should be even more prevalent in the future. As the demand for space structures increases, the desire for more rapid methods of analysis will also increase. Therefore, it is not presumptuous to assume that any additional information pertinent to the analysis of determinate and indeterminate space structures may some day be of benefit. This assumption motivated the development of the three-dimensional displacement diagram.

As is true in planar structural analysis, determinate and indeterminate space frames can be analyzed, and deflections can be computed by many of the classical methods involving work and strain energy principles.[3] One of the better systems for computing joint deflection is the virtual work method wherein a virtual or imaginary unit load is placed upon the structure in the direction of the desired deflection. The ensuing calculations for stress and deformation result in the desired deflection.[4] These calculations, particularly in space structure analysis, are inordinately long, even if only one deflection is desired. The computations, furthermore, increase in complexity if the space structure is indeterminate to any degree. If all components of displacement are required, the individual calculations must be performed three times at each joint.

Another method of analyzing the panel-point displacements within a space structure was presented in 1946.[5] The calculations for deflection involve the solution of as many simultaneous equations as there are unknown displacements within the structure. Actually, the solution for deflection is but one of the necessary steps in the solution for member stresses within the frame and is not meant to be a solution for deflection alone. This system can be used advantageously in the simpler space frames where few joints give rise to but few unknown displacements. The method becomes laborious as the required equations (to be solved simultaneously) increase in number. Consequently, although many decry the use of graphical methods because of the inaccuracies involved, it is evident that a workable and comparatively rapid method of finding space structure deflections in one graphical construction can lessen the lengthy computations encountered in space frame analysis.

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[2] "Williot Equations for Statically Indeterminate Structures," by Charles A. Ellis, *Transactions*, ASCE, Vol. 100, 1935, p. 580.

[3] "Theory of Structures," by S. Timoshenko and D. H. Young, McGraw-Hill Book Co., Inc., New York, N. Y., 1945, p. 326.

[4] "Theory of Modern Steel Structures," by L. E. Grinter, Macmillan Co., New York, N. Y., 1937, Vol. II, p. 70.

[5] "Theory of Indeterminate Structures," by L. C. Maugh, John Wiley & Sons, Inc., New York, N. Y., 1946, p. 300.

THE SPACE DISPLACMENT DIAGRAM

Fundamentally the construction of the space displacement diagram involves the following steps:

1. The solution of stress or axial force in all members of the space frame, whether determinate or indeterminate in nature—axial forces are found by one of the analytical methods of analysis.

2. The computation of strain within each member of the frame—values of $\frac{PL}{AE}$ are computed and considered either elongations or contractions, dependent upon whether tension or compression exists in the member.

3. The construction of a plan and elevation view of the space frame concerned—these diagrams serve to ascertain slope of individual members.

4. The selection of a fixed point and fixed member with which to start the space displacement diagram—the fixed point and line are represented orthographically in plan and elevation views.

5. The construction of strain lines to some scale, of lengths, equal to the strains of those members emanating from the fixed point and fixed member—

(*a*) These lines are represented orthographically in the plan and elevation views and their slopes are determined from the orthographic representation of the actual space frame.

(*b*) Foreshortened strains are plotted where necessary, if the member itself is foreshortened in its orthographic representations.

6. The erection of planes perpendicular to the ends of the strain lines.

7. The intersection of the planes perpendicular to the ends of the strain lines—

(*a*) The simultaneous intersection of at least three planes is necessary for the determination of a point of intersection.

(*b*) Where the three planes are perpendicular to a common plane, their intersection with a fourth plane may be required to afford a point of intersection.

8. The location of the displaced joint—the intersection of the three planes, represented orthographically in all views, will be the location of the displaced joint. (In a planar Williot diagram the directions of only two strains are required to determine the new location of a displaced joint. Perpendiculars at the ends of these strain lines representing arc lengths are intersected to locate the adjacent panel point. In space, a minimum of three strain lines

of known direction is necessary to locate, geometrically, the position of an adjacent panel point. Perpendicular planes rather than perpendicular lines are required to determine the location of a displaced joint for there is no assurance that the resulting intersection will fall in the original plane of any two strain lines.)

9. The continuation of the procedure of erecting strain lines from known or located joints, of constructing planes perpendicular to their ends, and of intersecting these planes.

10. The correction of the completed diagram—a rotational correction diagram is required provided an original assumption as to direction of any member is made.

Deflections in a Space Pedestal

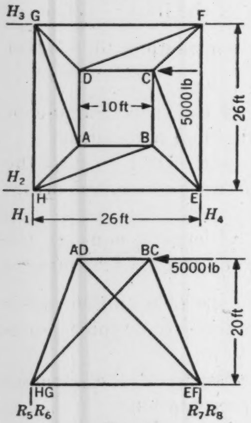

FIG. 1

In order to illustrate the method of construction involved in a three-dimensional displacement diagram, the simple space pedestal shown in Fig. 1 has been chosen as the first example. The frame has sixteen members and eight joints, four of which are points of application for reactions. The reactions are eight in number, four holding the pedestal on a horizontal datum plane and four maintaining equilibrium and stability within this horizontal plane.

A brief study will indicate that the frame is a statically determinate one because the twenty-four unknowns involved are not in excess of the available equations of statics. Although this particular quality is not a prerequisite for constructing a space displacement diagram, it is considered desirable for the primary illustration of the graphical method.

For the purpose of this example, the frame has been subjected to only one horizontal force of 5,000 lb, applied at the upper joint C. This simply gives rise to a number of inactive members—members carrying no axial load. Since this recourse has the effect of simplifying the graphical construction of the displacement diagram, it is advantageously used in the initial demonstration.

The space pedestal chosen has the added advantage of being fixed completely at one point and of having its lower horizontal members fixed in direction by the reactions. Thus, the necessity for a rotational correction diagram has been eliminated since no assumption of direction need be made for the original members meeting at the fixed point.

In Table 1, the axial forces in the various members are listed together with the $\frac{L}{A}$-values assumed for each member. (The axial force analysis can be found in the complete research on file in the Engineering Societies Library.[1a]) The deformation factors $P\,L/A$ are proportional only to the actual strains since the modulus of elasticity has been neglected. This device merely affords a larger than actual value of deformation, which is plotted more readily in the space displacement diagram. Displacement values, scaled from the completed diagram, also will be proportional only to the actual values.

The Space Displacement Diagram for the Space Pedestal

The graphical procedure used in the construction of the three-dimensional diagram has been illustrated in three separate phases (see subsequently in Figs. 2, 3, and 4). The development of the diagram in progressive phases has been resorted to in order to clarify this initial illustration. The processes of descriptive geometry, inherent in a problem of this nature, are of temporary significance only and, consequently, have been omitted wherever possible in ensuing phases of the construction.

TABLE 1.—Deformation Factors for the Space Pedestal, Fig. 1

Bar	P	L/A	$P\,L/A$
AB	0	2	0
BC	0	2	0
CD	−5,000	2	−10,000
DA	0	2	0
HE	0	5.2	0
EF	−1,540	5.2	− 8,000
FG	−3,460	5.2	−18,000
GH	+1,540	5.2	8,000
HA	0	4.54	0
HB	0	5.6	0
EB	0	4.54	0
EC	0	5.6	0
FC	0	4.54	0
FD	+5,390	5.6	+30,200
GD	−4,420	4.54	−20,050
GA	0	5.6	0

The notation employed in the three-dimensional displacement diagram is as follows:

A^v, A^h = the final position of any displaced joint;

A^vb^v, A^hd^h, etc. = the strain lines plotted orthographically with space directions identical to those in the corresponding truss member;

H-trace of $A'b'$ = the horizontal trace of a plane $A'b'$ perpendicular to the strain line Ab;

Locus of A = all feasible positions of a displaced joint A prior to its definite location;

HV = the intersection between the horizontal and vertical projection planes;

x, y, and z = the coordinate axes of space; and

x_A = deflection of displaced joint A in the x-direction.

Procedure.—To any convenient scale, the plan and elevation views of the framed pedestal are drawn as shown in Fig. 2. These views are used to determine the slope of the strain lines in the displacement diagram.

Joint H is selected as the fixed reference point and separated in plan and elevation by an assumed HV-line. In all cases, unless otherwise specified, the plotting of any point, line, or plane is performed in both plan and elevation views.

The axial deformation Hg is plotted parallel to its corresponding member in the actual frame. The plotting is in the direction of the movement of the free end. No assumption has been made as to the direction of Hg since joints G, H, E, and F are held to a horizontal datum plane by vertical reactions. Furthermore, members HG and HE are fixed in direction on this horizontal plane. The g-end of the strain line Hg consequently becomes the final position of the displaced joint G.

The strain He is plotted in its direction, zero deformation in the bar HE causes point e to coincide with point H, locating the final position of the displaced joint E.

From joints G and E strains Gf and Ef are plotted in the direction of the movement of the free ends. Perpendicular lines are erected at the ends of these strain lines and intersected to locate joint F. Joint F is the last point to remain on the horizontal datum plane.

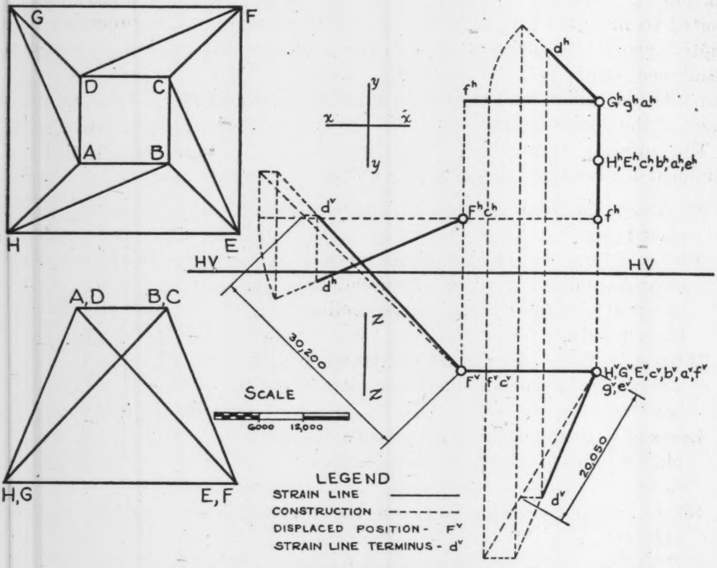
G
F
D
C
A
B
H
E
A,D
B,C
H,G
E,F
y
x
x
y
f^h
d^h
$G^h g^h a^h$
$H^h, E^h, c^h, b^h, a^h, e^h$
d^v
$F^h c^h$
f^h
HV
HV
d^h
Z
Z
30,200
SCALE
6000
12,000
F^v
$f^v c^v$
$H^v, G^v, E^v, c^v, b^v, a^v, f^v$
g^v, e^v
20,050
d^v
LEGEND
STRAIN LINE
CONSTRUCTION
DISPLACED POSITION - F^v
STRAIN LINE TERMINUS - d^v

FIG. 2.—THREE-DIMENSIONAL DISPLACEMENT DIAGRAM, PHASE 1

Referring to the plan and elevation views of the actual pedestal, it is seen that from none of the joints thus far located can the final position of joints A, B, C, or D be determined. This is evident since at least three members are required to fix any one point in space. All the space joints A, B, C, and D are positioned by only two members emanating from the basal joints H, G, E, and F. The final position of any of these space joints is determined by its movement relative to another space joint.

Joint D is related to the base by members GD and FD. The locus of all possible positions of joint D is derived from the relationship between points D, G, and F. Since members GD and FD are three dimensional in direction, the corresponding strain lines Gd and Fd are also three dimensional in direction. The axial strains in GD and FD are of values +30,200 and −20,050, respectively, as seen from Table 1. These values are necessarily foreshortened in the displacement diagram representation since the actual frame members are foreshortened in both plan and elevation views. To plot these values correctly, it is necessary to find a true length projection of the proper strain line. For Fd and Gd this is done by revolving the horizontal projection of the lines to show true length in the vertical projection. On these true length projections, the true values of strain are plotted and then projected back into foreshortened plan and elevation views. This procedure locates the foreshortened strain lines Fd and Gd.

In similar fashion, the axial strains Ec and Fc are plotted parallel to their respective frame members. The strains in EC and FC are both zero, as indicated in Table 1, which causes c to coincide with joints E and F in the space diagram. Likewise, strains Eb, Hb, Ha, and Ga are zero and the free ends a and b coincide with the fixed joints E, H, and G.

Now, since all the strains rising from the basal points E, H, G, and F are three dimensional in nature, even though some are zero, perpendicular lines, plotted at the free ends of the strain lines and presumably intersected, are of no significance. Rather, it is necessary to erect perpendicular planes to the ends of each of these strain lines and to intersect these planes. An intersection of two planes will yield a straight line which, in this case, will represent all possible positions of a displaced joint.

Fig. 3 shows the geometric procedure required for the erection of the eight planes perpendicular to the strain lines Gd, Fd, Ga, Ha, Hb, Eb, Ec, and Fc. The constructions required to plot foreshortened strain values for strains Gd and Fd have been omitted in Fig. 3 since they were of only temporary significance.

In order to erect the plane G′d′ perpendicular to the d-end of Gd, a horizontal line is drawn through point d perpendicular to the strain line Gd. This line, as indicated in Fig. 3, shows its perpendicularity in the horizontal projection and runs parallel to the HV-reference line in the vertical projection. The perpendicular line is continued in space until its horizontal projection intersects the HV-line, thus determining the vertical piercing point of the line. Through this vertical projection of the vertical piercing point the V-trace of the plane G′d′ is drawn perpendicular to the vertical projection of the strain line $G^v d^v$. The intersection of the vertical trace of G′d′ with the HV-reference line also indicates one point on the horizontal trace of G′d′. Through this intersection the H-trace of G′d′ is erected perpendicular to the horizontal projection of the strain line $G^h d^h$. The plane G′d′ represented by its horizontal and vertical traces is the required plane perpendicular to the strain line Gd at point d. Any descriptive geometry textbook will validate this method of construction.

In similar manner the plane F′d′ is erected perpendicular to the end of the strain line Fd, and is represented by its horizontal and vertical traces. The same method of construction is used to erect planes perpendicular to the strain lines Ga, Ha, Hb, Eb, Ec, and Fc which are all equal to zero in length. In these cases, even though the strain lines are equal to zero in length, potential space directions are indicated by the plan and elevation views of the actual space pedestal.

These potential space directions must be recognized in order to erect the perpendicular lines which, together with the zero strain lines, are instrumental

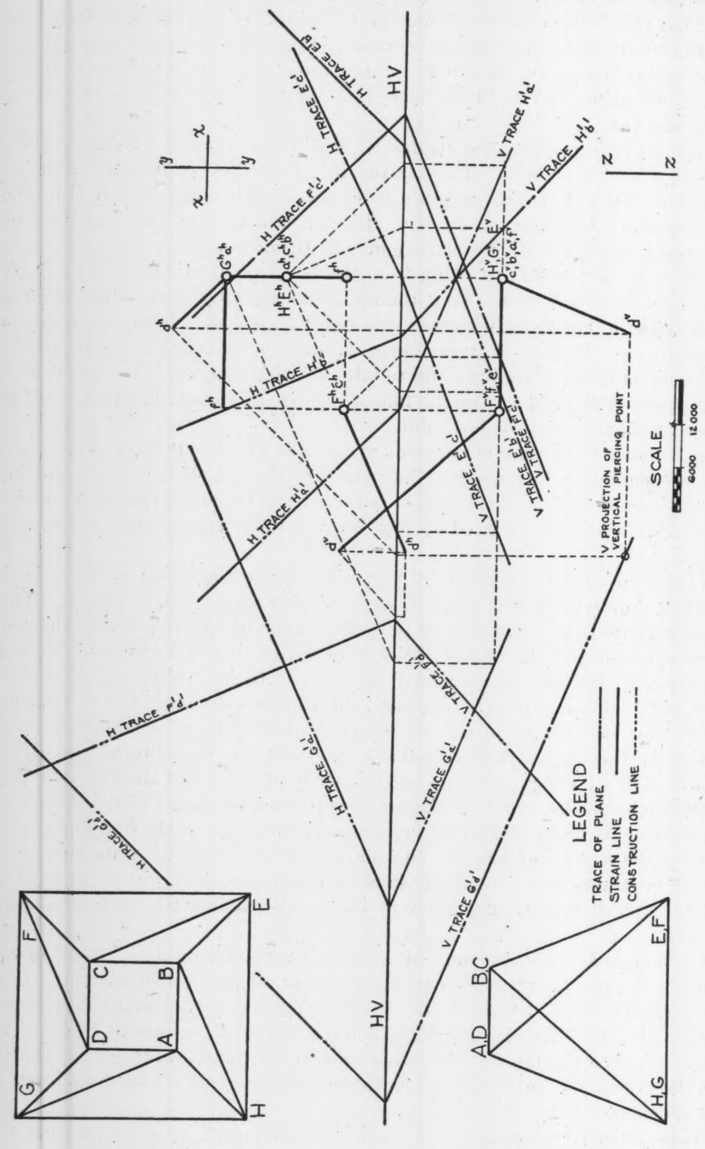
HV
H TRACE E'B'
H TRACE E'C'
V TRACE H'A'
V TRACE H'B'
x
y
z
H TRACE F'C'
G^h d^h
H^h E^h a^h c^h b^h
e^h
d^h
f^h
H TRACE H'D'
F^h C^h
H'V G'V E'V
c'v b'v a'v f'v
d'v
F^v I^v e^v
V TRACE E'C'
V TRACE E'B'
V TRACE F'C'
H TRACE H'A'
d'v
d^h
V PROJECTION OF VERTICAL PIERCING POINT
SCALE
6000
12000
H TRACE F'D'
V TRACE F'D'
H TRACE G'A'
V TRACE G'A'
H TRACE G'D'
V TRACE G'D'
LEGEND
TRACE OF PLANE
STRAIN LINE
CONSTRUCTION LINE
G
F
C
D
E
B
A
H
BC
AD
E,F
H,G

FIG. 3.—THREE-DIMENSIONAL DISPLACEMENT DIAGRAM, PHASE 2

in the construction of the planes. This completes the second phase of construction as shown in Fig. 3.

All dependence on the basal joints H, E, F, and G has now been reckoned with. There remains the determination of the loci of all feasible positions of joints A, B, C, and D and thence the location of their final positions.

In Fig. 4 it is evident that the intersection of the two planes G′d′ and F′d′ will indicate a line which serves as the locus of all feasible positions of point D. The locus of point D is found by realizing that the intersection of the H-traces of G′d′ and F′d′ and the intersection of the V-traces of the same planes locate two points on the line of intersection between the two planes. These two points are sufficient to determine the horizontal and vertical projections of the locus of point D. In like fashion, the locus of joint B is found in both horizontal and vertical projections.

The horizontal projections of the loci of joints A and C are found by locating the intersection of the H-traces of the proper planes. The vertical projection of the locus of A is then drawn parallel to the V-traces of planes G′a′ and H′a′ which are nonintersecting. In like manner the V-locus of joint C is drawn parallel to the V-traces of planes F′c′ and E′c′.

To find the final positions of joints A, B, C, and D, the relative movements of these space joints must be considered. As in the planar Williot diagram, the space diagram not only shows the movement of a particular joint with respect to its original position, but also indicates the movement of that joint with respect to any other joint.

From Table 1 it is seen that the following strains or deformations exist in the top horizontal members: AB = no deformation; AD = no deformation; DC = − 10,000 units of deformation; and CB = no deformation.

Since the relative x-displacement of joint A with respect to joint B is zero, joint A is located in the horizontal projection at the intersection of the H-locus of A with the H-locus of B. The final position of joint A in the vertical projection is obtained by projecting A^h perpendicular to the HV-reference line, to the V-locus of A. Point D is located in the horizontal projection at the intersection of the H-loci of A and D. This location is possible since the relative y-displacement of D with respect to A is zero. Joint D is located in the vertical plane by projecting D in the plane G′d′ to its proper position. The elements of descriptive geometry are necessary to effect this projection.

The relative x-movement between joints C and D is equal to 10,000 units. From the locus of D in the horizontal projection, point C is plotted along the H-locus of C, a distance of 10,000 units in the direction of the movement of the free end, C. Joint C is located in vertical plane by projecting a perpendicular from the HV-reference line to the V-locus of C.

At the intersection of the H-loci of C and B, the horizontal projection of joint B is located. This location is possible since the relative y-movement between C and B is equal to zero. Point B is then projected in the plane E′b′ to its proper location in the vertical view.

The space displacement diagram is thereby completed, and the true positions of all joints without recourse to a rotational correction diagram are furnished. All values of displacement are scaled parallel to the coordinate axes x, y, and z

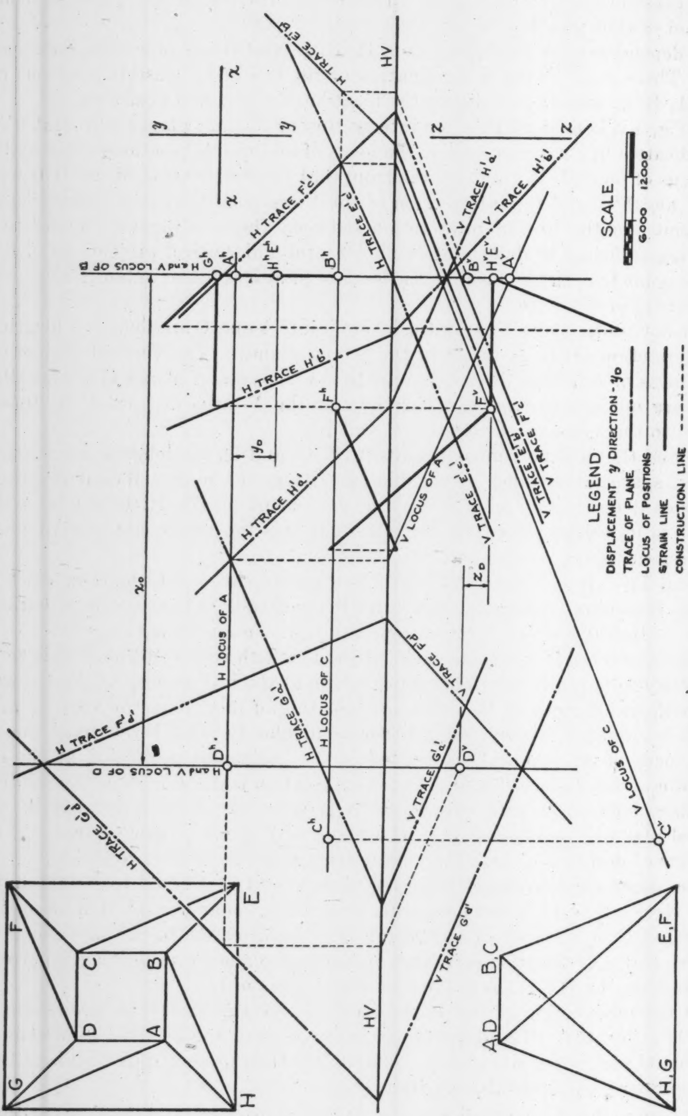
G
F
D
C
A
B
H
E
AD
B,C
E,F
H,G
HV
HV
H TRACE G'd'
V TRACE G'd'
H TRACE F'd'
H TRACE G'a'
H LOCUS OF C
H LOCUS OF A
V TRACE G'a'
V TRACE F'a'
V LOCUS OF C
H and V LOCUS OF D
H and V LOCUS OF B
H TRACE H'a'
H TRACE H'b'
H TRACE F'c'
H TRACE E'b'
H TRACE E'c'
V LOCUS OF A
V TRACE E'c'
V TRACE E'b'
V TRACE F'c'
V TRACE H'a'
V TRACE H'b'
x₀
y₀
z₀
C^h
C^v
D^h
D^v
G^h
A^h
H^h E^h
B^h
F^h
F^v
B^v
H^v E^v
A^v
x
y
z
LEGEND
DISPLACEMENT y DIRECTION - y₀
TRACE OF PLANE
LOCUS OF POSITIONS
STRAIN LINE
CONSTRUCTION LINE
SCALE
6000 12000

FIG. 4.—THREE-DIMENSIONAL DISPLACEMENT DIAGRAM, PHASE 3

from the fixed joint H. The values of the displacements x, y, and z of joint D from its original position are shown in Fig. 4 by x_D, y_D, and z_D.

To effect a check on the graphical procedure involved in the three-dimensional diagram, values of displacements x, y, and z were computed by the virtual work method as explained in the complete manuscript.[1a] These computations involved: (1) The solution for axial forces within the space pedestal under sixteen separate and distinctly different applications of unit load at the various joints; (2) the proper combination of these axial forces with the forces resulting from the 5,000-lb load; and (3) the algebraic summation of all $\frac{P\,u\,L}{A\,E}$-values pertinent to a particular displacement. The time required for the analytical method was at least tenfold that required for the graphical analysis. The values of displacement obtained graphically were within 1% of those calculated by the virtual work principles.

Deflections in a Space Truss

As a second and more generalized application of the principles involved in space displacement diagram construction, the deflections are found graphically for the space truss shown in Fig. 5.

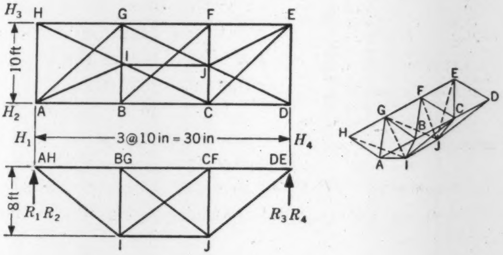

Fig. 5

This particular truss has been selected as an illustration because it embodies all the characteristics which ordinarily tend to complicate the construction of displacement diagrams. The construction of an accurate three-dimensional displacement diagram for this truss should indicate accordingly the applicability of the method to any type of space structure, however complicated.

Primarily, the space truss is an indeterminate one. The eight external reactions required to maintain horizontal and vertical stability consist of four horizontal and four vertical components. Furthermore, the truss consists of ten joints and twenty-four bars, none of which lie in a common plane at any of the ten joints. Applying the test for redundancy in space, it is observed that the number of unknown elements is in excess of the number of available static equations by two. Consequently, the space truss considered is statically indeterminate to the second degree.

The application of 10-kip, panel-point concentrations at each upper panel point is in keeping with the standard method of applying dead load. This loading tends to reduce the number of inactive members to a minimum, which in turn has the effect of making the final displacement of any joint dependent upon a number of internal deformations. Such a dependence upon internal deformations necessitates the construction of strain lines for practically every member of the truss.

Although the truss has been loaded symmetrically with 10-kip concentrations, the nonsymmetry of the internal member and the reaction arrangement cause an inequality in axial force values on either side of the center line. This inequality, in turn, eliminates the possibility of choosing a member that will maintain a fixed direction under loading. If an assumption as to the direction of any member must be made at the outset of the construction, a rotational correction diagram must be incorporated within the construction in order to arrive at the true values of displacement. Since the displacement diagram for this space truss requires an initial assumption as to the direction of one of its chords, a rotational diagram is needed. This rotational diagram tends to generalize the construction further.

TABLE 2.—NUMERICAL EXAMPLE, FIG. 5

Bar	P	$P\,L/A$
AB	−13.60	−13.60
BC	−11.77	−11.77
CD	−11.22	−11.22
DE	− 5.06	− 5.06
EF	−11.95	−11.95
FG	−10.12	−10.12
GH	− 9.57	− 9.57
HA	− 5.61	− 5.61
AG	− 2.57	− 2.57
BG	+ 8.08	+ 8.08
BF	− 2.57	− 2.57
FC	+ 8.08	+ 8.08
CE	− 2.57	− 2.57
HI	+15.48	+15.48
AI	+18.95	+18.95
GI	−10.60	−10.60
BI	−11.80	−11.80
IJ	+25.28	+25.28
IC	− 1.75	− 1.75
JG	− 1.75	− 1.75
JF	−11.80	−11.80
JC	−10.60	−10.60
JE	+18.95	+18.95
JD	+15.48	+15.48

In Table 2 the axial forces in the various members of the indeterminate space truss are listed together with the resulting deformation factors ($P\,L/A$) in each member. The complete computation is included in the original manuscript on file in the Engineering Societies Library.[1a]

The $\frac{L}{A}$-term has been assumed unity for each member because of the facility it affords in indeterminate force computations. Although the resulting deformations and displacements are disproportionate under an assumption of a unit $\frac{L}{A}$-value for all members, none of the principles of stress analysis has been violated; nor have any of the difficulties in displacement diagram construction been circumvented. The modulus of elasticity has been neglected simply to yield larger values of deformation which are more easily plotted.

Subsequently, Figs. 6, 7, and 8 show, in three phases, the three-dimensional displacement diagram for the space truss. Since many of the internal members of the truss are parallel to the profile plane, it has been considered advisable to represent the diagram in profile projection as well as in horizontal and vertical projections.

The notation is essentially the same as was used for the space pedestal; any additional notation resulting from the use of a profile projection is self evident.

PROCEDURE

To any convenient scale the space truss is represented orthographically in all three of the common views as shown in Fig. 6. All views are used to determine the corresponding slopes of the strain lines in the displacement diagram.

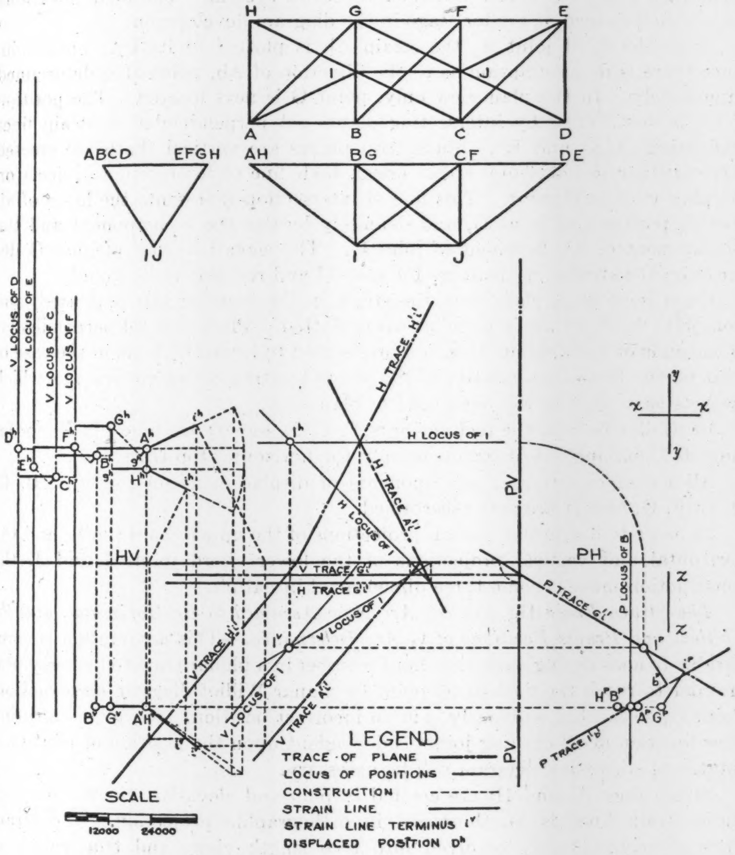
H
G
F
E
I
J
A
B
C
D
ABCD
EFGH
AH
BG
CF
DE
IJ
I
J
V LOCUS OF D
V LOCUS OF E
V LOCUS OF C
V LOCUS OF F
H TRACE H'i'
H LOCUS OF I
H TRACE A'i'
H LOCUS OF I
V TRACE G'i'
H TRACE G'i'
V LOCUS OF I
V TRACE H'i'
V LOCUS OF I
V TRACE A'i'
HV
PH
PV
PV
P LOCUS OF B
P TRACE G'i'
P TRACE I'D'
x
x
y
y
z
z
LEGEND
TRACE OF PLANE
LOCUS OF POSITIONS
CONSTRUCTION
STRAIN LINE
STRAIN LINE TERMINUS i'
DISPLACED POSITION D^h
SCALE
12000
24000

FIG. 6.—THREE-DIMENSIONAL DISPLACEMENT DIAGRAM; SPACE TRUSS, PHASE 1

Phase 1.—Since joint A is the only completely fixed joint in the truss, it is used as the origin of the displacement diagram. Joint A is plotted and separated in the three views by reference lines HV, PV, and PH.

Referring to the orthographic representation of the space truss it is seen that the truss consists of two upper chords, AD and HE, separated by lateral bracing members, HA, GA, GD, etc., all of which lie in the same horizontal plane. As far as the x-direction and y-direction of space are concerned, the true displacements of all upper chord joints in this horizontal system may be determined by an ordinary Williot diagram. This is essentially the initial step. The final elevation of each of these joints is dependant upon the deformations in the lower chord and the connecting system. The final elevations may be determined at a later stage in the diagram development.

From the fixed joint A, the strain Ah is plotted in its true projection. Since there is no assumption as to the direction of Ah, point H is determined immediately. In the plan view only, point G is next located. The position of G^h is determined by intersecting planes set perpendicular to strain lines originating at A^h and H^h. Since both planes are vertical (both are erected perpendicular to horizontal strain lines), their line of intersection projects on the plan view as a point. This line of intersection represents the locus of all feasible positions of joint G, and definitely locates the x-component and the y-component of displacement of joint G. The elevation view of joint G depends on the strains in members IG and GJ and remains to be found.

From joint G in plan view, the strain in the member GB is plotted, and from joint A, the strain in member AB is plotted. Planes are set perpendicular to the ends of these strain lines and intersected to locate B^h. As in the case of joint G, the line of intersection of the planes locating B^h is a vertical line which projects as a point in the horizontal or plan view.

In similar fashion the loci of joints F, C, E, and D are found in the order indicated, making use of strains in only the horizontal top truss.

All x-components and y-components of displacement for joints A, B, C, D, E, F, G, and H are now ascertained.

In order to locate the vertical projections of the upper chord joints, and the horizontal and vertical projections of the lower chord joints I and J, the construction must become three dimensional in nature.

The Strain Lines Hg *and* Ag *Are Now Assumed to Be Horizontal, and the Vertical and Profile Positions of* G *Are Determined.*—This assumption is comparable to considering an initial chord member in a planar truss to be horizontal in order to reach the next panel point by planar Williot diagram construction. Point G, so located, obviously is in an incorrect position. Since the elevation view location of all ensuing joints is dependent upon the position of joint G, a rotational correction diagram will be necessary.

Strain lines Ai and Hi are erected in plan and elevation views. Each of these strain lines is foreshortened in orthographic projection. The strain lines are consequently revolved into true length views and true values of deformation are plotted on the true length views. The values of deformation are then projected back into the foreshortened horizontal and vertical views.

Planes H′i′ and A′i′ are erected perpendicular to these strain lines at their free ends. The planes are represented by their horizontal and vertical traces.

The strain in GI is plotted in the three-dimensional displacement diagram parallel to its corresponding truss member, and the plane G′i′ is erected perpendicular to the end of Gi. This procedure is carried out most advantageously

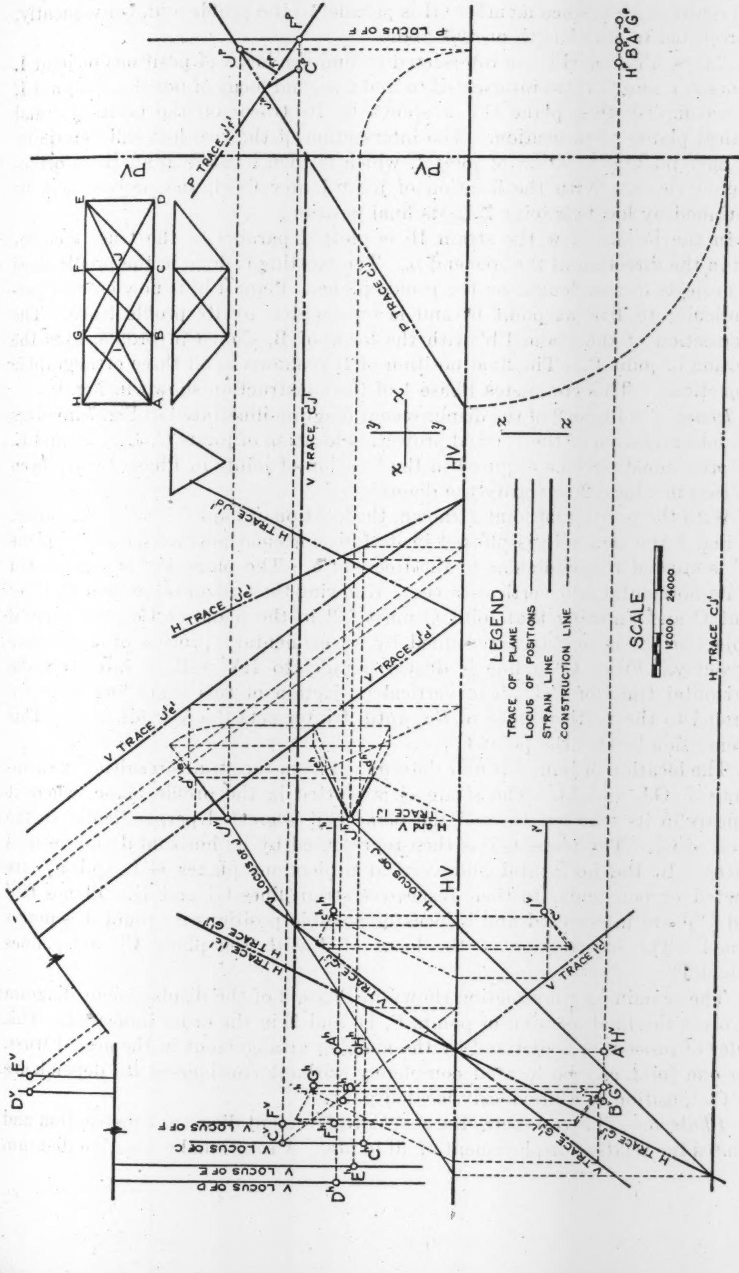
V LOCUS OF D
V LOCUS OF E
V LOCUS OF C
V LOCUS OF F
D^V E^V
D^h E^h C^h F^h
C^V F^V
B^V G^V A^H^V
H TRACE G J'
V TRACE G J'
V TRACE I J'
H TRACE I J'
H TRACE C J'
V TRACE C J'
HV
H and V TRACE I J
V LOCUS OF J'
H LOCUS OF J'
V TRACE J'e'
H TRACE J'e'
V TRACE J'd'
H TRACE J'd'
V TRACE C J'
P TRACE C J'
P TRACE J'f'
P LOCUS OF F
PV
H^P B^P A^P G^P
J^P F^P C^P
LEGEND
TRACE OF PLANE
LOCUS OF POSITIONS
STRAIN LINE
CONSTRUCTION LINE
SCALE
12000 24000
H TRACE C'J'
H G E F D A B C J I
x y z

FIG 7.—THREE-DIMENSIONAL DISPLACEMENT DIAGRAM; SPACE TRUSS, PHASE 2

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in the profile view since member GI is parallel to the profile and, consequently, is projected in true length on this plane.

Planes A′i′ and H′i′ are intersected to find the locus of positions of joint I. Planes A′i′ and G′i′ are intersected to find a second locus of position for joint I. To accomplish this, plane G′i′ is shown by its traces on the horizontal and vertical planes of projection. The intersection of the two loci will determine the only possible position of joint I, which is then located in all three orthographic views. With the location of joint I now fixed, the process may be continued by locating joint B in its final position.

In the profile view the strain Ib is plotted parallel to the truss member and in the direction of the free end b. The plotting is done in the profile since IB projects in true length on the profile plane. Plane I′b′ is now erected perpendicular to $I^p b^p$ at point b^p and is represented by its profile trace. The intersection of the plane I′b′ with the locus of B, shown in profile, fixes the position of joint B. The final position of B is shown in all three orthographic projections. This completes Phase 1 of the construction shown in Fig. 6.

Phase 2.—Phase 2 of the displacement diagram illustrated in Fig. 7 involves the determination of the vertical projection location of joints C, J, F, D, and E. All prior constructions required in the location of joints in Phase 1 have been omitted in Phase 2 to clarify the diagram.

With the position of joint I known, the location of joint C is a simple matter. In Fig. 7 the strain Ic is plotted in its foreshortened positions, and the plane I′c′ is erected perpendicular to the free end c. The plane I′c′ is represented by its horizontal and vertical traces. Knowing the horizontal projection C^h of joint C and knowing that joint C must fall in the plane of I′c′, the vertical projection C^v is readily determined by an elementary process of descriptive geometry. From C^h a line is drawn parallel to HV until it intersects the horizontal trace of I′c′. The vertical projection of this same line is drawn parallel to the vertical trace of I′c′ until it intersects the V-locus of C. This intersection locates the point C^v.

The location of joint J is now determined from the strains resulting in members CJ, GJ, and IJ. The strain cj is plotted in the profile plane, where it appears in its true length, and the plane C′j′ is erected perpendicular to the j-end of Cj. The plane C′j′ is then represented by its horizontal and vertical traces. In the horizontal and vertical projections planes G′J′ and I′j′ are erected perpendicular to their respective strain lines Gj and Ij. Planes G′j′ and C′j′ are intersected and a locus of possible positions for point J is determined. The intersection of the locus of J with the plane I′j′ determines J^v and J^h.

The remaining construction shown in Phase 2 of the displacement diagram involves the final location of points F, D, and E in the order indicated. This order of procedure is dictated by the member arrangement in the actual truss. No one joint can be located completely without considering its dependence on the position of a previously located joint.

Phase 3.—This completes the space displacement diagram construction and locates the relative displacement of all joints. A rotational correction diagram

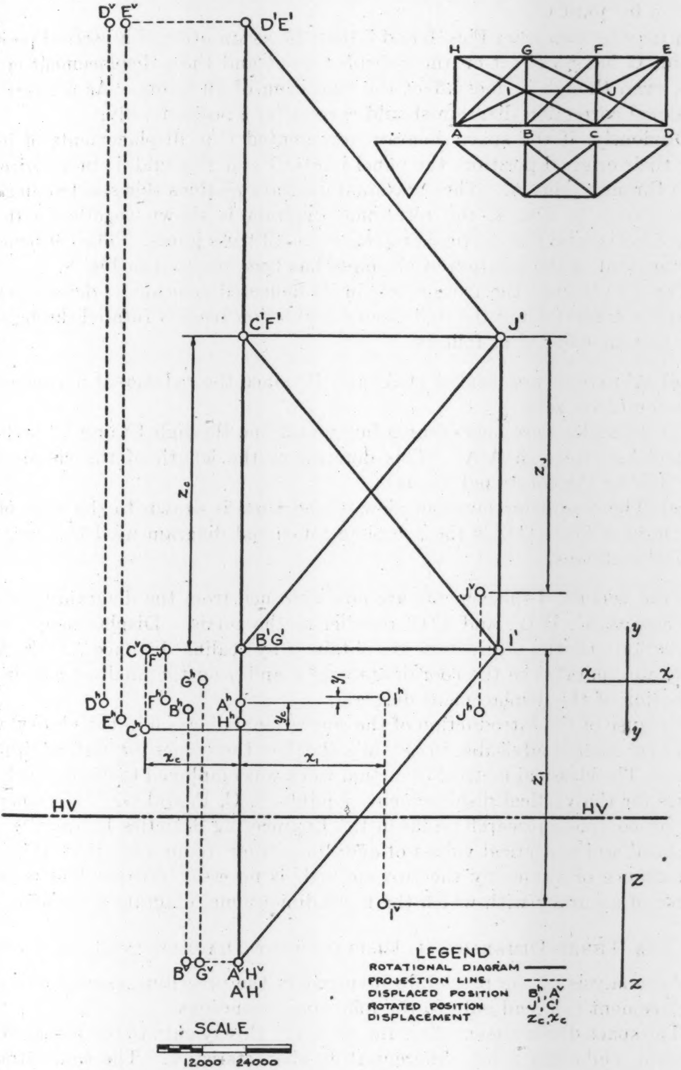
D^v E^v
D'E'
H G F E
I J
A B C D
C'F'
J'
Z_c
Z_J
J^v
C^v F^v
B'G'
I'
G^h
D^h F^h B^h
E^h C^h
A^h
H^h
y_c
y_1
I^h
J^h
x_c
x_1
y
x x
y
z_1
HV HV
Z
I^v
Z
B^v G^v A^v H^v
A'H'
LEGEND
ROTATIONAL DIAGRAM
PROJECTION LINE
DISPLACED POSITION B^h, A^v
ROTATED POSITION J', C'
DISPLACEMENT z_c, x_c
SCALE
12000 24000

FIG. 8.—THREE-DIMENSIONAL DISPLACEMENT DIAGRAM; SPACE TRUSS, PHASE 3

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is now required to eliminate the fallacies arising from the initial assumption of position for joint G.

It may be seen from Figs. 6 and 7 that the assumption of a vertical position of joint G has no effect on the x-displacement and the y-displacement of any joint, even though it does affect the z-position of all joints. As a result, the rotational correction diagram should correct for z-position only.

Obviously, if the space diagram represented true displacements of joints from their original position, the panel joints D and E would lie in a horizontal plane through joint A. The rotational diagram rectifies this shortcoming.

In Phase 3, Fig. 8, the rotational diagram is shown together with the located horizontal and vertical projections of all truss joints. The construction instrumental in the location of all joints has been omitted in Fig. 8.

For convenience the truss is left in its deflected position as determined by the space diagram, and the undeflected correction truss is rotated through 90°. This may be effected as follows:

(*a*) A′ and H′ are located at A^v and H^v since the rotational movements of these joints are zero.

(*b*) E′ and D′ are located on a horizontal line through E^v and D^v and on a vertical line through A^vA'. This determines the length of the chords A′D′ and H′E′ in the rotational diagram.

(*c*) The complete elevation view of the truss is drawn to the scale of the top chord A′D′. This is the complete rotational diagram used for correcting vertical positions.

True vertical displacements are now obtained from the diagram by scaling distances A^vA', B^vB', and C^vC' parallel to the z-axis. Displacements in the x-direction and the y-direction are obtained by scaling distances, A^hC^h, A^hB^h, A^vC^v, etc., parallel to the coordinate axes x and y on the unaltered horizontal projection of the displacement diagram.

Because of the introduction of the correctional diagram in the vertical view, it was considered advisable to obtain a check of the values for vertical displacement. The classical method of virtual work was employed to obtain analytical values for the vertical displacement of joints B, C, F, and G. (For computation see complete research on file in the Engineering Societies Library.[1a]) The graphical and analytical values of deflection differ by no more than 1%. The concurrence of values by the two methods is necessarily dependent upon the degree of accuracy with which the three-dimensional diagram is drawn.

The Three-Dimensional Displacement Diagram in Retrospect

An analysis of the procedure involved in the preceding examples of space displacement can lead only to the following deductions:

The space displacement diagram, although closely akin to the planar Williot diagram, embodies a few differentiating characteristics. The same straightforwardness of attack employed in the planar Williot diagram cannot always be used in the three-dimensional construction. Ordinarily, the planar construction proceeds progressively joint by joint until the complete picture is obtained. In the usual space frame the joints are interdependent upon one another for their exact location. This mutual relationship between joints eliminates the possibility of a step-by-step solution and dictates the use of a more circuitous method of attack. This deviousness is well illustrated by the first example of space diagram construction. The location of the upper panel points of the space pedestal cannot be fixed without recourse to a study of the relative movement between the upper joints. As long as the general stability of a space frame is contingent upon all its members, this roundabout type of solution cannot be avoided.

Other differences existing between the space and planar displacement diagram are minor. The use of perpendicular planes at the end of strain lines, in lieu of spherical surfaces, and the intersection of these planes to form loci of possible positions for joints, is in keeping with the reasoning employed by Williot. Proceeding from planar to spacial construction, the use of perpendicular planes rather than perpendicular lines follows as one of the natural consequences.

The use of a rotational correction diagram is not in itself a new idea, but its application to one or more views of a space displacement diagram might be considered unique.

Lastly, it may be deduced that any objection to the three-dimensional diagram on the basis of its complexity is groundless. The diagram is merely a sequence of superimposed constructions, all of which are fundamental in character. If the clarity of the diagram is impaired, at any time, by previous constructions, all located points, loci of points, and required planes may be transferred to a new and lucid picture. This is possible because the processes of descriptive geometry used in such a problem are of temporary significance only.

In conclusion, it should be pointed out that the principles involved in three-dimensional displacement diagram construction can be used in the deflection study of any space frame.

This graphical method of finding displacements will not precipitate any strong inclinations toward space frame analysis, but it will aid in those solutions that must be made. The planar Williot diagram has been put to many uses other than that of finding deflection. Someday, the space displacement diagram may have a similar number of applications. The development of these applications will follow, in natural sequence, the desire to simplify the analysis of space structures further.

DISCUSSION

W. Watters Pagon,[6] M. ASCE.—The method of determining displacements, offered in this paper, applies to structures that cannot be sufficiently treated by disregarding the influence which each plane system of members of a space truss exerts on other similar plane systems. Normally, a bridge truss can be resolved into four sets of plane systems, even though in a curved chord truss the lateral bracing systems are framed to the chord planes of a cylindrical surface, and the effect of the vertical stress components which the bracing members exert on the vertical trusses is not of importance.

The Williot diagram for determining displacements graphically is an old and tried method of determining the entire aggregation of displacements at one time, and the expansion of this method to space frames, therefore, offers all the advantages which the method possesses.

The author deserves the greatest commendation, therefore, for his exposition of the method. If there be no redundancy, the method can be followed well, despite the maze of lines (most of which to be sure are only light construction lines) that make difficult a clear comprehension of what is being performed. When there is redundancy, the confusion is much increased. The success of the method outlined, therefore, depends very largely on the symbolism adopted, such as the use of upper and lower case letters, and on the juxtaposition of the vertical, horizontal, and transverse planes. Being pressed for space, the author develops these planes along their intersection lines, HV, PV, etc., which tend to cloud and confuse; therefore, in practice he undoubtedly would open up his composite diagram through displacements of the planes at some distance from each other, since projections from one to the other would in no way be affected. Hence in actual use much of the confusion would disappear.

It would seem to the writer that the successful use of this method will rest on the symbolism used, and if this can be simplified so as not to use superscripts, primes, etc., then the greatest clarity would result. In any case, the author has given to the profession a valuable tool for the determination of displacements where simplified methods, such as have been used in the past, are insufficient.

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[6] Cons. Engr., Baltimore, Md.

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Paper No. 2449

DESIGN CHARACTERISTICS OF LOCK SYSTEMS IN THE UNITED STATES

A SYMPOSIUM

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With Discussion by Messrs. Basil Standing, and A. Frederick Griffin.

Note.—Published in January, 1950, *Proceedings*. Positions and titles given are those in effect when the Symposium or the discussion was received for publication.

FOREWORD

The lifting or lowering of vessels vertically from one water level to another by locks is not new. The earliest boatmen resorted to unloading, carrying, and reloading to overcome distance and differences in elevation between bodies of water. Necessity brought about the development of inclined planes and the use of waves resulting from controlled releases of stored water. The forerunners of the modern lock, possibly in the year 1000, consisted of single gates installed in dikes along the North Sea, where the fluctuations of tide were employed to lift and lower shipping. Records indicate that actual "chamber locks" were invented about the same time in Germany, Holland, and Italy, perhaps near the end of the fifteenth century. The great Italian engineer Leonardo da Vinci, about 1497, invented the insertion of filling valves and emptying valves in the lock gates, which system has been used for centuries and has been found adequate for the passage of small boats.

Today, there are in the United States locks in active operation which are not greatly different from those designed by da Vinci and also there are many which have been designed on the basis of exhaustive laboratory testing and years of experience in operating locks which pass many millions of tons of traffic each year.

It is the purpose of the Committee on Design, Construction, and Operation of Navigation Structures of the ASCE Waterways Division, which prepared the Symposium, to present papers setting forth the latest thought regarding the design of locks and other navigation structures. The present coverage, admittedly, is incomplete, but it is hoped that it will be a start toward publication of design bases which have not recently been presented to the engineering profession, and which are well known to a relatively few individuals.

It is to be hoped that discussion of these papers will bring to bear on the problems the thoughts of a larger group of engineers, producing new ideas on all specific subjects presented herein, and perhaps suggesting to the committee the most needed direction for future papers and discussions.

INFLUENCE OF MODEL TESTING ON LOCK DESIGN

By A. Frederick Griffin,[1] M. ASCE

Synopsis

Hydraulic systems of various types in use in locks on many American rivers and canals are described briefly, together with observations as to the manner in which they operate. The objectives of engineers engaged in lock design are stated and the important part played by the hydraulic laboratory as an aid in determining ways and means of improving the efficiency of lock-filling and lock-emptying systems is depicted. Some general criteria for guidance in design, based on results of comparatively recent prototype and model testing, together with examples of designs produced in conjunction with laboratory testing, are presented. The writer finds that the locks reported are used effectively in the transportation of traffic, yet practically none is operated as designed. Through experience, a valve-operation procedure has been developed for each lock which provides satisfactory lockage conditions. However, operating time has been increased in varying degrees over that planned in the design, and the over-all efficiency of the lock has been correspondingly decreased. Lock hydraulic systems should be designed so they can be operated as planned. The writer suggests that it is both possible and practicable to design systems that will provide fast and safe operation without creating intolerable conditions within the lock chamber or in the lock approaches. Lock models have been found efficient and practical in resolving many pertinent difficult hydraulic problems. Much in the field remains to be accomplished and the continued use of models is strongly recommended.

Introduction

The development of lock hydraulic systems has been largely predicated on economy of time and of materials—that is, on the value of time lost during lock operations balanced against the additional cost of expenditures required to minimize surges, turbulence, and adverse currents. As long as barges and towboats were small and slow moving, the value of the time expended in lifting or lowering vessels while passing from one pool to another was not consequential. Small craft and small tows could be accommodated in lock chambers so small that surges and turbulence were not too troublesome. Filling and emptying systems consisting of valves in the lock gates are both inexpensive and, for low-lift locks of small size, satisfactory. However, progress in the development of shipping and barge transportation, the size and draft of vessels and of barge tows, the increase in the speed and power of vessels, the volume of water transportation, and the size and lift of locks—have all contributed to the need for further development, and perhaps refinements, in the operating features of navigation locks.

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[1] Chf. Engr., Office of Div. Engr., Upper Mississippi Valley Div., Corps of Engrs., U. S. Army, St. Louis, Mo.

It is the purpose of this paper, through references to various locks of which the writer has direct or indirect knowledge, to call attention to: (*a*) The features of a variety of hydraulic systems used or proposed; (*b*) observed performance at representative locks; (*c*) some prototype and model tests employed to discover methods of improving the operating characteristics of lock hydraulic systems built or contemplated; and (*d*) the practical use of lock models as a preliminary to lock design. Much has been written by numerous observers and investigators on various phases of the subject of lock hydraulic systems. In general, such papers have dealt with specific problems of the subject. In this paper an effort is made to summarize and to generalize, with a view toward affording the profession a compendium and discussion of salient features of modern lock design and performance. Obviously, such a treatment results in the inclusion of much information which is neither new nor novel, but which is considered essential for a balanced treatment of such a broad subject.

TYPES OF LOCK-FILLING AND LOCK-EMPTYING SYSTEMS

Lock systems currently in use may logically be classified into the following basic types:

A. Those filling and emptying over, between, or around the lock gates;

B. Those filling and emptying by valves in the gates, or through short culverts around the gate bays, through the lock walls, or in the gate sills; or

C. Those filling and emptying through longitudinal culverts in the walls or floor, connected to the lock chamber through either wall ports or floor laterals.

Type A. Filling and Emptying Over, Between, or Around the Lock Gates.—With lock gates of the sector or Tainter type it is possible to eliminate the need for special filling and emptying valves and conduits and to use the lock gates for this purpose. Sector gates (see Fig. 1) are in general use as lock gates where it is desired that they be opened or closed in flowing water or against a static head and, in addition, where they can be used to fill and to empty the lock. This type of gate consists of two leaves, each composed of a sector of a vertical cylinder which rotates about a vertical axis at or behind the face of the lock

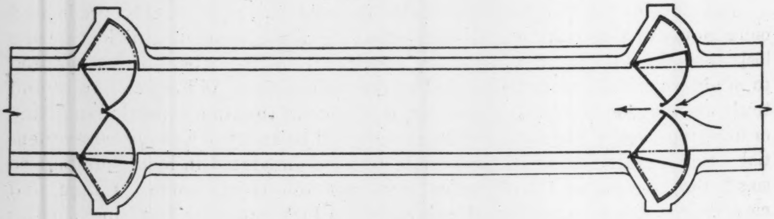

FIG. 1.—SECTOR GATES

wall. The two leaves abut against one another at the center line of the lock when closed, and are withdrawn into recesses in the lock walls when opened. They were developed for use on the Södertalje Canal in Sweden, where it was desired to pass storm water through the lock. In the United States, sector gates have been used in the lock at the mouth of the Chicago River in Illinois, where differentials in water levels cause reversals of flow; in Bayou Sorrel Lock at the junction of the Intracoastal Waterway and the East Atchafalaya Basin Protection Levee, in Louisiana; in the St. Lucie Locks in Florida; and on several large drainage canals. To fill a lock equipped with sector gates, water is admitted between the edges of the gate leaves and between the leaves and the upstream corners of the wall recesses. It has been found by experiment or testing that, if the major part of the flow is through the recesses and if the curve of the recess walls directs the stream from each side into the chamber in a direction perpendicular to the axis of the lock, turbulence in the lock chamber is not excessive for relatively small heads.

The application of Tainter gates for use as lock gates has not been extensive. They are receiving consideration as upper lock gates, however, in connection with the design of several proposed locks. The Tainter gate would be opened for lockages by lowering below the lock sill and would provide for filling by flow over the top of the gate. Model tests indicate that turbulence and surge in the lock chamber can be controlled to a reasonable extent by using baffles between the gate trunnions and by employing the upstream end of the lock floor as a stilling basin. Provision of satisfactory lockage conditions for medium lifts appears feasible.

Type B. Filling and Emptying by Valves in the Gates, or Through Short Culverts Around the Gate Bays, Through the Lock Walls, or in the Gate Sills.—Most of the locks constructed in the late nineteenth century and early twentieth century on the Ohio River in Pennsylvania and bordering Ohio, West Virginia, Indiana, Kentucky, and Illinois; the Monongahela River in Pennsylvania; the Cumberland River in Tennessee and Kentucky; and other rivers were of this type. The system consists of either valves integral with the lock gates or a number of short culverts passing directly through the river wall, or through or around the gate sill, each controlled by a separate valve. Filling and emptying by valves in the miter gates may be exemplified by the Deep Creek and South Mills Locks in the Dismal Swamp Canal near the Virginia–North Carolina state line. These locks are small—52 ft by 300 ft. They have a lift of about 12 ft and were designed to accommodate movement of small craft. Turbulence in the upper end of the lock chamber and both turbulence and high velocities experienced below the lower gates have not been found detrimental to the type of traffic using the waterway.

Many of the locks on the Ohio River are of low lift and are filled or emptied through a series of ports in the river lock wall controlled by butterfly valves (see Fig. 2). The filling valves, in the upstream section of the wall, discharge directly from the upper pool into the lock chamber. The emptying valves, in the downstream section of the same wall, discharge directly into the lower pool. Filling time averages from about 5 min to 8 min, and emptying requires from 4 min to 6 min, depending on the head. During the filling operation, flow through the filling ports creates a turbulent eddy in the lock chamber with a downstream flow along the lock land wall. Somewhat similar currents are experienced during the emptying operations. These conditions are not too noticeable for the lower lifts but are troublesome when the lift approaches from 8 ft to 10 ft. The action of the currents described is ordinarily controlled by judicious opening of the filling and emptying valves, the times for filling and emptying for lockages being lengthened ordinarily by as much as 3 min.

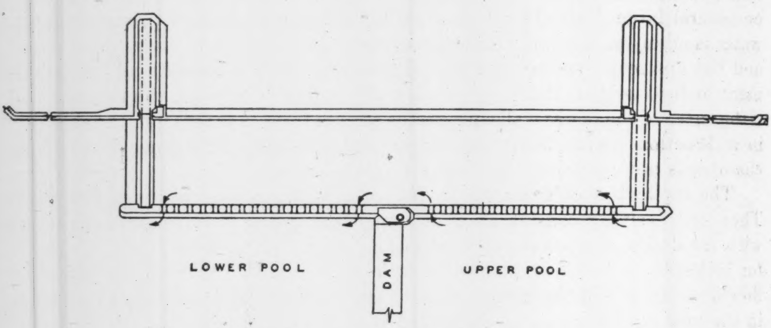

FIG. 2.—VALVES IN RIVER WALL

Locks on the Monongahela River (where the hydraulic systems consist of short culverts either around or in the upper sill and valves in the lower gates) are of relatively low lift and it is reported that hydraulic operations have not resulted in undue turbulence or in other conditions adverse to traffic.

On the Cumberland River, filling valves are located in culverts under the upper miter sill. Lifts for the 52-ft by 280-ft locks vary from 5.5 ft to 19.5 ft. Filling time is normally from 5 min to 8 min. Turbulence in the locks is not severe but it is general practice in filling to use a one-third valve opening until the lock chamber is one third full, a two-thirds opening until the lock chamber is two thirds full, and a complete opening for the remainder. This practice results in slightly longer filling periods.

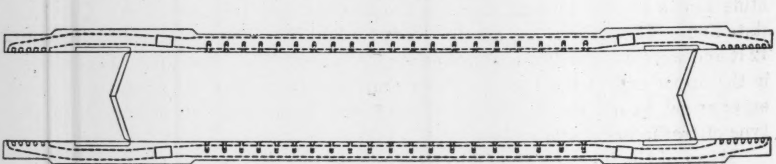

FIG. 3.—"CONVENTIONAL" TYPE

Type C. Filling and Emptying Through Longitudinal Culverts in the Walls or Floor, Connected to the Lock Chamber Through Either Wall Ports or Floor Laterals.—What may be termed the "conventional American type" of hydraulic system (see Fig. 3) consists of wall intakes in the upper approach walls, longitudinal culverts in the lock walls, wall-chamber filling and emptying ports distributed more or less evenly throughout the length of the lock, and wall-discharge ports below the lower lock gates. This type is found in the more recently constructed locks on the Allegheny and the Monongahela rivers in Pennsylvania, the Ohio River, the Kanawha River in West Virginia, the Tennessee River in Tennessee and Kentucky, and the Upper Mississippi River bordering Minnesota, Wisconsin, Iowa, Illinois, and Missouri; the Illinois Waterway; the New York State Barge Canal; the Intracoastal Waterway along the Atlantic and Gulf coasts; and also in the locks in the Welland Canal in Canada. In general, the conventional system has performed satisfactorily for relatively low lifts.

EXPERIENCE IN THE OHIO VALLEY

At the Gallipolis Lock on the Ohio River in Ohio and the three locks on the Kanawha River, which have "conventional" systems, the lifts vary from 24 ft to 28 ft. When filling these lock chambers in the minimum possible time, noticeable turbulence and surge action obtain. In actual practice this action is substantially reduced by incremental valve opening, the filling time being increased possibly 2 min. While emptying these locks in minimum time, pronounced turbulence and surge action are present in the lower approach, particularly just below the lower gates. It is necessary to slow down the discharge by proper manipulation of the valves to prevent the strong currents from breaking mooring lines. Minimum emptying periods for these locks are increased as much as 4 min.

At Lock No. 41 on the Ohio River, where the maximum lift is 37 ft, adverse conditions obtain in both the upper and lower approaches and in the lock chamber. (The dams on the Ohio River are numbered westward[3] beginning at a point west of Pittsburgh, Pa.; dam No. 41 is west of Louisville, Ky.). The upper approach is the Portland Canal, in which, during filling operations, current velocities of approximately 2 ft per sec and troublesome surges exist. These disturbances are sufficiently severe to necessitate special care in mooring the first sections of upbound double-lockage tows securely to avoid the possibility of barges breaking loose and damaging the upper gates. The hydraulic system for the main lock, conventional as to side culverts and lock-wall ports, deviates from the "conventional" in that the filling and emptying systems are entirely separate, the filling being restricted to the upper half of the lock and the emptying to the lower half (see Fig. 4). Excessive turbulence exists during filling operations, transverse currents are strong enough to move tows from side to side of the chamber, and longitudinal surge continues until the lock is about one third full. During emptying operations turbulent conditions of less consequence exist in the lock chamber but current velocities of from 6 ft per sec to 7 ft per sec may obtain in the lower approach. These currents and the accompanying turbulence require tows to be securely moored at a considerable distance below the lock to avoid parting of lines. Minimum filling and emptying times for "main lock" No. 41 are 20 min and 18 min, respectively. To overcome adverse operating conditions, at least partly, the filling and emptying times are ordinarily 27 min and 21 min, respectively, a lengthening of 7 min and 3 min in filling and emptying periods, respectively.

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[3] "Some Notes on the Location and Construction of Locks and Movable Dams on the Ohio River, with Particular Reference to Ohio River Dam No. 18," by William M. Hall, *Transactions*, ASCE, Vol. LXXXVI, 1923, p. 94, Fig. 1.

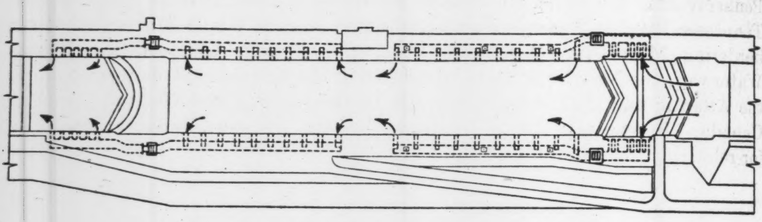

FIG. 4.—LOCK NO. 41, OHIO RIVER (LOUISVILLE, KY.)

Experience at auxiliary lock No. 41, an example of the conventional side-culvert, wall-port system, indicates that it is possible to effect substantial reductions in the time required for filling and emptying lock chambers by better designs of the filling and emptying systems.

EXPERIENCE IN THE TENNESSEE VALLEY

There are 11 locks in use on the Tennessee River,[3] ten of which operate under high lifts—that is, from 39 ft to 80 ft. All these locks have conventional type filling and emptying systems, the more modern locks offering systems modified as to spacing, size, and shape of ports in accordance with the results of model testing. Model studies were used extensively in determining the type and action of the lock hydraulic systems, some being performed at the Hydraulic Laboratory of the University of Iowa, Iowa City, and others at the Hydraulic Laboratory of the Tennessee Valley Authority (TVA), Knoxville, Tenn. Velocity tests were made to determine the size and shape of intake and discharge ports, the size of culverts, and the spacing of ports, and to obtain uniform flow into the culverts and through ports. Surge in the lock chamber, filling and emptying times, and open and closed systems—that is, vented or unvented—were investigated.

Turbulence in these locks during filling operations, although quite noticeable, is reported to have little effect on tows, but for small boats it must be reduced by slowing down of the valve-opening period. Turbulence is held to a minimum by controlling the inflow until there is sufficient water over the floor of the chamber to prevent excessive disturbance. Valve-opening time for Fort Loudoun and Watts Bar locks in Tennessee is from 3 min to 4 min and results in relatively satisfactory conditions. At other locks the operator usually delays the valve-opening period by holding the valve in a partly open position for varying periods, depending on the size of the tow. Filling times are lengthened slightly but are considered satisfactory. There is a slight tendency for tows to move upstream at all locks. Transverse currents are minor. Turbulence below the lower gates is considerable at locks where discharge is into the lower approach, but has little apparent effect on upbound tows, which normally stay well below the gates until the chamber is ready for lockage. It would be troublesome in the event of double lockages, in all probability necessitating a considerable lengthening in emptying time.

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[3] "Allocation of the Tennessee Valley Authority Projects," by Theodore B. Parker, *Transactions*, ASCE, Vol. 108, 1943, p. 175, Fig. 1.

EXPERIENCE ON THE MISSISSIPPI RIVER

Locks on the Upper Mississippi River, except No. 1 (in Minnesota) and No. 19 (in Iowa),[4] are 110 ft by 600 ft, with lifts varying from 5.5 ft to 25 ft. The hydraulic systems consist of the conventional longitudinal side culverts and lock-wall ports as shown in Fig. 3. Wall ports are spaced evenly throughout the length of the lock, those in one wall being directly opposite the corresponding ports in the other wall. Depth over the lock floor is only 13 ft at low-water stage. Filling and emptying times vary from 5 min to 8 min, depending on the head. Variable degrees of turbulence exist in all locks during filling and below the lower gates during emptying operations, depending on the operating head. Both transverse and longitudinal currents exist in the locks, particularly during the first stage of filling, and are more noticeable at the lower third of the chamber. Turbulence and adverse currents are reduced to suit the type of traffic by regulating and extending the time of opening valves. The increase in filling time varies from 2 min to 4 min. During double lockages, the emptying process must be lengthened by slower valve operation to diminish turbulence in the lower approach when unattended tows are moored to the lower guide wall. The increase in emptying time varies from 2 min to 6 min, depending on existing conditions. As modified, hydraulic operations are quite satisfactory.

EXPERIENCE ON THE ILLINOIS WATERWAY

The 7 locks on the Illinois Waterway are each 110 ft by 600 ft. Lifts at low water vary from 10 ft to 40 ft. The conventional hydraulic system is used. Wall ports are directly opposite each other. The average time of filling varies from 12 min to 20 min; and the time of emptying, from 9 min to 16 min. These periods, in general, result in satisfactory operations, with no noticeable turbulence in lock chambers. Transverse and longitudinal currents are negligible. However, turbulence below the lower gates at the Lockport Lock[5] (lift, 40 ft) is such that small boats and light tows are kept at least 300 ft downstream during emptying operations.

EXPERIENCE ON THE INTRACOASTAL WATERWAY

The Harvey Lock connects the Mississippi River with the section of the Intracoastal Waterway that lies west of New Orleans, La. It has a usable lock chamber, 75 ft by 425 ft. The lift varies from 0 ft to 20 ft, depending on the Mississippi River stage. The filling and emptying systems are the conventional types. Time of filling or emptying at maximum stage is 8 min. Model tests were not made in connection with the design. As originally constructed, considerable turbulence resulted in the lock chamber, together with longitudinal surge. These conditions were substantially corrected by reducing the size of the filling ports in the lower end of the chamber. Transverse currents are such that caution must be exercised in filling the lock chamber to prevent the breaking of mooring lines, particularly with smaller tows. The inflow must be regulated to have the stronger flow enter from the side to which the tow is moored. Turbulence in the lower approach is such that boats are not allowed to approach closer than 150 ft until outflow has ceased.

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[4] "Canalization of the Upper Mississippi," by E. L. Daley, *Civil Engineering*, February, 1936, p. 105, Fig. 1.

[5] "Engineering Features of the Illinois Waterway," by Walter M. Smith, *Transactions*, ASCE, Vol. 98, 1933, p. 310, Fig. 1.

EXPERIENCE AT THE WELLAND CANAL LOCKS

The 7 locks in the Welland Canal overcome a lift of 325.5 ft, an average of 46.5 ft. Usable dimensions of locks are 80 ft by 820 ft. Filling and emptying systems consist of longitudinal culverts in lock walls and side ports spaced on 30-ft centers near the bottom of the walls. The depth over the sills is 30 ft. The hydraulic system as designed permits 8-min filling and emptying. The actual operation period is 15 min, during which valves are opened one fourth, one half, three fourths, and fully, during a total period of about 3 min. This slower rate of filling results in negligible surges and turbulence. Considerable turbulence obtains below the lower gates during normal discharge.

EXPERIENCE AT THE LOCKS AT SAULT STE. MARIE, MICH.

Among the first locks in the United States was one built by the North West Fur Company, in the St. Marys River at Sault Ste. Marie. This lock was only 8.75 ft wide and 38 ft long, with a lift of 9 ft. The importance of traffic between Lake Superior and Lake Huron required enlarged facilities at this location from time to time—first by the State of Michigan in about 1853, and since 1873 by the federal government. The latter has constructed 5 locks at this location: The Weitzel Lock in 1873 to 1881 followed by the Poe Lock in 1887–1896, the Davis Lock in 1908–1914, and the Fourth Lock, 1913–1918.

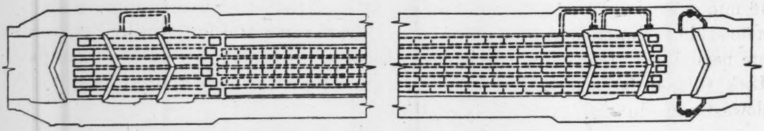

FIG. 5.—FOURTH LOCK, SAULT STE. MARIE, MICH.

The Weitzel Lock was replaced in 1943 by the MacArthur Lock. The filling and emptying systems for the first four of these locks consist of longitudinal culverts under the lock floor with ports in the culvert roof permitting direct upward discharge into the lock chamber (see Fig. 5). Inflow and outflow are controlled by butterfly valves in the culverts, at the place where they pass through the upper and lower sills. Operating experience with the first lock (Weitzel) was satisfactory, and so hydraulic systems similar but increased in size were developed and used in the Poe, Davis, and Fourth locks. Depths on the floor at lower pool are 16 ft, 23 ft, and 23 ft, respectively. These systems have proved to be quite satisfactory in so far as disturbance within the lock chamber is concerned. Considerable disturbance is present in the lower lock entrances as a result of direct discharge from floor ports, but this has little detrimental effect on navigation, as approaching upbound ships stay well below the gates until locks are ready for entrance and erosion is not a problem. However, a serious problem did develop in the upper approach to the locks from the surge, back and forth, in the upper canal. Waves of considerable magnitude were experienced and the channel depth was affected as much as 1.5 ft. Ships were grounded and lines of ships moored to the upper approach walls, awaiting lockage, were snapped because of the longitudinal forces during lock filling. The original times for filling and emptying the large locks were 10 min and 8 min, respectively. Valves opened at a uniform rate resulted in a rate of inflow too high for existing approach conditions—6,500 cu ft per sec. The situation and the prototype tests made to ease it were described in some detail in 1937 by Horace M. Edmands.[6] The tests resulted in operational changes so that valve openings are governed by a filling curve having equal and gradual rates of acceleration and retardation of flow, so that there is a maximum rate of inflow of about 5,600 cu ft per sec, and so that time of filling is lengthened 3 min—that is, to 13 min.

EXPERIENCE ON THE COLUMBIA RIVER

The Bonneville Ship Lock in Oregon, opened to navigation in 1938, was designed with the aid of hydraulic models, which were used where practicable to test the basic design assumptions which, because of lack of precedent for high-lift locks, were necessarily based on analytical and theoretical reasoning.

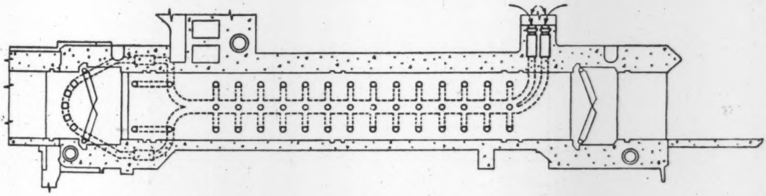

FIG. 6.—BONNEVILLE LOCK, COLUMBIA RIVER, IN OREGON

The lock is 76 ft wide and 500 ft long. Its maximum lift is 66 ft. Its hydraulic system, shown in Fig. 6, consists of a double intake through one lock wall, a single longitudinal culvert beneath the lock floor along the center of the lock, fourteen branches thereto, and forty-two evenly distributed floor ports permitting vertical flow into the lock chamber. The minimum depth on the lock floor is about 26 ft. The discharge passes through the lock-floor system and around the lower gates via loop culverts with a valve chamber in each lock wall, to five vertical discharge ports in the floor of the lower approach. Normal filling and emptying take about 12 min.

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[6] "Lock Filling at St. Marys Falls Canal," by Horace M. Edmands, *The Military Engineer*, January–February, 1937, pp. 62–64.

Model tests indicated that the lock could be filled and emptied best with a slow valve opening at the beginning and with an accelerated movement near completion. Although the Bonneville model did not indicate excessive turbulence, subsequent model studies on a somewhat similar filling system for another lock showed individual boils of strong intensity over each floor port. This "boiling" is duplicated in the prototype to such an extent that the operating force throttles the flow, particularly when small craft are being locked. Inflow is not evenly distributed, being intensified in the lower end of the lock chamber, which causes a considerable longitudinal slope in the water surface. Concentration of discharge just below the lower gates results in a "fountain effect" that raises the water surface as much as 6 ft. A comprehensive prototype test determined the maximum line stress, when mooring the floating plant of 383 tons displacement, to be 10 tons, the maximum occurring at time of greatest rate of inflow.

Experience at the Keokuk (Iowa) Lock, Mississippi River

The 110-ft by 358-ft lock at dam No. 19 utilizes a filling system (see Fig. 7) somewhat similar to that at Bonneville. The maximum lift at this lock is 38.2 ft. The water flow enters through intakes in the upper end of the river wall and discharges vertically into the chamber through a longitudinal culvert in that wall, eight laterals beneath the floor, with seven ports in the roof of each lateral. The depth of water cushion at the beginning of the filling operation is normally about 10 ft. The minimum filling time is 15 min. With such operation, excessive turbulence is experienced, individual boils over each floor port mounting as much as 2 ft or 3 ft above the normal water surface. Lockage of small craft and even small tows is dangerous without judicious filling operations. The ordinary filling period is lengthened to 18 min. This lock discharges through the river wall directly to the river, thus eliminating adverse conditions in the lower approach chargeable to lock discharge.

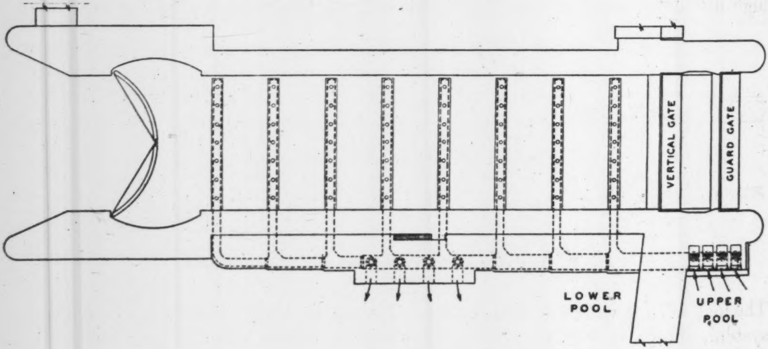

Fig. 7.—Keokuk Lock, Mississippi River, in Iowa

SUMMARY OF EXPERIENCE

The locks reported in this paper are used effectively in the transportation of traffic. However, practically none is operated as designed, particularly in the case of upbound lockages or when sections of tows are moored to the lower guide walls. Through experience, a valve-operation procedure has been developed for each lock which provides satisfactory lockage conditions. However, in doing so, the operation time has been increased from 50% to 100% more than that planned in the design.

DESIGN OBJECTIVES

The objectives of those engaged in lock design have been, and are, to design the lock hydraulic system so as to:

(1) Permit filling or emptying the chamber in the briefest period compatible with unobjectionable turbulence, line stress, crosscurrents, or surges within the lock chambers;

(2) Minimize disturbances in lock approaches, thus permitting more efficient use of guide walls for mooring vessels; and

(3) Obtain an even distribution of inflow and discharge across the lock entrances to reduce the concentration of velocities, surges, eddies, and erosion, giving due consideration to the economics of the solution.

THE HYDRAULIC LABORATORY "ENTERS THE PICTURE"

The complexity of a lock hydraulic system is such that it does not lend itself well to rational analyses; and, in consequence, the hydraulic laboratory has played an important part in determining how to improve the efficiency of lock-filling and lock-emptying systems. In recent years models of lock hydraulic systems have been used extensively to predict operational characteristics of the prototype, to check the design before construction, and to study operating conditions of existing structures. It has been found practicable to reproduce, in the model, effects and conditions comparable to those that will be experienced in the prototype. The practicability of models was demonstrated in 1923 and 1924, in the River Hydraulics Laboratory at Karlsruhe, Germany, by identical tests on the model with a scale ratio of 1:50, of a group of locks at Steenenhoek, Netherlands, on the Merwede Canal, and on the prototype. Comparison of observations of water level, discharge, and slopes determined the mean deviation for sixteen sets of observations to be but 3.5%.

In the United States similar tests were run on Wheeler Lock in Alabama, the first project completed by the TVA. The model was constructed in 1936 to a scale ratio of 1:20 with exact geometric similarity to the prototype that was then in operation. Observations were made on both the prototype and model of the (*a*) water level in the lock chamber, (*b*) velocities and discharges from the lock-chamber ports, (*c*) flow conditions in the intake ports, and (*d*) pressures in one of the culverts. In addition, photographic records were obtained of the turbulence in the lock chamber and in the discharge section. Comparison of results shows a reliable agreement between the model and prototype in filling time, slope of the water surface in the lock chamber, and maximum values of rise and fall of the water surface. The emptying time for the prototype was 8% less than that for the model, which reflects a higher hydraulic efficiency in the prototype system. The model did show that the behavior of the prototype could be predicted with reliability.

Additional tests were obtained by the Division Engineer of the Ohio River Division on the Guntersville (in Alabama) and Pickwick (in Tennessee) locks of the TVA for verification of model tests to determine the reliability of the model as a unit or of its individual parts. The results of such tests, together with observations and the limited prototype tests made on the MacArthur Lock at the "Soo," have indicated such excellent agreement between model and prototype that it is felt the reliability of the hydraulic laboratory work in the study of the more complex phases of lock hydraulic systems has been established.

Although hydraulic tests have been made on lock-emptying and lock-filling systems in many hydraulic laboratories in the United States, such testing (pertaining largely to the side-culvert or floor-culvert systems and their modifications) has been more or less concentrated with the laboratories of the federal government at Knoxville, Bonneville, Iowa City, Vicksburg, Miss., and Balboa Heights, Panama Canal Zone. Some of the projects tested at these laboratories, since 1938, are as follows:

	Name of structure tested	Location
Knoxville (TVA).	Guntersville Lock	Tennessee River in Alabama
	Gilbertsville Lock	Tennessee River in Kentucky
	Watts Bar Lock	Tennessee River in Tennessee
Bonneville (Corps of Engineers)....	Willamette Falls Lock	Willamette River in Oregon
	Dam 1, Ice Harbor Site	Lower Snake River in Washington
	McNary Lock	Columbia River in Oregon
Vicksburg.......	Algiers Lock	Intracoastal Waterway west of New Orleans
	New Jersey Ship Canal-Locks	Intracoastal Waterway
Iowa City.......	Lock No. 2	Mississippi River in Minnesota
	Lock No. 19	Mississippi River in Iowa
	Lock No. 27	Mississippi River in Illinois
	New Cumberland Locks	Ohio River in Ohio
	MacArthur Locks	St. Marys Falls Canal in Michigan
Balboa Heights..	Third Locks	Panama Canal

GENERAL RESULTS OF PROTOTYPE AND LABORATORY TESTS

Experience from prototype and model studies has developed three general criteria as a guide for design, based on currents, turbulence, and conditions in approach channels:

Currents.—Longitudinal currents in lock chambers, created by rapidly accelerated inflow, may be minimized if the first chamber port or bottom lateral is located a distance below the upper miter gate equal to from 25% to 30% of the lock-chamber length. Operation of the filling valves to eliminate a sudden inrush of water at the beginning of the filling period is important. Gradually accelerating and retarding the inflow will reduce ship movement and line stress, but at the expense of filling time. In practice lock operators use this method to reduce line stresses.

Uniform distribution of inflow along the lock walls will do much to eliminate the formation of currents in the lock chamber. Prototype experience and tests in the hydraulic laboratory support theory in indicating that, in the interest of uniform distribution of flow, port or lateral spacing should be varied by decreasing the spacing distances toward the upstream end of the chamber.

Bellmouth entrances and diverging side walls of lock-chamber ports result in decreased chamber inflow velocity and improved entrance flow conditions without decreasing the port-discharge capacity. Transverse currents can be minimized in a lock chamber fed through a single side culvert by use of properly designed floor laterals.

Turbulence.—Turbulence in the lock chamber is largely the result of improper conditions for the most efficient dissipation of the energy contained in the inflow. Actual operation of prototypes shows that an important feature affecting turbulence and oscillating surges is the time required for valve opening. If the valves are opened slowly enough, no turbulence will develop and surging will be reduced. Of course, a longer filling time results. A bottom lateral system will reduce turbulence under a quick valve opening and thus permit a shorter filling time.

Prototype tests indicate that the safe rate of inflow for side-filling systems is a function of the water-cushion depths in the lock chamber. The greater the depth the larger will be the rate of inflow that may be accommodated without causing undue turbulence and surges.

A decided improvement in side-port-filling operations can be realized by staggering the location of ports in one wall with respect to those in the opposite wall. This action permits adjacent jets to pass each other and to travel the entire width of the lock before being deflected, rather than to impinge on each other in the center of the lock, thus effecting greater diffusion and dissipation than are possible when ports are located on coincident center lines. Transverse flows are less pronounced and the residual surface flow is toward the center of the lock.

The entrainment of air in the inflow causes severe turbulence in the lock chamber as the air is expelled from the system in large bubbles under pressure. The culverts should preferably be placed well below the elevation of the lower pool, particularly for high-lift locks, so that the hydraulic gradient is above the roof of the culvert, to reduce the amount of air entrained during the valve-opening period and to minimize the possibility of valve chatter. In addition, the tendency for air to be entrained in a system equipped with Tainter valves can be minimized by facing the skin plate downstream from the trunnions and sealing the downstream stop-log recess.

A larger culvert area (about 20% larger) is required in a bottom-filling system to obtain the same operation schedule as from an efficient side-culvert system.

Turbulence in a lock chamber, particularly where the depth of water over the floor of the lock is small or where locks are subject to relatively high lifts, can be reduced by the use of floor laterals discharging horizontally through side ports. The floor laterals should be placed with the upstream lateral about 25% or 30% of the lock-chamber length downstream from the upper lock gate. The laterals should be spaced to provide a distance between them approximately equal to the width of a lateral. Side ports in any lateral should be so spaced that they alternate with those in adjacent laterals to permit discharges to impinge on opposite walls rather than on each other.

In the more effective dispersion of the jets, the bottom-filling system has an advantage over the side-port system for locks having shallow depth over the floor. However, turbulence caused by vertical jets in a bottom-filling system augments the disturbance caused by any uneven distribution of discharge. To minimize this adverse condition, the jets should be horizontal.

Approach Channels.—In addition to turbulence and surges within the lock chamber, the characteristics of the filling and emptying systems will have a predominant influence on conditions in the approach channels. The effects are particularly aggravated where the size of the approach channels is limited, as is the case where the lock is located in a canal. Pondage in the vicinity of the upper lock entrance has been found to be a practical method of reducing the surges within the canal.

Below the lock, the discharge from the emptying system during lockage may necessitate an increase in the length of the lower guide wall because of the increase in hazards to traffic, particularly during double lockages. A diffuser in the lower entrance of a lock, immediately below the downstream miter gates, provides uniform distribution of the discharge from the lock across the lower approach, localizes the turbulence in a small area near the gates, and consequently permits more efficient use of the lower guide wall than if the water is discharged in the conventional manner through horizontal ports in the lower approach walls.

THE MACARTHUR LOCK, ST. MARYS RIVER, SAULT STE. MARIE

General.—To present an example of recent lock design in the United States, the MacArthur Lock will be discussed in some detail. This design was based on a general mathematical approach, available information, and experimental model data on similar systems; and refinements were made in accordance with the results of model tests in the lock-testing laboratory of the Corps of Engineers (United States Army) at Iowa City. The model simulated, to a scale of 1:25, the complete lock and parts of the upper and lower approach channels. In prototype dimensions the lock is 80 ft wide by 800 ft long between service gates; it has a lift of 22 ft and a depth of water cushion over the lock floor of 31 ft. The construction of the prototype was begun in April, 1942, and was completed in July, 1943. Since changes in design were anticipated, the model was constructed so that the parts would be accessible and as easily removable as practicable.

The hydraulic system adopted (see Fig. 8) consists essentially of a longitudinal culvert in each side wall of the lock, terminating with intake and discharge manifolds in the upper and lower pools, respectively; transverse lateral culverts beneath the floor of the lock alternately connected to the wall culverts; and vertical ports through the cover slab of each lateral. Flow of water through the system from the upper pool into the lock chamber and from the lock chamber into the lower pool in the filling operation and the emptying operation, respectively, is controlled by interposed valves of the reversed Taintergate type.

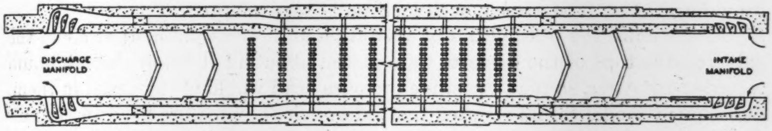

Fig. 8.—MacArthur Lock, Sault Ste. Marie, Mich.

Design Criteria.—The basic criteria that governed the design were

(*a*) The filling and emptying system was to be of sufficient capacity to fill the lock in 8 min and to empty it in 7 min at a lift of 22 ft. In actual operation the maximum rate of inflow was not to exceed about 5,400 cu ft per sec to prevent adverse surging in the upper approach.

(*b*) Disturbances in the lock chamber which would tend to increase line stresses were to be kept to a minimum.

(*c*) Any transverse slopes existing in the lock chamber should tend to hold vessels against the wall of the lock rather than to move them to the center of the lock.

These criteria were established after an investigation of the lock-approach conditions and the characteristics of the waterway traffic. The difficulties experienced in the past in the approach channel (see previous observations on the "Soo" locks) dictated that every precaution should be taken to insure that conditions would not be aggravated.

Conduits.—The approximate size of the lock culverts required to fill or to empty the lock was determined from the relation between the volume of the lock chamber, the lift, and the contemplated efficiency of the lock-filling system, with cognizance taken of the maximum permissible filling rate. This relation has been expressed as

$$A_C = \frac{2\,A_L\,\sqrt{h}}{C\,t\,\sqrt{2\,g}}$$

in which A_C is the total culvert area, in square feet; A_L is the area of lock chamber, in square feet; h is the lift, in feet; C is a coefficient—a measure of the efficiency of the filling and emptying systems; t is the time required to fill or empty the lock, in seconds; and g is the acceleration of gravity, in feet per second per second. The foregoing equation does not take into consideration either the reduced effective culvert area available during the valve-operating period or the dynamic head operating on the system during a large part of each filling and emptying operation. Allowances were made for these factors. It was determined that, to satisfy the emptying-time and the filling-time criteria, a total culvert area of approximately 300 sq ft would be required. Although friction losses in circular culverts are somewhat less than those in rectangular culverts of equal area and roughness, in the case of lock hydraulic systems rectangular culverts permit the construction of more efficient entrance and exit sections at intake, lock chamber, and discharge ports; and, since the energy lost by turbulence at inefficient entrances and exits would greatly exceed that lost by friction, culverts, rectangular in shape and 11 ft by 14 ft in size, providing 154 sq ft of area in each lock wall, were adopted.

To minimize the possibilities of cavitation and of entrainment of air in the culverts, the tops of the culverts were placed about 14 ft below the minimum recorded tailwater, so that a positive pressure head would always exist in them. Subsequent model tests indicated that, although the pressure head in the culvert near the Tainter valve would drop somewhat below the tailwater elevation for a short period immediately after the start of the filling and emptying operations, the selected elevation would assure that positive pressure would always exist in the culverts.

Intake Ports.—Intake ports were patterned after a streamlined intake-port manifold developed as a result of previous model testing (see Fig. 8). This type of intake-port manifold had been incorporated in 2 prototype locks constructed on the Tennessee River and had performed satisfactorily. To obtain an equal distribution of inflow through each of the ports, it is necessary that the port throats be progressively decreased in width, proceeding downstream from the first port. The ratio of the total throat area of the intake ports to the total culvert area for the MacArthur Lock is 1.39.

Laterals and Lock-Chamber Ports.—Because a greater degree of turbulence is likely to exist in the laterals than in the culverts and because, as a result, a smaller proportion of the lateral area is effective than in the case of culverts, it has been found desirable to provide a total lateral area somewhat greater then the total culvert area. Consequently, twenty-four laterals, rectangular in shape and 4 ft by 4 ft in size, alternately connected to opposite culverts by an elbow-shaped transition section, were adopted, providing a total lateral area of 384 sq ft. Tests later conducted on the model, during which the amount of total lateral area was varied, indicated that the basic design in this respect was satisfactory. The ratio of lateral area to culvert area is 1.25.

There were few design data on the spacing of the laterals along the lock chamber. However, experiments conducted at the Hydraulic Laboratory of the Corps of Engineers, at Iowa City, on a model of a side-port filling and emptying system indicated (and checked the theory) that during the filling operation the pressure gradient in the culvert would have an upward slope in the direction of flow. Consequently, a greater pressure head would exist at ports or laterals near the downstream end of the culvert and they would carry an appreciably larger volume of flow per unit of area than would ports or laterals near the upstream end of the culvert. To compensate for this effect and to attain more even longitudinal distribution of flow within the lock chamber, the distances between the individual laterals were varied, laterals near the upstream end of the lock being more closely spaced than those near the downstream end.

The actual spacing of the laterals in the MacArthur Lock model was based on the data obtained from the aforementioned tests on a side-port system. Although the desirability of this type of spacing was confirmed by tests later conducted on the model, the unbalance of flow distribution proved to be somewhat less for the floor-lateral system than for the side-port system previously tested, and pertinent adjustments based on quantitative tests in the model were incorporated in the final prototype design. The laterals adopted are uniform in size throughout their length and are alternately connected at one end only to culverts in the lock walls.

To reduce the velocity of jets entering the lock chamber and to minimize the turbulence created in the lock chamber during the filling operation, it was considered desirable to provide a total area of lock-chamber ports considerably larger than the total lateral area. In the preliminary design, from which the model was built, twelve rectangular ports, 1 ft by 1.83 ft, were provided for each lateral. This size gave a ratio of port area (lock chamber) to lateral area of 1.37. However, experiments in the model indicated that the former area should be increased and that the shape of the ports should be revised somewhat. After tests were made of several sizes, shapes, and spacings of lock-chamber ports, in conjunction with tests on both tapered and uniform laterals and on laterals connected to the culverts at both ends, as well as at alternate ends, a design for lock-chamber ports was adopted providing eleven rectangular ports, 9.5 in. by 4 ft, for each lateral. The ports are equally spaced along the lateral, and well streamlined, to reduce turbulence to a minimum. The arrangement gives a desirable transverse distribution of flow in the lock with a minimum of turbulence and surface disturbance resulting from jet action.

The ratio of port area (in lock chambers) to lateral area is 2.76.

Discharge Ports.—The discharge-port manifolds for the MacArthur Lock were first patterned after a type of manifold developed for Pickwick Lock on the Tennessee River. This type of manifold tapered toward its downstream end to counteract a tendency of the pressure gradient to rise in the direction of flow. Such a tapered discharge manifold was constructed in the model. However, the model tests revealed that the emptying system as originally designed would not be capable of emptying the lock in the desired time and that a considerable improvement in emptying efficiency was necessary. Consequently, it was decided to expand or flare the discharge manifold in the direction of flow.

However, the expansion of the manifold accentuated the previously described tendency of the pressure gradient in the manifold to rise in the direction of flow, and it was necessary to extend the discharge-port piers into the culvert and to shape them so that they would act as flow deflectors or vanes, to obtain an approximately equal distribution of flow through the discharge ports. A discharge manifold and ports, so modified, were installed in the model and, after some minor modifications, were found to operate very satisfactorily.

The final design (see Fig. 8) has four discharge ports for each culvert, providing a ratio of total port-throat area to total culvert area of 1.55.

Results.—The model tests resulted not only in obtaining better lockage and channel-approach conditions by important reductions in surface currents and turbulence, but also in increasing the over-all efficiency of the system. Valve schedules were developed for the recommended design to fill and empty the lock in varying periods to fit contemplated conditions, and anticipated prototype line stresses were determined to be not greater than about 10 tons.

In operation, the prototype has behaved quite like the model. Surface currents in the prototype appear to be remarkably similar to those observed in the model and are extremely minor. The lock coefficients for final design were found in the model to be 0.71 for filling and 0.61 for emptying. Preliminary prototype tests indicate these coefficients to be 0.72 and 0.67, respectively.

THE LOCKS PROPOSED FOR THE "CHAIN OF ROCKS" REACH, MISSISSIPPI RIVER

In a canal planned to by-pass the troublesome "Chain of Rocks" reach in the Mississippi River just upstream from St. Louis, Mo., there are 2 parallel locks. The design of the adopted hydraulic systems (see Fig. 9) was determined by local conditions and the results of intensive model experiments.

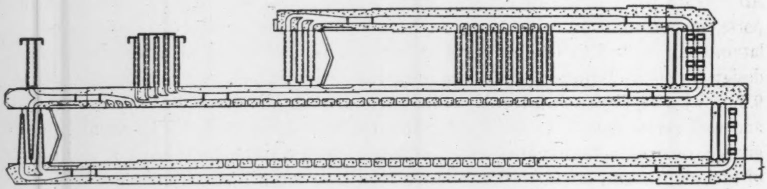

FIG. 9.—CHAIN OF ROCKS LOCKS NEAR ST. LOUIS, MO.

The normal lift is about 12 ft; the maximum, 21 ft. For the main lock, 110 ft by 1,200 ft, floor intakes, longitudinal wall culverts, and wall ports are used. The floor intakes upstream from the upper sill provide against entrainment of air. Wall ports are used because the water cushion over the floor of the lock will be not less than 25 ft. The ports are streamlined and have rounded entrances and exits to improve flow conditions. They are grouped in the center section of the length of the chamber, the ports in one wall being staggered with respect to those in the opposite wall to permit adjacent jets to pass each other—all according to the results of the model testing. A vertical-lift gate was used as the upper lock gate because of the necessity of passing ice from the 7-mile upper canal, because of the height of the lock walls above the upper sill (58 ft), and because of the favorable elevation of the rock foundation. This type of gate is advantageous in filling the lock. Flow over the gate will be added to, and synchronized with, the flow through the culverts, the filling time for the conditions under a 21-ft lift being thereby shortened 2 min, to a period of 7.5 min. Discharge from the main lock is to be controlled by three valves, one in each of the longitudinal culverts and an additional valve in a stub culvert at the lower end of the intermediate wall. The three exits are required to give an emptying period comparable to the filling period; and the flow discharges into the lower pool through lateral diffusers, as indicated, to provide uniform distribution across the canal prism and to obtain minimum turbulence in the lower approach to the auxiliary lock. Details of a lateral diffuser are shown in Fig. 10.

The auxiliary lock will provide a chamber 110 ft by 600 ft. To minimize surge and longitudinal currents and crosscurrents, it will be filled and emptied by a single wall culvert, fed through a floor intake similar to that provided for the main lock and by a group of floor laterals in the center part of the length of the chamber. The laterals will discharge horizontally through side ports to minimize turbulence further. Discharge will be controlled by a single valve in the side culvert, the flow emptying below the lower miter gates through a lateral diffuser to minimize turbulence and to distribute the flow uniformly across the lower approach.

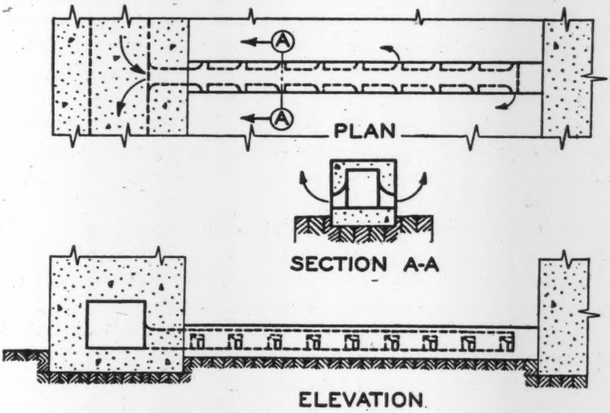

FIG. 10.—LATERAL DIFFUSER, CHAIN OF ROCKS LOCKS, NEAR ST. LOUIS, MO.

Very satisfactory operating conditions were obtained in the model as finally modified. Similar, or even better, operating conditions are confidently expected in the prototype.

PRESENT AND FUTURE USE OF MODELS

Entrance and discharge conditions of the various components of the lock hydraulic system have been greatly improved through the use of lock models. As a result, lock-filling time and lock-emptying time have been reduced in spite of a general increase in the size and the lift of locks, and significant improvement has been obtained in hydraulic conditions within and without the lock chamber. The following data serve to illustrate the use of models in solving some perplexing problems:

(*a*) With a 1:25 scale model, the Bonneville Hydraulic Laboratory has obtained test data on several types of lock-filling and lock-emptying systems to permit the selection of the most satisfactory hydraulic system for a lock 675 ft long and 86 ft wide, with an average head of 85 ft. This lock was designed as a part of the McNary Power and Navigation Project in Umatilla, Ore. The results of the first series of tests indicate that a system incorporating bottom longitudinal culverts is not as satisfactory as one consisting of longitudinal side culverts with fourteen lateral culverts in the lock-chamber floor. Maximum line stresses of 14 tons were developed using the longitudinal system while operating with a 92-ft head and a 4-min valve-opening period as compared to a maximum stress of 2.5 tons with the lateral system.

(*b*) Using a 1:20 scale model, the Waterways Experiment Station at Vicksburg has investigated the feasibility of filling a lock 800 ft long, 75 ft wide, with a minimum depth of 12 ft over the sills, and subject to reverse heads up to a maximum of 18.5 ft, by sector gates. The lock, known as "Algiers Lock," was designed to connect the Intracoastal Waterway from the west with the Mississippi River at New Orleans. Lock-filling tests have been conducted to determine the best distribution of flow between the gate leaves and through the gate recesses, and the optimum gate-travel characteristics. Preliminary test data indicate that it is desirable to pass as large a percentage of the flow as is possible through the gate recesses. It was also determined that, to prevent undue surges in the lock chamber, the initial operation of the gates must be very slow. By use of a variable gate-speed travel to open the gates 5 ft in 5 min, it was found practicable to fill the lock in about 7 min without creating undue turbulence.

(*c*) With a 1:25 scale model, the Hydraulic Laboratory of the Corps of Engineers at Iowa City has investigated the over-all and component parts of proposed hydraulic systems for a main lock 1,200 ft long and 110 ft wide with a difference in pool elevations of 22.6 ft, to develop improvements in the originally selected systems. These locks are planned as a part of a replacement and improvement program on the Ohio River near New Cumberland, Ohio. Longitudinal culverts in the river and middle walls connected to intake manifolds in the sides of the approach walls, to lock-chamber ports in the sides of the walls, and to outlet manifolds discharging into the tail bay below the lower miter gates are proposed for the main lock. The auxiliary lock is filled and emptied by an intake manifold, a longitudinal culvert, and a discharge manifold in the land wall, the longitudinal culvert being connected to laterals with side ports in the chamber floor. Tests on the main lock have shown that modifying the lock-chamber ports so that the jets from the ports strike the floor of the lock chamber at the base of the opposite wall, instead of at the midpoint of the floor, results in a reduction in hawser stresses by an average of 50%. With an assumed 2-min time of culvert-valve operation, satisfactory filling in about 8 min and emptying in about 8.4 min were obtained. It was also found that reducing the inside area of the laterals in the auxiliary lock chamber, in steps from the first port to the end, would result in a more equal distribution of flow during filling operations.

(*d*) In addition to the foregoing model studies, a study is currently being conducted at the St. Anthony Falls Hydraulic Laboratory of the University of Minnesota in Minneapolis, by the St. Paul (Minn.) District of the Corps of Engineers in connection with a proposed extension of the Upper Mississippi River canalization project above St. Anthony Falls. The project will incorporate two locks 56 ft wide and 400 ft long. The upper lock will have a 50-ft lift and the lower lock, a short distance downstream, will have a lift of 25 ft. A model scale of 1:22.4 is being used. The purposes of the investigation are to determine the hydraulic characteristics of the Tainter gates (upper lock gates) and chambers in the proposed locks when operated as spillways, to study the filling conditions in the chambers when used for lockages, and to develop a one-culvert, bottom-filling system for the lower lock.

(*e*) The need for extensive generalized information on lock-filling and lock-emptying systems has long been recognized. The St. Paul District has been delegated to make generalized model tests on high-lift locks for the Corps of Engineers. It is contemplated that the program will furnish data on the operating characteristics of locks for heads as high as 100 ft. These data will include the limiting rates of rise versus cushion depth for various types of lock-filling systems, the loss of head for various units of the hydraulic system, methods of minimizing vortices at the intake section in the upper bay, surge and pressure tests on culvert Tainter valves, dimensions and spacing of lock-chamber laterals with side-wall ports, proportionment and expansion of culvert flow in the discharge section, and the formation of waves and surges in the lock chamber. Supplementing the foregoing program, prototype tests are contemplated to obtain stage-time data from filling and emptying operations, head-loss data for computing coefficients for component parts; and, for the over-all systems, data on distribution of flow, turbulence, surges, waves, vortices, and other disturbances, measurements of forces acting on typical craft in lockage, and data on the cushion depths required to obtain satisfactory operating conditions for various rates of filling. It is anticipated that the gathering and analyzing of these data will require several years.

Concluding Remarks

Unless obviously adverse operating conditions can be tolerated, or unless economic studies indicate a cost prohibitively greater than that of a less desirable system, it is considered that the hydraulic system offering the maximum efficiency consistent with economy and safe and relatively quiet lockage should be adopted.

Experience proves that lock operators and navigators will soon forget the economic considerations that may have determined the selection of a lock hydraulic system, but will evaluate the system, and those who designed it, from the manner in which the lock operates.

Lock hydraulic systems should be designed so that they can be operated as planned without creating intolerable lockage conditions in the lock chamber or intolerable mooring conditions in the upper and lower approaches. It is believed to be both possible and practicable to design systems that will provide fast and safe operation under various pertinent conditions. The side-culvert and the side-port system will fulfil the requirements for safe and fast lock filling where the initial water cushion in the chamber is adequate. On the other hand, for high-lift, shallow-draft locks, where fast operation is required, the bottom-lateral system with side ports offers a solution to the problem. In the event that turbulence and currents in the lower approach, created by lock discharge, need correction to permit safe mooring of vessels not too far from the lower lock gates, a diffuser is recommended. Lock models can be used to solve difficult hydraulic problems efficiently and practically. Much is yet to be done in this field and the continued use of models is strongly recommended.

ACKNOWLEDGMENTS

The writer wishes to acknowledge his deep appreciation of the fine spirit of cooperation shown by all who contributed their time and effort to the preparation of this paper. Many of the findings recorded herein were obtained through model and prototype tests at, or under the direction of, personnel connected with the Hydraulic Laboratory of the Corps of Engineers, at Iowa City. Other results were obtained from tests made at other laboratories, such as those at Bonneville and Panama and at the TVA Hydraulic Laboratory, and from experiences of, and information given by, those cooperating with the writer in furnishing basic data for this paper.

STRUCTURAL ELEMENTS FOR EMERGENCY CLOSURES AND UNWATERING OPERATIONS

By C. E. Blee,[7] M. ASCE

Introduction

A review of past and present practice indicates that the provisions with which this paper is concerned, made at navigation locks, fall into one of the following three classifications:

1. Provisions for unwatering the lock and its operating equipment for ordinary routine inspection and maintenance, in which case closure can be made under conditions of balanced head;
2. Provisions for safeguarding, or preventing damage to, lock operating gates and other equipment; or
3. Provisions for making an emergency closure of the lock under an unrestricted flow of water resulting from damage of disastrous proportions to the lock gates.

These classifications are not sharply defined and in some cases facilities may overlap into two or more groups. The extent to which any or all of these provisions are employed at a given location is dependent upon the importance of the waterway, the natural site conditions, or similar governing limitations, although there may be fundamental differences in concept of the provisions necessary for reliable operation.

As may well be imagined, practice has varied widely over the years. During the early development period, lockage facilities were constructed principally for the purpose of connecting large bodies of navigable water of natural origin, such as lakes. The works at Sault Ste. Marie are a notable example of this type of development. The vital importance of this waterway had long been recognized and the wisdom of using every device to assure reliability of operation has never been questioned. At the time of initial development, much thought was evidently given to the problem of devising methods for stopping the unrestricted flow that would result from direct communication of water levels—that is, for emergency operations. This attitude has persisted throughout the development, and over the years several types of structures for accomplishing this purpose have been designed and built. Apparently there has been no occasion to use them for the designed purpose on the United States side; and, on the Canadian side, there has been only one such occasion.

Again in the design of the new Welland Canal connecting Lake Erie and Lake Ontario, also a waterway of great importance, the same problem apparently received much thought. Here, however, consideration led to placing the emphasis on protective measures against damage, rather than to controlling the damage after it had occurred. As a result, reliability in operation is secured through the use of guard gates or protective fenders, or both, at vulnerable locations in the waterway. As an added feature for maintenance purposes only, wherever necessary an additional set of gates of timber construction is provided to permit the unwatering of the locks.

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[7] Chf. Engr., TVA, Knoxville, Tenn.

With the intensive, although somewhat less spectacular, development of rivers as inland waterways (the Ohio, Upper Mississippi, and Tennessee rivers, among others) a fundamentally different attitude has developed. Although attention is still paid to the methods of controlling an unrestricted flow of water through the lock (as reflected by the installation of an emergency dam of the wicket type at Pickwick Landing Dam on the Tennessee River, 206 miles above the mouth), the trend has been toward the use of types of structures suitable only for maintenance purposes, and of the simplest construction. These structures usually take the form of a needle dam or a Poirée dam, although other forms are used to suit special site conditions. They are economical in first cost but are cumbersome and slow in operation. The trend toward their use is the result of a belief, proved by experience, that the probability of accidents leading to the connection of water levels is much less than previously anticipated. Therefore, the emergency gates will seldom be used and the greater expense incurred in operating these somewhat cumbersome types is more than offset by the saving in capital cost.

STRUCTURES INTENDED FOR USE IN THE ROUTINE UNWATERING OF A LOCK

Needle Dams.—The needle dam (see Fig. 11) consists of a horizontal girder spanning the lock opening and seated in recesses in the lock walls above pool level, together with approximately vertical beams, or needles, extending between the horizontal girder and a seat on the lock sill. The horizontal girder is a member of substantial proportions, the dimensions of which increase rapidly with an increase in width of the lock opening. It has usually been built of structural steel, but efforts to reduce its weight to reasonable limits for handling have led to proposals for the use of structural aluminum. The needles may be of wood, of steel shapes or of built-up members—or a combination of each. The trend is toward larger members of built-up steel sections. The needle dam is suitable for locks of the lesser and moderate widths and the relatively shallow depths. For the larger locks the weight and the size of the members generally preclude its use. It is slow in operation and generally requires treatment with some form of sealing medium, such as cinders, to reduce leakage to a reasonable amount.

Poirée Dams.—Poirée dams comprise a series of trestles, usually in the form of A-frames, spaced at regular intervals across the lock opening (see Fig. 12). These trestles support horizontal beams, or "walers," to which are attached the approximately vertical sheathing elements. The trestles are anchored to the sill by devices embedded in the concrete. This type of dam partakes of all the advantages and disadvantages of the needle dam with the added disadvantage that, in operation, it is even more slow and cumbersome and requires the services of a diver to seat the trestles in the anchor block and attach the framing members below the water level. It is used for locks of greater width where the needle type dam is not practicable.

Horizontal Trusses with Skin Plate.—At the Fort Loudoun Lock, 45 miles downstream from Knoxville on the Tennessee River (which represents the latest design for the TVA system), the sectionalized bulkhead to permit unwatering operations is formed of horizontal trusses with skin plate (see Fig. 13).

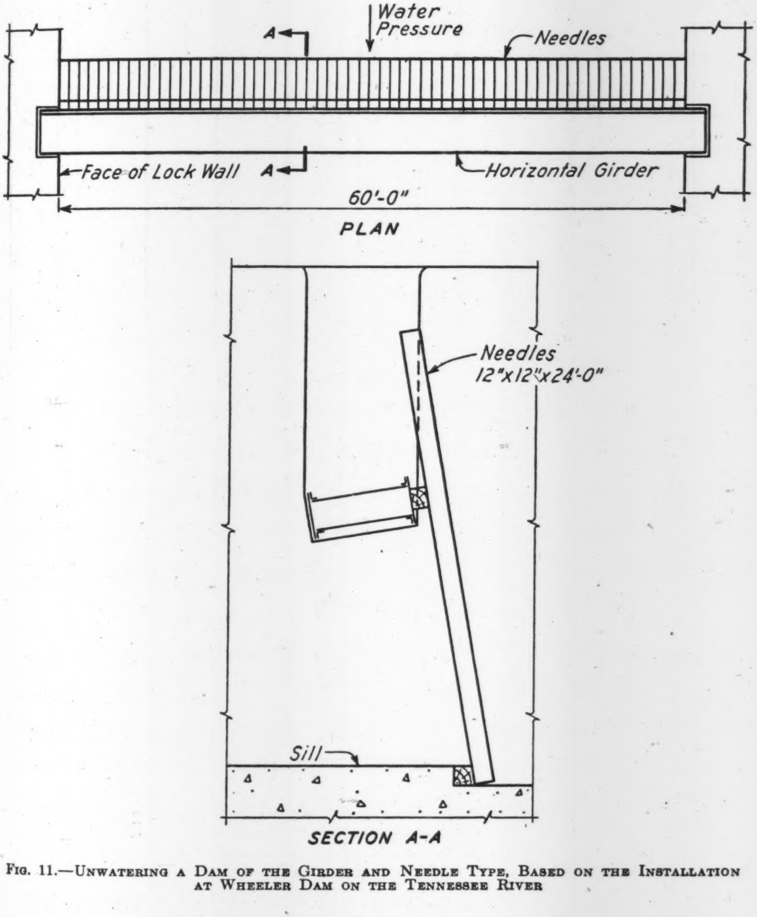
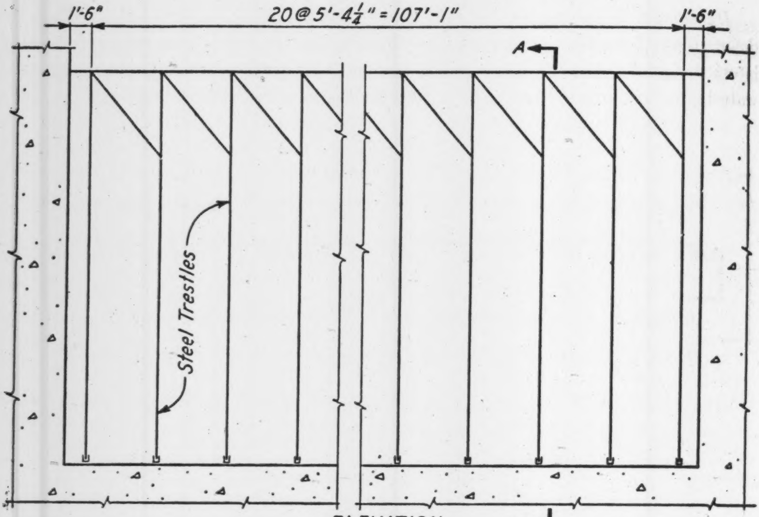
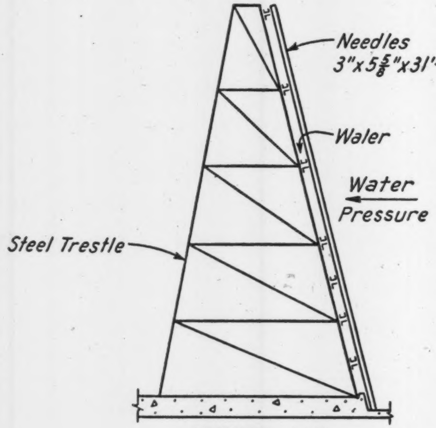

Water Pressure
A
Needles
Face of Lock Wall
A
Horizontal Girder
60'-0"
PLAN
Needles
12"x12"x24'-0"
Sill
SECTION A-A

FIG. 11.—UNWATERING A DAM OF THE GIRDER AND NEEDLE TYPE, BASED ON THE INSTALLATION AT WHEELER DAM ON THE TENNESSEE RIVER

Each section consists of two horizontal trusses, an upstream skin plate, and a vertical bracing between trusses. The clear width of the lock is 60 ft and vertical slots in the lock walls form the seats for the trusses. The horizontal trusses have their upstream chord curved in plan and their downstream chord straight. The skin plate is carried on the upstream chord. When in place,


1'-6"
20@5'-4¼"=107'-1"
1'-6"
A
Steel Trestles
A

ELEVATION
View from within lock looking downstream


Needles
3"x5⅝"x31'-4"
Waler
Water
Pressure
Steel Trestle

SECTION A-A

Fig. 12.—Unwatering a Dam of the Poirée Type, Based on the Lower Installation, Pickwick Landing Dam on the Tennessee River

the trusses are spaced close together near the bottom and wider apart near the top, the spacing varying with the head.

The bulkhead sections are stored on land and when needed can be placed by a floating derrick as no permanent derrick is provided. In accordance with general practice on the TVA system of locks, the same bulkhead can be used at either the upper end or the lower end of the lock and only one bulkhead is provided.

Other Types.—On the inland river waterways the needle and Poirée types of dams are most commonly used for emergency closure of the lock openings, especially where mitering types of service gates are used. At locks where special conditions are encountered, other types of emergency closures have been provided or proposed. Two of these will be mentioned briefly, as follows:

1. A Caisson and Temporary Navigation Gate.—A number of locks on the Ohio River are equipped with service gates of the rolling type—that is, each gate is mounted on a large number of wheels and is opened or closed by rolling it on a straight steel track in and out of a recess built into the landward side of the lock. Originally it was planned to repair or renew the track on the floor of the lock chamber by unwatering the lock with Poirée dams at each end. Concern for the safety of the lock walls and the possibility of leakage under those built on piles caused abandonment of this plan to unwater the locks. Instead, a floating coffer or caisson and a temporary navigation gate are provided. The caisson is designed as a buoyant vessel having a broad base and

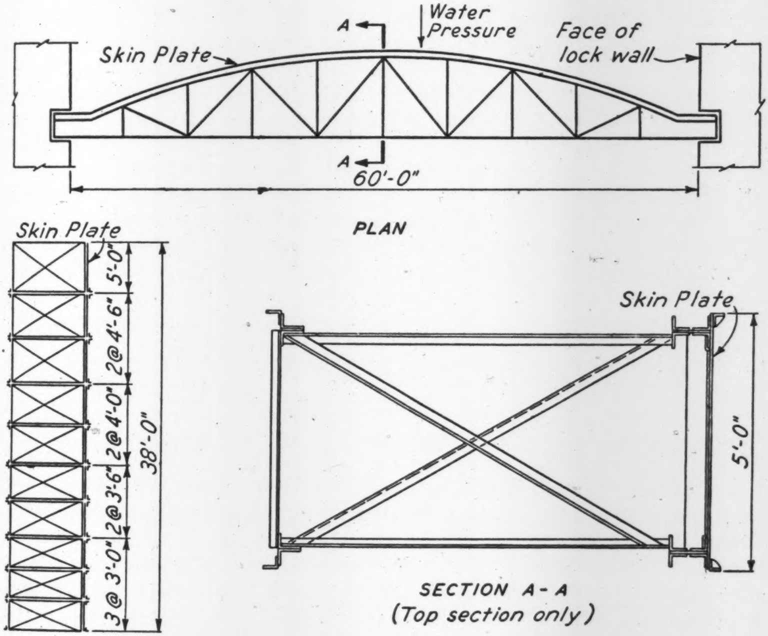


Fig. 13.—Unwatering a Dam of the Horizontal Truss and Skin Plate Type, Based on the Installation at Fort Loudoun Dam on the Tennessee River

a vertical access superstructure. When sunk in place over a gate track, it extends halfway across the lock. The temporary gate is used across the other half of the opening. By pumping out the central part of the caisson, access is provided to that part of the gate track covered by the caisson. The other part can be inspected and repaired by reversing the position of the caisson and temporary gate. When the caisson is on the land side of the lock chamber, it bulkheads the gate recess, thus permitting work on the gate. The caisson and the temporary navigation gate are handled by suitable derrick boats. Navigation through one side of the lock is maintained by handling the temporary gate. When the caisson and temporary navigation gate are required at another lock, they are loaded on a barge for transportation. These units are reported to give very satisfactory service and to reduce, materially, the time required for maintenance.

2. A Caisson.—An application proposed for the Kentucky Lock, 22 miles above the mouth of the Tennessee River (but never built), called for a caisson similar to the type employed in closing graving and dry docks. This caisson was designed as a buoyant vessel of trapezoidal cross section which could be floated into position in front of a gate and then sunk to place by the addition of water ballast. It was designed to span the full lock opening and to be supported at its vertical edges by slots in the lock walls, and along its keel by the miter sill of the lock. It was to be provided with the necessary pumps, valves, piping, and watertight compartments for manipulation of the water ballast. This lock as now operated is unwatered with the aid of a Poirée dam.

STRUCTURES INTENDED TO PREVENT OR MINIMIZE DAMAGE

Guard Gates.—Guard gates are provided for the purpose of presenting a second barrier (the first being the service gate) to the direct communication of water levels through the lock. Generally they are placed at the upstream approach to the lock chamber where the hazard is the greatest, although they have also been installed to protect the lower gates at vulnerable locations. Lower guard gates are required to provide complete protection by this method throughout the locking sequence; but it may be argued that once the vessel is within the lock chamber it will be under control and, hence, not likely to damage the lower gates. Guard gates are usually duplicates of the service gates, operated with them in the same manner and in such sequence that they will bear the brunt of collision with a vessel. Neither the service gates nor the guard gates are operable under unbalanced water pressures, but the possibility of an accident serious enough to damage both sets of gates to an extent which would permit any substantial flow of water through the lock is remote. With this in mind, guard gates may be considered effective and adequate protection against the intercommunication of water levels through the lock chamber. Of course, they are expensive; their cost includes not only that of the gates and operating machinery but also that of the necessary extension of the heavy gate thrust block of the lock walls.

Fenders.—Fenders have been built in many forms. They are placed ahead of the gates with the idea of absorbing the shock of collision with a vessel and arresting its motion before it can cause damage to the gate. The more primitive form consists of nothing more than a chain rigidly attached to the lock walls, hanging just above the water surface when in working position, and dropped to the sill to permit the passage of vessels. Improvements have been made from time to time to increase their resistance to a blow and to facilitate their handling. There are inherent limitations to their use in their lack of resiliency and snubbing action; and, although they do provide a measure of protection, their use on locks of modern construction has largely been discontinued. On the other hand, the idea of the fender has been developed to a high degree in designing the locks of the Welland Canal. Here the fender comprises a 3.5-in. (diameter) wire rope, extending across the lock opening and attached on each wall to snubbing devices which provide a dampening action against the motion of the vessel in cases of collision. The rope is supported on a light bridge structure conforming to the general design of a rolling-lift bascule bridge. At one of the lock-wall faces the rope is separated and provided with matching rope sockets through which a retractable pin is inserted to make the rope continuous when in working position in front of the gates. In case of collision the supporting structure is destroyed, but its replacement amounts to only a minor item. A fender of this type represents a substantial expense. Undoubtedly, it provides a large measure of protection.

Protective measures such as those described are expensive but they are apparently effective in preventing serious damage, judging from the lack of evidence to the contrary. They undoubtedly contribute to a greater sense of security and confidence on the part of both lock operators and navigators. In inverse ratio their existence may lead, in some cases, to a lessening of caution, with unfortunate results to the protective equipment itself, if not to the main operating equipment. Their use will result in slowing the time of transit of a vessel through the lock.

Structures Intended for Use in Closing Off Unrestricted Flow

Emergency Dam of the Bridge Type.—The dam in this design consists of a swing bridge with center of rotation on the lock wall. Under this bridge are mounted girders that rotate in vertical planes about a horizontal axis close to the bottom chords of the bridge trusses. These vertical girders support wickets of some form. In operation the bridge is rotated into position across the lock and seated so that the bottom chord system spans the lock walls. The vertical girders are then rotated to a bearing against the lock sill and the wickets placed. The reactions for the water load are distributed between the lock sill and the bottom chord system of the bridge. This type is suitable for use across openings of moderate width and presumably would function as intended. However, few, if any, of those built have been operated under the design conditions. It is cumbersome in operation and requires considerable space for mounting and storing on the lock wall. Notable examples are the early dams at Sault Ste. Marie.

Emergency Dam of the Lift Type.—This dam consists of bulkhead sections, built as trusses with skin plates, or as box girders, which span the lock entrance and, when placed one upon another, form a continuous dam from the sill to the water surface. Rollers are mounted on the ends of the bulkhead and the rollers bear against tracks embedded in slots in the side walls of the lock. The bulkheads are thus, in effect, sections of wheeled gates and can be operated under unbalanced water pressures (see Fig. 14). The bulkheads are handled by derricks mounted on the lock walls. This dam is less complicated in design than the bridge type and less cumbersome in operation. Storage facilities for the girders and mounting of the derricks, however, do require substantial space allowances on the lock walls. Notable examples are the later dams at Sault Ste. Marie and the closures at Bonneville Dam on the Columbia River.

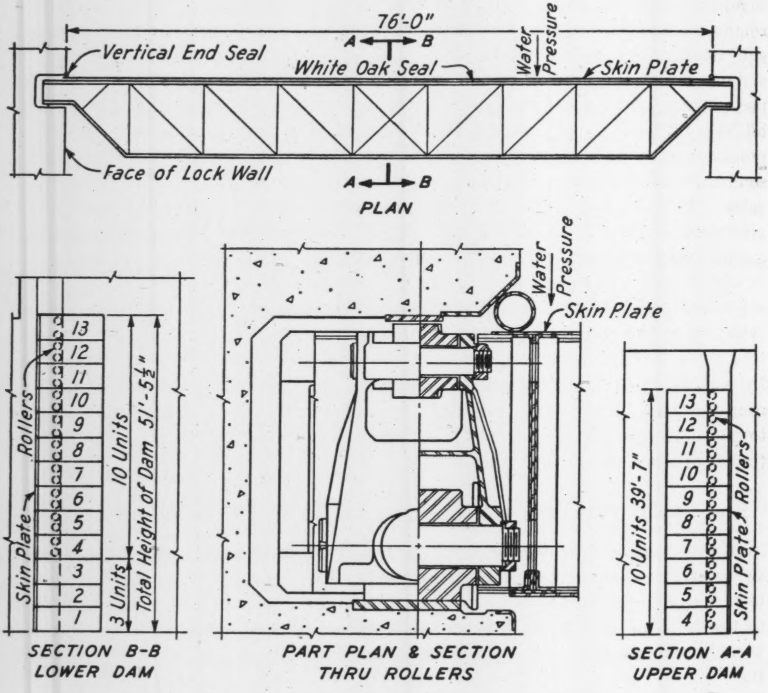
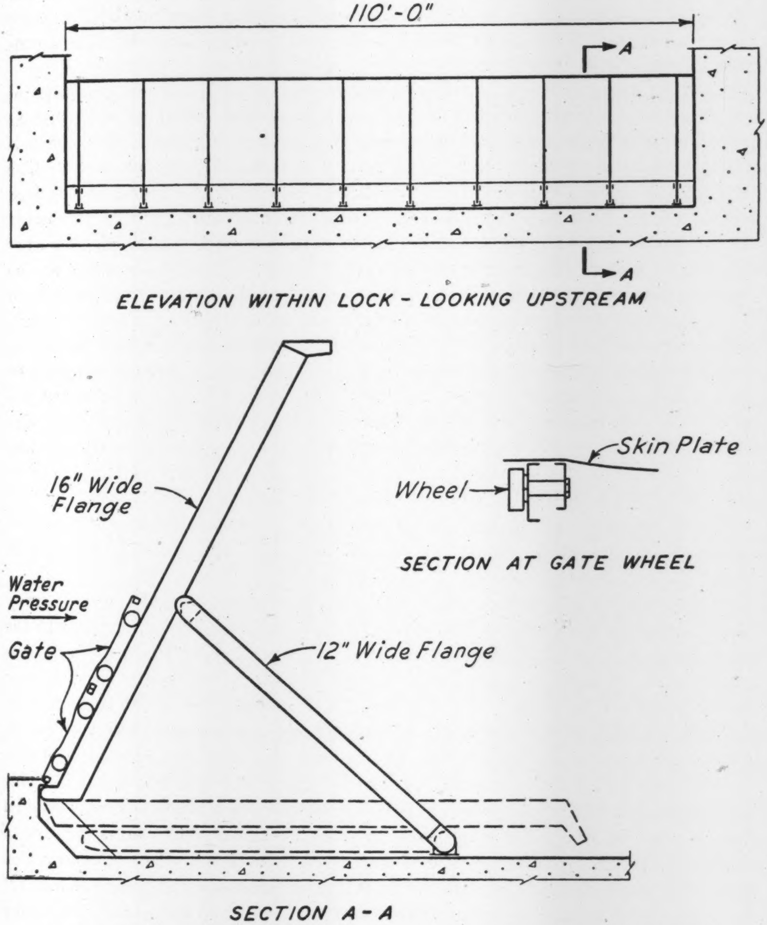

76'-0"
A
B
Water Pressure
Vertical End Seal
White Oak Seal
Skin Plate
Face of Lock Wall
A
B
PLAN
Water Pressure
Skin Plate
13
12
11
10
9
8
7
6
5
4
3
2
1
Rollers
Skin Plate
10 Units
3 Units
Total Height of Dam 51'-5½"
10 Units 39'-7"
Rollers
Skin Plate
SECTION B-B
LOWER DAM
PART PLAN & SECTION
THRU ROLLERS
SECTION A-A
UPPER DAM

FIG. 14.—EMERGENCY DAM OF TRUSSED BULKHEADS WITH ROLLERS, BASED ON THE INSTALLATION AT BONNEVILLE DAM ON THE COLUMBIA RIVER

Emergency Dam of the Wicket Type.—This dam consists of a series of wickets stored in slots in the lock sill. When raised, it supports sectionalized gates designed to operate under unbalanced water pressures. In the raised position the wicket forms a modified A-frame (see Fig. 15). In operation the wickets are raised into position by a pull from the lines of the handling derricks exerted through chains permanently attached to the wickets. The derricks also serve to handle and lower the gates. This type is suitable for the operation of locks up to the maximum width, but of rather shallow depths, such as occur on the waterways developed for barge traffic. The use of portable derricks, stored with the gates in recesses of the lock walls, provides clear working spaces on the lock walls and presents a much cleaner appearance. The component parts of this dam are not large, and operation is less cumbersome than for the other types. The wickets are continually immersed in the water when stored and are therefore subjected to severe corrosive action without practicable



ELEVATION WITHIN LOCK - LOOKING UPSTREAM

SECTION AT GATE WHEEL

SECTION A-A

FIG. 15.—EMERGENCY DAM OF THE WICKET TYPE, BASED ON THE UPPER INSTALLATION, PICKWICK LANDING DAM ON THE TENNESSEE RIVER

methods of repair and maintenance. The slots are subject to silting. The degree to which operation is affected is dependent on the progress of these adverse conditions. Care is necessary also in returning the wickets to the storage slots after they have been used to prevent silt or gravel deposits from causing an incompletely lowered wicket to project into the channel clearance. Examples are the emergency dams at Pickwick Landing Dam on the Tennessee River and at the Montgomery Island Locks on the Ohio River, 32 miles downstream from Pittsburgh.

Another type of structure that has been used with success on locks of comparatively narrow width is the adaptation of the radial gate for use as a service lock gate. In this adaption the gate is installed with the axis of the trunnions in a vertical plane near the faces of the lock walls. The gate rotates around this axis from its storage position in a recess in the lock wall to its working position mitered against its matching gate in somewhat the same manner as the conventional miter gate. Because of the circular contour of the face of the gate, concentric with the axis of rotation, there are no significant unbalanced moments of the water loads against the rotation of the gate, thus permitting operation under any conditions of water pressure, either balanced or unbalanced. When both upper and lower lock gates are of this design, there is adequate assurance that the one may act as an emergency gate in case of serious damage to the other. As far as is known, no gates of this type have been used as gates or dams for emergency use only; but it does offer a possibility of a service gate that could act as an emergency gate to close against free flow. It is cited as a possible solution to some of the problems which have long confronted the designers of navigation locks.

Other types of emergency dams have been built; but, since they have appeared only in isolated instances and since the designs probably will not be repeated, they are not included in this paper.

Except where a long approach canal permits the location of the emergency dam far enough upstream to be outside the zone affected by an accident in the immediate vicinity of the lock gates, they are installed on the upper sills immediately upstream from the gates. In this position it is quite possible that the wreckage of the gates or of the vessel will interfere with the operation of the dams. Since this is the region in which accidents are most likely to occur, there arises the question of whether or not even the most mechanically reliable and efficient dam will accomplish its purpose under these circumstances. This consideration and the infrequency of their need have combined to create a belief that, except under special conditions, provision of emergency dams capable of closing under an unrestricted flow of water is not justified. In this connection it may be noted that in the case of the 9 highlift locks on the Tennessee River there has never been an occasion that would have required the use of emergency equipment to close the lock against a condition of unbalanced head.

SUMMARY

In any discussion of this subject, there emerges the ever recurrent question of the control of flow through the lock chamber in the event of direct intercommunication of water levels. During the early development of large-scale lock construction, it apparently was considered to be of far greater importance than it is today. At present, it occupies a prominent place principally in connection with additions to installations where previous practice has hardened into a set pattern, or where the great importance of the waterway justifies the installation of this rather expensive equipment. This long-standing problem was reviewed comprehensively, in 1945, by the late Isaac De Young,[8] M. ASCE.

In more recent times the trend has been toward reliance on precautionary and protective measures on waterways of great importance. On waterways where interruption of traffic would not cause a crippling loss of transportation facilities, the trend is toward the more simple and less expensive types of closure suitable only for ordinary maintenance work. This practice seems fully justified, and may well have been encouraged, by the rarity of accidents of crippling proportions with modern locking facilities.

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[8] "A Plan for a Movable Dam," by Isaac De Young, *Transactions*, ASCE, Vol. 111, 1946, p. 253.

LOCK SIZES FOR INLAND WATERWAYS

By Ralph L. Bloor,[9] M. ASCE

Synopsis

Data on the location, size, and date of construction of navigation locks on the inland waterways of the United States are presented, together with historical information serving to establish, in some important cases, the reasons for the sizes selected in the past. Conclusions regarding past practice are drawn. Factors governing the selection of lock sizes are enumerated and data are presented regarding the size of existing barges and towboats, and the make-up of tows. Trends in barge and towboat sizes and towing practices are discussed. Standard sizes for locks which are considered to fit best all the controlling elements are suggested.

Introduction

The size of a navigation lock to be constructed on a stream or waterway is an important subject worthy of careful study in each individual case. It is important because the structure is expensive to build, in the first place, and is extremely difficult to alter later. The size has a major effect on the success of the waterway. It may be so small as to stifle development of the traffic it is built for; or it may be so unnecessarily large as to place fixed and operating charges beyond the economic limit. It is hoped that the following observations will serve to promote a more uniform and complete consideration of the subject.

Scope

The paper presents data on the horizontal dimensions of the chamber between lock walls and between the upper and lower gates of navigation locks. The facts set forth, and the comments and conclusions derived from the facts are concerned primarily with navigation locks on inland waterways of the United States where the cargoes are carried in barges which are assembled with a power towboat into a tow. They apply only incidentally and partly to navigation locks in harbors or canals, which essentially serve ocean ships or lake ships. The lock widths are true distances between lock walls except for 2 in. or 3 in. which often are added to allow for construction inaccuracies. The lock lengths stated are the so-called usable lengths—that is, the lengths that can be utilized by tows occupying full widths without being struck by the movement of the lock gates. For the most common type of lock with mitering lock gates, the usable length is measured from the downstream side of the upper miter sill where it joins the lock wall to the upstream point of the lower gate when it is in the open position. The stated lengths may be accurate only to within 1 ft or 2 ft because of the common practice of spacing the pintle points of upper and lower mitering gates an even number of feet apart.

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[9] Chf., Structures Branch, Corps of Engrs., U. S. Army, Washington, D. C.

Existing Locks

Fig. 16 is a partial map of the United States on which are marked all the waterways that contain operating navigation locks. There are about 360 locks, a few of which are older than 100 years. There are other streams, of course, navigated by other methods, and because of the small scale of the map it has not been practicable to designate the location of individual locks. Some idea can be obtained, however, of the geographical distribution of the canalized waterways and some estimation can be made of one question germane to this paper: Is there a need for considering the characteristics of connecting waterways when studying lock dimensions on any one stream?

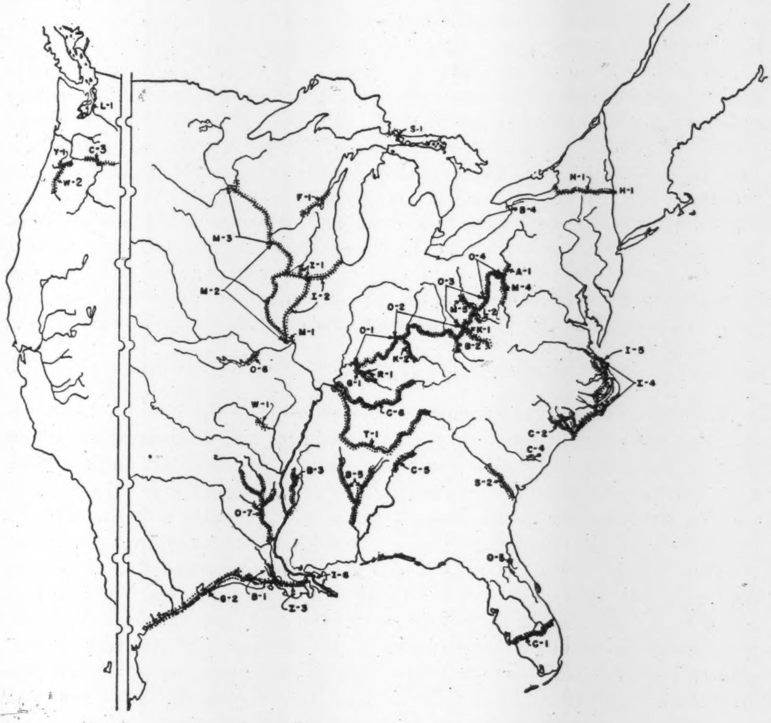


Fig. 16.—Canalized Waterways of the United States
(Letters and Numerals Refer to Table 1)

The Mississippi and Ohio rivers and their tributaries form the most extensive system of canalized waterways. This system connects at its southern end with the Intracoastal Waterway along the north and west shores of the Gulf of Mexico, and the latter waterway in turn joins the Alabama, the Tombigbee, and the Warrior rivers in Alabama, and the Coosa River in Alabama and Georgia as tributaries to the same system. Other canalized streams on the east and west coasts, including the Erie Canal in New York, are quite isolated from the Mississippi-Ohio system at present. Relatively little new construction would be necessary to effect a continuous system along the east coast which would include the Erie Canal, and there is no insurmountable obstacle to connecting the east coast system to the Mississippi-Ohio system at one or more points. It appears, therefore, that in making long-range plans—and all lock-size planning should be at long range—consideration must be given to facilities on actual or possible connecting streams if the project being studied is in the Mississippi River Basin or to the east or south of it. The west coast developments will doubtless remain isolated and the only attention that needs to be given the eastern streams is an appraisal of the experience developed from them during the many years they have been navigated.

Table 1 furnishes certain information on all the operating navigation locks in the United States. There are 364 of them exclusive of parallel or auxiliary locks, and the oldest was built in 1836. Of course, many of them have supplanted older locks on the same streams and it is known that navigation locks were built in the United States well before the beginning of the nineteenth century. Fig. 17 shows the date of construction of all the existing locks plotted against the area of the lock chamber. It serves to indicate the variations in lock sizes constructed in the past and to illustrate the usual trend in American structures from the small sizes of many years ago to the largest sizes in recent years. The lock size on the Ohio River is a notable exception. (Not shown in Fig. 17 are 57 New York State Barge Canal locks, 45 ft by 300 ft, completed in 1918.)

HISTORY

It is interesting and instructive to review some of the history of lock construction with special attention to the reasons for adopting certain lock sizes. Some of the reasons, of course, can be expected to be different from those which exist today, but one of the engineering problems remains much the same—to adopt a size that will be adequate (but not extravagantly large) to serve the traffic that will develop during the expected life of the structure.

The New York State Barge Canal is one of the oldest canalized waterways. Action to build it started in 1768 when freight costs from New York, N. Y., to points along the route were $75 per ton to $100 per ton and New York City was losing fur trade to Canadian cities and streams. Under these circumstances the problem that overshadowed everything else was to create some kind of waterway from the Hudson River in New York to the Great Lakes. It was needed quickly and its considerable length and use of private capital discouraged any long-range planning. By 1796 locks 10 ft wide and 70 ft long, permitting a 2-ft draft, had been built and freight costs dropped to $32 per ton one way and $16 per ton the other way. By 1825 the State of New York had completed 84 locks between Albany and Buffalo 15 ft wide and 90 ft long for a waterway with a 4-ft draft. This number had been reduced by 1862 to 72 locks 18 ft wide and 110 ft long. Some of these locks were lengthened in 1897–

TABLE 1.—NAVIGATION LOCKS OF THE UNITED STATES

Waterways	Lock No.	Year opened	DIMENSIONS (FT)		
			Width	Length	Lift
(1)	(2)	(3)	(4)	(5)	(6)
GROUP A-1. PENNSYLVANIA AND NEW YORK; DEPTH, 9 FT					
Allegheny River	2	1934	56.0	360.0	11.0
	3	1934	56.0	360.0	13.5
	4	1927	56.0	360.0	10.5
	5	1927	56.0	360.0	11.8
	6	1928	56.0	360.0	12.2
	7	1930	56.0	360.0	13.1
	8	1931	56.0	360.0	17.9
	9	1938	56.0	360.0	22.0
GROUP B-1. BAYOU TECHE, LA.; DEPTH, 8 FT					
Keystone Lock	..	1913	36.0	160.0	1-8
GROUP B-2. WEST VIRGINIA AND KENTUCKY; DEPTH, 6 FT					
Big Sandy River	1	1905	54.6	160.0	13.5
	2	1905	54.8	160.0	12.6
	3	1897	51.9	158.0	10.6
Levisa Fork	1	1910	55.0	160.0	11.0
Tug Fork	3	1910	55.0	160.0	12.0
GROUP B-3. MISSISSIPPI; DEPTH, 4.5 FT					
Big Sunflower River	1	1918	36.0	160.0	...
GROUP B-4. BUFFALO, N. Y.; DEPTH, 21 FT					
Black Rock Channel	..	1914	68.0	625.0	5.0
GROUP B-5. ALABAMA; DEPTH, 9 FT					
Tombigbee River	1	1908	52.0	281.9	12.0
	2	1914	52.0	286.0	9.0
	3	1914	52.0	286.0	10.0
	4	1908	52.0	285.6	10.0
Warrior River	5	1908	52.0	285.6	10.0
	6	1908	52.0	285.6	10.0
	7	1903	52.0	284.7	10.0
	8	1903	51.7	284.7	10.0
	9	1903	52.0	284.7	11.0
	13	1905	52.0	285.5	11.0
Black Warrior River; Tuscaloosa Lock	..	1939	95.0	460.0	30.0
	14	1910	52.0	282.1	14.0
	15	1910	52.0	282.1	14.0
	16	1915	52.0	285.5	21.0
	17	1915	52.0	285.5	72.0
GROUP C-1. FLORIDA;[a] DEPTH, 6 FT TO 12 FT					
New St. Lucie Lock[b]	..	1943	50.0	250.0	14.0
Old Lock No. 2[b,c]	2	1925	30.0	130.0	14.0
Moore Haven Lock	..	1935	50.0	250.0	2.5
Ortona Lock	..	1937	50.0	250.0	11.5

TABLE 1.—(*Continued*)

Waterways (1)	Lock No. (2)	Year opened (3)	Dimensions (Ft) Width (4)	Dimensions (Ft) Length (5)	Dimensions (Ft) Lift (6)
Group C-2. North Carolina; Depth, 8 Ft to 30 Ft					
Cape Fear River	1	1916	40.0	200.0	11.0
	2	1917	40.0	200.0	9.0
	3	1935	40.0	300.0	9.0
Group C-3. Columbia River, Oregon; Depth, 27 Ft					
Bonneville Lock	..	1938	76.0	500.0	63.8
Big Eddy Lock	..	1941	45.0	265.0	45.0
Fivemile Lock	..	1915	45.0	265.0	10.1
Tenmile Lock	..	1915	50.0	265.0	6.0
Celilo Lock	..	1915	45.0	265.0	6.0
Group C-4. South Carolina; Depth, 4 Ft					
Congaree River	..	1904	55.0	170.0	10.0
Group C-5. Georgia and Alabama; Depth, 4 Ft					
Coosa River	4	1914	52.0	280.0	8.0
	3	1890	40.0	176.0	8.0
	2	1890	40.0	176.0	5.51
	1	1890	40.0	176.0	5.33
Lock at Mayos Bar	..	1913	40.0	176.0	12.0
Group C-6. Kentucky and Tennessee; 6 Ft[d]					
Cumberland River	F	1923	52.0	280.0	11.5
	E	1922	52.0	280.0	10.1
	D	1916	52.0	280.0	10.9
	C	1918	52.0	280.0	12.3
	B	1916	52.0	280.0	11.8
	A	1904	52.0	280.0	13.3
	1	1904	52.0	280.0	5.5
	2	1907	52.0	280.0	9.0
	3	1908	52.0	280.0	11.1
	4	1909	52.0	280.0	13.3
	5	1909	52.0	280.0	10.7
	6	1910	52.0	280.0	13.7
	7	1910	52.0	280.0	10.3
	8	1924	52.0	280.0	12.7
	21	1911	52.0	280.0	19.5
Group F-1. Fox River, Wisconsin; Depth, 4 Ft to 6 Ft					
De Pere Lock	..	1936	36.0	146.0	8.9
Rapide Croche Lock	..	1934	36.0	146.0	8.3
Kaukauna Lock:					
Little	..	1938	36.0	146.0	7.2
Fifth	..	1898	35.6	144.0	9.1
Fourth	..	1879	36.6	144.1	10.2
Third	..	1879	36.6	144.0	10.2
Second	..	1903	35.0	144.0	9.6
First	..	1883	35.1	144.4	11.0
Little Chute Locks:					
Lower Combined	..	1879	35.4	146.5	10.9
Upper Combined	..	1879	36.3	144.1	10.6
Second	..	1881	35.0	144.2	13.8
Cedars Lock	..	1888	35.0	144.0	9.8

TABLE 1.—(*Continued*)

Waterways (1)	Lock No. (2)	Year opened (3)	Dimensions (Ft) Width (4)	 Length (5)	 Lift (6)
Group F-1.—(*Continued*)					
Appleton Locks:					
Fourth	..	1907	35.0	144.0	7.6
Third	..	1900	35.0	144.0	8.7
Second	..	1901	35.1	144.6	9.6
First	..	1884	35.0	144.7	10.0
Menasha Lock	..	1899	35.4	144.0	8.5
Eureka Lock	..	1876	35.0	148.6	2.0
Berlin Lock	..	1878	34.8	148.6	1.3
White River Lock	..	1878	34.5	148.5	1.9
Princeton Lock	..	1878	34.9	148.4	1.5
Grand River Lock	..	1878	34.7	148.3	1.4
Montello Lock	..	1901	35.3	137.0	3.8
Governor Bend Lock	..	1931	35.0	137.0	3.8
Fort Winnabago Lock	..	1936	34.7	137.0	6.4
Portage Lock	..	1928	35.2	146.0	2.3
Group G-1. Kentucky; Depth, 5 Ft					
Green River	1	1840	35.5	139.0	11.8
	2	1895	36.0	141.0	14.3
	3	1836	35.8	137.5	17.0
	4	1839	35.8	138.0	16.4
	5	1934	56.0	360.0	15.2
	6	1905	36.0	145.0	9.2
Barren River	1	1934	56.0	360.0	15.2
Group G-2. Gulf Intercoastal Waterway;[a] Depth, 12 Ft					
Harvey Lock	..	1934	75.0	425.0	0–20
Plaquemine Lock	..	1909	55.0	260.0	0–38
Vermilion Lock	..	1933	56.0	1,182.0	0–5
Group H-1. Hudson River, Troy, N. Y.; Depth, 27 Ft					
Federal Lock	..	1917	44.4	492.5	17.3
Group I-1. Illinois; Depth, 7 Ft					
Canal between Illinois River and Mississippi River	1	1907	35.0	143.08	6.7
	2	1907	35.0	143.08	9.0
	3	1907	35.0	143.08	9.0
	4	1907	35.0	143.08	9.0
	5	1907	35.0	143.08	8.0
	6	1907	35.0	143.08	10.0
	7	1907	35.0	143.08	8.0
	8	1907	35.0	149.75	8.0
	9	1907	35.0	149.75	8.0
	10	1907	35.0	149.75	9.0
	11	1907	35.0	149.75	9.0
	12	1907	35.0	149.75	8.0
	13	1907	35.0	149.75	10.0
	14	1907	35.0	149.75	10.0
	15	1907	35.0	149.75	10.0
	16	1907	35.0	149.75	11.0
	17	1907	35.0	149.75	10.0
	18	1907	35.0	149.75	9.0
	19	1907	35.0	149.75	10.0
	20	1907	35.0	149.75	11.0
	21	1907	35.0	149.75	11.0

TABLE 1.—(*Continued*)

Waterways (1)	Lock No. (2)	Year opened (3)	DIMENSIONS (FT)		
			Width (4)	Length (5)	Lift (6)
GROUP I-1.—(*Continued*)					
Guard Lock	..	1907	35.0	149.0	1.8
Rock River Lock	..	1907	35.0	143.08	9.2
Canal between Illinois River and Mississippi River	22	1907	35.0	143.08	9.0
	23	1907	35.0	143.08	11.0
	24	1907	35.0	143.08	11.0
	25	1907	35.0	143.08	8.0
	26	1907	35.0	143.08	9.0
	27	1907	35.0	143.08	8.0
	28	1907	35.0	143.08	8.0
	29	1907	35.0	143.08	11.0
	30	1907	35.0	149.00	0.0
	31	1907	35.0	145.42	6.0
	32	1907	35.0	143.08	7.7
GROUP I-2. ILLINOIS WATERWAY; DEPTH, 9 FT					
La Grange Lock	..	1939	110.0	600.0	10.00
Peoria Lock	..	1939	110.0	600.0	11.00
Starved Rock Lock	..	1933	110.0	600.0	18.70
Marseilles Lock	..	1933	110.0	600.0	23.95
Dresden Island Lock	..	1933	110.0	600.0	21.75
Brandon Road Lock	..	1933	110.0	600.0	34.00
Lockport Lock	..	1933	110.0	600.0	40.00
Lockport Lock	..	1933	22.0	130.0	40.00
Blue Island Lock	..	1933	50.0	360.0	0
GROUP I-3. INLAND WATERWAY, FRANKLIN, LA., TO MERMENTAU RIVER, LOUISIANA; DEPTH, 5 FT					
Hanson Lock	..	1907	26.5	937.0	1–3
Schooner Bayou Lock	..	1913	36.0	300.0	1–2
GROUP I-4. INLAND WATERWAY, NORFOLK, VA., TO BEAUFORT INLET, N. C.; DEPTH, 12 FT					
Tidal Guard Lock	..	1932	75.0	600.0	2.7
GROUP I-5. INLAND WATERWAY, NORFOLK, VA., TO THE SOUNDS OF NORTH CAROLINA; DEPTH, 10 FT					
Deep Creek Lock, Va.	..	1940	52.0	300.0	12.0
South Mills Lock, N. C.	..	1941	52.0	300.0	12.0
GROUP I-6. INLAND WATERWAY, NEW ORLEANS, LA.; DEPTH, 30 FT					
Inner Harbor Lock	..	....	75.0	640.0	0–20
GROUP K-1. KANAWHA RIVER, W. VA.; DEPTH, 9 FT					
Twin Locks:					
Winfield	..	1937	56.0	360.0	28.0
Marmet	..	1934	56.0	360.0	24.0
London	..	1934	56.0	360.0	24.0

TABLE 1.—(*Continued*)

Waterways	Lock No.	Year opened	Dimensions (Ft)		
			Width	Length	Lift
(1)	(2)	(3)	(4)	(5)	(6)
GROUP K-2. KENTUCKY RIVER, KENTUCKY; DEPTH, 6 FT					
Lock Nos.	1	1882	38.0	145.0	18.23
	2	1882	38.0	145.0	13.94
	3	1882	38.0	145.0	13.16
	4	1882	38.0	145.0	13.22
	5	1886	38.0	145.0	15.00
	6	1894	52.0	147.0	13.95
	7	1897	52.0	147.0	15.30
	8	1900	52.0	146.0	18.66
	9	1903	52.0	148.0	17.34
	10	1904	52.0	148.0	17.00
	11	1906	52.0	148.0	18.00
	12	1910	52.0	148.0	16.00
	13	1915	52.0	148.0	18.00
	14	1917	52.0	148.0	18.00
GROUP L-1. LAKE WASHINGTON SHIP CANAL, SEATTLE, WASH.; DEPTH, 30 FT					
Large Lock	..	1916	80.0	760.0	6.5–25
Small Lock[e]	..	1916	30.0	123.0	6.5–25
GROUP L-2. LITTLE KANAWHA RIVER, WEST VIRGINIA; DEPTH, 4 FT					
Lock Nos.	1	1874	23.0	125.0	6.4
	2	1874	23.0	125.0	11.3
	3	1874	23.0	125.0	11.0
	4	1874	23.0	125.0	11.9
	5	1891	25.5	125.0	12.4
GROUP M-1. UPPER MISSISSIPPI RIVER; DEPTH, 9 FT					
Lock Nos.	26	1938	110.0	600.0	23.0
	..	1938	110.0	360.0	23.0
	25	1939	110.0	600.0	15.0
	24	1940	110.0	600.0	15.0
GROUP M-2. MISSISSIPPI RIVER; DEPTH, 9 FT					
Upper River	22	1938	110.0	600.0	10.2
	21	1938	110.0	600.0	10.5
	20	1936	110.0	600.0	10.0
	19	1913	110.0	358.0	38.2
	18	1937	110.0	600.0	9.8
	17	1939	110.0	600.0	8.0
	16	1937	110.0	600.0	9.0
	15	1934	110.0	600.0	16.0
	..	1934	110.0	360.0	16.0
Le Claire Lock (Canal)	..	1922	80.0	320.0	11.0
Upper River	14	1939	110.0	600.0	11.0
	13	1938	110.0	600.0	11.0
	12	1938	110.0	600.0	9.0
	11	1937	110.0	600.0	11.0
GROUP M-3. UPPER MISSISSIPPI RIVER; DEPTH, 9 FT					
Lock Nos.	10	1936	110.0	600.0	8.0
	9	1938	110.0	600.0	9.0
	8	1937	110.0	600.0	11.0
	7	1937	110.0	600.0	8.0
	6	1936	110.0	600.0	6.5

TABLE 1.—(*Continued*)

Waterways (1)	Lock No. (2)	Year opened (3)	Dimensions (Ft) Width (4)	Length (5)	Lift (6)
Group M-3.—(*Continued*)					
Lock Nos.	5-A	1936	110.0	600.0	5.5
	5	1935	110.0	600.0	9.0
	4	1935	110.0	600.0	7.0
	3	1938	110.0	600.0	8.0
	2	1930	110.0	600.0	12.2
	..	1930	110.0	500.0	12.2
	1	1917	56.0	400.0	35.9
	..	1917	56.0	400.0	35.9
Group M-4. West Virginia and Pennsylvania; Depth, 7 Ft to 9 Ft					
Monongahela River Lock Nos.	2	1905	56.0	362.0	8.7
	..	1905	56.0	362.0	8.7
	3	1907	56.0	360.0	8.2
	..	1907	56.0	720.0	8.2
	4	1932	56.0	360.0	10.6
	..	1932	56.0	720.0	10.6
	5	1909	56.0	360.0	12.4
	6	1916	56.0	360.0	13.1
	7	1925	56.0	360.0	15.0
	8	1925	56.0	360.0	15.0
	10	1903	56.0	182.0	10.2
	11	1903	56.0	182.0	10.7
	12	1903	56.0	182.0	10.7
	13	1904	56.0	182.0	10.7
	14	1903	56.0	182.0	10.7
	15	1903	56.0	182.0	10.7
Group M-5. Muskingum River, Ohio; Depth, 4 Ft to 5 Ft					
Lock Nos.	1	1890	55.8	179.0	4.29
	2	1840	35.8	160.0	10.76
	3	1840	35.8	160.0	14.19
	4	1840	35.8	160.0	9.32
	5	1840	35.8	160.0	10.76
	6	1840	35.8	160.0	12.33
	7	1840	35.8	160.0	10.10
	8	1840	35.9	160.0	10.96
	9	1891	35.8	160.0	11.01
	10	1840	35.4	156.5	15.43
	..	1840	35.4	158.5	15.43
	11	1910	36.0	160.0	11.75
Group N-1. New York State Barge Canal; Depth, 14 Ft					
Erie Canal	2	1918	45.0	300.0	33.55
	3	1918	45.0	300.0	34.50
	4	1918	45.0	300.0	34.50
	5	1918	45.0	300.0	33.25
	6	1918	45.0	300.0	33.00
	7	1918	45.0	300.0	27.00
	8	1918	45.0	300.0	14.00
	9	1918	45.0	300.0	15.00
	10	1918	45.0	300.0	15.00
	11	1918	45.0	300.0	12.00
	12	1918	45.0	300.0	11.00
	13	1918	45.0	300.0	8.00
	14	1918	45.0	300.0	8.00
	15	1918	45.0	300.0	8.00
	16	1918	45.0	300.0	20.50
	17	1918	45.0	300.0	40.50
	18	1918	45.0	300.0	20.00
	19	1918	45.0	300.0	21.00

TABLE 1.—(*Continued*)

Waterways (1)	Lock No. (2)	Year opened (3)	Dimensions (Ft)		
			Width (4)	Length (5)	Lift (6)
Group N-1.—(*Continued*)					
Utica Harbor Lock	..	1918	45.0	300.0	7.00
Erie Canal	20	1918	45.0	300.0	16.00
	21	1918	45.0	300.0	25.00
	22	1918	45.0	300.0	25.00
	23	1918	45.0	300.0	6.90
Oswego Canal	1	1918	45.0	300.0	10.20
	2	1918	45.0	300.0	17.80
	3	1918	45.0	300.0	27.00
	5	1918	45.0	300.0	18.00
	6	1918	45.0	300.0	20.00
	7	1918	45.0	300.0	14.50
	8	1918	45.0	300.0	11.10
Group N-1. New York State Barge Canal; Depth, 12 Ft					
Erie Canal	24	1918	45.0	300.0	11.00
	25	1918	45.0	300.0	6.00
	26	1918	45.0	300.0	6.00
	27	1918	45.0	300.0	12.50
	28-A	1918	45.0	300.0	19.50
	28-B	1918	45.0	300.0	12.00
	29	1918	45.0	300.0	16.00
	30	1918	45.0	300.0	16.40
	32	1918	45.0	300.0	25.10
	33	1918	45.0	300.0	25.10
	34	1918	45.0	300.0	24.60
	35	1918	45.0	300.0	24.50
Cayuga and Seneca Canal	1	1918	45.0	300.0	7.50
	2	1918	45.0	300.0	24.50
	3	1918	45.0	300.0	24.50
	4	1918	45.0	300.0	14.50
Champlain Canal	1	1918	45.0	300.0	14.30
	2	1918	45.0	300.0	18.50
	3	1918	45.0	300.0	19.50
	4	1918	45.0	300.0	16.00
	5	1918	45.0	300.0	19.00
	6	1918	45.0	300.0	16.50
	7	1918	45.0	300.0	10.00
	8	1918	45.0	300.0	11.00
	9	1918	45.0	300.0	16.00
	11	1918	45.0	300.0	12.00
	12	1918	45.0	300.0	15.50
Group O-1. Ohio River; Depth, 9 Ft					
Lock Nos.	53	1929	110.0	600.0	13.4
	52	1928	110.0	600.0	12.0
	51	1929	110.0	600.0	8.0
	50	1928	110.0	600.0	10.0
	49	1928	110.0	600.0	11.0
	48	1922	110.0	600.0	7.0
	47	1928	110.0	600.0	9.0
	46	1928	110.0	600.0	11.0
	45	1927	110.0	600.0	9.0
	44	1925	110.0	600.0	7.0
	43	1921	110.0	600.0	9.0
	41	1921	110.0	600.0	37.0
	..	1929	56.0	360.0	37.0

TABLE 1.—(*Continued*)

Waterways (1)	Lock No. (2)	Year opened (3)	Dimensions (Ft)		
			Width (4)	Length (5)	Lift (6)
Group O-2. Ohio River; Depth, 9 Ft					
Lock Nos.	39	1921	110.0	600.0	6.0
	38	1924	110.0	600.0	7.3
	37	1911	110.0	600.0	7.8
	36	1925	110.0	600.0	7.9
	35	1919	110.0	600.0	6.4
	34	1925	110.0	600.0	5.6
	33	1921	110.0	600.0	7.0
	32	1926	110.0	600.0	7.5
	31	1919	110.0	600.0	7.5
	30	1923	110.0	600.0	7.5
	29	1916	110.0	600.0	8.0
Group O-3. Ohio; Depth, 9 Ft					
Ohio River	28	1915	110.0	600.0	7.1
	27	1923	110.0	600.0	6.4
Gallipolis Locks	..	1937	110.0	600.0	26.0
	..	1937	110.0	360.0	26.0
Ohio River	23	1921	110.0	600.0	5.6
	22	1918	110.0	600.0	7.8
	21	1919	110.0	600.0	5.6
	20	1917	110.0	600.0	7.5
	19	1916	110.0	600.0	7.7
	18	1910	110.0	600.0	6.2
	17	1918	110.0	600.0	8.2
	16	1917	110.0	600.0	7.8
	15	1916	110.0	600.0	7.8
	14	1917	110.0	600.0	8.3
Group O-4. Ohio; Depth, 9 Ft					
Ohio River	13	1911	110.0	600.0	7.3
	12	1917	110.0	600.0	8.4
	11	1911	110.0	600.0	7.3
	10	1915	110.0	600.0	8.4
	9	1914	110.0	600.0	7.4
	8	1911	110.0	600.0	6.4
	7	1914	110.0	600.0	6.9
Montgomery Island Locks	..	1936	110.0	600.0	19.4
	..	1936	56.0	360.0	19.4
Dashields Locks	..	1929	110.0	600.0	10.0
	..	1929	56.0	360.0	10.0
Ensworth Locks	..	1936	110.0	600.0	18.0
	..	1936	56.0	360.0	18.0
Group O-5. Oklawaha River, Florida; Depth, 6 Ft					
Moss Bluff Lock	..	1925	30.0	125.0	11.5
Group O-6. Osage River, Missouri; Depth, 3 Ft					
Lock	1	1906	42.0	220.0	16.0

TABLE 1.—(*Continued*)

Waterways (1)	Lock No. (2)	Year opened (3)	Dimensions (Ft)		
			Width (4)	Length (5)	Lift (6)
Group O-7. Ouachita and Black Rivers, in Arkansas; Depth, 6.5 Ft					
Lock Nos.	2	1918	55.0	268.0	14.40
	3	1919	55.0	268.0	14.80
	4	1915	55.0	268.0	8.90
	5	1924	55.0	268.0	6.90
	6	1913	55.0	268.0	9.70
	8	1912	55.0	268.0	13.50
Group R-1. Rough River; Kentucky; Depth, 4 Ft					
Lock	1	1896	27.0	123.0	9.9
Group S-1. St. Marys River, Michigan; Depth, 21 Ft to 25 Ft					
MacArthur Lock	..	1943	80.0	800.0	21.7
Poe Lock	..	1896	95.0	800.0	21.7
Davis Lock	..	1914	80.0	1,350.0	21.7
Sabin Lock	..	1919	80.0	1,350.0	21.7
Group S-2. Savannah River, Georgia; Depth, 6 Ft					
Lock	..	1937	56.0	360.0	15.0
Group T-1. Tennessee River in Tennessee, Alabama, and Kentucky; Depth, 9 Ft					
Kentucky Lock	..	1942	110.0	600.0	56.0
Pickwick Landing	..	1937	110.0	600.0	55.0
Tennessee River	1	1926	60.0	300.0	2.3
Wilson Lock:					
Upper Lift	..	1927	60.0	292.0	91.7
Lower Lift	..	1927	60.0	300.0	91.7
Wheeler Lock	..	1934	60.0	360.0	48.0
Guntersville Lock	..	1937	60.0	360.0	39.0
Hales Bar	..	1913	60.0	265.0	34.0
Chickamauga Lock	..	1937	60.0	360.0	51.0
Watts Bar Lock	..	1941	60.0	360.0	58.0
Fort Loudoun Lock	..	1943	60.0	360.0	72.0
Group W-1. White River, Arkansas; Depth, 4 Ft					
Batesville Lock	1	1904	36.0	147.0	15.6
Earnharts Lock	2	1905	36.0	147.0	12.8
Walls Ferry Lock	3	1908	36.0	147.0	14.5
Group W-2. Willamette River, Oregon; Depth, 6 Ft					
Lock Nos.	1	1873	37.0	175.0	22.5
	2	1873	37.0	175.0	8.7
	3	1873	37.0	175.0	10.9
	4	1873	37.0	175.0	8.1
Group Y-1. Yamhill River, Oregon; Depth, 4 Ft					
Lafayette Lock	..	1900	40.0	175.0	16.0

[a] Caloosahatchee River, Lake Okeechobee, Florida. [b] At Stuart, Fla. [c] Auxiliary. [d] Project depth at numbered locks, 4.6 ft. [e] From Apalachee Bay, Florida, to the Mexico-United States border.

1898 to 208 ft or 230 ft. Still later the number of locks was reduced to 57 and the lock dimensions were (and still are) 45 ft by 300 ft. Thus, in a little more than 100 years the locks were enlarged four times. Of course, in a waterway such as this the locks form a much more minor part of the total system than is the case where large natural waterways are used to a greater extent. For this reason there is more justification than usual for increasing lock sizes only after a demand for a larger channel has developed, rather than for constructing locks to a size believed desirable for future anticipated requirements.

The Monongahela River development is one of the oldest and still one of the busiest waterways. Originally it was important because it was the first navigable stream west of the Allegheny Mountains (in the eastern United States) which was reached by the national road (present highway, United States route 40). The first traffic was by flatboats and rafts of westward moving settlers and, for the protection of this traffic, there was a law prohibiting the construction of dams with lifts greater than 4.5 ft. There are records of armed resistance to the construction of locks and dams. However, 4 locks and dams were built between 1840 and 1844 with the locks 50 ft wide and 158 ft long. This is a rather large size for such an early date and it may have been influenced by the fact that many flatboats were built on the Monongahela River, floated downstream as far as New Orleans with various cargoes, and then broken up and the lumber and timbers sold for the construction of other things. It may well have been that planking about 25 ft long was generally available in the Monongahela forests of that day and consequently the greatest width of barges and flatboats was fixed at that length. At any rate, a barge width of from 26 ft to 27 ft is widely used today and is generally known as the Monongahela River size. All the locks that have been built on the river vary in width from only 50 ft to 56 ft. Lock lengths were increased over the years as the coal trade developed. A little before the beginning of the twentieth century the engineers developing the river believed that, considering channel

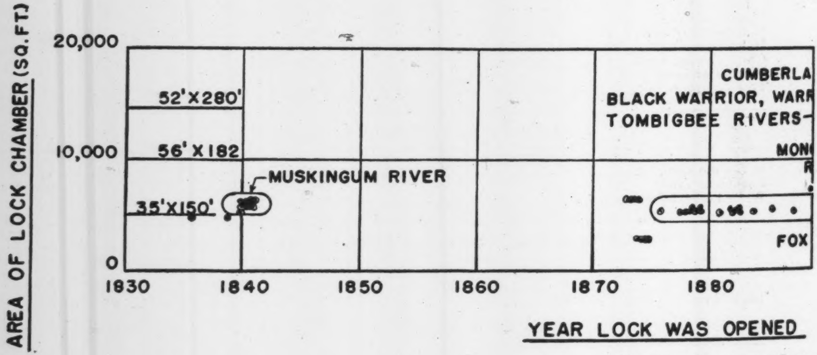
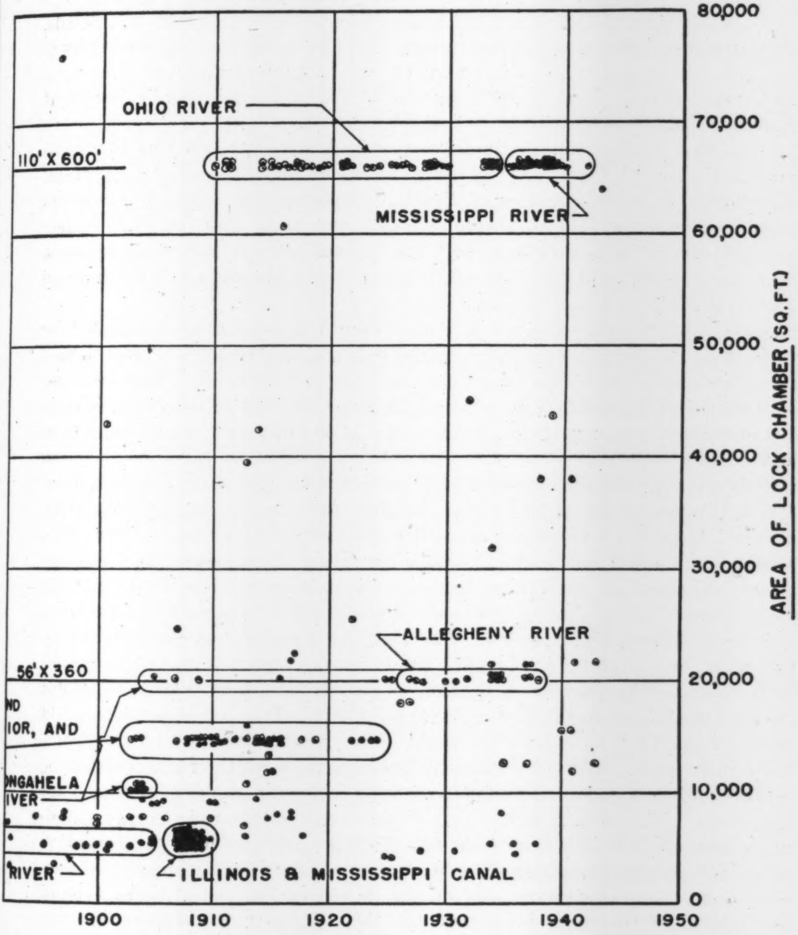


FIG. 17.—NAVIGATION LOCKS

limitations, experience indicated that a fleet carrying 3,000 tons on an 8-ft draft was about the limit of safety and that disposition in three coal boats, from 26 ft to 27 ft wide by 175 ft long, was most convenient. Such a tow needed a lock 350 ft long. As a result, a lock size of 56 ft by 360 ft was fixed. This size has been used on many other streams and has been varied on the Monongahela only by doubling its length to 720 ft where traffic is densest.



TO NAVIGATION

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Similar reasoning was applied in planning locks on the Cumberland, Warrior, and other rivers. Lock sizes of 52 ft by 280 ft were adopted on these two streams to accommodate three barges 25 ft by 130 ft and a towboat 150 ft long. On the Tennessee River there was another influencing factor. The State of Alabama in 1830 had built lock and canal systems at Muscle Shoals, the worst restriction to open-river navigation on that stream. This early work fell into disuse and when the complete canalization of the river was later undertaken a lock width of 60 ft was adopted because the original canals at Muscle Shoals were 60 ft wide.

The records show repeatedly that many lock sizes were selected to accommodate the largest boats and tows using the stream under open-river conditions. There may be some criticism of this basis inasmuch as a canalized stream should be reliably able to accommodate larger tows than could be safely operated under open-river hazards. The records also reveal, in some cases, that the engineers did not overlook this latter consideration but did have to neglect it for the purpose of curtailing costs.

The Ohio River has a long active navigation history. It was navigated for more than 100 years before it was canalized, and the tonnages it has carried have increased steadily during the 60 or more years since the first lock and dam were built. By 1870, or thereabouts, the possibility of securing a reliable 6-ft channel throughout its length of nearly 1,000 miles by the continuation of open-channel methods seemed remote. On the other hand, the navigation interests at that time were reluctant to accept the alternative of canalization. The traffic was predominately coal, moving downstream, and the operation involved large fleets of barges operating during the spring rises in the river. The type of dam developed for the Ohio River was a compromise between two needs—(1) dams to provide pools for navigating during low water, but (2) dams which could be lowered to the bed of the river to permit boats to pass over them during higher stages. The lock size selected was also affected by this need for compromise. In 1874 the engineers reported that locks should be large enough to pass a coal fleet of ten barges 24 ft by 130 ft, one fuel flat 22 ft by 100 ft, and one steamboat 48 ft by 230 ft. Two barges abreast would require locks 50 ft wide, but this would be too narrow for some of the steamboats then in use. If the barges were four abreast, a width of 100 ft would be needed, and it was doubted that a mitering lock gate could be designed for this width. Therefore, a lock 75 ft wide seemed desirable and the corresponding length needed was about 600 ft. To the coal carriers, accustomed to open-river operations, a lock 75 ft wide seemed hopelessly restrictive. In an effort to reach a satisfactory solution the engineers designed a rolling type of lock gate which they considered usable on locks wider than 100 ft, and in 1878 recommended a lock size 110 ft wide by 600 ft long. Thus was originated the largest commonly used lock size in the United States. By reference to Fig. 17 it will be seen how much larger it was than any other lock built up to that time and, furthermore, how the size has continued to be the largest used since then, including the quite recent and very important Upper Mississippi River development. It may be said that the engineers planned well for the future on the Ohio River; but it must be admitted that they were influenced by navigation interests which did not want any locks at all.

CONCLUSIONS ON THE BASIS OF EXPERIENCE

In trying to learn from the foregoing experience, there is no need to be critical of the decisions made in the past. There are many diverse influences that can be imagined to have existed but that cannot be confirmed in any specific case without much more research than is worthwhile. It is quite sufficient simply to judge the results produced and to use the conclusions so drawn as a basis for guiding the planning of the future.

Experience indicates that the most successful waterways are those in which liberal lock sizes were adopted at a relatively early date. It is not true, of course, that large lock sizes will insure an active waterway; but it does appear that relatively large locks are necessary to attract and hold traffic which otherwise would revert to other modes of transportation.

No record has been uncovered of a lock being replaced because it was worn out. On the other hand, many have been replaced or augmented because they were too small. Locks are structures capable of long life. There are several locks that have been in service more than 100 years. They have been repaired and supplied with new gates and sills, but the main structure is the original structure. There are many locks approaching 50 years of age. Some on the Monongahela River have operated at very nearly their full capacity for many years, and in addition were subjected to freezing weather, acid waters, and countless heavy blows from floating plant; yet, if they were large enough, they could be rehabilitated at relatively small cost to serve many more years. Therefore, on the basis of experience, it seems proper to conclude that a lock size which is determined by the size of boats or tows then commonly using the waterway is almost certain to be too small before it is worn out.

Experience suggests that a similar attitude should be adopted regarding (*a*) the relation of lock size to width and alinement and (*b*) the character of adjacent navigable channels. It has been relatively simple to enlarge channels in response to traffic demands, but the enlargement of locks in this manner is so much less feasible that the usual procedure appears to have been the construction of new locks. Furthermore, the canalization of a stream reduces navigation hazards as compared to open-river conditions, and this fact in itself supports the use of larger tows.

Finally, regarding experience, it appears that since the needs of the United States, and its transportation patterns, have changed with growth, many locks have fallen into disuse and in these instances the smaller locks represent smaller economic losses. Furthermore, other changes have rendered locks obsolete before they became inoperable because of age. Progress in engineering design and materials has made higher and higher lifts acceptable so that a reduction in the number of lock and dam structures with consequent saving in lockage time has been possible. Just below Pittsburgh several of the original Ohio River dams were replaced for the principal reason that a different type of dam was needed, one that would maintain more stable pool levels in the important harbor area. These and similar developments in the past make it desirable to consider carefully whether a proposed lock will have the long usage of which it, as a structure, is capable.

FACTORS GOVERNING LOCK-SIZE SELECTIONS

Costs.—Certainly one of the most important elements in selecting a lock size is cost. The writer has attempted to establish some rough, general relationship between increases in lock sizes and increases in costs but found it impossible; it must be concluded that the relation between size and cost varies widely and is completely dependent on the local conditions at the site. Under favorable conditions an increase of a certain percentage in the lock length or in the lock width may be accompanied by an increase in the cost of only one fourth as much. At other sites, the percentage increase in the cost might even be more than the percentage increase in the length or the width. The reasons for these variations are rather obvious, but it is important in making comparative estimates that none of them be overlooked. Lock gates and operating machinery become heavier, per unit of gate area, as lock widths are increased; and, furthermore, filling and emptying valves become larger with increases in either length or width if the same operating time is to be maintained. The larger locks also require more expensive treatment of lock-filling ports and discharge points because larger volumes of water are moved in a limited operating period. Locks situated in open rivers require guide walls along the riverbank upstream and downstream from the lock, and generally these must be as long as the lock chamber itself. Therefore, if a lock chamber is lengthened, the two chamber walls and also both the upper guide wall and the lower guide wall should be lengthened by an equal amount. For the best operating conditions, the landward lock wall and the guide walls must be in one straight line, and in many natural streams it will be difficult to find a reach straight enough to accommodate such a structure with satisfactory alinement of approach in each direction. The only alternative is to set the lock and approaches back into a convex shore, and the accompanying excavation often causes the cost of the longer locks to increase disproportionately. Additional width also has a similar effect if (as is often the case) nearly the full width of the natural channel is needed for the dam. Foundation conditions, of course, are likely to be more nearly at an optimum for a small lock than for a large one. Extra length and width in cofferdams, where they are required, entail extra costs for larger locks. In spite of the aforementioned factors, the basic requirement of only two lock gates (regardless of length of lock) and only two lock walls (regardless of width) tends in many cases to keep the cost increases accompanying the lock-size increases within acceptable limits.

Economics.—The practice of making economic studies in the selection of a lock size is now rather generally accepted. Details of the methods of evaluating the delay time of boats and tows using a smaller lock as compared to using a larger lock are not unusual and need not be treated in this paper. However, it is important that basic assumptions be correct to insure an acceptable result. The assumed traffic density should correspond to the computed amount of traffic which will develop, and usually this will involve the waiting time at locks (especially of the smaller sizes). In addition to time lost in smaller locks due to the necessity of breaking tows into several lockages, there will be time lost waiting for a turn to lock. Even at locks that operate below their capacity, delays in waiting time occur when several tows arrive at the lock at the same time. Another factor that is sometimes overlooked is that a tow of a given size can be brought safely into a larger lock more quickly than into a smaller lock. Therefore, there is some delay involved in a smaller lock even for tows that would not fill a larger size. Time studies should recognize the fact that breaking a tow transversely is a simpler and faster operation than breaking it longitudinally.

Even after all tangible factors are considered in an economic study of lock-size requirements, some thought should be given to other cogent circumstances which are practically impossible to evaluate. Smaller locks increase the wear and tear on floating plant and are more often damaged and put out of service by floating plant. Shippers hesitate to use a service that may be delayed even when shipping costs are materially lower. For these and similar reasons, therefore, economic studies of tangible costs and values should not be the sole basis for lock-size determination.

Boats and Barges.—The size of boats and barges that will use a lock is a vital consideration in selecting the size of the lock. Figs. 18, 19, and 20 show the number of barges, of various widths, in operation on several of the main waterways of the United States in 1945. These illustrate the great preponderance of barges from 26 ft to 27 ft wide (all of which will hereafter be designated as 26-ft barges). More than 70% of the barges of this width are 175 ft long and carry from 750 tons to 1,250 tons each, with drafts ranging from 8 ft to 11 ft. A few have been built as long as 210 ft, but most of the remaining 20% or so are less than 175 ft. The next most common barge is from 34 ft to 35 ft wide (which hereafter will be called a 35-ft barge). About one half of these vessels are 195 ft long and carry 1,500 tons on a 9-ft draft. A few of this width have been built longer than 200 ft but most of the others are less than 195 ft.

Fig. 18 shows quite clearly the effect of long established lock sizes on barge construction. The 26-ft barge width is designed to fit the 56-ft lock widths on the Monongahela River, the Kanawha River, and other Ohio River tributaries. These same barges also fit the 110-ft-wide locks on the Ohio River as do also the 35-ft barges. Other barge widths on the Ohio River are notable by their scarcity. On the Mississippi River system (Fig. 19) there are locks 110 ft wide from St. Louis to Minneapolis and downstream from St. Louis navigation is by open river. Although most of the barges on the Mississippi River are either 26 ft wide or 35 ft wide, there are more barges 40 ft wide (or wider) than there are on the Ohio River. Fig. 20 for the Louisiana-Texas Intracoastal Waterway shows less tendency toward standardization in barge widths, doubtless because much of the traffic need not proceed through locks.

An examination of the size of barges constructed during the 8-year period from 1937 to 1945 as compared to the total in operation (see Fig. 21) shows that on the Ohio and Mississippi rivers a slightly smaller percentage of new barges are 26 ft wide than is indicated by the total in operation. On the Ohio River alone the percentage of new barges in this width has risen as may be noted by comparing the data in Table 2. Barge construction in the 35-ft widths has risen sharply in the same 8-year period. There are fewer new barges in most other widths. Both the total number, and the increased widths in new barges (40 ft, 48 ft, and 50 ft), are too small to be significant. It seems quite evident that there is a definite tendency to standardize the 26-ft-wide and the 35-ft-wide barge sizes, and that even the open-river reach, more than 1,000 miles long, in the Lower Mississippi River, has not caused any substantial adoption of wider barges.

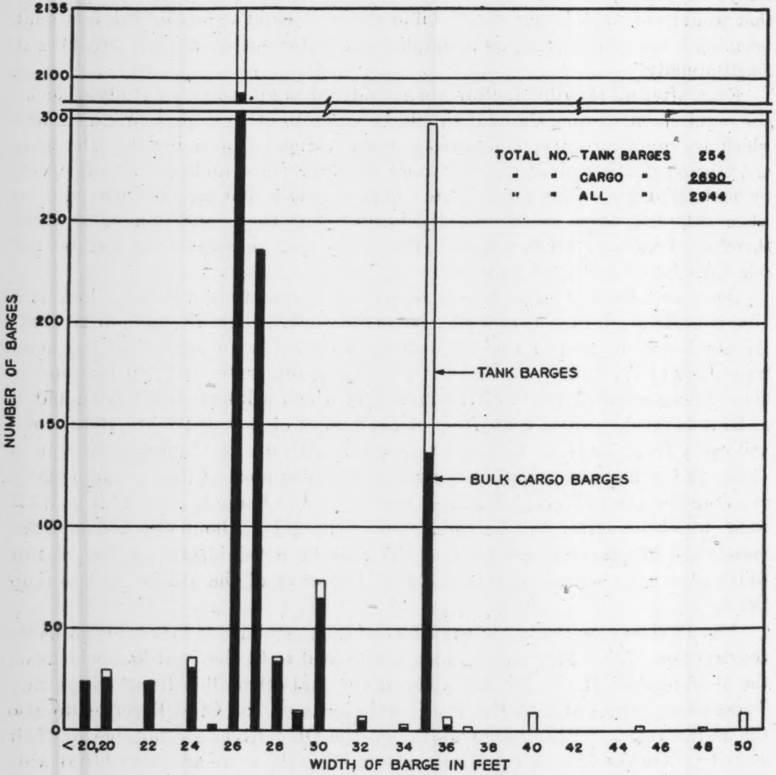

Fig. 18.—Width of Barges on the Ohio River

There are a few factors in the construction and operation of barges which might be mentioned as having an effect on optimum barge size. In general, the cost of a barge, per ton of cargo, reduces as the size increases. The ends of the barge represent more expensive construction than the midship sections and, as a result, there is an advantage in building long barges. Large barges have several disadvantages from an operational point of view. It is difficult

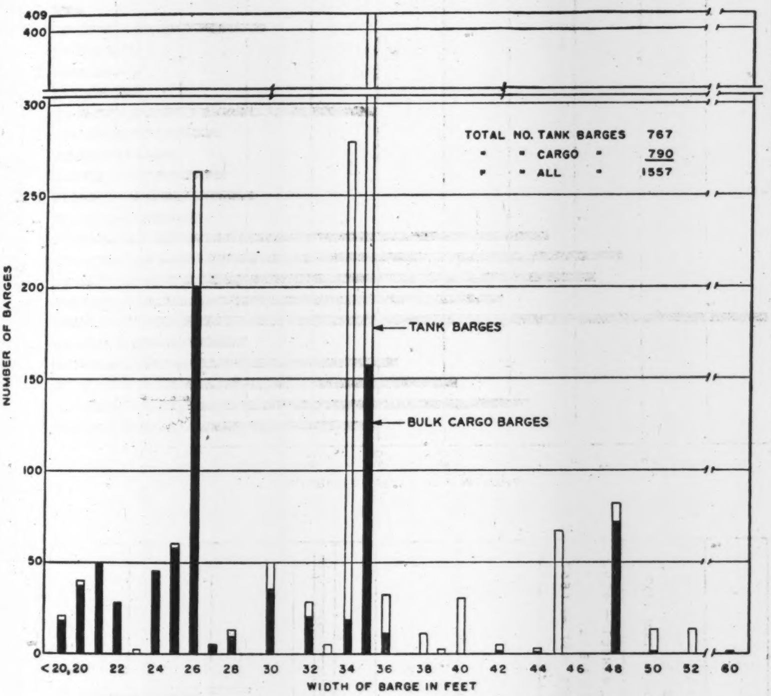

FIG. 19.—WIDTH OF BARGES ON THE MISSISSIPPI RIVER

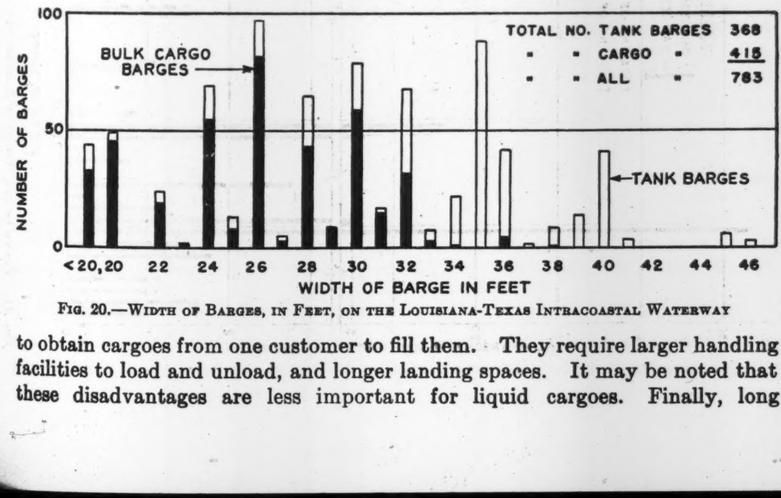

FIG. 20.—WIDTH OF BARGES, IN FEET, ON THE LOUISIANA-TEXAS INTRACOASTAL WATERWAY

to obtain cargoes from one customer to fill them. They require larger handling facilities to load and unload, and longer landing spaces. It may be noted that these disadvantages are less important for liquid cargoes. Finally, long

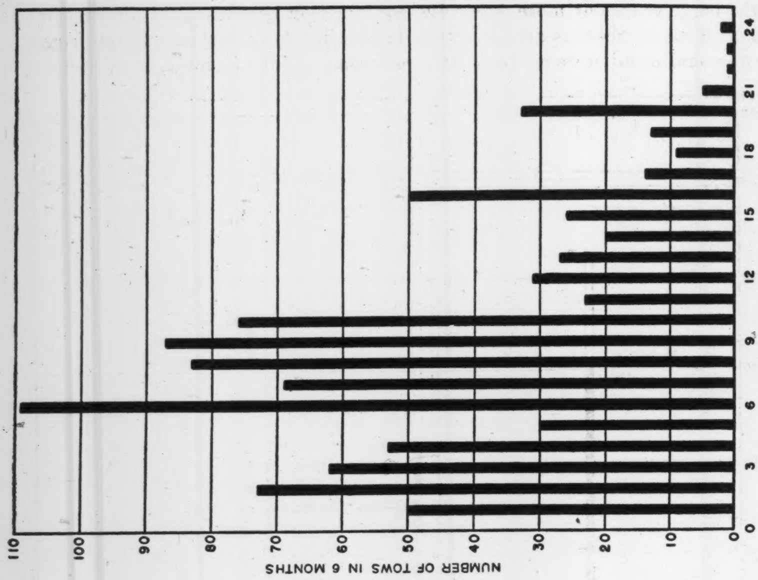

FIG. 22.—BARGES PER TOW, LOCK 36, OHIO RIVER

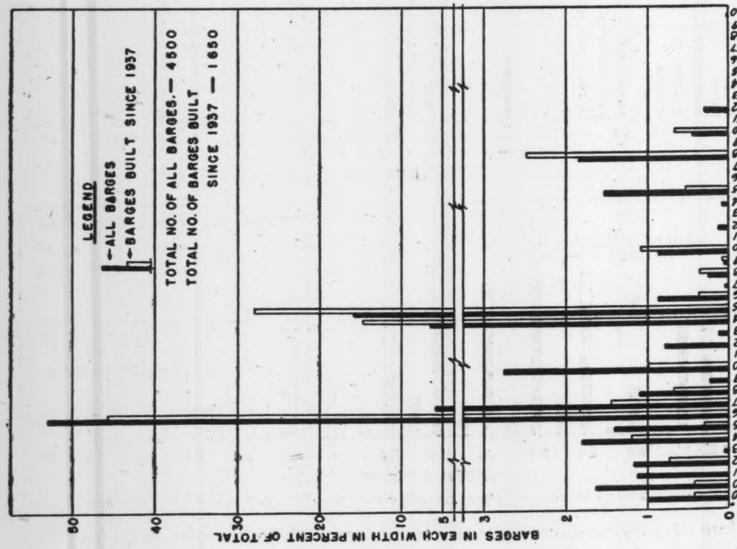

FIG. 21.—TREND IN THE WIDTH OF BARGES, MISSISSIPPI AND OHIO RIVERS

narrow barges that permit the assembly of long, narrow tows have the advantage of requiring less towing power. Model towing tests indicate, for instance, that from 25% to 50% more power is required for equal speeds if a tow of six barges is arranged three wide and two long rather than two wide and three long. The tests also indicate that short barges assembled into a tow of a certain width and length would have added towing resistance over long barges assembled into a tow of equal dimensions. The additional resist-

TABLE 2.—BARGES OPERATING ON THE MISSISSIPPI AND OHIO RIVERS, IN 1945

Width, in ft	BULK CARGO BARGES				TANK BARGES			
	No. in Operation		New Since 1937		No. in Operation		New Since 1937	
	Ohio	Mississippi	Ohio	Mississippi	Ohio	Mississippi	Ohio	Mississippi
<20	24	18		5		3		2
20	26	37	5		4	3		2
21		50						
22	24	28	7	5				
23						2		
24	31	45	6	14	5			
25	3	58		5		2		
26	2,094	200	640	69	41	63	17	28
27	234	5	24		1			
28	34	9	2	3	2	4	2	4
29	9				1			
30	65	35	4	4	9	15	1	7
31								
32	5	20		4	2	8		
33					5			
34	2	18		2	1	261		239
35	136	157	35	102	162	252	128	193
36	3	11		1	4	21		5
37					2			
38						11		6
39						2		1
40					9	30		18
41								
42		1				4		
43								
44		1				2		
45					2	67	2	7
46								
47								
48		72		35	1	10		6
49								
50		1			8	12	3	8
51								
52						13		
53								
54								
55								
56								
57								
58								
59								
59								
60		1						
Totals	2,690	767	723	249	259	785	153	526

ance would come from the drag produced in the larger number of transverse joints in the tow where the raked ends of the barges come together.

The manner in which barges will be assembled into tows is the subject that must be considered in proceeding from barge sizes to lock sizes. It has been mentioned that long narrow tows require less power for equal speeds, and barges will be so assembled if navigating conditions permit. Curvature and width of the navigating channel have a great effect on the practical length of a tow. Although it may be desirable to operate a tow of twelve barges on the Ohio River under very favorable conditions arranged two wide and six long, such a tow could not operate on any Ohio River tributary and probably would not be considered safe for open-river navigation because of higher stream velocities. In restricted channels, having radii of curvature as small as 1,200 ft, it may be necessary to ignore all considerations of towing resistance and arrange tows two barges wide and one barge long for sufficient steerability to negotiate the sharp turns without backing and flanking. Of course, there will be many combinations between these two extremes. Tows are being built up to as many as twenty-four barges on the Ohio and Mississippi rivers. Fig. 22 shows the number of barges making up tows passing through a lock on the middle Ohio River during a 6-month period. The most recurrent size of tow is one that contains six barges, and there is a sharp decrease in the number of tows containing more than ten barges. This trend represents a lock-size effect since most tows of ten barges passed with a single lockage (lock size, 110 ft by 600 ft), whereas a larger number of barges nearly always required a double lockage. There are enough tows containing sixteen or twenty barges to indicate that double lockages are considered worthwhile for this number and possibly to indicate that longer locks or twin locks might be desirable.

Towboats.—The size of towboats has a minor effect on lock size, not only because there is only one of these per tow, but also because improved designing has made it possible to produce towboats of greater power and smaller hull size than in the past. Towboats of fairly recent design, of the class from 2,000 hp to 2,400 hp, now vary in width from 38 ft to 58 ft and in length from 180 ft to 240 ft. It appears likely that boats of this power will be 50 ft by 200 ft or less in the future. Furthermore, slack-water navigation conditions such as are produced by locks and dams are favorable to the use of less powerful towboats than the foregoing, and on waterways of limited width and sharp curvature horsepowers of more than half of those mentioned would be of no value.

Make-Up of Tow.—On inland waters of the United States barges are almost always pushed by the towboat. In narrow channels of excessive curvature or in rough water, barges are sometimes towed astern; but it is a dangerous practice because of the uncontrollability of the barges. In the ordinary tow 1 ft for fenders should be allowed between barges in computing the width of the tow and 1.5 ft should be allowed on each side of the tow in computing lock width in order to facilitate entrance. At least 10 ft, and preferably 20 ft, should be allowed in lock length to minimize the possibility of gate damage.

STANDARDIZATION OF LOCK SIZES

There has been no particular effort to standardize lock sizes in the United States; but, as waterways become interconnected and as transportation patterns become more fixed, there would be some advantage in doing so. A lock size of 110 ft by 600 ft has established itself as a kind of standard because it is wide enough to accept either the 26-ft barge or the 35-ft barge without wasting space. Since these sizes include the vast majority of barges and since (even on the large navigable waterways) there is an advantage to longer tows rather than to tows of greater width, the size is tolerable even when double lockages are necessary. The lock size of 56 ft by 360 ft has also been widely used although it accepts only 26-ft barges without waste of space. There is nothing that can be foreseen which suggests a change in the 110-ft lock width. If greater tonnages are to be carried in the future, the first step will be to increase depths, and if larger barges become desirable there is no important reason why they should not be approximately 50 ft wide and thus substantially fill a lock 110 ft wide. Transportation experience would seem to indicate that barge sizes will not increase except for certain special cargoes because of the desirability of keeping loads within the limit of individual customer requirements and facilities. Therefore, a lock width of 110 ft is considered an acceptable standard for the future. The length of 600 ft is satisfactory for barge sizes since barge lengths generally are close to, but less than, 200 ft; but for the larger rivers a length of 1,200 ft must be considered since tows of this length have already been operated to a limited extent and future increases in traffic will make them more common. On slightly more restricted waterways a length of 800 ft may have a place. The 56-ft-wide lock is not so promising for the future because it will not accept 35-ft barges properly. To accommodate these and also the 26-ft barges, with the least waste, the next width less than 110 ft should be 83 ft. A still smaller but acceptable width, although it would be accompanied by more wasted space, would be 74 ft, and, since the difference between this and 83 ft is so small, it appears that a minimum lock width of 83 ft would be justified in most cases. Corresponding lengths, depending on the nature of the waterway, would be 400 ft, 600 ft, or 800 ft on the basis that barges as long as 195 ft must be accommodated.

CONCLUSIONS

Five general conclusions can be drawn from this paper:

1. Navigation locks are structures capable of long life;
2. Lock sizes adopted in the past frequently have been too small to utilize the full life of the structure;
3. Lock sizes should be determined on the basis of economic studies with consideration given to additional, and sometimes intangible, factors such as (*a*) future increases in traffic, (*b*) undesirability of delays at locks, (*c*) future increases in size of tows, and (*d*) high cost, in many cases, of providing additional or augmented locks;
4. Barge and towboat sizes will not change enough in the future to render the presently acceptable lock sizes obsolete providing the lock sizes are large enough to accept larger tows with less delay;

5. Acceptable standard lock sizes for future construction, in feet, are:

Width	Length
110	600
110	800
110	1,200
83	400
83	600
83	800

REFERENCES AND ACKNOWLEDGMENTS

Data have been extracted from publications[10,11] prepared by the Board of Engineers for Rivers and Harbors, Corps of Engineers, and by L. A. Baier.[12]

Advice regarding barge construction was furnished by the Dravo Corporation in Pittsburgh. Statistics regarding tow make-up and movements were supplied by the office of the Division Engineer, Ohio River Division, Corps of Engineers.

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[10] "Locks and Dams of the United States, Navigation Data," prepared by the Board of Engrs. for Rivers and Harbors, Corps of Engrs., U. S. Army, Washington, D. C., December 31, 1945.

[11] "Transportation Lines on the Mississippi River System, 1946," *Transportation Series No. 4*, prepared by the Board of Engrs. for Rivers and Harbors, Corps of Engrs., U. S. Army, U. S. Govt. Printing Office, Washington, D. C., 1946.

[12] "The Resistance of Barges and Flotillas," by L. A. Baier, *Transactions*, Soc. of Naval Archts. and Marine Engrs., 1947.

DISCUSSION

Basil Standing,[13] Assoc. M. ASCE.—When turbulence is likely to occur in the lock chamber as the result of entrained air expelled in the flow, Mr. Griffin recommends placing the culverts well below the elevation of the lower pool, so that the hydraulic gradient is above the roof of the culvert. If this were to entail considerably increased costs for extra excavation, an alternate measure might be the fitting of bleeder valves, operated manually or automatically, at selected points along the culvert.

A model to be tested at the Whitworth Research Laboratories, University of Manchester (Manchester, England), will comprise a number of culverts in a lock 800 ft long and 100 ft wide, with standpipes at several predetermined points along each culvert to permit entrained air to escape. It would be interesting if Mr. Griffin would discuss the relative potential merits of such air valves.

Another point that deserves serious consideration is the question of the settlement of any silt carried through the culverts when a lock is being filled from, say, a tidal river. Depending on the degree of turbulence produced, the silt may be either: (*a*) Deposited in its entirety at some place in the vicinity of the inlet ports, eventually building up into banks or mounds sufficiently high to form serious obstructions, at low water, to vessels in the lock, or (*b*) distributed, more or less evenly, over the entire floor area of the lock. Thus the two requirements—(1) a minimum of turbulence of the kind tending to cause motion of vessels and thus increasing live stresses, and (2) a flow, from the entry ports, of such type as to avoid the creation of uneven silt deposits or the disturbance of even silt deposits already in existence—may be conflicting, and model tests may indicate the need for a compromise between the two.

The Symposium provides a complete picture of the many considerations involved in the design of lock systems.

A. Frederick Griffin,[14] M. ASCE.—The pertinency of the two problems presented by Mr. Standing in his discussion is fully realized. At the Alton Locks, No. 26 in the Upper Mississippi River series of locks and dams, the tops of the filling and emptying culverts are very close to extreme low water. At such times the lift at the locks is about 24 ft. The main lock is 110 ft wide and 600 ft long. The auxiliary lock is 110 ft by 360 ft. The problem of air entrainment was carefully considered during the design of the locks and, as a result, vent systems with 12-in. risers spaced some 75 ft apart were installed in each lock wall. The vents in the river wall open to the atmosphere on the river side of the wall, slightly above upper pool elevation. Those in the landward and intermediate walls are each connected to a 36-in. header parallel to and above each culvert at approximately upper pool elevation. The 36-in. pipes discharge into the bullkhead recesses at the downstream valves. The probable degree of efficiency of the vents is unknown as varying amounts of turbulence are evident in the lock. The lockmaster at Lock and Dam No. 26 advises that water and air are jetted from the vents and 36-in. headers during low tailwater and he considers that the systems are efficient because of the hissing sound that occurs at most other stages. All valves at this installation are the Tainter-gate type, sealing against the upstream side of the valve recess.

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[13] Lecturer in Engineering, Manchester Univ., Manchester, England.

[14] Chf. Engr., Office of Div. Engr., Upper Mississippi Valley Div., Corps of Engrs., U. S. Army, St. Louis, Mo.

Model tests for the Pickwick Lock (110 ft by 600 ft) on the Tennessee River, made in the Hydraulic Laboratory, University of Iowa (Iowa City), demonstrated that best results were obtained with the Tainter valves facing downstream, the reverse of previous installations. The top of the culvert is 5 ft below normal lower pool in this instance and the lift is about 57 ft. The report on model tests states

> "* * * with the tainter valve faced downstream, the water surface in the valve recess drops to the top of the culvert for only a brief period, just at the completion of the opening of the valve, then rises as the pressure in the culvert increases."

These tests also demonstrated the value of sealing the valve-bulkhead recesses to prevent ingress of air at those points, thus obtaining what has been termed "a closed system." In connection with model tests for the Guntersville Lock (60 ft by 360 ft), another Tennessee River lock (the top of culvert below normal lower pool is 9 ft and the lift about 39 ft), it was found that the prototype with valves faced downstream and equipped with vents, could be operated satisfactorily with the open culvert system. With the closed culvert system, vents are unnecessary. Experience indicates that it is best to keep the ingress of air into the culvert system to an absolute minimum. It is difficult, if not impossible, to remove entrained air before it reaches discharge ports. It is evident that vent systems can be of considerable benefit where it is impracticable to place culverts below the hydraulic gradient or install a "closed system," as described.

Silting in the vicinity of filling ports in the lock chambers has not proved to be a serious problem. Silting upstream from the miter gates is of some concern. Experience has indicated that movement of miter gates to and fro near the gate recesses, together with some flushing, can be of considerable help in overcoming this problem. Silting in the lock approaches is taken care of by occasional dredging. It is thought that the use of a small dredge, clamshell excavator, or perhaps a drag, supplemented by flushing via the filling culverts, would provide a reasonable method of removing silt deposited in the lock chamber.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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TRANSACTIONS

Paper No. 2450

MEASUREMENT AND ANALYSIS OF SUSPENDED SEDIMENT LOADS IN STREAMS

By Martin E. Nelson,[1] M. ASCE, and Paul C. Benedict,[2] Assoc. M. ASCE

Synopsis

The measurement and the analysis of fluvial sediment have been the subjects of a cooperative study conducted since 1939 by several agencies of the United States government. In this paper the various phases of the study which pertain to suspended sediment, particularly the history of sediment research and the equipment and methods used in making suspended sediment measurements, are reviewed. Theoretical and experimental analyses of the phenomena of sediment transportation, and the basic principles for accurate sampling, are set forth. During the course of the study a new laboratory apparatus, a sedimentation tube for making particle size analyses of sediment samples, and two new types of suspended sediment samplers were developed. The US D-43, depth-integration sampler, is designed for sampling throughout the depth of the stream whereas the US P-46, point-integration sampler, is used principally for sampling at a point, but may be used equally as well for depth integration.

Rôle of Fluvial Sediment Measurements in Engineering

Rivers are classed among the most important basic natural resources in almost every country on the globe. Throughout historical times they have been exploited and developed for water supply, navigation, irrigation, water power, flood control and soil conservation, and other purposes for public benefit. Improvements on sediment bearing streams have generally been made in the past without due consideration of the effects of sedimentation on the life and utility of the projects. As a result, the efficiency and value of important engineering works have depreciated rapidly and in many instances maintenance costs have been exorbitant. Difficulties in operation and maintenance have developed because of reservoir sedimentation, deposition or degradation in channels, and erosion or gullying in agricultural areas. Not

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Note.—Published in September, 1950, as *Proceedings-Separate No. 31*. Positions and titles given are those in effect when the paper was received for publication.

[1] Engr., Corp. of Engrs., U. S. Army, St. Paul, Minn.

[2] Dist. Engr., U. S. G. S., Dept. of the Interior, Lincoln, Nebr.

anticipated in the planning stage, these difficulties become evident only after the projects are completed.

The solution of any fluvial sediment problem must be based on factual information. The water discharge in the principal rivers of the world has been measured systematically for many years and, in most instances, adequate data are available to permit satisfactory analyses of the hydrologic and hydraulic characteristics required in planning and designing river developments. Only within the past few years has an attempt been made to obtain systematic sediment load records of the same quality as those of water discharge. Furthermore, the techniques and equipment used in sediment sampling have been improvised without due regard to the fundamentals of sediment transportation or the technical requirements for taking representative samples of a water-sediment mixture.

Sediments transported as bed load, or saltation load, merge with the suspended load near the bottom of the stream. However, only the problems connected with sampling of suspended sediment will be considered in this paper.

FEDERAL CONSTRUCTION AND ENGINEERING AGENCIES ORGANIZED TO STUDY PROBLEMS OF SEDIMENT MEASUREMENT

Recognizing the desirability of perfecting the methods for measuring the quantity and for determining the character of sediment loads in streams, several agencies of the United States government organized an informal Interdepartmental Committee in 1939 to sponsor an exhaustive study of all problems encountered in collecting sediment data and, eventually, to standardize accepted methods and equipment. The agencies of the federal government which have actively participated in this investigation are as follows: The Corps of Engineers, United States Army; the Soil Conservation Service (SCS), United States Department of Agriculture; the Geological Survey (USGS), the Bureau of Reclamation (USBR), and the Office of Indian Affairs (OIA), United States Department of the Interior; and the Tennessee Valley Authority (TVA). This study was conducted in cooperation with the Iowa Institute of Hydraulic Research at the Hydraulics Laboratory, State University at Iowa in Iowa City.

The scope of the general project (entitled, "A Study of Methods Used in the Measurement and Analysis of Sediment Loads in Streams") is indicated by the following titles of reports that have been published under the direction of the Interdepartmental Committee:

Report No.	Title
1	"Field Practice and Equipment Used in Sampling Suspended Sediment"
2	"Equipment Used for Sampling Bed-load and Bed Material"
3	"Analytical Study of Methods of Sampling Suspended Sediment"
4	"Methods of Analyzing Sediment Samples"
5	"Laboratory Investigations of Suspended Sediment Samplers"
6	"The Design of Improved Types of Suspended Sediment Samplers"
7	"A Study of New Methods for Size Analysis of Suspended Sediment Samples"
8	"Measurement of the Sediment Discharge of Streams"
9	"Density of Sediments Deposited in Reservoirs"

Based on this series of reports, and on progress reports submitted to the Interdepartmental Committee periodically during the course of the investigation, this paper is essentially a compilation of the more significant facts and information which are considered to be pertinent to the interests of the ASCE Joint Division Committee on Sedimentation in Reservoirs.

HISTORY OF SEDIMENT SAMPLING

Records of past civilizations in China, Mesopotamia, and Egypt indicate that man has always experienced difficulties and hazards from the sediment carried by natural streams. However, the manner in which fluvial sediment is transported and deposited has been investigated only in comparatively recent years.

The first sediment measurements of record[3,4] were made in the Rhone River in 1808 and 1809. Other early observations were made by Baurath Blohm[5,6] in the Elbe River at Harburg, Germany, from 1837 to 1854 and by M. Baumgarten[6,7] in the Garonne River in southwestern France, from 1839 to 1846.

The earliest measurements of sediment loads in the United States were made in the Mississippi River by A. Talcott[3,4] in 1838. C. G. Forshey[3,4,6,8] made extensive observations in the lower Mississippi in connection with the studies of the late A. A. Humphreys, Hon. M. ASCE, and Henry L. Abbott in 1851 and 1852. Sediment samples were taken also near the mouth of the Mississippi from 1877 to 1898[8,9,10] and at several stations along the Mississippi and Missouri rivers[6,7] from 1879 to 1881. The development of irrigation along the rivers of the southwestern part of the United States was difficult because of the heavy sediment loads carried by these streams. To solve this problem, sediment measurements were begun in many of these rivers near the close of the nineteenth century. Samples were first taken in the Rio Grande[11] in 1889 and 1890. Observations have been made regularly in the Rio Grande since 1897, in the lower Colorado River since 1909, and in the upper Colorado River Basin since 1925. A detailed investigation of sediment loads in the Missouri River[12] and its tributaries was conducted in 1929 and 1930. Beginning in about 1930 extensive sediment surveys have been made in many other streams of the United States. Interest in sediment information and the practice of making observations of sediment loads have been growing steadily in nearly all countries of the world since the turn of the century.

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[3] "Report on the Physics and Hydraulics of the Mississippi River," by A. A. Humphreys and Henry L. Abbott, *Professional Paper No. 13*, Corps of Engrs., U. S. Army, 1876, pp. 132–148.

[4] *Ibid.*, Appendix A.

[5] "Ueber die in fliessenden Wasser Suspendiert erthaltenen Sinkstoffe," by Baurath Blohm, *Zeitschrift des Architekten und Ingenieur-Verein zu Hanover*, 1867, p. 276.

[6] "The Suspension of Solids in Flowing Water," by Elon Huntington Hooker, *Transactions*, ASCE, Vol. XXXVI, 1896, p. 239.

[7] "Navigation Fluviale; Garonne," by M. Baumgarten, *Annales des Ponts et Chausées*, Vol. 2, 1848, pp. 1–157.

[8] "Sediment Investigations on the Mississippi River and Its Tributaries Prior to 1930," *Paper "H,"* U. S. Waterways Experiment Station, Vicksburg, Miss., 1930, pp. 4–18.

[9] "Report of the Chief of Engineers, U. S. Army," U. S. Govt. Printing Office, Washington, D. C., 1894, p. 1345.

[10] "Report of the Chief of Engineers, U. S. Army," U. S. Govt. Printing Office, Washington, D. C., 1898.

[11] "Water Resources of the Rio Grande Basin," by Robert Follansbee and H. J. Dean, *Water-Supply Paper No. 358*, U. S. Geological Survey, U. S. Govt. Printing Office, Washington, D. C., 1915.

[12] "Missouri River Report," by L. G. Straub, *House Document No. 238*, 73d Cong., 2d Session, 1935, pp. 1189–1202.

THEORY OF SEDIMENT SUSPENSION

Recent studies of fluid turbulence, reported in 1937, made in connection with other phases of hydraulics, have been used by E. W. Lane and A. A. Kalinske,[13] Members, ASCE, to explain the sediment suspension phenomenon. The analytical basis for the turbulence concept of the transportation of suspended sediment has been presented in engineering literature. For the purpose of this paper it will suffice to present only a rational analysis of this concept. The turbulence concept of sediment suspension is now generally accepted, and it is believed that an understanding of it will provide an effective aid in planning and conducting a sediment measurement program.[14]

In turbulent flow the direction of the current at a given point changes rapidly and haphazardly. Although the flow at the point has a general forward motion, in a short space of time small areas of flow, or eddies, fluctuate in horizontal and vertical directions. These fluctuations are irregular and spontaneous and do not follow any definite sequence. The magnitude of the current also changes, fluctuating about a mean value in a manner similar to the variation in the direction of flow.

Sediment particles are raised from the stream bed by the shear or tractive force of the stream exerted on the boundaries of the channel and are moved upward by momentary upward eddy currents. The upward movement is counteracted by the force of gravity, which tends to settle the particles, and by the momentary downward eddy currents. When particles are caught in a current moving upward at a rate greater than their settling velocity, they will be transported upward. On the other hand, the particles will move downward if they are surrounded by water moving downward or by water moving upward at a rate less than their settling velocity. Normally the sediment concentration in a stream increases toward the bottom. Consequently, upward currents travel from a region of higher concentration to one of lower concentration, and thus have a greater potential of sediment transport than downward currents. Moreover, since the quantity of water moving upward and downward must be equal to maintain a given water surface elevation, more sediment will be transported by upward currents than by downward currents. The sediment load distribution in a stream vertical is said to have reached equilibrium when the downward rate of sediment movement through any horizontal plane (due to the force of gravity and downward components of eddy currents) is equal to the upward rate, resulting from the upward eddy components.

The settling velocity of sediment particles is a function of their apparent specific gravities. Consequently, coarse particles will settle faster than fine particles; and, in order to maintain a stable concentration distribution, the excess of sediment transported upward over that carried downward must be greater for coarse particles than for fine particles. This condition prevails in a natural stream because the vertical distribution of sediment varies with particle size, the concentration increasing more rapidly toward the bottom for coarse sediments than for fine sediments. For particles with low settling rates equilibrium will be attained with a very small differential concentration with respect to depth; and, therefore, the finer the particles, the more gradual becomes the concentration gradient, and, conversely, the coarser the particles, the steeper becomes the concentration gradient.

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[13] "The Relation of Suspended to Bed-Materials in Rivers," by E. W. Lane and A. A. Kalinske, *Transactions*, Am. Geophysical Union, 1939, pp. 637–641.

[14] "Engineering Calculations of Suspended Sediment," by E. W. Lane and A. A. Kalinske, *ibid.*, 1941, pp. 603–607.

The settling rates of solid particles increase with size but not in a uniform ratio. The settling rates for particles smaller than $\frac{1}{16}$ mm vary approximately as the squares of the diameters, whereas, for extremely large particles, the settling rates vary approximately as the square roots of the diameters. The $\frac{1}{16}$-mm size is usually considered to represent the approximate division point between sediments classed as silts and those classed as sands. Because of their high settling velocity, sand particles and those considerably larger than $\frac{1}{16}$ mm are more concentrated near the bed than near the surface, the concentration gradient increasing with the size. In general, clay and silt particles are uniformly distributed in a stream. The distribution of sediment in a stream vertical is an important factor to be considered in conducting a sampling program, especially in the selection of sampling points and in the analysis of the data.

METHODS OF MAKING SEDIMENT MEASUREMENTS

The sediment samples of scattered investigations recorded during the first half of the nineteenth century were taken at the surface of the stream at only one vertical in a cross section. Mr. Forshey, studying the Mississippi River at Carrollton, La., in 1851, took samples at the surface, middepth, and bottom of three verticals—one near each bank and one at midstream. Since that time, as the value of sediment data has become more apparent and the phenomenon of sediment transportation has become more clearly understood, observers have gradually expanded the network of samples taken in a stream cross section. Where a thorough examination of the sediment distribution is to be made, ten or more verticals may be sampled and ten samples or more may be taken from each vertical. In most sampling programs conducted on major streams since about 1908, three verticals have generally been used and about three samples have been taken in each vertical. The verticals have been variously spaced as at the quarter points, at the centroids of equal area, or at the centroids of equal discharge. The latter method has a rational basis and, more and more, it is coming into prominent use. At sampling stations where the sediment distribution has been correlated and is periodically checked by a complete network of samples, it is frequently the practice to take single samples in routine observations. These samples are taken at the surface, at 0.6 depth, or from a vertical integration. Correction factors based on complete cross section sediment surveys are usually applied to the single determination of sediment concentration.

If the sediment carried in suspension were uniformly distributed throughout a stream cross section, the determination of the total sediment load would be a comparatively simple procedure. A single sample taken at any point would then indicate the sediment concentration or the weight of sediment per unit volume of water, and the sediment discharge would be the product of the sediment concentration and the water discharge. The concentration of fluvial sediment, however, varies more or less throughout any cross section of a stream and to obtain an accurate measurement of the sediment discharge these variations must be considered.

The distribution of sediment in a stream cross section is generally found to vary more vertically than laterally. Consequently, greater importance should be placed on the spacing of sampling points in the vertical to provide an accurate measurement of the sediment load.

A suspended sediment sampler is not designed to measure the saltation load, or the bed load which rolls or slides along the stream bed. The latter types of sediment do not travel with the same velocity as the water, and therefore cannot be estimated correctly from the amount trapped in a unit volume of water. Moreover, it is difficult to measure the velocity of the water near the bottom where saltation or bed load movements take place. An approximate determination of the rate of sediment movement by rolling and sliding can be made by trapping this sediment in a bed load sampler at typical points across the width of the stream. However, the quantitative accuracy of such measurements is subject to question owing to uncertainties as to the effects of disturbances caused by the sampler resting on the stream bed, the position and shape of sand dunes (if any are present), and the calibration of the sampler.

Many suspended sediment samplers will trap some of the saltation load if placed so near the bottom that the intake is in the region of the saltation load movement; and, if the screens used in the construction of the bed load samplers are extremely fine, they may trap some of the suspended load. The errors which result from such overlapping of the sampling zones are usually regarded as being so small as to be negligible. An accurate determination of the sediment carried by a stream would require measurement of both the suspended load and the bed load. Usually, however, the nature of the problem under consideration is such that the suspended load plays a more prominent rôle than does the bed load. It is possible sometimes to construct a drop over which both the bed load and the suspended load pass, and where both can be sampled at the same time by passing a suitable sampler through the overfalling nappe.

In any type of sampling program the variability and characteristics of the sediment load and of the water discharge are the most important factors to consider in planning a sampling schedule. Both these variables are affected by the climatic, hydrologic, and topographic features of the stream and its watershed. The quantity, distribution, and intensity of precipitation; the ranges in temperature; the size, shape, topography, and geology of the drainage basin; and the soil and vegetal cover are all factors which influence the rate of sediment discharge. Other things being equal, the sediment discharge varies with the size of drainage area, and, obviously, the heavier and more intense the precipitation the greater will be the quantity of sediment removed from the watershed. A friable soil, unprotected by vegetal cover, will yield higher sediment runoff than a more stable soil. These and other factors must be considered in preparing a sampling schedule for any particular location.

The majority of streams pass more than 50% of the total annual flow during flood periods, at which time the sediment concentration also is greater than during periods of nearly normal stage. Both the water discharge and the sediment concentration vary more rapidly on a rising stage than on a falling stage. Consequently, it is important that sediment samples be taken more often during flood periods than at other times, and more frequently on the rising side of a flood hydrograph than after the crest has passed. (Various sampling schedules which have been used in actual field practice are described in detail in reports No. 1 and No. 8 of the sediment series.)

SUSPENDED SEDIMENT SAMPLERS

In the earliest attempts to measure sediment loads in streams the very simplest types of samplers were used. An open container or a pail was used by J. L. Riddell[3] in sampling the Mississippi River at New Orleans, La., from 1843 to 1848. Similarly crude and unscientific equipment was used until about 1900. Since that time much thought and ingenuity have been exercised in developing sediment sampling equipment. Various individuals engaged in sediment work have devised samplers which operate in accordance with their respective interpretations of the sediment transportation phenomenon. Consequently, many different instruments and apparatus have been introduced. The sampling action and the techniques adopted for operating the samplers differ widely. As a result, the data obtained with one sampler are not necessarily comparable to those of another, and in the light of the studies made by the Interdepartmental Committee, many samplers used extensively in the past are now known to be decidedly unsatisfactory.

The various samplers that have been investigated in this study can be classified, as to their mode of operation, into two general types—instantaneous and time integrating. Many samplers do not actually qualify for either classification because their design, sampling action, or method of operation eliminates them from the categories of true instantaneous or integrating samplers. As the name implies, the instantaneous sampler is designed to trap a specimen of the water-sediment mixture passing the selected sampling point at a desired instant. The time-integrating sampler, on the other hand, takes the sample more slowly over an extended period of time to obtain a specimen in which the momentary fluctuations in sediment load are averaged. The majority of samplers in use at present are designed to operate on the principle of time integration. Although this principle is basically sound and conducive to accurate sampling, the samplers are often constructed or operated in such a way as to preclude time integration dominating the sampling procedure.

Time-integrating samplers may be subdivided again into two types—point integrating and depth integrating. The point-integrating sampler is held stationary at a point in the sampling vertical while the sample is taken. The depth-integrating sampler is lowered to the bottom of the stream and raised again to the surface at a uniform rate, sampling continuously during both periods of transit, or it may be designed to sample only from the surface to the stream bed. The depth-integrating sampler is designed to take a mean sample in the vertical, giving uniform weight to the increments of the water-sediment mixture taken at the various levels.

The requirements of an ideal sediment sampler are summarized as follows:

1. The velocity within the cutting circle of the intake should be equal to the stream velocity;
2. The intake should be pointed into the approaching flow and should protrude upstream from the zone of disturbance caused by the presence of the sampler; and
3. The sample container should be removable and suitable for transportation to the laboratory without loss or spoilage of the contents.

Furthermore, the sampler should:

4. Fill smoothly without sudden inrush or gulping;
5. Permit sampling close to the stream bed;
6. Be streamlined and of sufficient weight to avoid excessive drag;
7. Be rugged and simply constructed to minimize the need for repairs in the field; and
8. Be as inexpensive as possible, consistent with good design and performance.

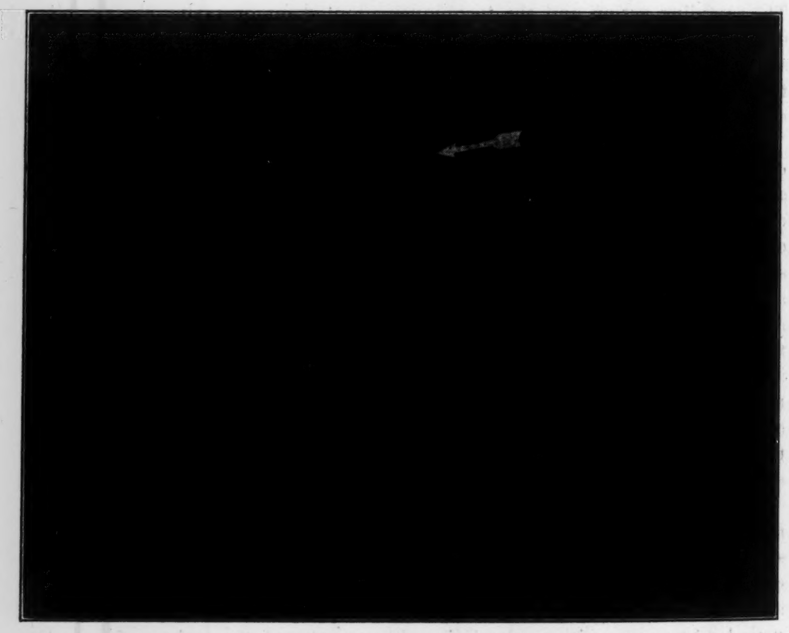

FIG. 1.—LABORATORY SEDIMENT SAMPLING APPARATUS

In the early phases of the project the Interdepartmental Committee authorized an exhaustive laboratory and analytical study of existing integrating samplers to locate faulty design and operation procedures that would contribute to inaccuracies in the collection of sediment samples. Basically, the principle of integration sampling requires that a continuous filament of the water and sediment be taken from the flowing stream in the same relative proportions as exist in the immediately surrounding filaments of flow. The sample must be taken from the stream without modifying the velocity and with minimum disturbance to the adjacent flow lines. In order to avoid any disturbing effects due to the presence of the sampler itself, the instrument should be streamlined as much as practicable; and the point of entry of the sample should be in the form of a nozzle projecting upstream beyond the zone where there is any appreciable disturbance in the flow pattern. With these criteria in view, a laboratory apparatus was constructed so that the essential features of the samplers could be tested under conditions simulating as nearly as possible those encountered in the field to determine the effects of deviations from ideal sampling conditions on the accuracy of the sediment data. The apparatus consisted essentially of a loop conduit in which a column of water could be circulated at constant values of velocity, sediment concentration, and temperature. The test section of the 10-in. square conduit, with transparent sides and top for observation, is shown in Fig. 1.

It was recognized that the velocity at which the sample is withdrawn from the stream relative to that of the stream itself, at the sampling point, should be unity and that deviations from this relationship would result in erroneous sampling. Should the sample be withdrawn at a slower velocity than that of the stream, the flow lines in the filament being sampled would diverge while the sediment particles in the filament, being of higher specific gravity than the water, would tend to follow paths that diverge less than the flow lines. Consequently, the sample would contain a greater concentration of sediment than the stream. Conversely, if the intake velocity were greater than that of the stream, the approaching flow lines would converge while the sediment particles would converge to a lesser degree and the sample would contain a lower concentration than the stream. The magnitude of error due to incorrect sampling rates in suspensions of varying concentrations of 0.45-mm sediment is shown in Fig. 2(*a*). Figs. 2(*b*) and 2(*c*) show a decreasing effect of incorrect sampling rates with decreasing sediment size, the error becoming insignificant for very fine sediment. The magnitude of the stream velocity does not materially alter the results indicated by incorrect sampling rates as shown in Fig. 3(*a*).

The effect of rotating the axis of the intake nozzle slightly with respect to the approaching flow lines is illustrated in Fig. 3(*b*). An angular deviation of 20° has an inappreciable effect on the sampling accuracy, whereas a 30° approach velocity results in an error of about 7% in the sediment concentration at a relative sampling rate of unity. It is apparent from these tests that the sampler should be sufficiently sensitive to current direction to orient its intake within 20° of the direction of the flow.

The standard nozzle used in the laboratory tests had a cutting diameter of 0.25 in. Nozzles of 0.15-in. diameter and 0.375-in. diameter were also tested. The size of the nozzle appeared to have no effect on the sediment concentration obtained as long as a unity sampling rate was used, as shown in Fig. 3(*c*).

One of the more serious faults of existing samplers is the lack of streamlining. Parts of the samplers obstruct the intakes and disturb the flow

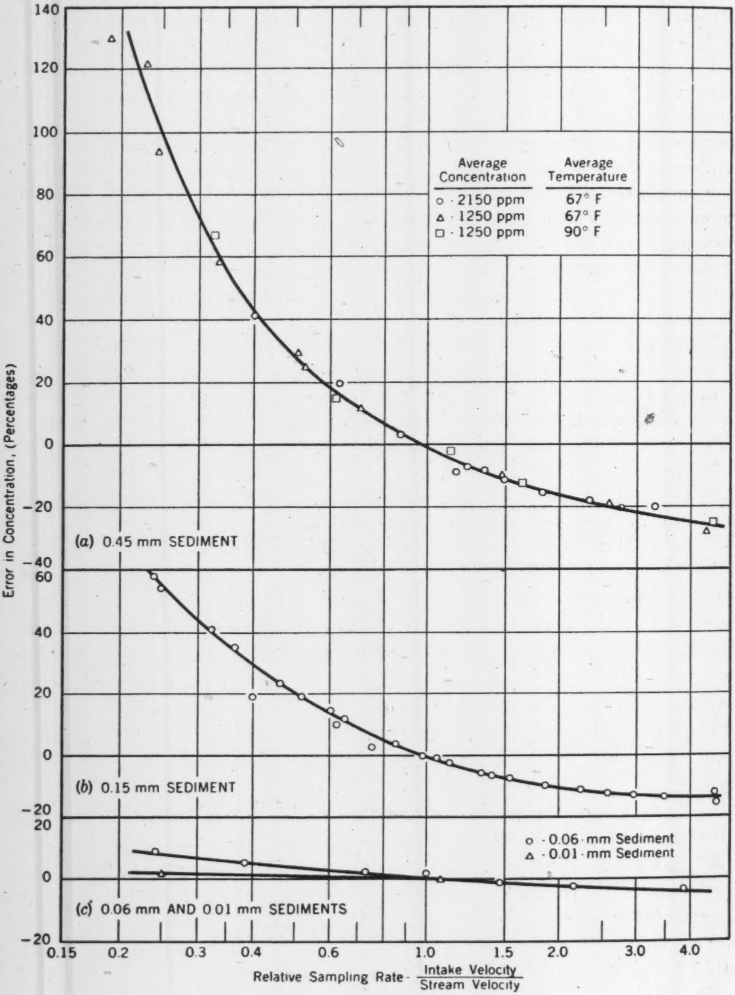

FIG. 2.—EFFECT OF SAMPLING RATE ON SEDIMENT CONCENTRATION (STANDARD NOZZLE; STREAM VELOCITY, 5 FT PER SEC)

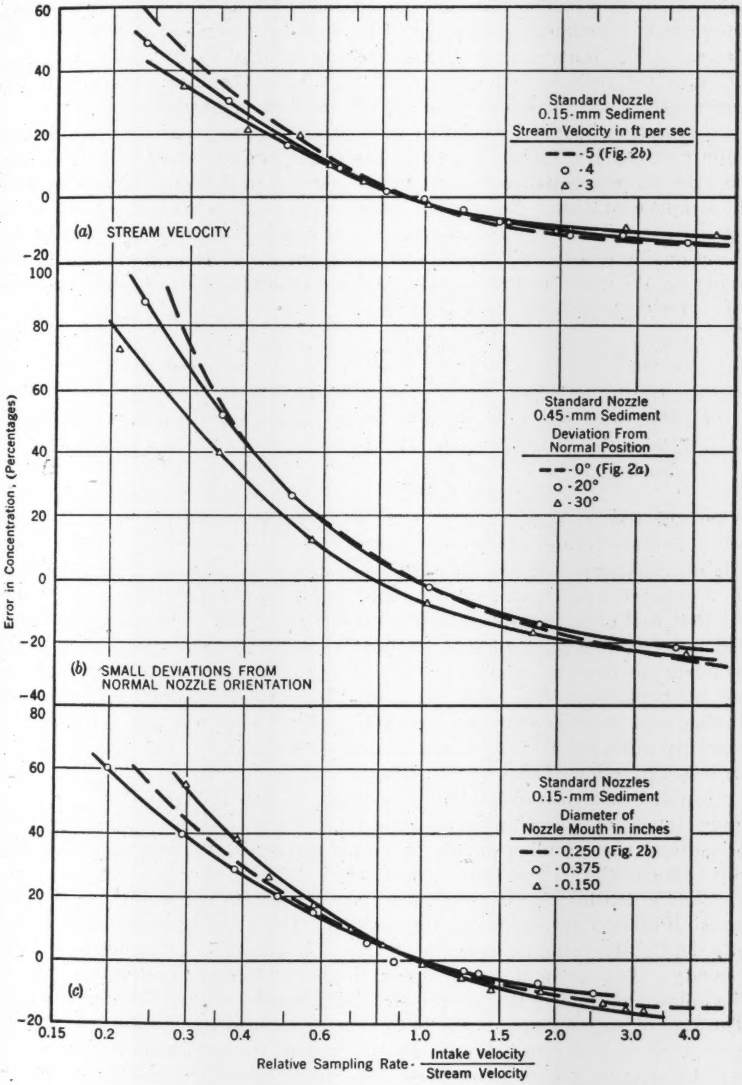

FIG. 3.—FACTORS THAT AFFECT THE ERRORS IN SEDIMENT CONCENTRATION

filaments being sampled. In many instances the intakes are normal to the direction of flow. The flow lines must bend 90° to enter the intake and, consequently, sediment particles will tend to be separated from the sample filament. The laboratory tests indicated an error of about 16% in sampling 0.15-mm sediment and 0.45-mm sediment with the intake normal to the current, the error decreasing to about 2% for 0.06-mm sediment.

A sampler which is designed to open below the water surface of a stream is subject to so-called "initial inrush" when the intake is opened unless provision has been made to equalize the air pressure inside and the water pressure outside the sample container. None of the integrating samplers in use by the cooperating agencies at the beginning of this study was designed to eliminate this objectionable feature. The effect of initial inrush on the rate of sampling is shown by the graph in Fig. 4. For each set of curves, the approximate filling rate after the initial inrush is as follows:

Samplers (Fig. 4)	Cubic centimeters per second	Remarks
(*a*) Straub	15	....
(*b*) Frazier	21	Except the 4-ft and 8-ft depths
(*c*) Colorado	17	....
(*d*) Rock Island	23	....

When the Frazier sampler (Fig. 4(*b*)) was opened at a depth of 36 ft in still water, half the volume of the container was filled in 1 sec or less. The sampling rate during this period was about 225 cu cm per sec, whereas the remainder of the sample was taken at a rate of about 21 cu cm per sec or less. Obviously, the first half of the sample which is taken during the period of initial inrush will not be integrated and, consequently, will not be representative of the sediment suspension existing at the sampling point due to segregation of sediment particles from the sampled filament. Furthermore, momentary fluctuations in sediment concentration and velocity will not be integrated correctly during the initial inrush period. As shown by Fig. 4 the degree of error is a function of the submergence.

Initial inrush in point-integrating samplers can be avoided by providing some method of balancing the air pressure in the container and the hydrostatic pressure at the sampling point. A simple method of automatically obtaining this balance will be described subsequently.

Simple depth-integrating samplers are subject to effects similar to initial inrush if lowered into a stream too rapidly. The velocity in the intake nozzle of a depth-integrating sampler must always be equal to the immediate stream velocity. The air in the sample container is compressed instantaneously by the inflowing liquid so that its pressure balances the external hydrostatic head. The rate of air contraction due to increasing hydrostatic pressure must never exceed the normal rate of liquid inflow. Hence, the maximum speed of lowering the sampler occurs when these two factors are equal and no air escapes from the container.

At any instant during the operation of a depth-integrating sampler, the air volume in the container is a function of the hydrostatic head and the prior rate of filling. If the reduction in air volume necessary to balance a given change in hydrostatic head at any instant is greater than the normal inflow, the actual inflow will occur at a rate higher than the local stream velocity and some inflow may occur through the air exhaust. Samples collected under these conditions will not be weighted according to the vertical velocity curve.

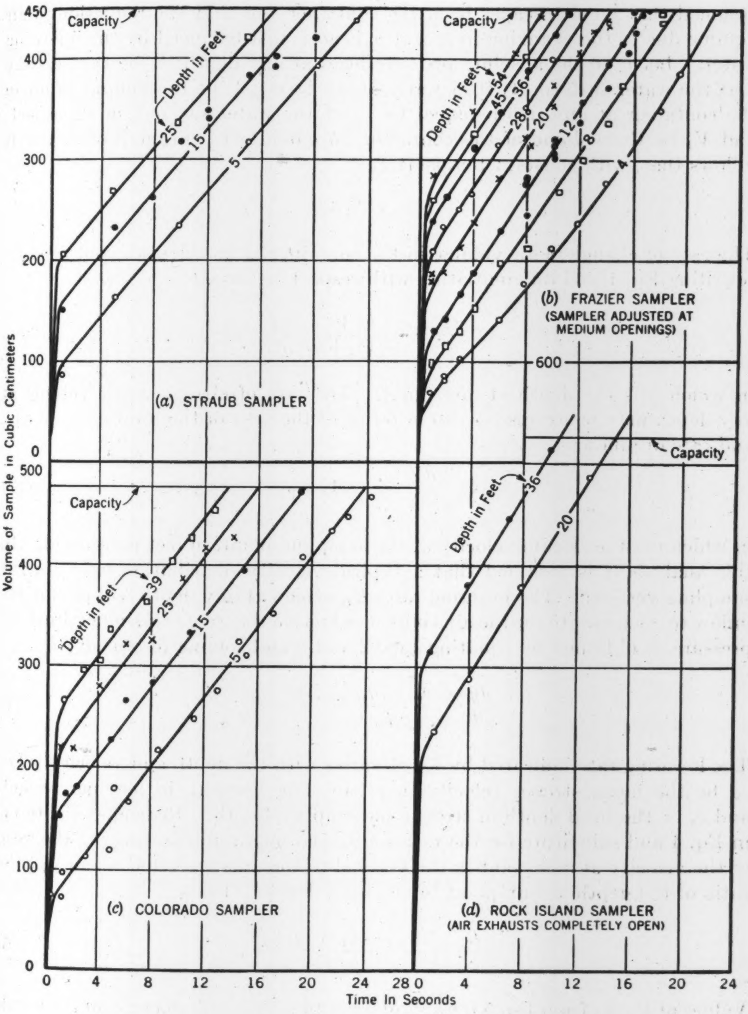

FIG. 4.—FILLING CHARACTERISTICS OF SAMPLERS IN STILL WATER

On the other hand, if the normal inflow exceeds the air volume reduction necessary to balance a given change in hydrostatic head, air will escape from the sample container so as to permit the actual inflow to occur at a rate equal to the stream velocity and the sample will be correctly weighted according to the vertical velocity curve.

In determining the limits of the permissible sampler transit rates, it will be assumed that no air escapes from the container and that the reduction in air volume due to the changing hydrostatic head is just balanced by the inflowing water. Let h be the absolute pressure head at any depth; h_1 be the pressure h at the water surface (34 ft of water at sea level); V be the volume of air in the container at any depth (equal to V_1 at the water surface) in cubic feet; and V_1 be the volume of the container, in cubic feet. From Boyle's law it follows that, with constant temperatures,

$$h\,V = h_1\,V_1 \tag{1}$$

The rate of change in air volume in the container at any depth is obtained by rewriting Eq. 1 and differentiating with respect to time, t

$$\frac{dV}{dt} = \frac{h_1\,V_1}{-\,(h_1+y)^2}\,\frac{dy}{dt} \tag{2}$$

in which y is the depth at any point. The rate of change in air volume at any depth may be expressed also in terms of the area of the intake nozzle, A_n, and rate of inflow:

$$\frac{dV}{dt} = -\,A_n\,v \tag{3}$$

in which v is the stream velocity at the sampling point, in feet per second. In this analysis it is assumed that a typical velocity distribution exists in the sampling vertical. The lowering rate, R_L, which at any depth will permit the inflow to compensate for the air volume reduction due to increasing hydrostatic pressure, is obtained by equating Eqs. 2 and 3 and solving for dy/dt,

$$\frac{dy}{dt} = \frac{A_n\,v\,(h_1 + y)^2}{h_1\,V_1} = R_L \tag{4}$$

The lowering rate indicated by Eq. 4 varies with the depth and velocity. Let v_m be the mean stream velocity in a sampling vertical, in feet per second; and y_s be the total depth of stream, or sampling depth. Rearrange the terms in Eq. 4 and substitute for the ratios v/v_m and y/y_s the following: v_r, the ratio of the velocity at any point in the vertical to the average velocity; and y_r, the ratio of the depth at any point to the total depth. Then,

$$\frac{R_L}{v_m} = \frac{A_n\,v_r\,(h_1 + y_s\,y_r)^2}{h_1\,V_1} \tag{5}$$

Values of R_L/v_m from Eq. 5 for a sampler with a $\frac{3}{16}$-in. (diameter) intake nozzle and a 1-pt sample container for various stream depths are shown in Fig. 5.

As the lowering rate must be uniform in order to obtain a true depth-integrated sample, the maximum permissible uniform lowering rate is the lowest that is shown in Fig. 5 for any depth. That is, for depths greater than 19 ft the maximum permissible uniform lowering rate is that shown for the surface of the stream. For depths less than 19 ft, the maximum rate (which is that indicated for the bottom of the vertical) varies with the depth. The maximum permissible uniform lowering rates with respect to the mean stream velocity, R_L/v_m, to be used with intake nozzles of various diameters ($\frac{1}{8}$ in., $\frac{3}{16}$ in., and $\frac{1}{4}$ in.) in streams of various depths, are shown in Fig. 6. If a uniform lowering rate equal to, or less than, the maximum indicated by Fig. 6 is selected for given sampling conditions, air will escape from the container while the sampler is in transit. As the sampler descends, the pressure in the container will always be equal to the hydrostatic pressure surrounding the sampler. The velocity in the nozzle at the point of intake will be equal to the local stream velocity, air will be expelled at a rate proportional to the inflow and to the change in hydrostatic head, and a true depth-integrated sample will be taken.

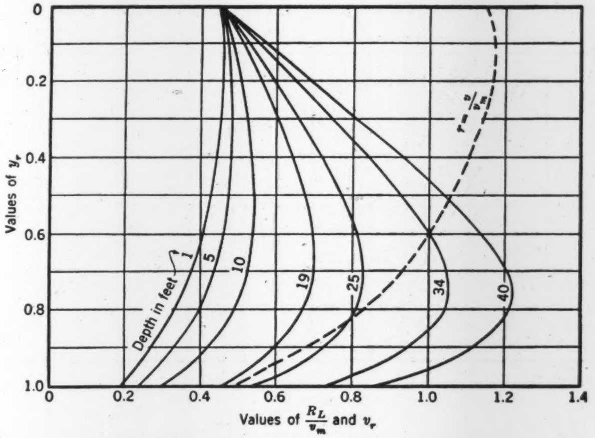

FIG. 5.—VALUES OF $\frac{R_L}{v_s}$ FOR A DEPTH-INTEGRATING SAMPLER, FOR A TYPICAL VERTICAL VELOCITY CURVE FOR ANY DEPTH

The minimum lowering rate that will allow the sample container to become full just at the instant the sampler reaches the bottom is given by the equation:

$$R_L = \frac{A_n v_m y_s}{V_1} \quad \text{.......................} (6)$$

In computing the minimum values shown in Fig. 6 it was assumed that the stream would be sampled throughout the entire depth. If lower transit rates than the indicated minimum are used, the sample container will be filled before the integration is completed, circulation will take place, and, consequently, the sample will indicate greater concentration than exists in the stream. When a simple depth-integrating sampler is used, on both the descending and ascending trips, the minimum transit rates must be twice those indicated in Fig. 6.

The maximum uniform lowering rate that can be used in streams deeper than 19 ft is the rate for the surface of the stream from Eq. 5. By substituting this rate in Eq. 6 a maximum computed sampling depth of 39 ft is obtained. That is, by closing the intake and air exhaust when the sampler makes contact with the stream bed, a depth of 39 ft can be sampled when a typical velocity distribution exists and the container is completely filled. When the velocity distribution is uniform in the vertical section, v_r is unity at any point and the maximum sampling depth is 34 ft. If the sample is taken during both the descending and ascending trips, the maximum depths that can be sampled would be one half of the foregoing values for the respective velocity distributions. In practice the sample container should not be completely filled in order to avoid circulation and spillage. Therefore, the practical maximum sampling depths should be limited to about 15 ft and 30 ft for the two-way and one-way integrations, respectively.

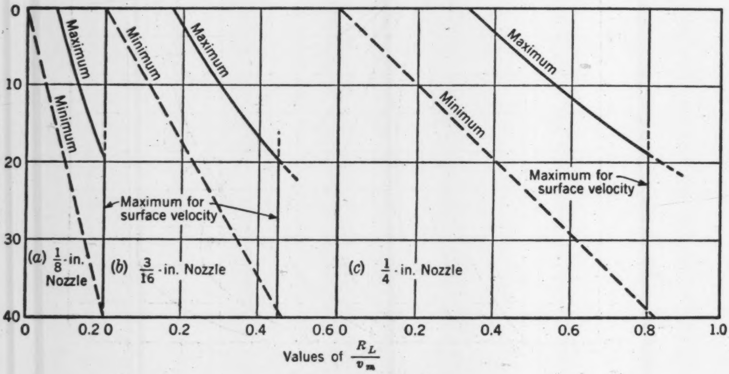

FIG. 6.—MAXIMUM AND MINIMUM VALUES OF $\frac{R_L}{v_m}$ FOR A DEPTH-INTEGRATING SAMPLER WITH A CONTAINER OF 1-PT CAPACITY

When a sample is taken during both the descending and ascending trips of the sampler, the maximum lowering rate should not exceed that indicated in the foregoing analysis. However, since the hydrostatic head decreases on the upward trip, an ascending rate greater than the lowering rate may be used provided the angle of relative velocity is not excessive. If different descending and ascending rates are used, they must be constant throughout their respective trips in order to obtain a properly weighted sample.

The times necessary to fill a 1-pt container at various intake velocities, with three diameters of nozzles, are shown in Fig. 7. Under some conditions, it might be impractical to use the maximum permissible transit rates indicated by the preceeding analysis. The ordinary stream gaging reels of 1-ft and 2-ft circumferences will not permit transit rates greater than about 2 ft per sec

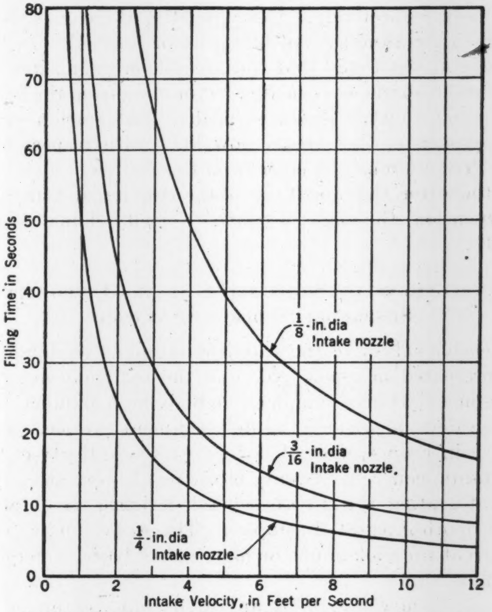

FIG. 7.—RELATION OF FILLING TIME TO INTAKE VELOCITY FOR A 1-PT CONTAINER

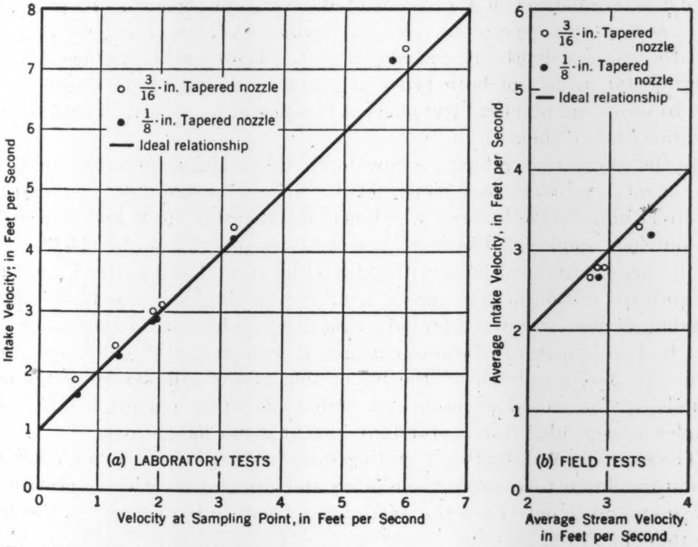

FIG. 8.—INTAKE CHARACTERISTICS OF FIFTH EXPERIMENTAL DEPTH-INTEGRATING SAMPLER

and 3.5 ft per sec, respectively. Consequently, these may be the limiting speeds with which the sampler can be raised or lowered. In the laboratory study of samplers, it was found that considerable error in sampling resulted if the axis of the intake nozzle was oriented 30° or more from the line of approaching flow. A similar effect is produced in depth-integration sampling due to the angular approach of the stream flow lines with respect to the moving intake nozzle. To minimize the error from this source (which is a function of the velocity of the water, the transit rate of the sampler, and size of the sediment particles in suspension) the angle of relative velocity should be restricted to a maximum of 20°.

DEVELOPMENT OF THE UNITED STATES INTEGRATING SUSPENDED SEDIMENT SAMPLERS

A comprehensive survey of the sediment sampling equipment used during past years is presented in report No. 1 of the sediment series. This study discloses that some sixty-five samplers, embodying a number of different designs, have been developed since the first sediment samples were taken from the Rhone River in France, more than 138 years ago; thirty of these samplers were of the instantaneous type, twenty of the point-integrating type, and eight of the depth-integrating type, with several designed to obtain both point-integrated and depth-integrated samples. The survey indicated that all the samplers in current use violate one or more of the basic principles of accurate sediment sampling.

In considering a new design, it appeared that a sampler of the point-integrating type would cover the greatest range of field conditions in the routine determination of the sediment discharge of streams. However, the initial cost of this type of sampler was estimated to be considerably greater than the simpler depth-integrating sampler. Therefore, it was decided that experimental models of both types of samplers should be developed with a view to providing practical and efficient equipment for as wide a range of field conditions as possible.

In the course of developing a new depth-integrating sediment sampler, five experimental samplers were constructed. Successive samplers were designed to correct undesirable features which field or laboratory tests had indicated in the previous models. The final sampler, designated the US D-43, has a cast bronze streamlined body with integral horizontal and vertical tail vanes. The forward section of the sampler is hinged to provide access to the sample container recess, and it is adapted to receive nozzles with diameters of from $\frac{1}{8}$ in. to $\frac{1}{4}$ in. which can be interchanged if damaged or if a different size is required. A spring latch on the under side of the sampler holds the head securely in the closed position and permits it to be opened readily. The sampler is suspended from a standard current meter hanger bar.

The experimental depth-integrating samplers were tested in a glass-walled laboratory flume to observe their poise and orientation with respect to the flow lines, and to determine the relationship between the velocity in the flume and the filling rate.

In field use, the depth-integrating sampler is lowered and raised at a uniform rate; consequently, the sampled filament will enter the intake nozzle at an angle. This flow condition can be simulated by holding the sampler in a fixed position at an angle to the current. Preliminary laboratory tests made with the sampler oriented with respect to the flow showed some variation in the relative sampling rate depending on whether the nozzle was tilted upward or downward. The tests were not conclusive, but they indicated that the relative sampling rate determined from tests made with the sampler in a fixed horizontal position should exceed unity slightly in order to obtain an approximate average rate of unity when collecting a depth-integrated sample.

A few field tests were made with the experimental samplers in streams near the hydraulics laboratory to determine the practicality of the samplers as field instruments and their general limits of use in streams of medium depths and velocities. The intake characteristics determined in laboratory and field tests on the fifth experimental depth-integrating sampler are shown in Fig. 8. Photographs of the 38-lb and 50-lb US D-43 samplers are shown in Fig. 9.

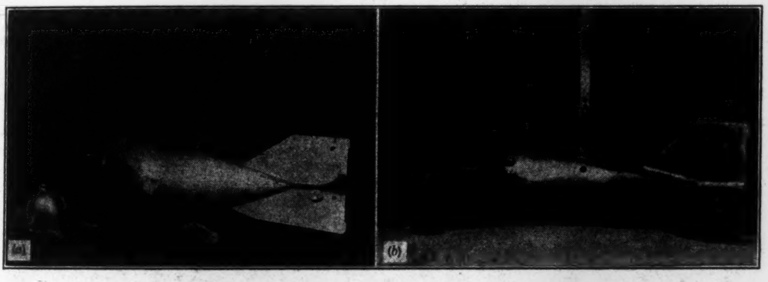

FIG. 9.—US DEPTH-INTEGRATING SUSPENDED SEDIMENT SAMPLER, US D-43

(a) Thirty-eight Pound
(b) Fifty Pound

The point-integrating suspended sediment sampler developed in this project embodies all the features of the depth-integrating sampler; but, in addition, it is designed to accumulate a water-sediment sample which is representative of the mean sediment concentration at any selected point in a stream vertical during a short interval of time.

The body of the US point-integrating sampler consists essentially of a streamlined cast bronze shell, an inner recess to hold the sample container, and an air chamber having a volume about five times that of the sample container. The air chamber and the sample container are interconnected by tubing and a passage through a spring-actuated valve. When the intake and the air exhaust, which are also controlled by this valve, are closed prior to lowering the sampler into a stream, the pressure equalizing passage is open. The air chamber has a permanent opening at the bottom of the sampler through which water can enter, thereby compressing the air in the sample container as the sampler is submerged. Consequently, initial inrush does not occur when the intake and the air exhaust passages are opened.

Three experimental point-integrating samplers were constructed during the course of the investigation. The results of the calibration tests of the second sampler indicated that the intake characteristics approached the ideal, as shown by the data presented in Fig. 10.

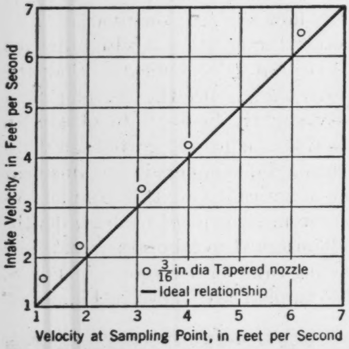

Fig. 10.—Intake Characteristics of Second Experimental Point-Integrating Sampler

The third experimental sampler embodies other improvements in addition to those developed on the first two samplers. The valve is driven by a flat clock spring and is controlled by an escapement for three positions: (1) Intake and air exhaust closed, equalizing passage open; (2) intake and air exhaust open, equalizing passage closed; and (3) all passages closed. A solenoid, energized by two or more 6-volt lantern batteries, is used to trip the escapement allowing the valve to turn from one position to the next. From the batteries located on the operating rig, the current is supplied to the solenoid through a two-conductor current meter cable. The third experimental point-integrating sediment sampler, designated US P-46, weighs about 100 lbs, unsubmerged (see Fig. 11).

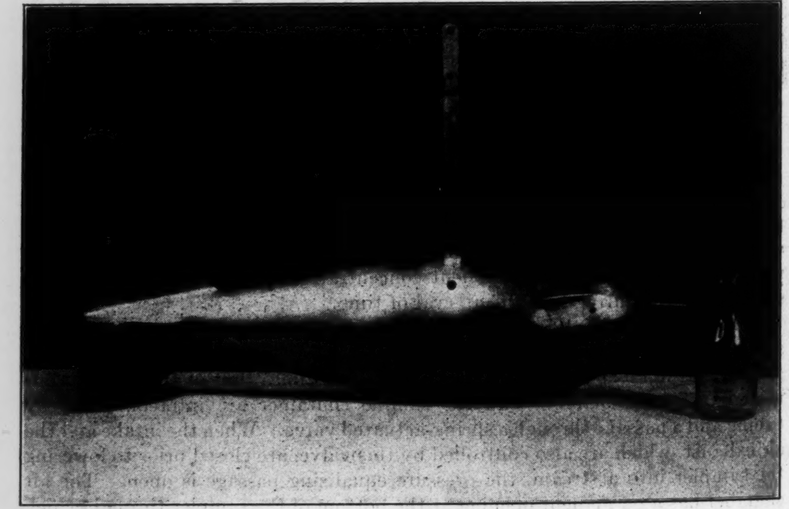

Fig. 11.—US Point-Integrating Suspended Sediment Sampler, US P-46

The US P-46 sampler is primarily a point integrator, but can be used also as a depth integrator. With the valve in the open position a sample can be integrated from the surface to the bottom and return, or the valve can be closed at the bottom so that the sample is taken only on the downward trip. The sampler may also be lowered to the stream bed with the valve in the closed position; then it is opened and the sample taken on the ascending trip only; or, the sampler may be used to integrate any part of a sampling vertical. Thus, in extremely deep streams the vertical may be sampled either by the point method or by successively integrating parts of the vertical.

As part of a program of field testing and standardization of sampling equipment, three US D-43 samplers were placed in routine operation in a cooperative sediment project in Iowa in September, 1943. Since that date, approximately fifty-five US D-43 samplers (50 lb) have been constructed for several of the agencies for routine sampling operations in streams of medium depths and velocities.

The progress being made in the development of the improved point-integrating sampler has been somewhat slower owing to the inherent mechanical difficulties involved and the initial construction cost of a small number of samplers.

FIELD TESTS ON SUSPENDED SEDIMENT SAMPLERS

In May, 1943, several suspended sediment samplers of the US D-43 model were distributed to field offices of the cooperating agencies for examination and trial. The second experimental model of the point integrator, US P-43, was made available for 60-day periods to field offices of the agencies desiring to make similar tests with this sampler. Thirteen field agencies have submitted reports, test data, and discussions of the experiences encountered in these tests.

The field tests brought to light many inaccuracies in existing sediment samplers, some of which had been anticipated on the basis of previous laboratory analyses. Disparities in sediment data were due to either faulty sampling techniques or faulty sampler design, or both. A summary of the results of the comparative field tests is presented in Table 1. Most of these tests were made in streams with relatively shallow depths, moderate velocities, finely divided sediments, and uniform vertical distributions of both velocity and sediment concentration—conditions which tend to minimize the effects of faulty sampler action and improper sampling methods. In such streams a departure from the optimum intake velocity, the streamlining, and the sampling procedure would not materially affect the representativeness of the sample. On the other hand, sampler design and operating technique become increasingly important as the physical characteristics of the stream depart from the favorable sampling conditions which prevailed during most of the tests. For instance, in comparative tests made with the Texas and the US D-43 samplers in streams where velocities and sediment concentrations were relatively low, a mean sampling ratio of 1.04 was obtained; whereas in another series of tests with the same two samplers and identical sampling methods, but with higher stream velocities, coarser sediment, and greater concentrations, the sampling ratio dropped to 0.48.

Comparative tests made with the Colorado River and US D-43 samplers in the San Juan River near Bluff, Utah, where the sediment was relatively coarse, indicated a sampling ratio of 0.82. Most depth-integrating samplers that were tested collected samples too low in concentration. However, the Omaha and Rock Island samplers collected an excess of sediment, the mean sampling ratios being 1.16 and 1.13, respectively.

A striking illustration of the error that may result from taking a single sample in a vertical is given by a further analysis of the data obtained in the comparative tests with the Faris and the US D-43 samplers. Seven series of comparisons were made in which the Faris samples were taken at the 0.6 depth and the US D-43 samples were obtained by the usual method of depth integration. The tests were made in the Rio Grande at Albuquerque, New Mex., in May and June, 1944. The average sediment concentration in the Faris samples was 0.48 of that in the US D-43 samples. Size analyses indicated that in the range of particle sizes smaller than 0.0312 mm, the ratio of concentration in the Faris samples to that in the US D-43 samples was about 0.75. However, this ratio decreased rapidly with increasing particle sizes. For particles 0.5 mm in diameter, the ratio dropped to 0.12. The relative percentages of the various sizes of particles taken with the two samplers are shown in Fig. 12. It is apparent that the coarser material present in the

TABLE 1.—SUMMARY OF SAMPLING RATIOS; FIELD TESTS ON SUSPENDED SEDIMENT SAMPLERS

Federal agency	Samplers compared	No. of comparisons	SAMPLING RATIO		
			Minimum	Maximum	Average
(1)	(2)	(3)	(4)	(5)	(6)
(a) UNITED STATES POINT-INTEGRATING SUSPENDED SEDIMENT SAMPLER D-43					
Bureau of Reclamation	Tait-Binckley[a]	9	0.66	1.00	0.81
Corps of Engineers					
Tulsa, Okla.	Texas[a,b]	78	....	....	1.04
	Vicksburg[a]	30	....	....	1.10
Albuquerque, N. Mex.	Wading sampler	6	0.77	1.05	0.98
	Faris[a,b]	7	0.76	1.02	0.91
	Faris[a,b]	7	0.31	0.82	0.48
Sacramento, Calif.	Sacramento	6	0.50	2.50	1.10
Rock Island, Ill.	RI[c]	8	0.68	1.45	1.13
Omaha, Nebr.	Omaha	6	0.97	1.37	1.16
	Omaha[a]	116	....	....	1.18
Geological Survey	Colorado River	22	0.59	1.00	0.82
(b) UNITED STATES POINT-INTEGRATING SUSPENDED SEDIMENT SAMPLER P-43					
Bureau of Reclamation	Tait-Binckley	24	0.67	2.14	1.26
Corps of Engineers:					
Huntington, W. Va.	ORD[d]	5	0.89	1.01	0.94
Tulsa, Okla.	Texas	23	....	....	0.97
	Vicksburg	52	....	....	1.03
Omaha, Nebr.	Omaha	115	....	....	1.18
Little Rock, Ark.	Texas	19	....	....	0.95
Geological Survey	Colorado River	4	0.53	1.00	0.80

[a] Sampling ratio is affected by both sampling method and sampling action. [b] Faris and Texas samplers are identical. [c] Rock Island. [d] Ohio River Division.

US D-43 samples was obtained largely from the vertical below the 0.6 depth, that part of the vertical which was not represented in the samples taken by the Faris sampler.

In May, 1945, samples were taken with the US P-43 point-integrating sampler at the same station in the Rio Grande. The sediment concentration and velocity in the vertical are shown in Fig. 13, and the size gradation of the sediment samples taken at various depths is shown in Fig. 14. It is evident from these data that samples taken at the 0.6 depth are not representative of the total sediment concentration in the vertical and that they do not indicate correctly the variation of particle sizes. Regardless of the type of sampler used, the 0.6 depth or any other single point method of sampling is not reliable where particle sizes and concentrations vary appreciably from the surface to the bottom of the stream.

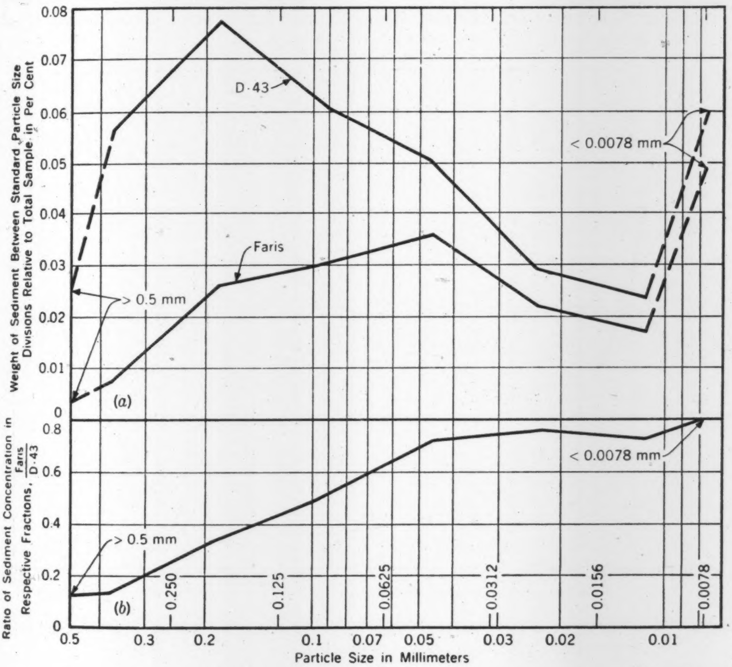

FIG. 12.—COMPARATIVE TESTS WITH FARIS AND US D-43 SAMPLERS, ALBUQUERQUE, N. MEX.; DISCHARGE, 5,000 CU FT PER SEC

LABORATORY ANALYSIS OF SEDIMENT SAMPLES

In conducting a sediment sampling program it is desirable that the methods employed in analyzing sediment samples should be equally as accurate and efficient as those used in taking the samples from the stream. For this reason the Interdepartmental Committee undertook an exhaustive study of the laboratory devices, instruments, apparatus, and methods developed by various investigators for sediment analysis work. This study is covered in report No. 4 of the series, under the title, "Methods of Analyzing Sediment Samples."

The determination of total sediment concentration and size distribution of relatively coarse particles (larger than about $\frac{1}{16}$ mm in diameter) generally does not present any great problem and usually can be accomplished by methods employed in soils or materials laboratories. In addition to the determination of the weights of the original sample and the residue, the most common method of determining total sediment concentration involves three basic steps—sedimentation, decantation, and evaporation. This process may be supplemented by filtering through filter paper or crucibles to expedite the operation and to eliminate discrepancies due to dissolved solids.

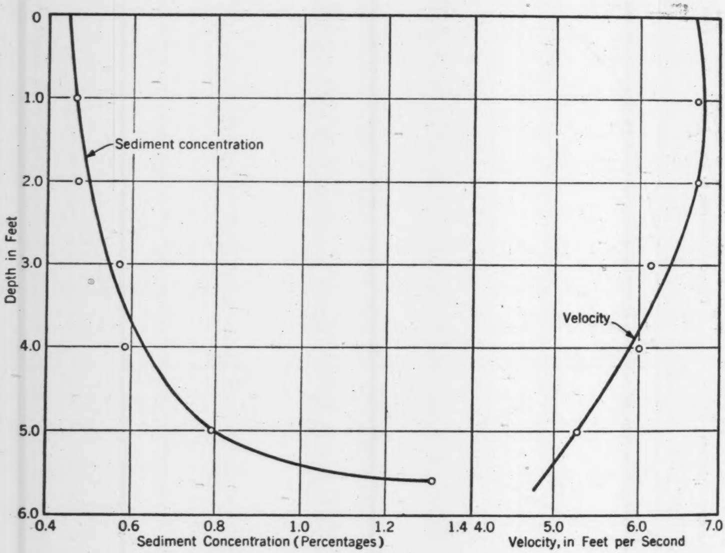

FIG. 13.—SEDIMENT CONCENTRATION AND VELOCITY TESTS SAMPLER US P-43; RIO GRANDE RIVER, ALBUQUERQUE, N. MEX.; MAY 4, 1945; DISCHARGE 4,027 CU FT PER SEC

The size distribution of sediment particles larger than $\frac{1}{16}$-mm diameter is determined, in most instances, by sieving. This method has reached a high degree of development in the United States due, no doubt, to its simplicity and reliability and to the fact that very accurate sieves are available. The behavior of sediment particles in streams, and their deposition in reservoirs and detention basins, depend to a greater degree upon their settling properties than upon their geometric size. Consequently, attempts have been made to develop sedimentation methods for size analysis of coarse particles and this promises to be an important field for future research.

The size analysis of finer sediments has stimulated much interest and many ingenious devices and processes have been developed. In analyzing soils, such common methods as elutriation, pipette, hydrometer, manometer, and plummet have been tried, but inherent limitations in their capacity to determine particle size and concentration often preclude their use in analyzing suspended sediment samples. The Interdepartmental Committee sponsored a special study of sedimentation methods of size analysis with a view to developing apparatus and techniques of sufficient scope and capacity to satisfy the requirements of suspended sediment investigations. An apparatus called the bottom-withdrawal tube was developed which enables the gradation of particles to be determined for a wider range of concentration and a wider range of size than is possible with any other of the commonly used methods. This device (see Fig. 15) is a graduated glass tube about 100 cm long and 2.54 cm in diameter with a quick acting outlet at the lower end. The sample is first uniformly dispersed in the bottom-withdrawal tube, and then the tube is supported in an upright position. The sediment particles begin to settle to the bottom immediately, the settling velocity being a function of the apparent specific gravity of individual particles.

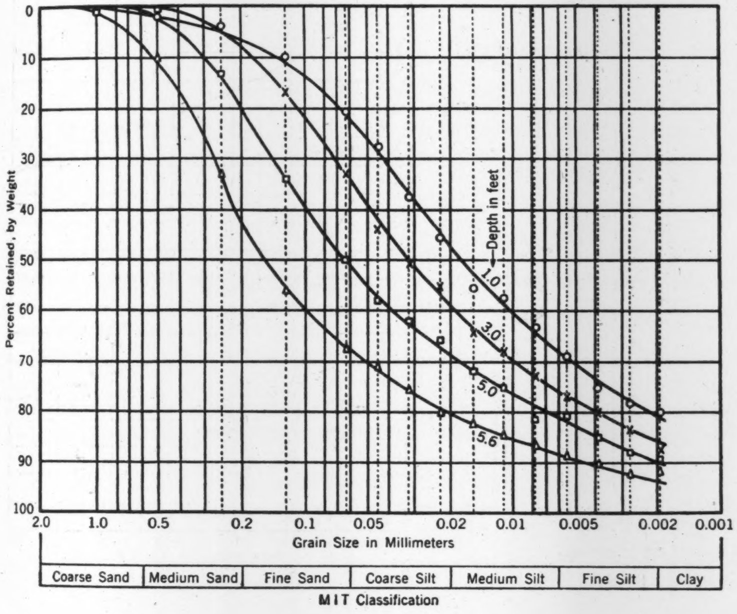

FIG. 14.—SIZE GRADATION OF SAMPLES AT VARIOUS DEPTHS

In a uniformly dispersed suspension such as is employed in the bottom-withdrawal tube, the sediment concentration at any level remains constant until the largest particle in suspension will have had time to settle from the surface to the level in question. Fractions of the sample are withdrawn intermittently from the bottom of the tube, the time element of withdrawals corresponding to the time required for particles of given sizes to settle throughout the length of the water column. Each fraction is then dried and the weight of accumulated sediment determined. The weight of sediment of the same concentration that would remain in the suspension is computed from these data, assuming that the water column remained at the original height. Since the height of the water column actually decreases with each withdrawal, a pro rata correction is made to obtain the amount of suspended material that would remain in a column of constant height. Similarly, the hypothetical settling time for each fraction in a column of constant height is also computed. The computed percentage of material remaining in the suspension at the moment of withdrawal, relative to the total amount of material present in the original sample, together with the corresponding settling time, provide the

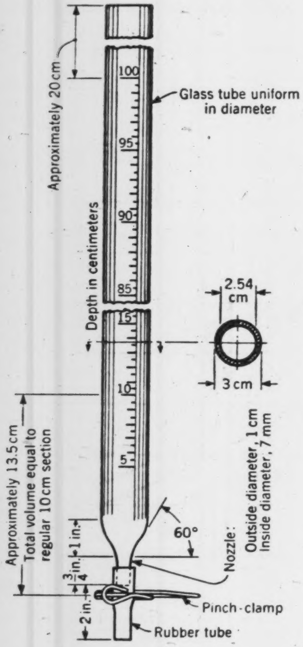
Approximately 20 cm
Glass tube uniform in diameter
Depth in centimeters
100
95
90
85
15
10
5
2.54 cm
3 cm
Approximately 13.5 cm — Total volume equal to regular 10 cm section
60°
1 in.
$\frac{3}{4}$ in.
2 in.
Nozzle: Outside diameter, 1 cm Inside diameter, 7 mm
Pinch-clamp
Rubber tube

FIG. 15.—PROPOSED BOTTOM-WITHDRAWAL SEDIMENTATION TUBE

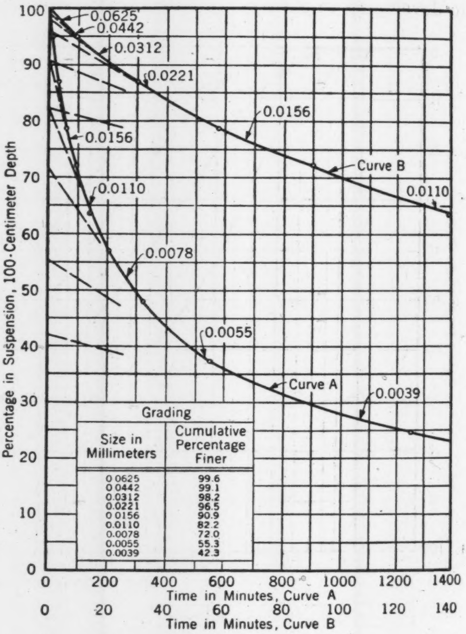

FIG. 16.—ODÉN CURVE, CEDAR RIVER; STATION 80, SAMPLE CC52, MAY 6, 1942 (TEMPERATURE 26° C, CONSTANT DEPTH, VARIABLE TIME)

data necessary to construct an Odén curve[15] (Fig. 16). Extending a tangent from any point on the curve to the ordinate scale will indicate the amount of material in the sample that is finer than the particle size represented by the corresponding time abscissa at the point of tangency. Gradation of the sediment according to any desired scale of particle sizes can be determined in this manner. Since the fall velocity is sensitive to temperature changes in the medium, variations in temperature must be considered in correlating particle sizes and settling time.

The development of the bottom-withdrawal tube and suggested methods of operation are described fully in report No. 7 of the series, under the title, "A Study of New Methods for Size Analysis of Suspended Sediment Samples." This device has been used with marked success in preliminary studies by several field agencies of the federal departments cooperating in this study.

Future Program of Sediment Investigation

The Interdepartmental Committee has received wide recognition for bringing to the attention of the engineering profession the need for accurate sediment data and the inadequacy and inefficiency of present sampling equipment and techniques. The laboratory and field investigations sponsored by the committee provide convincing proof that needed extension and improvement of sediment records can be materially aided by the development and standardization of sampling equipment, methods of measurement, and laboratory analysis, somewhat comparable to the commonly accepted procedures of stream gaging.

At a meeting of the Interdepartmental Committee in April, 1946, it was agreed to dissolve the committee and to transfer its activities and functions to the Federal Inter-Agency River Basin (FIARB) Subcommittee on Sedimentation. The Inter-Agency Committee is composed of representatives of the Corp of Engineers, the U. S. Department of the Interior, the U. S. Department of Agriculture, and the Federal Power Commission, and has as one of its objectives the coordination of the hydrologic activities of these federal departments through the assistance of its several subcommittees. The Subcommittee of FIARB, on Sedimentation, formally took over the activities and the unfinished program of the Interdepartmental Committee in June, 1946. Many phases of the sediment problem fall within the scope and interest of both the federal Inter-Agency River Basin Committee and the ASCE Joint Division Committee on Sedimentation in Reservoirs.

Acknowledgments

The sediment project inaugurated in 1939 has been directed by an informal committee composed of representatives of interested federal agencies and the Iowa Institute of Hydraulic Research. The active membership of the committee has changed from time to time and committee meetings have been open to other personnel interested in and associated with sediment investigations. The following representatives of the cooperating federal agencies have been identified with the committee at one time or another since its organization: Col. A. B. Jones and G. A. Hathaway, Members, ASCE (Corps of Engineers); C. G. Paulsen, M. ASCE, and the late G. L. Parker, M. ASCE (USGS); G. C. Dobson and C. B. Brown, Assoc. M. ASCE (SCS); A. S. Fry, M. ASCE (TVA); C. H. Southworth and J. C. McCaskill (OIA); and W. R. Nelson and W. E. Corfitzen, Members, ASCE (USBR). The possibility of deriving vast mutual benefits from a joint study of fluvial sediment problems of common interest to the federal agencies was conceived by Mr. Hathaway and Mr. Lane. Mr. Hathaway, who was largely responsible for organizing the Interdepartmental Committee, has served as chairman and secretary and has presided over many of the meetings. Mr. Lane, in the capacity of consultant for the Corps of Engineers and representative of the Iowa Institute of Hydraulic Research, supervised and directed the work for the committee at the University of Iowa Hydraulics Laboratory until June, 1942.

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[15] "Manual of Sedimentary Petrography," by W. C. Krumbein and F. J. Pettijohn, D. Appleton-Century Co., Inc., New York, N. Y., 1938.

The following personnel were assigned by their respective cooperating agencies to conduct the library and laboratory research and the investigations at Iowa City: Vernon J. Palmer (SCS); Cleveland R. Horne, Jr., Assoc. M. ASCE, Morgan D. Dubrow, Jun. ASCE, and Mr. Nelson (Corps of Engineers); Victor A. Koelzer, Assoc. M. ASCE, and Mr. Benedict (USGS); Clarence A. Boyll (TVA); Philip M. Nobel, Donald E. Rhinehart, and John W. Stanley (USBR); and Frank W. Parker (OIA). Assistance in the theoretical analysis of sediment transportation, review of reports, compilation of pertinent information and data, and other phases of the project were contributed by the following: Mr. Kalinske (State University of Iowa); S. K. Love, W. D. Collins, L. C. Crawford, M. ASCE, C. S. Howard, Affiliate, ASCE, and the late R. G. Kasel, M. ASCE (USGS); M. A. Churchill, Assoc. M. ASCE (TVA); V. A. Vanoni, M. ASCE, H. A. Einstein, and G. Rittenhouse (SCS); and J. J. Hartigan (Corps of Engineers). The cooperation of field offices of the sponsoring agencies in providing information on samplers and sampling conditions, in making field tests on various samplers, and in developing practical laboratory techniques for the bottom-withdrawal tube is acknowledged.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

TRANSACTIONS

Paper No. 2451

INDUSTRIAL STREAM POLLUTION ABATEMENT

By L. L. Hedgepeth[1]

With Discussion by S. Leary Jones.

Synopsis

In the abatement of industrial pollution of streams a program based on forthrightness, tact, good will, and good faith will be found profitable. This view is derived from the writer's experience on stream sanitation while in the employ of industry and of the public. The industrial waste problem is so complex as to defy standardization. Each case requires informed handling, and actual abatement may best be secured through study of each individual case, rather than the issuance of blanket orders.

The writer's work on major pollution sources in Virginia is cited in some detail in support of his regulatory thesis. The problems of a pulp mill, a wool scouring plant, and a rayon finishing and dyeing plant are used as examples. Also outlined is his method of attack of the complex problem presented by the concentration of industrial and municipal pollution on the South River at Waynesboro, Va.

Tasks of considerable size lie ahead of stream sanitarians in their work for the protection of presently clean streams and the restoration of those now polluted. Some of those awaiting performance are of sobering magnitude. Many, and probably the great majority, will respond to a sound technical approach. On others there is much to learn before success may be expected.

This paper relates principally to the industrial waste phase of stream pollution. Essentially it is an outline of one approach to the solution of this problem —by simplification and working together. It has been found that the employment of utter practicability in the interpretation of law and data, and in the determination of practicable goals of attainment, is a profitable method of attack on this problem.

A program based on forthrightness, tact, good will, and good faith is infinite in its possibilities. Successfully prosecuted, it discourages the development of widely divergent and incompatible views. By keeping discussions on a

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Note.—Published in August, 1950, as *Proceedings—Separate No. 26*. Positions and titles given are those in effect when the paper or the discussion was received for publication.

[1] Staff Technical Consultant, Calco Chemical Div., Am. Cyanamid Co., Bound Brook, N. J.; formerly Executive Secy., Virginia State Water Control Board, Richmond, Va., through December, 1948.

practical level, it also discourages wishful thinking by government and industry and thus avoids much misunderstanding.

These concepts have developed as a result of the observation that the actual accomplishment of stream pollution abatement is primarily a "selling" job. Principles of human persuasion have been applied with gratifying results to the stream pollution problems of the Commonwealth of Virginia which in 1946 started its first comprehensive effort[2] toward protection of presently clean streams and the restoration of those now fouled by sanitary sewage and industrial wastes.

Essentially, the Virginia State Water Control Law is a regulatory approach to the stream pollution problem. All discharges, whether new or old, are placed under individual permit control. In cases of established pollution, the owner may be required to reduce the pollution from time to time under threat of revocation of the permit. In cases of new or increased pollution, the permit is predicated on treatment of the wastes or sewage so as to render each one harmless to the receiving stream. Discharges without permits are forbidden. Penalties are provided.

According to the records of the Virginia State Water Control Board (VWCB), 226 permits to discharge sewage, mostly untreated, were issued in Virginia during the two years ending June 30, 1948 (meeting minute 3, VWCB, November 10, 1948), and 203 permits were issued for established industrial wastes. The latter may be harmless, or polluting, or devastating, depending on the case. Collectively they are massive and, for the most part, destructive in their effects. For example, the following conditions exist: A pulp mill with a B.O.D. (biochemical oxygen demand) load equal to 250,000 people; a rayon plant whose toxic wastes have destroyed virtually all marine life for 60 miles downstream; a chemical plant with wastes equaling 60,000 people in B.O.D. loading; another chemical plant with 110 tons of H_2SO_4 sewered daily; another plant discharging, daily, 2 million gal of waste brine containing 14% solids by weight; and an oil plant, one of several in the neighborhood, whose wastes equal 72,000 people.

These and other crushing loads issue from plants which are important elements in Virginia's economy. The problem is to reduce this pollution and to prevent new loading, without upsetting the state's manufacturing balance. It is an elementary fact that people must have an opportunity to work in order that they may be self-supporting. When industrial pollution abatement interferes with employment, stream restoration becomes most difficult.

On this problem, Mr. Justice Archibald C. Buchanan of the Virginia Supreme Court of Appeals has stated in part:[3]

> "The history of the Water Control Law and the terms of that law leave little room for doubt or speculation as to how its purposes are to be attained. Some waters of the State are clean, others are polluted. Pollution is a recognized evil and is against public policy when caused by one not having a certificate issued by the Board. * * * The end desired is to keep the clean waters clean and to reduce the pollution in the unclean waters. Some of this

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[2] State Water Control Law, Code of Virginia, Richmond, Va., 1946, Chapter 63-b.

[3] American Cyanamid Company Versus Commonwealth of Virginia, Virginia Supreme Court of Appeals, Richmond, Va., June 14, 1948, Record No. 3368.

> pollution comes from industrial waste, discharged by industries invited into the State and furnishing employment to some of its people. The problem is to be dealt with so as to give fair treatment to the industries, to its employees and to the public. That requires a measuring and balancing of the interests involved. Some waters should be kept pure. A measure of pollution in others is necessary. Not all pollution can be abruptly stopped. On the agreed facts here, for example, this industry would have to shut down if immediately required to cease discharging its acid waste into the river. The aesthetic and recreational features involved in the pollution problem are important, but the opportunity to make a living may be even more so."

Waste disposal should be regulated by control measures which are practical. Sound law, impartially enforced, is essential in placing stream pollution control on a "must" basis. Once such a law is provided, and all individuals and agencies are placed under its control, progress is best and most easily accomplished through a good technical analysis of each case and a determination of goals to suit the local needs. This opinion is based on the facts that actual abatement is secured by working on the individual cases, that compulsory action may be directed only against the individual party, and that, if the requirements for any given discharge are selected to suit that discharge, lessened resistance from the owners may be expected.

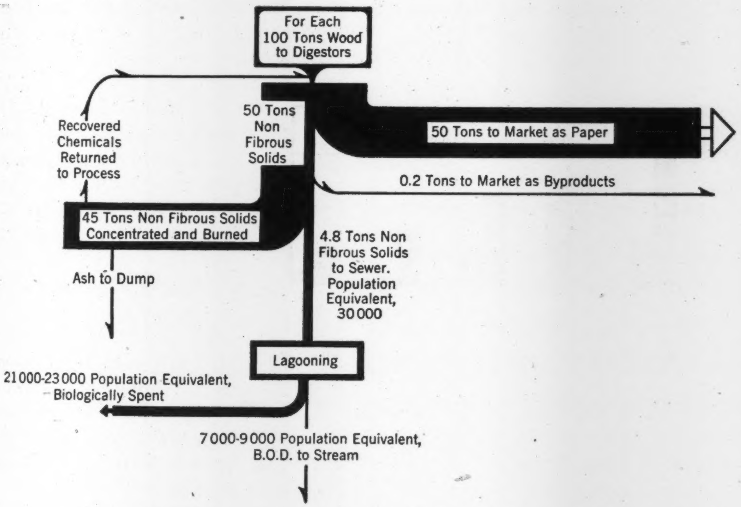
For Each
100 Tons Wood
to Digestors
50 Tons
Non
Fibrous
Solids
50 Tons to Market as Paper
Recovered
Chemicals
Returned
to Process
0.2 Tons to Market as Byproducts
45 Tons Non Fibrous Solids
Concentrated and Burned
4.8 Tons Non
Fibrous Solids
to Sewer.
Population
Equivalent,
30000
Ash to Dump
Lagooning
21000-23000 Population Equivalent,
Biologically Spent
7000-9000 Population Equivalent,
B.O.D. to Stream

FIG. 1.—SIMPLIFIED ILLUSTRATION OF A STREAM POLLUTION PROBLEM AT A SOUTHERN PULP MILL

To illustrate the practical individual approach, the waste water problems of a pulp mill, a woolen mill, and a rayon finishing and dyeing plant have been selected. All these are large plants with significant pollution problems. All are of considerable economic importance to their communities. The management of each has been partly or entirely successful in efforts to find a workable answer to the problem.

Fig. 1 depicts a problem (meeting minute 17, VWCB, June 3 and 4, 1948), involving a typical pulp mill which cooks nonfibrous matter from the wood by the use of a liquor consisting essentially of a mixture of caustic soda and sulfides. Since the soda and sulfur for the cooking liquor make-up are derived from sodium sulfate, this type of mill is also called a sulfate mill.

It is sometimes stated that this type of mill does not pollute streams because the cooking liquor is reprocessed for the recovery of its soda values. This is not true since only the cooking liquor and the stronger washing waters are customarily recovered. The stream pollution load from the remainder of the washing waters may be quite heavy. Examination of Fig. 1 shows that about 10% of the nonfibrous refuse (about 5% of the weight of pulp wood) and also about 10% of the chemicals used in the cooking process are sewered.

In the mill whose problem is depicted by this graph, 500 tons, more or less, of wood is normally pulped daily to produce 250 tons of paper. The result is a daily loss of 25 tons, more or less, of nonfibrous wood solids borne to the sewer in 8 mgd of water. In addition to its odor, color, and resistance to assimilation, this waste has a B.O.D. of 150,000, population equivalent.

The solution to this problem, provided by the owner after considerable voluntary study, consisted of: First, eliminating unnecessary wastes at their source; and, second, supplying aging and biological treatment in very large lagoons for all the sewered wastes. When first reported, these lagoons were reducing the 5-day B.O.D. about 22%.[4] Today they are yielding an over-all reduction of from 70% to 75% in the 5-day B.O.D. load to the stream and the effluent is otherwise less troublesome than before (meeting minute 17, VWCB, June 3 and 4, 1948).

Thus, the owner has greatly improved the stream; and, except during drought conditions, there is not much of a stream pollution problem remaining below this mill. Additional reductions are now being sought through a change in pulp washing equipment designed to reduce further the sewered wastes.

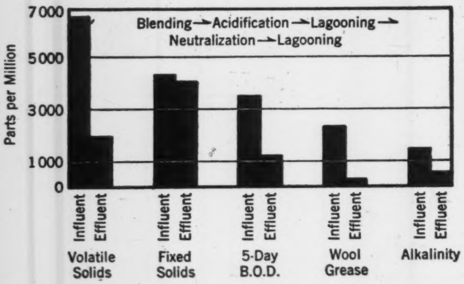

Fig. 2.—Results of Emergency Treatment for Wool—Scouring, Finishing, and Dyeing Wastes

Fig. 2 illustrates what was done with an "unsolvable" problem (meeting minute 15, VWCB, February 4, 1947; meeting minute 20, VWCB, May 19–20, 1947; and meeting minute 13, VWCB, October 1–2, 1947). A conflict was probable between the "no new pollution" requirements of the law and the obstinate fact that a year or more would be required to design and construct a treatment plant to remove the heavy content of wool grease from these wastes, to remove the sludge and scum, to provide an effluent pH-value within the range of from 6.0 to 8.0, and to reduce the flowing load B.O.D. to the level of safety for the receiving stream.

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[4] "Ponding of Sulfate Paper Mill Wastes," by Stuart C. Crawford, *Sewage Works Journal*, July, 1947, p. 621.

Officially a deadlock was imminent. The owner felt that he should be allowed to run his own manufacturing plant, and that time must be granted to build the treatment plant. The authorities felt that it could not be granted under the law. Therefore, it was undertaken unofficially and cooperatively with the owner and his engineers. The engineers risked their professional reputation on an emergency treatment plant employing makeshift apparatus and lagoons; the owner risked his money. Providence helped considerably by producing higher than normal river flows for the next 3 months, thus avoiding stream damage during the no treatment period.

The treatment provided consisted essentially of blending of the several wastes, then acidification, then lagoon storage for 2 weeks, and finally neutralization followed by 3 weeks of lagooning before discharge of the supernatant into the river (see Fig. 2).

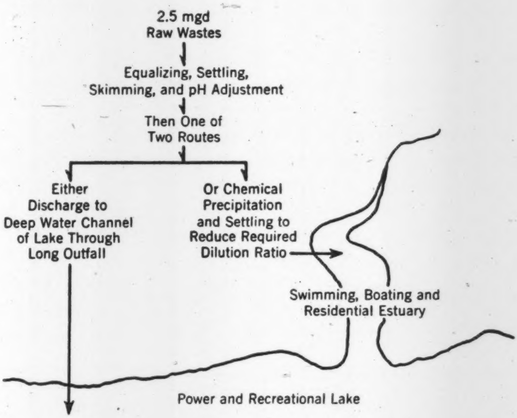
2.5 mgd
Raw Wastes
Equalizing, Settling, Skimming, and pH Adjustment
Then One of Two Routes
Either Discharge to Deep Water Channel of Lake Through Long Outfall
Or Chemical Precipitation and Settling to Reduce Required Dilution Ratio
Swimming, Boating and Residential Estuary
Power and Recreational Lake

FIG. 3.—PROPOSED DISPOSAL METHODS FOR RAYON—FINISHING AND DYEING WASTES

For more than a year this emergency plant retained substantially all the wool grease, adjusted the pH-value, removed the sludge and scum, and otherwise prevented stream damage. It also provided the owner time to build a proper plant, made it possible for more than 2,000 people to remain at work, gave the owner use of his manufacturing property, and prevented the law from compounding an absurdity.

Fig. 3 is an illustration of the alternate solutions devised for a rayon finishing and dyeing company which had purchased a property near a recreational and power lake and then had sought approval for the discharge of the colored waste waters into this normally clear lake (meeting minute 8, VWCB, February 10, 1948; and meeting minute 9, VWCB, September 14, 1948). Fortunately, partial operations for several months provided a period of valuable experience with existing treatment equipment for determining how much of the color could be removed by chemical precipitation and settling. The conclusion was that no combinations of iron or alum coagulants and chlorine would reduce the color so that the full flow of wastes could be absorbed during drought conditions without an esthetic nuisance in that arm of the lake nearest the plant—because at least 300 to 1 dilution with clear lake water was required for satisfactory absorption of the treated wastes being produced by this plant during the study period.

Furthermore, if the raw wastes were equalized, settled, skimmed, and kept between pH-values of from 6.0 to 8.0, they could be disposed by dilution, provided a 900 to 1 dilution ratio was available. Since this ratio was available in the lake proper, but not in the near-by arm, the owner elected to make the additional capital investment for the long pipe line and thus avoid the threat of a slowdown or shutdown of the dyehouse in dry weather.

This case has been selected because of the color problem. For example, 400 ppm of residual color has been proposed as a maximum standard for industrial waste effluents.[5] In the writer's opinion, such a standard will be unenforceable. Some of these dye wastes have been found to be totally resistant to biological treatment and to respond reluctantly to chemical treatment, even when massive doses of coagulants and bleaching agents are applied. Experience strongly indicates that there is no maximum color standard that can be enforced on dyeing and finishing plants and that they should always be located on large streams where high dilutions are available. Given such locations it appears proper and reasonable to limit treatment to settling, skimming, equalizing, and pH-control before dilution. A big muddy river is the ideal location for a plant producing highly colored wastes.

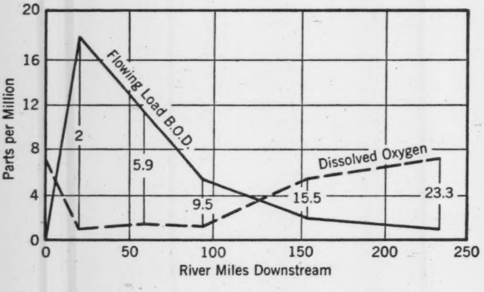

FIG. 4.—OXYGEN BALANCE OF SOUTH RIVER, DOWNSTREAM FROM WAYNESBORO, VA. (AVERAGES FOR JULY, AUGUST, AND SEPTEMBER, 1948)

There is also a workable approach to the group problem involving industrial wastes and sanitary sewage. An example is the stream pollution problem in 24 miles of the South River between Waynesboro and Port Republic, Va., where the South River joins the Middle and North rivers to form the South Fork of the Shenandoah River.[6]

Upstream from Waynesboro this stream is relatively clean and normal. On its 2 miles of flow through the city, it receives the wastes from a large rayon finishing and dyeing plant, a huge cellulose acetate plant, and the city sewage. Below the city, the stream's oxygen drops in summer to from 1 ppm to 2 ppm and at times to zero. The coliform organism population rises to astronomical values. The stream is dark and scum laden. The fishes try to reestablish themselves when cool weather and higher flows restore the oxygen. They are later killed by the recurrence of oxygen depletion and float downstream by the hundreds. Considerable ill feeling prevailed because of the condition of this stream. The problem was to separate fact from fancy by providing a simple and dependable analysis of this pollution for the city and its industries as well as for the VWCB.

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[5] "Industrial Waste Requirements," Maryland Water Pollution Control Comm., Baltimore, Md., 1948, Regulation IV.

[6] "Final Report South River Study Committee," Records, Virginia State Water Control Board, Richmond, Va., November 4, 1948.

Obviously several sources of pollution were jointly responsible; but it was not apparent for what kind of pollution they were severally responsible—and for how much of the total pollution. The VWCB consulted those involved, formed a cooperative task force, distributed the work load, and studied the stream for 3 months during its most polluted season. Then the collected data were reviewed, conclusions of fact were drawn as a basis for later work, and, its task having been completed, the task force committee was dissolved.

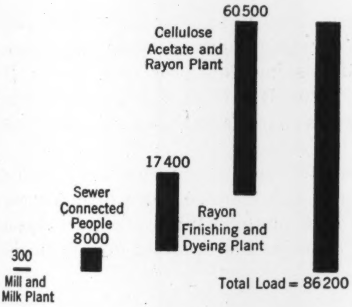

FIG. 5.—DISTRIBUTION OF B.O.D. LOAD (POPULATION EQUIVALENTS) FROM SANITARY SEWAGE AND INDUSTRIAL WASTES—JULY, AUGUST, AND SEPTEMBER, 1948; SOUTH RIVER BELOW WAYNESBORO, VA.

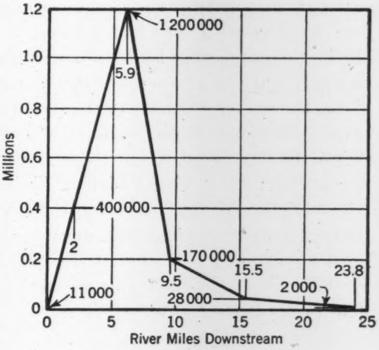

FIG. 6.—MOST PROBABLE NUMBERS OF COLIFORM GROUP DURING JULY, AUGUST, AND SEPTEMBER, 1948; SOUTH RIVER BELOW WAYNESBORO, VA.

The 3-month average of the oxygen balance of the 24 miles, as reported by the South River Cooperative committee, is shown in Fig. 4. The river is in foul condition for about 10 miles; but beyond that point its recovery is quite rapid, being well advanced at mile 15.5 and substantially complete at mile 23.8. The stream appears to make a complete recovery below mile 23.8 and no latent harmful effects have been noted downstream from this pollution. This opinion is based on some spot checks by biologists and on a knowledge of the nature of the pollutants upstream.

Distribution of responsibility for this heavy B.O.D. loading is shown in Fig. 5. The data were derived from samples from each principal industrial sewer composited over 24 hours on each sampling day during the 3-month study. The principle involved is that the B.O.D. loading reduction problem is primarily industrial.

The rise and fall of the coliform group population are shown in Fig. 6. Although Fig. 5 shows that the city is responsible for only about 7% of the B.O.D. loading, it obviously is responsible for substantially all the sewage bacteria phase of the stream pollution. The principle involved is that treatment of sanitary sewage is essential to the restoration of the health of this stream.

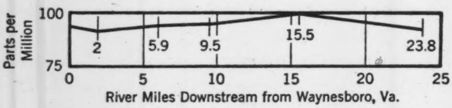

FIG. 7.—ALKALINITY VARIATIONS, DURING JULY, AUGUST, AND SEPTEMBER, 1948; SOUTH RIVER BELOW WAYNESBORO, VA.

Fig. 7 shows the alkalinity variations. Since the only significant changes occurred downstream, they appeared to be caused by downstream tributaries. In any case, the effect of the several pollutional discharges in Waynesboro was negligible on the alkalinity. This finding has exploded a popular belief that the fish were being killed by acid from the chemical plants. Similarly, the pH-variations at all stations were within the range covering normal biological activity and, therefore, were considered to be normal.

The average apparent color was increased from 17 ppm upstream to 30 ppm downstream. This increase was considered to be without practical significance. Thus, the principal phase of the dyeing and finishing plant problem is the B.O.D. load. Although there is hope of reducing the B.O.D., nothing can be done about the dye color except to decrease it at its source and then to dispose of the remainder by equalizing and by diluting the stream.

Collectively, Figs. 4 to 7 provide some badly needed information. Behind them is a story of good will, good faith, and much teamwork of which Waynesboro government, its industries, and the staff of the VWCB are quite proud. Fact has been separated from fancy—a considerable accomplishment in this field of conflicting interests.

CONCLUSION

In 1946 the writer proposed a rationalization of the industrial waste problem,[7] based on the premise that the stream pollution abatement program must be made attainable by industry if success were to be expected.

It is interesting to review this reasoning by examining the results of its application to a specific program. Some of the successes of this approach have been discussed. Its failures (and there have been several) have resulted from the refusal of angry and powerful groups to accept this scientific and cooperative approach as more suitable than the mandatory actions which they advocate.

On the "plus side of the ledger," manufacturing plant design, process control, waste treatment, and stream studies have gone hand in hand in progressive plants. This action by industry is most profitable and should yield great reductions in industrial pollution.

A case in point involves a pharmaceutical plant where, by correlation of the efforts of plant designers, process control men, and waste supervisors, management has internally reduced the B.O.D. load from a 66,000 population equivalent in early 1947 to a 20,000 population equivalent in late 1948 (meeting minute 10, VWCB, December 6, 1948). In addition to this load reduction, these process control measures have so simplified the disposal problem that primary treatment is all that is now needed for this effluent.

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[7] "Industrial Use of Streams," by L. L. Hedgepeth, *Journal*, A.W.W.A., June, 1947, p. 532.

Earnest and powerful forces in industry are at work on this problem. Its complexities are apparent. No two plants are alike. No two area or basin problems are the same. A program of considerable flexibility is needed if industrial stream pollution is to be corrected. Open minds and intellectual honesty will be needed.

DISCUSSION

S. Leary Jones,[8] Assoc. M. ASCE.—Industrial stream pollution abatement cannot be accomplished by the simple process of closing a valve on the discharge line. The problem is complex, and the technical solutions defy standardization. This is ably stated by Mr. Hedgepeth. Each industry must receive individual consideration, both in the evaluation of the need for treatment and in the design of the treatment (or recovery) works. Sound engineering is needed to define the problem, to determine the objectives of treatment, to anticipate the results to be obtained, and to achieve the final design of the needed works.

There are three stages in development that play an important part in successful pollution abatement. They apply to both industry and government, and it would be difficult to determine wihch stage is the most important. The first is the informative stage with recognition of the problem by the polluter and the state. Once the problem is understood and its challenge accepted, the ground work is laid for a cooperative approach. At the same time accurate information should be provided so that the public can avoid wrong impressions.

The second stage involves the assembly of the necessary technical data by the state to define the problem and to determine the effect of pollution on reasonable and necessary stream uses. From these data the allowable pollution load in the stream can be determined. The industry, at the same time, should obtain technical data on treatment or recovery methods for individual or combined wastes from the plant. These serve as a basis for the design of disposal works.

Consideration of finance (the third stage) cannot be disregarded, either as to extent or method. Although it is the responsibility of the polluter to finance the abatement program, sound consideration of the technical phases has marked effects on the final cost of abatement.

The Tennessee Stream Pollution Control Board (hereinafter called "the Board") has recommended that three steps to abate pollution be followed by municipalities and industries. After an evaluation of the problem by the staff of the Board, the polluter should:

(1) Study the problem and prepare preliminary plans which are submitted to the Director, Division of Sanitary Engineering, who is also technical secretary to the Board;

(2) Design adequate treatment or recovery works and submit the plans and specifications to the technical secretary for approval; and

(3) Construct the needed works and operate them thereafter.

The Board's staff assists the polluter in these steps to the extent of its legal, financial, and technical ability. The Board, however, cannot design the treatment works for the polluter. It is important that discussions with the control agency start at an early stage in the corrective program of an industry.

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[8] Principal Sanitary Engr., Stream Pollution Control, Tennessee Dept. of Public Health, Nashville, Tenn.

A cooperative program for pollution abatement (the aim of the Stream Pollution Act of 1945) has been established by the Board. This program is being carried out as is evidenced by the "General Regulations" and the "General Criteria of Water Quality" adopted by the Board. Tolerance permits are issued to allow time for the design and construction of abatement works provided that the polluter files with the Board a schedule of the work that will be done to abate pollution. Permits are issued when adequate treatment has been provided. The extent of treatment that is adequate varies in relation to the type of waste, the stream uses, the quantity of dilution water, and other factors. The usability of the receiving stream in relation to reasonable and necessary uses is the final criterion. This approach is quite far removed from that followed in the standardization of effluents.

The study program of the Board is well underway. This includes analyses of the sewage and wastes in each industrial and municipal sewer outfall as well as analyses of the stream water above and below each point of pollution. Additional stream samples are collected as needed to determine the quality of the stream water. The analytical results are correlated with calculated loads from the city or industry. In reality, these are screening surveys to determine the effect of the pollution, but the results which are furnished to the polluters have assisted them in planning their abatement program. Follow-up detail studies will be made as expediently as possible at each point of pollution indicated. Results of some surveys have indicated the feasibility of joint treatment works whereby industrial wastes can be treated with municipal sewage. This information is sent to the municipal and industrial officials concerned for their information and consideration.

Adequate consideration of the quantity and quality of the dilution water in the receiving stream is important. On those streams where long-term flow records are available from the United States Geological Survey (USGS), they have been a valuable aid. Considerable data are available on water quality in the Board files. These have been collected by the Division of Sanitary Engineering of the Tennessee Department of Public Health, by various municipal water supplies, and by the Tennessee Valley Authority (TVA). These data are used to supplement those collected by the Board since 1947. Although the data in some areas are not as complete as one might wish, they are sufficient to serve as a base for a continuing stream-sampling program that will reflect pollution conditions and the improvements in stream water quality through pollution abatement. Estimation of the available dilution water in streams where flows can be completely controlled by power dams presents other complex problems.

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Paper No. 2452

MAXIMUM LOAD CAPACITY OF BAILEY BRIDGES

BY ROBERT B. STEGMAIER, JR.,[1] JUN. ASCE

WITH DISCUSSION BY MESSRS. D. ALLEN FIRMAGE, AND ROBERT B. STEGMAIER.

INTRODUCTION

Full-scale ultimate capacity tests of the Bailey bridge by the Research and Development Laboratories of the Corps of Engineers, United States Army, in 1944, provided basic data for determining the capacity of the structure.[2] This paper explains how the test results were used to determine the maximum military loads that could cross various types of Bailey bridges. This method, developed as a result of the ultimate capacity tests, represents a departure from the strictly analytical methods generally used to determine the capacities of bridges. For the most part, maximum load capacities were increased as a result of the investigation.

The load characteristics and dimensions of military vehicles are known, so that the determination of the maximum load capacity of a military bridge, in terms of these vehicles, is based on more definite information than is ordinarily available to civilian bridge designers. Driver training, plus rigid traffic control at bridges, provides a check to insure that crossing regulations will be enforced. Whereas each civilian bridge is usually designed to satisfy a special set of conditions, military bridges are mass produced to rigid and exacting specifications, with continuous close inspection to guarantee uniformity. These factors, and the urgent necessity to make the maximum use of bridging materials, enable the military designer to reduce the margin of safety to the absolute minimum. Under these conditions and requirements, the most logical basis for the determination of the maximum capacity of a military bridge is full-scale tests to failure.

LOADS

The dead load of various spans and types of Bailey bridges was known accurately because the methods of assembly were standardized and the weights of the

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[1] Chf., Information Requirements Branch, Program Div., Research & Development Bd., Washington, D. C.

[2] "Testing Bailey Bridges to Failure," by D. Allan Firmage, *Civil Engineering*, April, 1946, p. 154.

component parts were known. Actual weights were used in the computations, and equal distribution of the dead load to the trusses on either side of the roadway was assumed.

In calculations involving live loads, the knowledge of vehicle dimensions, their weight, and their rated load capacities permitted accurate computation of shear and moment. The gross weights of the vehicles were determined on the basis of their rated pay loads. The concentrated load equivalents of 120 military vehicles were computed for each 10-ft increment of span from 40 ft to 190 ft.

An impact allowance of 10% of the live load was made for all types and classes of vehicles on all the spans. Limited dynamic impact tests, using electrical strain gages and recorders, indicated that a flat 10% impact allowance was adequate for military bridges. The maximum load capacity of these bridges was actually determined by relatively heavy, slow-moving, tracked vehicles, and the impact of these vehicles is reduced to a minimum by the combined cushioning effect of their track-laying action and the action of their bogie wheels. Additional dynamic impact tests are being conducted to substantiate or revise the present 10% allowance.

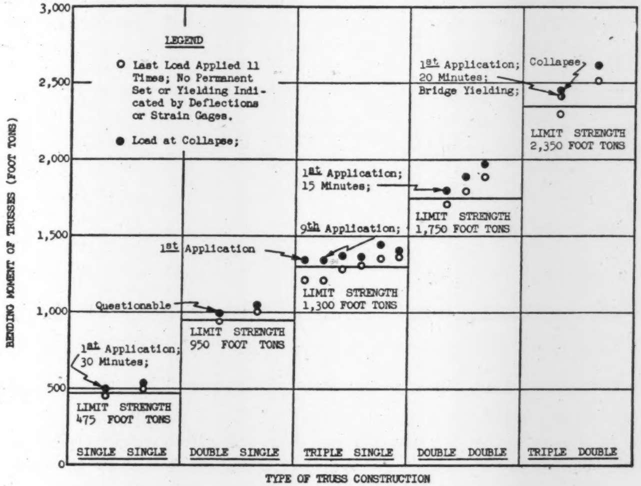

FIG. 1.—LIMIT STRENGTHS OF BAILEY BRIDGE TRUSSES AS DETERMINED FROM RESULTS OF ULTIMATE CAPACITY TESTS

A final load factor requiring consideration was the effect of eccentricity of the vehicle. Since the Bailey bridge has a single-lane roadway, the vehicles were placed against the curb to obtain maximum eccentricity. Using the floor beam as a simple beam supported by the side trusses at their center of gravity the load carried by the side trusses was computed. Doubts as to the correctness of this procedure were eliminated by the results of the ultimate capacity tests.

MARGIN OF SAFETY

The resisting moment of a bridge at failure, determined on the basis of ultimate capacity tests, was defined as its limit strength. The results of the tests and the chosen limit strengths are shown in Fig. 1. The word "chosen" is used because the choice was arbitrary, since in most cases only two bridges were tested. Notes made in Fig. 1 describe the failure of each type of bridge that carried the least load. Generally the limit strength was determined to the nearest 25-ft tons midway between the last load carried satisfactorily and the load causing failure. "Carried satisfactorily" meant that a given load was applied eleven times without either yielding or permanent set being indicated by electrical strain gage or deflection measurements.

Before selecting a value for the margin of safety, three types were considered (Fig. 2). Type I was a constant margin of safety on live load only. Type I had the disadvantage that on long spans, which would carry only light vehicles, the over-all margin of safety was very low. Even though the dead load was known accurately, and high efficiency to save materials was desirable, the possibility that overloading would occur simultaneously with an unusual impact ruled out this type of safety factor. Type II, a constant margin of safety on the dead and live loads, was adopted. It provided maximum efficiency in the short span range where vehicles were heavy, their weights accurately known, and impact was generally at a minimum because of the track-laying characteristics of the governing vehicles. In addition, it was adequate for the lighter vehicles on long spans and it was easy to apply. Type III was a compromise between Types I and II. The beginning of Type II could have been determined on the basis of span length or on the ratio of dead load to live load. As illustrated, Type I was used when the ratio was less than 1; and Type II, when it was greater. This type of safety factor was not used because it did not make efficient use of the materials on short spans and because, in practice, it was the most difficult to apply. The three types of margin of safety and their relation to total load and live load plus impact are compared in Fig. 3. The assumption was made that when the ratio of dead load to live load was 1, they were identical. The margin of safety of Type II which was adopted is a constant on the total load; it is low for large live loads; and it increases as the live load decreases and the span increases.

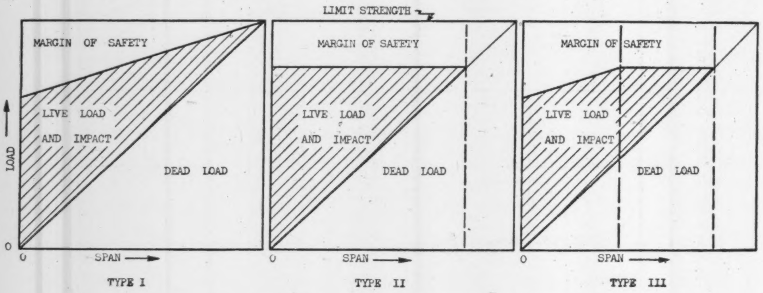

FIG. 2.—THREE TYPES OF MARGIN OF SAFETY

The actual value of the margin of safety was based on the consideration of three factors—the determination of the limit strength by ultimate capacity tests, the rigid traffic control regulations, and the effects of the load on the structure. These are summarized in Table 1 with the design criteria that resulted from their evaluation.

Traffic control regulations permitted three types of crossing. "Control" crossings were relatively unrestricted. Vehicles moved at normal convoy speed and spacing, and could use any part of the roadway. After use as a "control" crossing the bridge could be disassembled, and re-erected time and again at new sites. The several parts of the bridge were "standardized" so that new parts or used parts from other sites could be used interchangeably. This requirement fixed the maximum "control" load as one that would not cause permanent deformation or local stresses exceeding the yield point. These stress criteria also applied to "caution" crossings, which differ from "control" crossings in that the interval between vehicles is increased, speed is reduced, movement is continuous, and the vehicle must travel on the center line of the roadway. A military expedient, and still higher classification than the foregoing, was the "risk" type of crossing. It is all that its name implies, with the possibility that when it is used it may cause the bridge to collapse. Local stresses were permitted to exceed the yield point and permanent deformation could prevent re-use of some of the bridge parts. Traffic control regulations for this type of crossing are similar to the "caution" crossing but more rigid; speed is reduced to a minimum and only 1 vehicle is permitted on the bridge at a time.

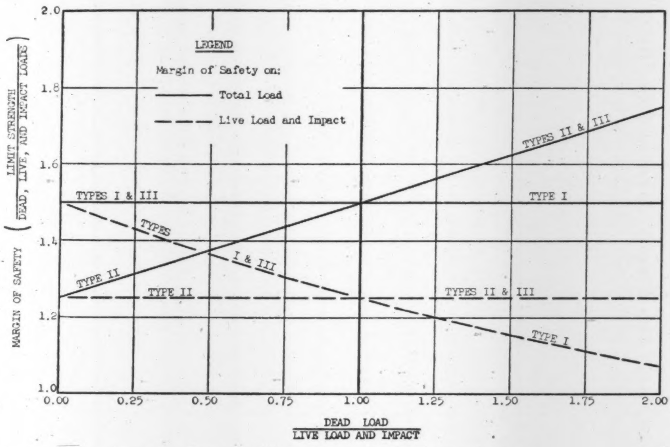

FIG. 3.—COMPARISON OF TYPES OF MARGIN OF SAFETY

The design criteria corresponding to "risk," "caution," and "control" crossings translated the traffic control regulations into simple specifications for bridge design—axle or track spacings for vehicles and an eccentricity of 6 in.

TABLE 1.—TRAFFIC CONTROL, DESIGN CRITERIA, AND BRIDGE EFFECTS FOR "RISK," "CAUTION," AND "CONTROL" LOADS

Manner of crossing	Traffic control regulations	Effects of load on bridge	Design criteria
"Risk"	Walking guide. Single vehicle. Speed, 2 miles per hr to 3 miles per hr. Continuous movement; no shifting, braking, or stopping. Vehicle on center line of bridge.	Localized stresses exceeding yield point practically a certainty. Remote possibility of collapse. Constant and repeated use of structure for "risk" loads may eventually result in collapse. Parts of structure will be deformed or actually broken so that re-use will not be possible.	Single vehicle on bridge. Eccentricity of 6 in. No impact. Weakest part of structure will be stressed to 90% of limit strength as determined by small number of ultimate capacity tests.
"Caution"	Vehicles spaced 50 yd, nose to tail. Speed, not more than 5 miles per hr. Continuous movement, no shifting, braking, or stopping. Vehicle on center line of bridge.	Localized stresses below yield point of material. No possibility of collapse if traffic control regulations are properly enforced. Repeated "caution" crossing will not result in collapse. All parts of structure will be available and serviceable for re-use unless damaged by accident.	Vehicles spaced 150 ft from rear axle of lead vehicle to front axle of second vehicle or from rear of ground contact of lead vehicle to front of ground contact of second vehicle. Eccentricity of 6 in. No impact. Weakest part of structure will be stressed to 80% of limit strength as determined by small number of ultimate capacity tests.
"Control"	Vehicles spaced 30 yd, nose to tail. Speed, not more than 25 miles per hr. Vehicles not required to stay on center line.	Localized stresses below yield point of material. No possibility of collapse if traffic control regulations are properly enforced. Repeated "caution" crossing will not result in collapse. All parts of structure will be available and serviceable for re-use unless damaged by accident.	Vehicles spaced 100 ft from rear axle of lead vehicle to front axle of second vehicle or from rear of ground contact of lead vehicle to front of ground contact of second vehicle. Maximum eccentricity; vehicle against curb. 10% impact. Weakest part of structure will be stressed to 80% of limit strength as determined by small number of ultimate capacity tests.

for vehicles theoretically on the center line. No impact allowance was used for "caution" and "risk" crossings. The effects of the load on the bridge were specified in terms of the margins of safety that allowed for uncontrollable variations, such as slight variations in material, fabrication, and field erection. They did not allow for variations that could be minimized through traffic control, such as overloading, impact, and eccentricity. For both "control" and "caution" crossings the maximum permissible load is 80% of the load that might cause collapse. This is 80% of the limit strength, or a margin of safety of 1.25 on failure. The margin of safety on "risk" loads is 1.11, or 90% of the limit strength.

These values of the margin of safety were based on the testing to failure of a limited number of bridges. However, if more bridges of one type were tested, the test data could be analyzed statistically to determine the limit strength; and, based on a limit strength determined in this manner, the margin of safety might possibly be lowered.

STRESSES

The margins of safety have been applied to the results of a bridge test in Fig. 4. This was the double-double truss bridge that carried the least load of the three bridges of this type tested. For "caution" and "control" crossings the average truss stress was 35,200 lb per sq in. and the maximum local stress was 43,000 lb per sq in. For "risk" loads these stresses were 39,500 lb per sq in. and 48,300 lb per sq in., respectively. At limit strength the computed truss stress was 42,000 lb per sq in. In the other two double-double bridges tested it was measured as 41,900 lb per sq in. and 42,200 lb per sq in. Similar data for all the test bridges are presented in Table 2. All the maximum local stresses at limit strength and half of those for "risk" loads exceeded the guaranteed yield

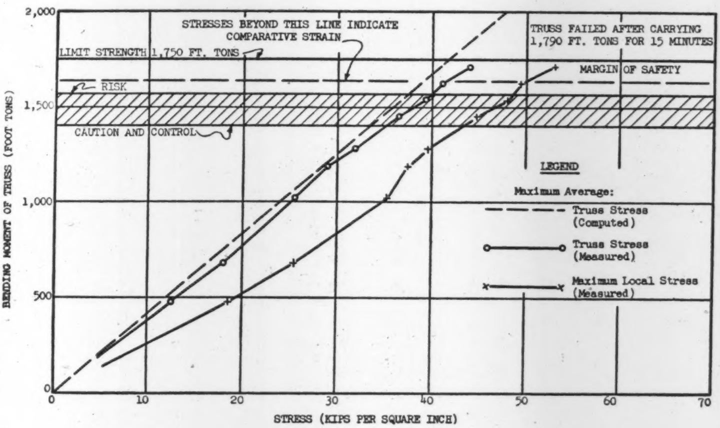

FIG. 4.—CRITERIA FOR MAXIMUM LOAD CAPACITY APPLIED TO TEST RESULTS OF 130-FT DOUBLE-DOUBLE TRUSS

point of the material, 50,000 lb per sq in. Test samples of the material showed actual yield points varying betweeen 50,000 lb per sq in. and 60,000 lb per sq in. The guaranteed minimum tensile strength of the material was 70,000 lb per sq in., and actual test sample results varied from 73,000 lb per sq in. to 87,000 lb per sq in. In general, the measured stresses were low compared to the computed average truss stresses, except where the strain gages indicated maximum local strains above the yield point. The only notable exception was the second 130-ft double-double bridge. Careful check of the original data indicated no explanation for the higher strains actually recorded. An important point to note in Table 2 is that, because of the nature of the built-up trusses in the Bailey bridge, and the use of the limit strength based on ultimate capacity tests, the computed maximum allowable average stress was not the same for all types of trusses. Prior to the ultimate capacity tests, the maximum "control" loads were determined from computed stresses with a maximum allowable average truss stress of 30,200 lb per sq in. This allowable stress has now been increased to 37,800 lb per sq in. for the single and double-single types of trusses. The ultimate capacity tests were primarily responsible for this increase in allowable capacity. Increased capacities were also permissible for the other types of trusses with only an insignificant decrease for the triple-double type. The

TABLE 2.—MAXIMUM COMPRESSION CHORD STRESSES, IN KIPS PER SQUARE INCH, MEASURED DURING ULTIMATE CAPACITY TESTS

Test bridge no.	LIMIT STRENGTH			"RISK"			"CAUTION" AND "CONTROL"		
	Average Truss Stress		Maximum local stress	Average Truss Stress		Maximum local stress	Average Truss Stress		Maximum local stress
	Computed	Measured		Computed	Measured		Computed	Measured	
60-Ft Single-Single:									
1	47.2	..	..	42.5	37.4[a]	57.0[b]	37.8	32.7	40.9
2	47.2	47.1[a]	86.5[b]	42.5	40.8[a]	51.5[b]	37.8	35.0	39.8
100-Ft Double-Single:									
1	47.2	..	..	42.5	36.0[a]	54.7[b]	37.8	31.2	34.3
2	47.2	48.6[a]	58.9[b]	42.5	39.9	46.5	37.8	35.0	40.0
110-Ft Triple-Single:									
1	43.0	47.0[a]	83.6[b]	38.8	36.7[a]	56.6[b]	34.5	31.9	45.5
2	43.0	37.7[a]	50.5[b]	38.8	32.6	39.6	34.5	28.8	33.5
130-Ft Triple-Single:									
1	43.0	38.4[a]	52.0[b]	38.8	34.6	42.6	34.5	31.4	39.0
2	43.0	..	..	38.8	35.5[a]	50.4[b]	34.5	31.8	43.4
140-Ft Triple-Single:									
1	43.0	47.9[a]	87.9[b]	38.8	40.4[a]	58.3[b]	34.5	35.7	48.1
2	43.0	..	..	38.8	33.8	43.0	34.5	30.0	37.0
130-Ft Double-Double:									
1	42.0	41.9[a]	57.0[b]	37.8	36.5	48.0	33.6	32.9	41.9
2	42.0	..	..	37.8	39.5	48.3	33.6	35.2	43.0
3	42.0	42.2[a]	..	37.8	36.6	..	33.6	33.3	..
150-Ft Triple-Double:									
1	37.6	37.7[a]	55.0[b]	33.9	32.4	43.1	30.1	28.4	36.6
2	37.6	43.9[a]	..	33.9	33.9[a]	55.7[b]	30.1	29.1	40.9

[a] These average values include individual stresses above yield point (50,000 lb per sq in.) and therefore are not true stresses but indicate comparative strain. [b] These values were converted from strains that exceeded the yield point and therefore are not true stresses.

ultimate capacity tests also permitted the introduction of "risk" crossings, a necessary and desirable military expedient. This classification was not possible on the basis of computations alone, because it was known that they were not sufficiently exact for that purpose.

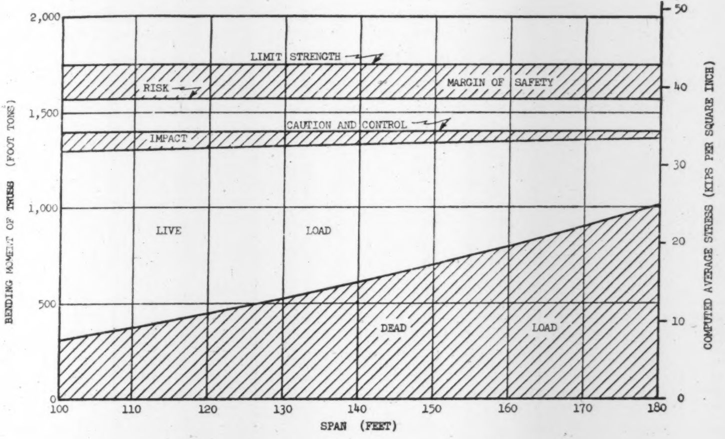

FIG. 5.—USE MADE OF MOMENT CAPACITY OF DOUBLE-DOUBLE TRUSSES

FIG. 6.—TOP CHORD FAILURES OF SINGLE TRUSS, DOUBLE TRUSS, AND TRIPLE TRUSS BAILEY BRIDGES

The use made of the truss capacity of a common type of Bailey bridge construction is shown in Fig. 5. For spans less than 100 ft this type of construction was uneconomical and for spans longer than 180 ft it could not be erected by the normal cantilever method of launching. Thus far, all discussion has dealt with moment capacity. For the Bailey bridge spans commonly used, moment governed, and failure occurred by buckling of the top chord of the trusses (see Fig. 6). Shear did govern in some types of trusses up to spans of 80 ft. Ultimate capacity tests were conducted, and limit strengths were determined for shear. The foregoing principles for moment were applied for shear with the same margins of safety.

TABLE 3.—SAFE AND "CAUTION" CAPACITIES OF VARIOUS TYPES OF BAILEY BRIDGES, IN TONS

Span (ft)	SINGLE-SINGLE		DOUBLE-SINGLE		TRIPLE-SINGLE		DOUBLE-DOUBLE		TRIPLE-DOUBLE	
	"Control"	"Caution"	"Control"	"Caution"	"Control"	"Caution"	"Control"	"Caution"	"Control"	"Caution"
40	32	45	..	..	..	..	..	..	..	..
50	28	38	80	100	..	..	..	..	..	..
60	24	32	67	84	..	..	..	..	..	..
70	20	27	57	71	..	..	..	..	..	..
80	17	23	48	60	75	94	..	..	..	..
90	13	18	40	50	62	78	..	..	..	..
100	10	13	31	40	50	63	75	94	..	..
110	..	..	23	31	40	50	61	76	80	100
120	..	..	18	23	31	41	50	63	67	84
130	..	..	13	18	23	32	40	51	57	71
140	..	..	10	13	18	23	31	40	48	61
150	..	..	..	..	13	18	23	32	39	50
160	..	..	..	..	10	13	18	25	31	40
170	..	..	..	..	..	..	13	19	23	31
180	..	..	..	..	..	..	10	13	18	23
190	..	..	..	..	..	..	..	..	12	16

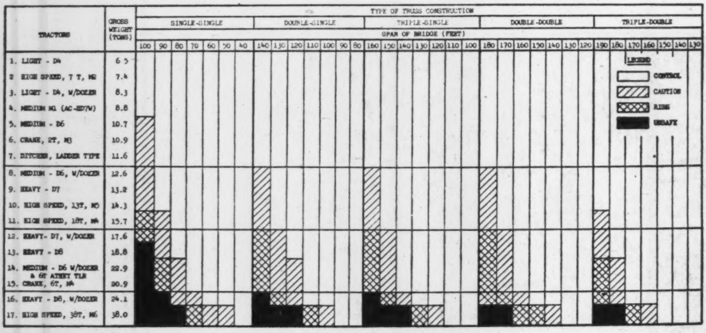

FIG. 7.—CAPACITY OF BAILEY BRIDGES

APPLICATION

For each type and span it was determined in what manner, "risk," "caution," or " control," each of 120 military vehicles could cross. Tables were set up for each bridge, giving live load, shear, and moment capacity for each type of crossing. Similar tables were set up for each vehicle in terms of its concentrated load equivalent. These tables accounted for the length of track or number of axles of the vehicle on the span (whether one or more vehicles were on the bridge), the impact, and the maximum (6-in.) eccentricity. Comparison of these tabulated data produced graphical bar charts such as that shown in Fig. 7. On the basis of the gross load system of vehicle classification used by the Army, the bar charts were reduced to tabular form as shown in Table 3. These data for "control" crossings were plotted as shown in Fig. 8.

The bridge capacity data were published in all three forms as bar charts, tables, and graphs. Each had a preferred use, and only the bar charts were complete. The tables did not contain data on "risk" loads, and the graphs showed only the "control" loads. The graph was used to determine the type of bridge to be constructed which would carry the maximum expected load for the given site. Table 3 had the same purpose, and if a "caution" rating was specified for the route it had to be consulted. The bar chart determined the safety of passage for individual vehicles and was the only published data on "risk" crossings. Bridges were posted with their "control" capacity in tons but the rounding off of these values, necessitated by the gross load system for classifying vehicles, prohibited the use of certain bridges to some vehicles that could cross as shown by the bar chart. In all cases the bar charts were the final authority. The general rule for overloaded vehicles was that they cross as the next higher restricted class of load. For example, an overloaded vehicle, normally a maximum "control" load, could cross as a "caution" load, a "caution" load could cross as a "risk" load, and a vehicle normally a "risk" load would be forbidden passage.

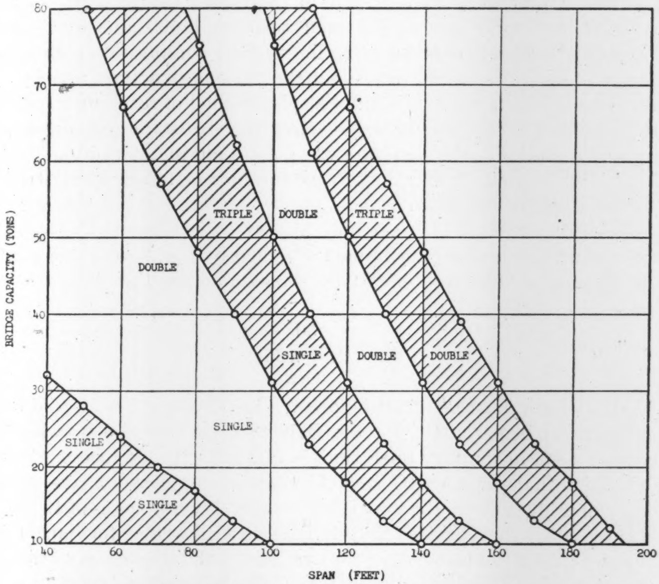

FIG. 8.—BAILEY BRIDGE TRUSS CONSTRUCTION FOR VARIOUS LOADS AND SPANS

DISCUSSION

D. Allan Firmage,[3] Jun. ASCE.—Although at first observation Mr. Stegmaier's paper appears to be only of academic interest to the designer of civilian bridges, it contains certain aspects that may be of considerable interest.

The design and rating of military bridges are somewhat along the line of structural design of aircraft. Because excess dead load cannot be tolerated, military bridges must be designed with small factors of safety and with a high degree of design precision. The nature of the Bailey bridge is such that it is impossible to determine (by purely analytical methods, and within the required degree of accuracy) the buckling limit of the compression chord and the distribution of load among the trusses for the various structural arrangements. Ultimate capacity tests were the author's only method of reaching a sufficiently accurate solution. The use of SR-4 electrical resistance strain gages to measure actual strains in the structure was a great aid in determining stress distributions; and it yielded information as to the maximum local stress that could be expected for any load.

The 10% allowance for impact may appear to be somewhat unsafe when compared with the allowance for impact specified by the American Association of State Highway Officials (AASHO). According to the AASHO impact equation—

$$I = \frac{50}{L + 125} \qquad (1)$$

—the loaded length must be 375 ft before a value as low as 10% could be used. The maximum span permitted for the Bailey bridge was 200 ft; whereas the average span used in military operations is about 75 ft. It has long been the writer's opinion that the AASHO impact limit was very conservative, especially for short span bridges where the upper limit of 30% is used (spans of 42 ft or less). It is on these short span bridges that an excessive allowance for impact results in the most waste of material, because live load is the larger proportion of the total design load. These structures make up the greater percentage of the total bridge mileage in the United States. A less conservative impact allowance would save the taxpayers many dollars for highway bridge construction. This belief in the excessive allowance for impact, as required by AASHO, is partly substantiated by the results of tests conducted in England[4] as reported by G. R. Mitchell in 1949. The results of these tests showed that the impact $\left(\frac{\text{dynamic stress} - \text{static stress}}{\text{static stress}}\right)$ varied from maximum values of +30% to −30%, the most frequently occurring value being 0%. Of a total of 114 occurrences only 20 had impact values greater than 10% and in only 8 occurrences were impact values greater than 20%. The bridges tested in England were all short span bridges and the results include

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[3] Associate Prof. of Civ. Eng., Univ. of Florida, Gainesville, Fla.

[4] "Problems of Impact and Fatigue and Their Effect on Permissible Stresses in Cast Iron Girder Bridges," by G. R. Mitchell, *Publications*, International Association for Bridge and Structural Engineering, Vol. 9, 1949, pp. 61–64.

> "* * * a number of tests made with a solid-tired vehicle on one bridge where there was a sunken trench near the mid-span line giving a rise to a one inch (2.5 cm) depression in the road surface."

The paper by Mr. Mitchell, however, fails to state whether the larger values of impact were from this condition of test or not. It is presumed that this rather severe condition would result in larger impact values. The results of the impact tests mentioned in Mr. Stegmaier's paper should be published to stimulate further research in the problem of impact on civilian bridges.

The concentrated load equivalent was used as a convenient method of comparing the capacity of the bridge (concentrated load capacity) against the effect of a vehicle. The longer the span and the shorter the wheel base, the closer will the concentrated load equivalent approach the gross weight of the vehicle. It was a very "handy" tool for comparing effects of vehicles. It was easier to compare concentrated load equivalents than bending moments because the values did not cover such a wide range in going from short span bridges to long span bridges.

Mr. Stegmaier's paper shows the great amount of effort and time-consuming research that was needed in only one small part of the total war effort. It shows, in a degree, that the problem of conducting a war in this present day and age is surely a technical problem. The eventual winner will be the one who can solve the technical problems most expeditiously.

ROBERT B. STEGMAIER, JR.,[5] JUN. ASCE.—In his discussion Mr. Firmage has touched lightly on the most important problem involved in the design of military bridges. His assertion that "* * * excess dead load cannot be tolerated * * *" might be better expressed by the statement that every pound of material in a military bridge should be used to 100% of its load-carrying capacity. Of the many variables involved in the development of criteria for the design of military bridges, there was considerable background data to serve as a basis for all assumptions except that of impact. Mr. Firmage has indicated that critical examination of AASHO specifications for impact proved them unrealistic for application to military bridges. A primary consideration in the choice of the relatively low value of 10% of the live load was that, on the usual range of bridge spans, heavy concentrated loads in the form of tanks and other track-type vehicles were the governing loads. Such vehicles carry their own roadways in the form of tracks, and these tracks, combined with the cushioning effect of "bogie" wheels, were assumed to minimize impact, as compared to the wheeled vehicles for which AASHO specifications were developed.

It is true that, on longer spans, wheeled tank transporters, which have a relatively long wheel base, were the critical vehicles, but for such spans the live load was a much smaller proportion of the total dead and live load on the structure. In such cases, therefore, the net effect of impact was reduced.

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[5] Reports and Statistics Branch, Resources Div., Research and Development Bd., Office of the Secretary of Defense, Washington, D. C.

Recent dynamic tests on a 120-ft span, Bailey-type bridge demonstrated that the maximum impact load on the trusses was 12% of the live load for both a 43-ton tank and a 60-ton, pneumatic-tired tank transporter. These values are comparable to the 10% originally assumed. The maximum impact loads on the floor system of the bridge (stringers and floor beams), using the tank as the test vehicle, were 15% of the live load. The impact load on the trusses was increased to 14% when the tank rode over a 2 in. by 4 in. timber placed on the bridge deck. There is no doubt that the study of impact effects on structures—(1) under peacetime civilian conditions (with well-maintained roads and high speeds) and (2) under military activity in wartime—is a fertile field for valuable contributions to the profession.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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TRANSACTIONS

Paper No. 2453

RETROGRESSION ON THE LOWER COLORADO RIVER AFTER 1935

BY J. W. STANLEY[1]

WITH DISCUSSION BY T. BLENCH, EMORY W. LANE, AND J. W. STANLEY.

SYNOPSIS

River surveys on the Colorado River below Hoover Dam in Arizona-Nevada and Arizona-California indicate that approximately 125,000,000 cu yd of material was scoured from the river bed between the dam and Havasu Lake from 1935 to 1946. Steepening of the river slope for many miles downstream has resulted. From the time of the Parker Dam closure in 1938 to 1946 approximately 135,000,000 cu yd was removed downstream from Parker Dam. The major part of this material was deposited in the Imperial Reservoir area until 1945, since which time nearly all the sediment has been carried past Imperial Dam. River surveys begun below Imperial Dam in 1940 indicate that approximately 65,000,000 cu yd of material was removed from the river bed between Imperial Dam and the International Boundary from 1940 to 1946. The suspended sediment load carried by the Colorado River at Yuma has been reduced to approximately 10,000 tons per day at a discharge of 10,000 cu ft per sec from the approximately 100,000 tons per day which it carried at the same discharge before Hoover Dam was closed.

INTRODUCTION

The completion of Hoover Dam in 1935 marked a turning point in the history of the development of the Colorado River.

Millions of tons of sediment eroded from the 242,000 sq miles of its watershed (see Fig. 1) were carried by the Colorado River to the Gulf of California (in Mexico) each year prior to completion of Hoover Dam. Depending partly on intensity and quantity of precipitation, and partly on distribution of precipitation over the watershed, the river bed underwent alternate periods of building up and wearing down. Occurrence of heavy precipitation in, and resultant high runoff from, the upper watershed caused immense loads of sediment to be

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[1] Office of River Control, Region 3, Bureau of Reclamation, U. S. Dept. of Interior, Boulder City, Nev.

carried down the river, and even caused removal of sediment previously deposited on the river bed. On the other hand, when excessive precipitation occurred over the drainage area of the lower river, heavy material was carried from the tributaries into the main channel and deposited there, forming barriers that resulted in building up the river. This building up would continue until another flood of sufficient magnitude to tear out the barriers descended from the upper river. Again a lowering would occur, until new barriers were formed. When Hoover Dam was placed in operation, all sediment from the upper basin was thenceforth trapped in the reservoir. In addition, the dam and reservoir have largely eliminated the annual floods, as far as the lower river is concerned.

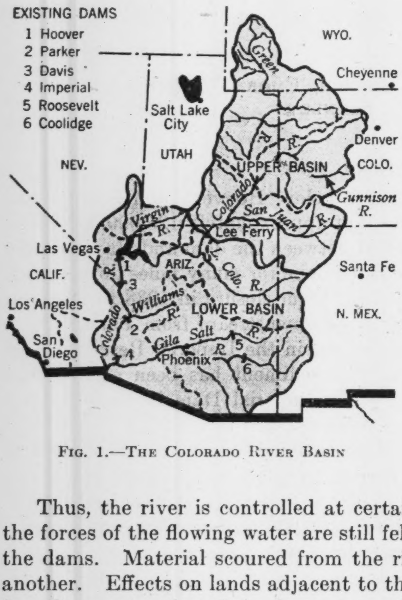
EXISTING DAMS
1 Hoover
2 Parker
3 Davis
4 Imperial
5 Roosevelt
6 Coolidge
WYO.
Cheyenne
Green R.
Salt Lake City
UTAH
Denver
NEV.
COLO.
UPPER BASIN
Colorado R.
San Juan R.
Gunnison R.
Virgin R.
Lee Ferry
Las Vegas
CALIF.
ARIZ.
L. Colo. R.
Santa Fe
Los Angeles
Williams R.
LOWER BASIN
N. MEX.
San Diego
Colorado R.
Gila R.
Salt R.
Phoenix
1
2
3
4
5
6

Fig. 1.—The Colorado River Basin

Parker Dam, about 150 miles downstream from Hoover Dam, was completed in 1938. In addition to providing a minimum water-surface elevation for diversion of water for municipal and industrial use in southwestern California, Parker Dam serves to minimize the effects of floods from the drainage area below Hoover Dam. Imperial Dam, about 305 miles downstream from Hoover Dam, also was completed in 1938. This dam provides a fixed water-surface elevation for diversion of irrigation water to agricultural lands in Arizona and California. Davis Dam, about 70 miles downstream from Hoover Dam, will be utilized to reregulate the flow of the river below Hoover Dam.

Thus, the river is controlled at certain points; but at intermediate points the forces of the flowing water are still felt, as they were before construction of the dams. Material scoured from the river bed at one place is deposited at another. Effects on lands adjacent to the river are as varied as they were before the construction of the dams—some lands are completely eroded, some are left "high and dry," some are isolated by the changing river course, and some are inundated. In areas where land is valuable, these tendencies of the river necessitate channelization to prevent erosion and construction of levees to prevent inundation.

Retrogression Between Hoover Dam and Davis Dam

With the closure of Hoover Dam, the Bureau of Reclamation, United States Department of the Interior, instituted a program of river surveys at selected sites downstream from the dam for the purpose of obtaining an approximate record of the volume of material carried by the river. At first, the measured reach extended only 13 miles, but 6 months later it was extended another 13 miles because scouring action had begun beyond the original reach. Measurements were being made 43 miles downstream a year after closure of the dam. The surveys have been extended in this manner from time to time until, in 1941, measurements were being made 10 miles into Havasu Lake, or 120 miles below Hoover Dam.

During the first 6 months of operation, the clear water released from the dam scoured material from the river bed to an average depth of 4 ft, 1 mile downstream from the dam, and to depths of from 2 ft to 6 ft in the first 13 miles, with river discharges of from 5,000 cu ft per sec to 15,000 cu ft per sec. Scouring of material continued in the following months; but, because of removal of the finer material, the river could not recharge itself to capacity with the coarser material remaining, and therefore began picking up small amounts of material farther downstream.

Not until Hoover Dam had been in operation for more than 6 years was it necessary to release water in appreciable quantities. In fact, the average monthly discharge seldom exceeded 11,000 cu ft per sec until May, 1941, when flood releases were begun. For the next 10 months the discharge from Lake Mead averaged 26,500 cu ft per sec. It was expected that this high average discharge (nearly three times the average for the previous 6 years) would dislodge and carry downstream large volumes of material, even from the first 13-mile stretch, which had become stabilized for discharges prevailing before that time. The expectation was realized for the lower reaches, but not for the first 13 miles. A lowering of about 9 ft which had already occurred at the lower end of the first 13-mile reach was not increased appreciably, indicating that a rock reef or large boulders had been uncovered. Similarly, at a point about 1 mile below the dam, a lowering of 5 ft, which had already occurred, was not increased. At intermediate points, additional lowering of from 2 ft to 5 ft took place, but the control uncovered at the lower end prevented material from being brought down in appreciable quantities.

During this 10-month period of high discharge, considerable scour took place farther downstream, however. At the lower end of the second 13-mile reach, a previous lowering of 4 ft was increased to 9 ft. (By the end of 1946 this lowering had increased to 11 ft and appeared to be stabilized.) Also, large quantities of bed material were removed from the reaches farther downstream. Some indication of the magnitude of quantities eroded to February, 1946, may be obtained from examination of Table 1, which shows estimated volumes of material removed (−) and deposited (+) between Hoover Dam and Havasu Lake. Volumes have been computed on the basis of area changes at river sections, assuming the sections to be representative of their respective reaches.

It was mentioned in a previous paragraph that a greater amount of scour occurred 13 miles from the dam than at the dam. The average bed slope, therefore, is now (2.0 ft per mile in 1950) greater than it was originally (1.7 ft per mile). The same phenomenon is noted for the next 13-mile reach, the original average bed slope being 1.6 ft per mile (2.0 ft per mile in 1950), which indicates more easily erodible material—that is, smaller amounts of coarse material at greater distances from the dam. Over the next 39 miles to Davis Dam the average slope of 1.7 ft per mile is about the same as at the beginning of measurements.

Now that actual measurements of rates of removal are available, it is of interest to compare the actual rate with the theoretical. It has long been known that the capacity of flowing water for carrying sediment usually varies approximately with some power of the discharge, other variables remaining constant. Early in the study of the sediment problem on the Colorado River it was estimated that, for the Colorado River, this power of the discharge was approximately the square. To test the applicability of the theory, data from Table 1 were used to compute the theoretical monthly rates of removal at 15,000 cu ft per sec. The equation is

TABLE 1.—Estimated Volumes of River–Bed Material Removed (−)
Section 43 (in Thousands

Approximate dates[b]	No. of months	Approximate mean[c] flow	Between			
			1–16	16–19	19–23	23–27
Distance, in miles	...	...	12.8	13.4	17.1	21.5
Year 1935	10.4	9,300	−4,081	− 1,591		
Year 1936	12.2	8,100	− 816	− 1,262	− 2,378	
Year 1937	12.0	8,300	− 774	− 1,682	− 2,221	− 3,732
Year 1938	12.7	9,300	− 470	− 1,656	− 1,729	− 4,258
January, 1939, to January, 1940	13.3	12,100	−1,994	− 869	− 1,572	− 4,648
February, 1940, to April, 1941	14.4	10,600	− 447	− 1,300	− 1,883	− 5,892
May, 1941, to February, 1942	10.5	26,500	− 774	− 3,590	− 7,587	− 5,440
March, 1942, to January, 1943	10.2	19,800	− 144	− 94	− 2,564	− 2,928
February, 1943, to July, 1943	6.0	17,300	− 71	+ 341	+ 85	− 5,818
August, 1943, to March, 1944	8.7	20,000	− 39	− 691	− 3,348	− 4,478
April, 1944, to February, 1946	21.9	18,200	+ 150	− 398	− 80	− 1,016
Total	...	...	−9,460	−12,792	−23,277	−38,210

[a] Material removed is denoted by "−"; and material deposited is denoted by "+." [b] Measurements were

$$q_{15} = \frac{15{,}000^2\, q_Q}{Q^2} \qquad (1)$$

in which Q is the actual average discharge; q_{15} is the theoretical rate of scour when Q is 15,000 cu ft per sec; and q_Q is the measured rate of scour. In Fig. 2, the plotted points represent values computed by Eq. 1 using the data in Table 1. It will be noted that Fig. 2 begins with the year 1937, this being the first year in which measurements extended as far as Davis Dam (section 27). Considering that all data used in Table 1 and Fig. 2 are from field measurements, without laboratory-type control, the points in Fig. 2 have a surprisingly small degree of scatter.

Aggradation of the River Valley Downstream from Davis Dam

Early in 1938 river surveys in the first 19-mile reach downstream from the present location of Davis Dam were begun in the regular course of extending measurements below Hoover Dam; and in 1939 they were extended an additional 15 miles, to a point opposite Needles, Calif.

As before, these extensions were made because scouring action was beginning at the downstream end of the reaches already measured. Thus far, the chief interest had been in the lowering of the river caused by the discharge of clear water from Hoover Dam. The question of what happened to the material the river had picked up and carried farther downstream was thought to be of lesser interest.

AND DEPOSITED (+)[a] BETWEEN HOOVER DAM, IN ARIZONA–NEVADA, AND OF CUBIC YARDS)

SECTIONS					Approximate dates[b]
27-31	31-33	33-34	34-40	40-43	
18.7	8.9	6.2	13.8	7.6	Distance, in miles
					Year 1935
					Year 1936
					Year 1937
− 2,589					Year 1938
− 1,283	−2,012	+ 255			January, 1939, to January, 1940
− 5,948	−2,168	+ 833			February, 1940, to April, 1941
− 5,160	+2,415	+ 2,973	+12,522	+ 333	May, 1941, to February, 1942
− 2,318	−2,868	+ 3,044	+53,598	+ 550	March, 1942, to January, 1943
− 3,003	−1,764	− 606		+1,049	February, 1943, to July, 1943
− 184	+1,508	+ 1,176		−2,154	August, 1943, to March, 1944
−11,718	−3,553	+ 4,122		+1,612	April, 1944, to February, 1946
−32,203	−8,442	+11,797	+66,120	+1.390	Totals

begun in the period in which the volume first appears. [c] Approximate mean flow, Q, in cubic feet per second.

By 1940, however, the water-surface elevation in the valley between Needles and Topock, Ariz., began to rise because of the deposit of material brought down from upstream. Examination of the records of earlier water-surface elevations revealed that a rise in water elevation in this area had been occurring gradually, although spasmodically, for many years. Records dating back to 1902 indicated a long-time average rise of about one third of a foot per year, there being short periods of rapid rise and others of no rise and, sometimes, even a period of lowering. Beginning in 1935, the rise was arrested, as it had been in earlier years. By 1940 the rise was resumed, also as in earlier years. By the end of 1944, the rise again was arrested and a lowering began. The rise in elevation which began in 1940 focused attention on the deposit of the material as well as on its removal. Therefore, in 1941, the river surveys were extended about 21 miles farther downstream, into Havasu Lake.

Referring again to Table 1, an indication of the magnitude of deposit in the valley from Needles to Havasu Lake may be obtained. The total of the deposit shown in the columns for sections 33 to 40 amounts to nearly 78,000,000 cu yd. The total volume of sediment carried downstream below section 40 (that is, into Havasu Lake) is estimated to be about 40,000,000 cu yd, or only about 4% of the original lake capacity.

RETROGRESSION BELOW PARKER DAM

At the time Parker Dam was closed, Hoover Dam had been in operation for more than 3 years. Only little change in the river at and below Parker Dam had occurred during those 3 years. Therefore, changes occurring in the river were essentially the same as if the river had been in its virgin state, lacking only the annual floods of former years.

When the lake was formed behind Parker Dam, all the sediment from upstream was trapped, and the clear water issuing from the dam attempted to regain its natural load by scouring material from the river bed downstream. During the first 6 months of operation of Parker Dam, the river bed for the first 15 miles downstream was lowered an average of nearly 4 ft. Material uncovered at this depth apparently was too coarse to be moved by the discharges that prevailed, because no further appreciable lowering occurred between that time and the end of 1940, when measurements in the first 15-mile reach were discontinued because of construction of the Headgate Rock diversion dam.

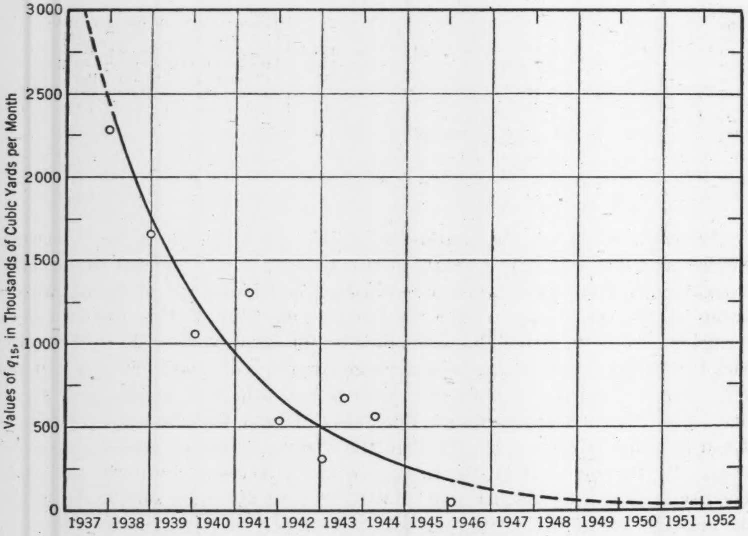

FIG. 2.—RATE OF SCOUR IN THE COLORADO RIVER FROM HOOVER DAM TO DAVIS DAM

Retrogression downstream from Headgate Rock has continued, however. Scour over the first 25 miles has occurred at a rather uniform rate, reaching a total depth of from 8 ft to 10 ft by 1945. Over the next 40-mile reach (within which is located the intake of the Palo Verde Irrigation District facilities in California), depth of scour has been from 4 ft to 6 ft. Within the next 15 miles, the depth of scour tapers rather uniformly to zero, at the upper end of the aggraded area about 55 miles above Imperial Dam.

An indication of the quantities of bed material that the river has transported in bringing about these many changes between Parker and Imperial Dams is furnished by Table 2. As with the quantities shown in Table 1, those in Table

2 can be regarded only as estimates, or approximations, because field measurements over the long reaches of the river necessarily cannot be sufficiently extensive to permit exact determination.

EFFECT OF RETROGRESSION ON THE DIVERSION INTO PALO VERDE IRRIGATION DISTRICT CANAL

The Palo Verde Irrigation District diverts water from the Colorado River about 60 miles downstream from Parker Dam. The history of river elevations at the intake, before the construction of Hoover and Parker Dams, is one of continual change. During the early years, difficulty in diverting sufficient water for irrigation was experienced at times, because of the lowering of the river bed. During cycles of aggradation, on the other hand, a plentiful supply was easily diverted. Periods of rising river elevations at the intake usually were caused by meandering of the river downstream, the slowing of the water being responsible for deposit of part of the sediment load of the river. When the meandering became excessive and threatened the levee system of the district, cutoffs were constructed to strengthen the channel With the shortened river course and increased slope, material was scoured from the river bed again, resulting in a lowered elevation at the intake. When Parker Dam was closed in 1938, the elevation was sufficiently high to permit satisfactory diversion; but, by early 1944, the scouring action of the river had lowered the bed opposite the intake by about 6 ft, and it became impossible to divert sufficient water for irrigation needs. As an emergency measure to restore gravity diversion during World War II, a temporary rock weir was constructed upstream from the old intake, no suitable site being available downstream. A new intake and headworks were constructed simultaneously; and by April, 1945, full gravity diversion had been resumed. Studies are continuing, looking toward a permanent

TABLE 2.—ESTIMATED QUANTITIES OF RIVER-BED MATERIAL REMOVED (−) AND DEPOSITED (+)[a] BETWEEN PARKER AND IMPERIAL DAMS, IN ARIZONA-CALIFORNIA (IN THOUSANDS OF CUBIC YARDS)

Period[b]	Approximate mean Q in cu ft per sec	BETWEEN SECTIONS					
		Parker Dam to section 31	Section 31 to section 23	Subtotal, Parker Dam to section 23	Section 23 to section 12	Section 12 to Imperial Dam	Subtotal, section 23 to Imperial Dam
Distance, in miles	...	15	33	88	39	22	61
Sept., 1937, to Jan., 1940	10,400	...	...	...	+25,064	+33,517	+58,581
Aug., 1938, to Jan., 1940	10,800	−19,783	− 1,057	−20,840	...	...	...
Jan., 1940, to Oct., 1940	11,900	− 6,678	− 3,404	−10,082	+ 1,685	+ 4,905	− 6,590
Oct., 1940, to Oct., 1941	17,700	−18,926	−15,742	−34,668	+ 3,307	−25,579	−28,886
Oct., 1941, to Oct., 1942	23,900	−23,754	−19,218	−42,972	− 2,611	−21,748	−19,137
Oct., 1942, to Aug., 1944	17,500	−12,675	−14,782	−27,457	+ 963	+22,047	+23,010
Aug., 1944, to Nov., 1945	18,700	+ 1,918	− 2,460	− 542	+ 1,397	+ 6,346	+ 7,743

[a] Material removed is denoted by "−"; and material deposited is denoted by "+." [b] Measurements were begun in period in which the volume first appears.

remedy by which the district can solve its problem of water supply, to replace the temporary solution provided by the weir.

AGGRADATION OF THE RIVER UPSTREAM FROM IMPERIAL DAM

The river bed has continued to scour for a distance of about 40 miles downstream from the weir, or 100 miles downstream from Parker Dam. Much of the material eroded from upstream has been deposited over the remaining 55 miles to Imperial Dam, with the result that the reservoir above the dam has been almost completely filled with sediment. This reaction was anticipated when Imperial Dam was designed, of course, and mechanical desilting works were designed and built at the same time as the dam. The desilting works are operated constantly, removing some 500 tons to 5,000 tons of sediment daily from water diverted into the All-American Canal (in California).

The deposit laid down for 55 miles above Imperial Dam is now (1950) on a comparatively steep slope. Whereas the "level pool" of water above the dam originally extended about 15 miles upstream at normal operating level, the pool now extends only a few thousand feet, the water level rising on an average slope of 0.7 ft per mile for the first 22 miles. For the next 22 miles the slope averages 1.0 ft per mile; and, for the next 11 miles, about 1.3 ft per mile. Depth of deposit in this 55-mile reach of the river ranges from zero at the upstream end to 16 ft at the dam.

An interesting phenomenon is noted in connection with this deposit which has occurred upstream from Imperial Dam. The river has spread out over a considerable width adjacent to its normal channel, encouraging plant growth and the accompanying swamp conditions in many places. This condition contrasts with the condition accompanying the aggradation of the river bed in the same area after completion of Laguna Dam in 1909, about 5 miles downstream from Imperial Dam. After completion of Laguna Dam, the river bed upstream was built up, as the river dropped a part of its sediment load in an attempt to readjust its slope and bring about a new balance. Unlike present conditions, however, the deteriorated channel and accompanying swamp conditions were not prevalent. The river, still in its natural state, was subject to flood flows in the spring and summer every year. These floods tended to clean out a channel through which the water could flow during the remainder of the year. Thus, the course of the river remained rather well defined over the major part of the aggraded area above the dam. Today the control afforded by Hoover Dam eliminates the flood flows of former years, and hence the annual cleaning out of the channel does not occur.

EFFECT OF AGGRADATION ON THE DRAINAGE OF LAND IN PALO VERDE AND CIBOLA VALLEYS

Flooding of much of the land in the river valley upstream from Imperial Dam was anticipated at the time the dam was designed. However, many hundreds of acres in this area are under the control of the federal government. Therefore, damage to individual owners along the river has been negligible. An old difficulty has continued, however, in connection with free discharge of drainage water from the lands of the Palo Verde Irrigation District. The main drainage ditch for the district discharges into the Colorado River in the aggraded area about 50 miles upstream from Imperial Dam. With the rise in the river bed, the water-surface elevation at the outlet of the drain has risen, thus further interfering with the proper drainage of the lands. This condition has been improved at least temporarily by rectifying the river channel at and below the mouth of the drain, to permit the water to escape more readily and thus to lower the elevation of the flow in the drain.

RETROGRESSION BELOW IMPERIAL DAM

Imperial Dam was placed in operation in April, 1938. The comparatively clear water issuing from the dam during the first few years of its operation scoured large volumes of sand and silt from the river bed downstream, as happened below Hoover and Parker Dams after their closure. Surveys for the determination of the magnitude of changes taking place below Imperial Dam were not begun until 1940, however. During the succeeding 6 years nearly 65,000,000 cu yd of material were picked up from the river bed and carried downstream past the International Boundary. As a result, the river bed in this area has been lowered slightly in the 5-mile reach between Imperial and Laguna Dams, an average of about 4 ft between Laguna Dam and Yuma, Ariz., and from 6 ft to 10 ft between Yuma and the lower International Boundary.

TABLE 3.—ESTIMATED VOLUMES OF RIVER-BED MATERIAL REMOVED (−) AND DEPOSITED (+)[a] BELOW IMPERIAL DAM IN ARIZONA-CALIFORNIA

Approximate dates[b]	No. of months	Approximate mean Q, in cu ft per sec	BETWEEN SECTIONS							
			Dam-2S	2S-6S	6S-8S	Subtotal, dam to 8S	8S-10S	10S-15S	15S-20S	Subtotal, dam to 20S
Distance, in miles	...	...	4.4	8.8	5.5	18.7	4.4	8.8	15.6	47.5
Jan., 1940, to Oct., 1940	9.4	6,900	− 72	− 568	− 780	− 1,420	− 420	− 281	+ 2,250	+ 129
Nov., 1940, to Oct., 1941	12.7	13,400	−570	−2,896	−1,321	− 4,787	− 537	−3,361	− 4,560	−13,245
Nov., 1941, to Nov., 1942	12.3	17,300	+451	− 799	−2,616	− 2,964	− 950	−1,425	− 7,669	−13,008
Dec., 1942, to May, 1943	5.8	9,700	+326	− 101	− 237	− 12	− 331	− 953	− 2,682	− 3,978
June, 1943, to July, 1944	14.4	11,800	+ 49	+ 137	−2,049	− 1,863	− 434	− 347	−10,994	−13,638
Aug., 1944, to Jan., 1946	17.6	9,800	−723	− 313	+ 48	− 988	+ 133	+1,154	−20,269	−19,970
Total, January, 1940, to January, 1946			−539	−4,540	−6,955	−12,034	−2,539	−5,213	−43,924	−63,710

[a] Material removed is denoted by "−"; and material deposited is denoted by "+." [b] Measurements were begun in the period in which the volume first appears.

Estimated quantities of material removed from the river bed between Imperial Dam and the International Boundary are shown in Table 3. No information is available regarding the quantities that may have been removed between the time of closure in 1938 and the beginning of measurements in 1940. However, if the rate of scour during that period equaled the average rate as measured since 1940, an additional 20,000,000 cu yd would have been removed, making a total, since closure of Imperial Dam, of about 85,000,000 cu yd. Measurements made below the International Boundary by the Mexican government indicate that the scouring action has continued for some distance into Mexico, the material being carried downstream toward the Gulf of California.

An indication of the reduction in suspended sediment load carried by the river at Yuma as a result of building upstream dams is given by Fig. 3, in which are shown total suspended loads, in tons per day, plotted against river discharges, in cubic feet per second. Before construction of Hoover Dam, roughly 100,000 tons per day was carried at discharges of 10,000 cu ft per sec. During the first few years after the closure of Hoover Dam in 1935, only a slight reduction occurred. Thus, in the 600 miles between Hoover Dam and Yuma, the river scoured material from its bed almost equal in total quantity to that carried in the natural state at a given discharge. Beginning in 1938, when Parker and Imperial Dams were closed, a reduction in load per unit discharge was noted; and by 1942, at a discharge of 10,000 cu ft per sec, a total load of only about 10,000 tons per day was in suspension at Yuma, compared with the 100,000 tons per day of earlier years.

Effect of Retrogression on Drainage of Yuma Project Lands

Lands of the Yuma Project lie adjacent to the Colorado River downstream from Laguna Dam. Water has been diverted for some of these lands since 1890. As acreage of the irrigated area increased, drainage of the lands became a more and more important problem. Therefore, early in the development of the project, pumping stations were established to pump the drainage water back into the river. It was necessary to operate the pumps for considerable periods nearly every year prior to the closing of Imperial Dam. As a result of the lowering of the river bed which began when the dam was placed in operation, natural drainage has been restored to much of the land.

Effect of Retrogression on Diversion Into Alamo Canal

Beginning in 1902, the Imperial Valley of California diverted water from the Colorado River into the Alamo Canal until the completion of Imperial Dam in 1938. The water supply for the valley now is diverted, at Imperial Dam, into the All-American Canal. Mexican water users still divert into the Alamo Canal, the intake for which is in the United States, about 6 miles downstream from Yuma. Before the flow of the river was controlled by Hoover Dam, the elevation of the river nearly always was high enough to permit satisfactory diversion into the canal during the spring and summer months when the river was in flood. At other seasons of the year it was sometimes necessary to resort to such measures as placing a temporary brush weir across the river to raise the water level high enough to divert into the canal. When Hoover Dam was placed in operation, the seasonal fluctuations in the quantity of flow were largely eliminated, and for several years water could be diverted during the entire year with little difficulty. As the river bed continued to be lowered by scouring action, however, it became increasingly difficult to divert water at any time. By late summer in 1944, it became necessary for the Mexican govern-

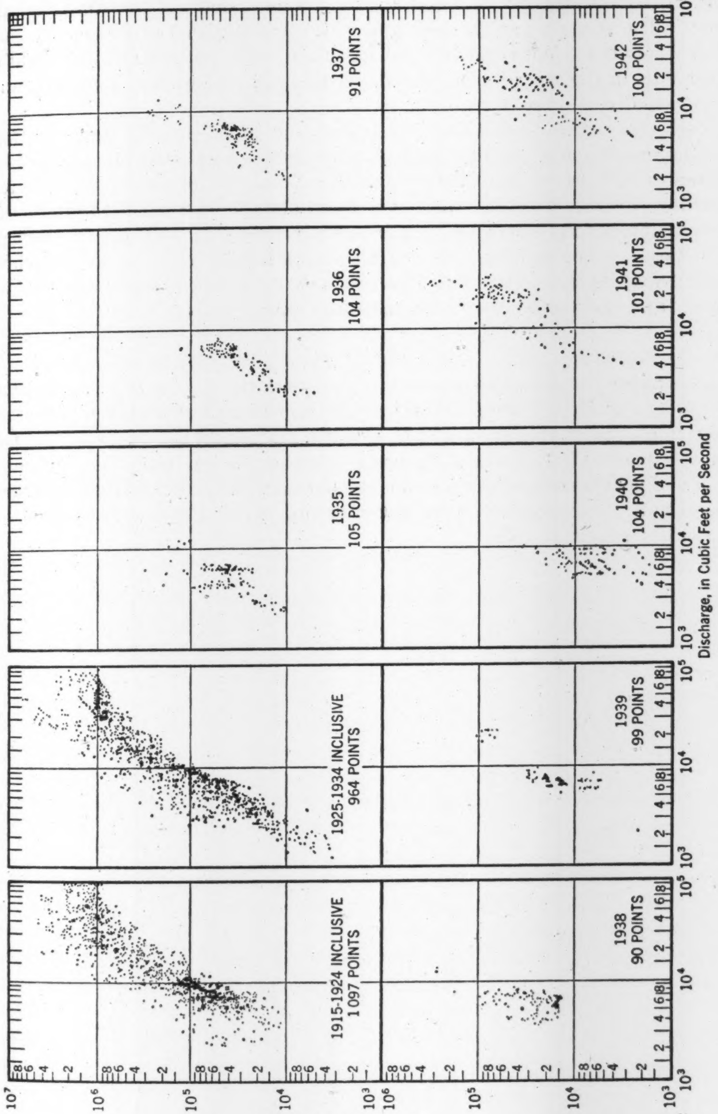

FIG. 3.—TOTAL SUSPENDED LOAD MEASURED AT YUMA, ARIZ., FROM 1915 TO 1942

ment to request delivery of water to the Alamo Canal via the All-American Canal. The request was granted, and water was supplied in this manner for about 2 months. With the reduced irrigation requirements and the slightly higher flow in the river during the succeeding fall and winter months, it was possible to divert directly from the river.

Each year since 1944 it has been necessary to resume deliveries via the All-American Canal. A permanent solution to this particular problem will be realized when the present treaty between Mexico and the United States regarding utilization of the waters of the Colorado and other international rivers becomes fully operative. The treaty provides for delivery of specified quantities of water each year via the All-American Canal. As is also required by the treaty Mexico is building a diversion dam on the Colorado River downstream from the northernmost International Boundary.

SUMMARY

This paper has discussed briefly the changes in the Colorado River since the construction of Hoover Dam. It is intended as a factual presentation, not as an evaluation of what changes might have taken place even had Hoover Dam not been constructed. Because of the interest in the development of the Colorado River and because of the future problems that may have to be solved, continued observation, accumulating an unbroken record of the physical history of the river, will be of great value.

DISCUSSION

T. Blench,[2] M. ASCE.—In India, government research organizations and other organizations have been collecting data of boulder rivers and torrents for several years. They expect that eventually the data will show how to modify the formulas of regime theory[3,4] to apply to river conditions. In their present form, the formulas are applicable to canals that have attained regime with constant discharge (or known formative discharge), with sides in fair condition (so that they are technically "smooth"), and that are free from meandering. With the aid of engineering common sense, the formulas can be applied to a limited range of river problems; they could be applied even more widely if more attention were paid to collecting and using river data. The paper suggests that, not only is a desire growing on the American continent to make use of river data for regime predictions, but also the special semicontrolled conditions on the Colorado River (in Arizona-Nevada and Arizona-California) may permit a scientific analysis in terms of regime theory which will yield some positive results.

The regime formula of interest to the subject of the paper is

$$S = b^{\frac{5}{6}} s^{\frac{1}{12}} Q^{-\frac{1}{6}} \div (2{,}080\,(10^{-5}/\nu)^{\frac{1}{4}}) \qquad (2)$$

In Eq. 2, Q is the steady or known formative discharge; b is the bed factor; s is the side factor; ν is the mean kinematic viscosity; and S is the regime slope. The bed factor depends on the bed material, on the relative bed load, and on the suspended load; but no useful formula exists to relate these quantities. However, a constant bed factor is selected by reference to a constant V^2/y (V being the mean speed at formative discharge and y being the corresponding depth), which is used to define it. Upper and lower limits to the side factor are set by the resistance of the banks to erosion and the tendency for transported material to form banks. In the Colorado reaches under discussion the limited discharge may permit an assessment of formative discharge to be used directly in Eq. 2. The banks of a river may approximate (in places at least) those of canals, so that s can be estimated. The very small index of s permits it to be estimated with poor accuracy. The correction factor to be applied to S, as found from Eq. 2, should depend mainly on the meandering of the river instead of on several causes, as in uncontrolled rivers. This factor could then be estimated from the data inserted in Eq. 2 (for comparison with values found from experiments on the effect of curvature of flow on rate of loss of head) if b were known. As the relative bed load in the river must now be small, as in canals, the existence of regime canals having the same bed material as river reaches would permit estimates to be made for b. Even if canal data were lacking to permit an estimate of *b*, the application of the formula to river reaches with different bed materials, but with about the same degree of meandering, would give positive information on the variation of *b* with bed material. This would be helpful since the present admittedly empirical and rough condition (that *b* varies as the square root of the mean diameter) is based on some information in the range of 0.25-mm size and some for material of about 4-in. size.

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[2] Cons. Engr.; Associate Prof., Civ. Eng., Univ. of Alberta, Edmonton, Alta., Canada; formerly Director of Irrigation Research, Punjab, India.

[3] "The Behaviour and Control of Rivers and Canals," by Sir C. C. Inglis, Govt. of India Publication, Yeravda Prison Press, Poona, India, 1949.

[4] "Théorie de L'Écoulement Turbulent," by T. Blench, *La Houille Blanche*, May, 1946, p. 163.

The principal items of information required for analysis of a reach are: (*a*) Longitudinal sections (along the thalweg) of the water surface from the period when damming starts until stability occurs; (*b*) hydrograph of the reach for that time; (*c*) mechanical analyses of mobile bed material at intervals along the reach, at various times up until stability occurs, together with a detail of parts of the rigid bed; (*d*) statement of the nature of the banks at the formative stage; and (*e*) plan of the reach to show the meandering and tortuosity. For the latter purpose an air photo is best. If the writer were favored with such information for a considerable reach (even if for ultimate allegedly stable conditions only) he could probably assess the possibilities of a full analysis.

The general river behavior indicated in the paper is consistent with regime theory. As long as any sediment is in motion, aggradation must occur above Imperial Dam until a regime slope is reached, everywhere, compatible with that sediment and the sediment already on the bed. Reduction of dominant discharge (for example, by taking the peaks out of the river hydrograph) tends to increase regime slope. Exclusion of potential bed material tends to reduce regime slope. It is possible to imagine an increase of relative bed load below a junction if one arm remains heavily charged and the other arm, which was relatively lightly charged, has its dominant discharge reduced by damming. Exclusion of fine suspended load may increase the bed factor by upsetting the previous velocity distribution. Accretion may begin again, downstream from Imperial Dam, for any one of three reasons: (*a*) The river regime being established upstream may permit bed load to move downstream in increased quantities; (*b*) the dominant discharge may be reduced; and (*c*) the sediment excluded from the All-American Canal may be returned to the river.

EMORY W. LANE,[5] M. ASCE.—The changes that have taken place in the bed of the Colorado River below Lake Mead are typical of what can be expected in the future below many other hydraulic developments. By a careful study of these actions, engineers can avoid many difficulties in future cases. Mr. Stanley has rendered a useful service in making information available on these changes.

The aggradation above Imperial Dam occurred in a very undeveloped region where it caused no damage and attracted little attention. In several ways, however, the action in this river section is important as an illustration of what will occur over a short period of time as the result of the raising of the water level by a low dam on a graded stream carrying a heavy sediment load. It is even more important as an indication of what will happen over a longer period of years on streams carrying moderate sediment loads.

The data given by Mr. Stanley indicate that the deposition caused by the construction of Imperial Dam extended nearly four times as far upstream as the length of the level pool caused by the dam, and that the limit of this deposition extended to an elevation above the crest of the dam of more than three times the dam height. A much greater extension both in elevation and in distance would no doubt have occurred if it had not been stopped by the degrading action caused by Parker Dam. An examination of the records indicates that this action occurred almost entirely in the short period between the 1938 and 1944 observations. Had the natural flow of the river not been equalized by Hoover and Parker Dams, even more rapid action would have occurred.

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[5] Consulting Hydr. Engr., Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo.

This is a very striking illustration of the fact that a dam on an alluvial river that is in equilibrium will cause deposits gradually extending farther and farther upstream, and the level of the deposit above the dam will tend to approach a slope equal to that of the river in its original condition, beginning at the crest of the dam. The rapidity of this trend depends largely on the storage volume available and the rate of sediment supply. In this case the deposits extended upstream from the crest of the dam and reached an average slope of about three fourths the original slope in about seven years. The action was rapid because the sediment load was great and because the storage volume below the crest level was comparatively small. Rough computations indicate that the sediment storage above the dam crest level was about 150% of that below the crest level, and that the deposits at the upper end of the level pool reached an elevation above the pool equal to about half the height of the dam.

The rapidity with which the Colorado River readjusted its slope above Imperial Dam should obviously serve as a warning to any one contemplating the construction of a dam on a stream carrying sediment loads where raising of the river bed above the level pool elevation will cause extensive damage. When one reflects that the same action will occur in many cases on other alluvial streams—but at a much slower rate—he cannot help but be glad that most streams carry much lower sediment loads than the Colorado River.

J. W. STANLEY[6].—The information on regime formulas furnished by Mr. Blench is timely. The profession should take heed of his suggestion that the formulas could be applied much more widely to river problems if more attention were paid to collecting the necessary river data. Eq. 2 should have a more general application than Eq. 1 as given in the paper. The purpose of quoting Eq. 1 was to illustrate how it might be used in predicting future scour in the particular reach of the Colorado River referred to. In other words, it was desired to determine whether Eq. 1 is applicable to the Colorado River to the extent that it could be used to develop the trend line of decreasing rate of scour shown in Fig. 2.

As pointed out by Mr. Lane, the deposit above Imperial Dam would no doubt have extended much farther upstream had it not been for the control of the natural flow of the river by Hoover and Parker Dams, and the depositional action would have been even more rapid without these dams. In this instance, then, the existence of the dams simplified at least one problem on the Colorado River.

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[6] Office of River Control, Region 3, Bureau of Reclamation, U. S. Dept. of Interior, Boulder City, Nev.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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TRANSACTIONS

Paper No. 2454

BUCKLING OF RIGID-JOINTED PLANE TRUSSES

By N. J. Hoff,[1] Bruno A. Boley,[2] S. V. Nardo,[3]
and Sara Kaufman[4]

With Discussion by Jacob Karol; Alfred S. Niles; Thomas C. Kavanagh; George Winter and Oliver G. Julian; N. J. Hoff, Bruno A. Boley, S. V. Nardo, and Sara Kaufman.

Synopsis

The buckling loads of two welded and six riveted frameworks were calculated by the convergence criterion of the method introduced by Hardy Cross, Hon. M. ASCE. The stiffening effect of the gusset plates was taken into account and the reduction in the effective modulus caused by stresses exceeding the elastic limit was considered. Values of stiffness coefficients and of carry-over factors are presented in graphs for bars with gusset plates of various length. Tests of eight frameworks are described, and the experimental buckling loads are compared with those calculated from the theory. Satisfactory agreement was obtained for frameworks buckling in the elastic range.

When the stresses exceeded the elastic limit, the choice of the correct value of the effective modulus presented a problem because large bending moments were observed in the bars before they buckled. These moments seemed to depend on the initial curvature and eccentricity of the bars and their magnitude could not be predicted theoretically. On the other hand, they influenced the effective modulus. In spite of the uncertainties involved in the choice of the effective modulus, reasonable agreement between theory and experiment was found with frameworks buckling in the inelastic range. Nevertheless, further investigation of this problem is indicated.

Introduction

The braced framework is one of the oldest types of structures and its analysis has been developed through centuries. However, comparatively little effort has been made to calculate its failing load accurately. If the framework

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Note.—Published in June, 1950, as *Proceedings-Separate No. 24.* Positions and titles are those in effect when the paper or discussion was received for publication.

[1] Prof. of Aeronautical Eng., Polytechnic Inst. of Brooklyn, Brooklyn, N. Y.

[2] Asst. Prof. of Aeronautical Eng., Polytechnic Inst. of Brooklyn, Brooklyn, N. Y.

[3] Instr. in Aeronautical Eng., Polytechnic Inst. of Brooklyn, Brooklyn, N. Y.

[4] Research Asst., Polytechnic Inst. of Brooklyn, Brooklyn, N. Y.

fails through rupture of the material in one of the members, the maximum load the structure can sustain may be calculated simply as the load at which the stress somewhere in the framework first reaches its ultimate value. However, if the bars of the framework are slender, failure is more likely to occur through buckling and the foregoing method of computing the maximum load is no longer applicable. It has been customary, in the determination of the buckling load of frameworks, to examine each bar individually and to calculate the buckling load by the Euler formula or by some short column formula. These calculations always involve an assumption of the value of the end fixity coefficient or of the effective length of the framework member.

In reality if any one bar in a rigid-jointed framework buckles, the rigidity of the joints implies that at the same time all the bars riveted to the ends of the first bar must distort. Since these adjacent bars are rigidly attached to other bars of the framework, a continuation of the reasoning shows that in the buckling of any one bar all the bars of the framework must be involved. The buckling load of a framework depends, therefore, not only on the mechanical and geometric characteristics of the highly compressed member but also on the amount of end restraint offered to this member by adjacent bars whose stiffnesses in turn are influenced by the stiffnesses of their neighbors. Depending on the amount of end restraint, the end fixity coefficient of a perfectly elastic bar varies from 1 to 4, corresponding to zero and infinite resistance to rotation of the ends of the member. As it is impossible to estimate correctly the actual amount of restraint, the method of calculating the buckling load of a framework by analyzing each bar individually leads to unreliable results.

An additional complication arises from the fact that the stiffness, or resistance to rotation, of a bar depends upon the load the bar is carrying. For instance, a tension member attached to the end of a compression member stiffens the latter, and this effect increases if the tensile load becomes higher. Thus the buckling load of the most highly compressed bar of a framework is influenced by the forces to which each member of the rigid-jointed framework is subjected.

The interaction of all these effects can be taken into account simultaneously if the buckling load of the framework is calculated by the convergence criterion of the Hardy Cross method. The criterion was proved by one of the writers (Mr. Hoff)[5,6,7] in various articles and in an address before the Second Annual Meeting of the Institute of the Aeronautical Sciences at Pasadena, Calif., on June 24, 1940. The purpose of this paper is an experimental verification of the theory and the development and presentation of stiffness coefficients and carry-over factors for bars whose ends are reinforced by gusset plates. In addition, the effects of stresses exceeding the elastic limit, of initial eccentricities and curvature, and of joint displacements are investigated.

THE CONVERGENCE CRITERION

The Hardy Cross Moment Distribution Method.—The rigid-jointed braced framework of Fig. 1 is supported in a statically determinate manner. If a

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[5] "Stable and Unstable Equilibrium of Plane Frameworks," by N. J. Hoff, *Journal of the Aeronautical Sciences*, January, 1941, p. 115.

[6] "The Proportioning of Aircraft Frameworks," by N. J. Hoff, *ibid.*, June, 1941, p. 319.

[7] "Stress Analysis of Aircraft Frameworks," by N. J. Hoff, *Journal of the Royal Aeronautical Society*, Vol. XLV, 1941, p. 241.

moment M acting in the plane of the truss is applied to one of the joints, for instance, to joint A, the joint rotates and the applied moment is distributed among the bars attached to the joint. Equilibrium of the truss requires that the moment be balanced by reactions at the points of attachment of the cantilever truss to the rigid wall. It can be shown that the axial loads caused in the various bars by the moment and its reactions are negligibly small because the bending rigidities of the individual bars are small as compared to the bending rigidity of the entire truss. Similarly, the linear displacements of the joints due to the moment and its reactions at the wall can be safely neglected.

Consequently, the framework of Fig. 1 can be analyzed as a problem in moment distribution disregarding the forces that are required at the joints to establish equilibrium. Analysis by the Hardy Cross moment distribution method involves a balancing moment distributed among the bars joined at point A in the ratio of their stiffnesses. The stiffnesses are calculated on the assumption that the far ends of the bars (at joints B, C, D, E, and F) are rigidly fixed against rotation (see Appendix 1). The moment required for fixing the far end is known as the carry-over moment. It is the product of the moment that became the share of any one bar in the balancing process and the carry-over factor of the same bar. The carry-over moment at any one of the joints B, C, D, E, or F is then considered as the unbalanced moment and is balanced. The balancing moments are distributed, the carry-over moments calculated, and the various steps of the procedure repeated at all the joints until the unbalanced moments that remain are considered negligibly small. The unbalances can always be made arbitrarily small if a sufficiently large number of suitable individual steps is undertaken. Moreover, it is always easy to find a suitable sequence of operations. One such sequence is established, for instance, by the rule that in each step the largest unbalanced moment should be balanced.

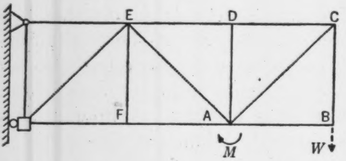

FIG. 1.—FRAMEWORK AND LOADS

Convergence of the Procedure.—A process in which the unbalanced moments can be reduced at will is known as a convergent process. The resultant bending moments at the ends of the bars converge to (that is, approach) finite values which are identical with those that can be obtained from an analytical solution of the moment distribution problem. The Hardy Cross process converges since the share of each bar after each distribution is smaller than the original unbalanced moment at the joint; and the carry-over factor is $\frac{1}{2}$ for bars of constant moment of inertia as long as the effect of the axial forces in the bars is neglected.

The situation is more complex when the individual bars of the framework are under the action of axial forces. Such axial forces can be caused, for instance, by a vertical downward load W applied to joint B. It is known that they can be calculated with sufficient accuracy by assuming ideal pin joints at the junctures of the bars even though in reality the bars are connected by sturdy gusset plates. The effect of the axial forces on the stiffness coefficients and the carry-over factors, however, should not be neglected. Values of these quantities have been calculated and presented in graphs and tables by Benjamin Wiley James[8] and A. S. Niles,[9] Assoc. M. ASCE, as well as by Eugene E. Lundquist, Assoc. M. ASCE, and W. D. Kroll[10] for different values of the governing parameter L/j, in which L is the geometric length of the bar, and j is defined by

$$j = \sqrt{\frac{E\,I}{P}} \qquad \text{(1)}$$

In Eq. 1 the product $E\,I$ is the bending rigidity of the bar and P is the absolute value of the axial force acting upon it. Since in these tables the stiffness factor has negative as well as positive values and the carry-over factor is often much greater than unity, the question of the convergence of the Hardy Cross procedure must be re-examined for the case when the bars are subjected to axial loads.

An investigation[5,6,7] of the convergence of the Hardy Cross method has revealed that the method is suitable for establishing the stability limit of a rigid-jointed framework. The procedure recommended can be explained with the aid of the framework shown in Fig. 1. It is assumed that the only external load is a vertical downward force W at joint B. When the force W is small, obviously the structure is stable; and, when the force is increased, a value is eventually reached at which the framework buckles. The problem is to calculate this critical value of the applied load.

Calculation of the Buckling Load.—The first step in the calculation is the determination of the axial loads in all the bars of the framework corresponding to an assumed value of the external load W. Next the $\frac{L}{j}$-ratios are computed for each bar. When the $\frac{L}{j}$-values are smaller than π in all the compression bars, the framework is stable because each of its bars would be stable even if the bars were attached to one another by ideal pin joints. When $\frac{L}{j}$ is greater than $2\,\pi$ in any one compression bar, the entire truss is unstable because the bar in question would be unstable even if its two ends were rigidly fixed. When some of the $\frac{L}{j}$-ratios are greater than π but none of them is greater than $2\,\pi$, further calculations are necessary to determine stability.

In such a case the stiffness coefficients have to be taken from the tables or graphs and the stiffnesses S must be calculated for each bar. The stiffness S

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[8] "Principal Effects of Axial Load on Moment-Distribution Analysis of Rigid Structures," by Benjamin Wiley James, *Technical Note No. 534*, National Advisory Committee for Aeronautics, Washington, D. C., July, 1935.

[9] "Airplane Structures," by A. S. Niles and J. S. Newell, John Wiley & Sons, Inc., New York, N. Y., 3d Ed., 1943, Vol. II, p. 121.

[10] "Tables of Stiffness and Carry-Over Factors for Structural Members Under Axial Load," by Eugene E. Lundquist and W. D. Kroll, *Technical Note No. 652*, National Advisory Committee for Aeronautics, Washington, D. C., June, 1938.

is defined as $\frac{L}{EI}$ multiplied by the stiffness coefficient. The sums of the stiffnesses must be positive at each joint. If at any one joint the sum ΣS is zero or negative, the subsystem comprising that particular joint and all the bars attached to it would be unstable even if the far ends of the bars were all rigidly fixed. Hence, the entire framework would also be unstable.

The conditions on $\frac{L}{j}$ and ΣS establish upper and lower limits for the critical value of the applied load. The condition on ΣS is correct for bars with or without gusset plates but the limits π and 2π stated for $\frac{L}{j}$ are increased by the gusset plate effect. The increases are moderate for frameworks of customary design. Use of the foregoing limits leads to conservative design when the bars have gusset plates. Within the upper and lower limits the stability can be checked by the convergence criterion. The criterion may be stated in the following form: Convergence to a finite unique set of end moments in a straightforward application of the Hardy Cross moment distribution method is proof of the stability of the framework provided the stiffness and carry-over factors are determined with due consideration of the axial loads.

The convergence can be checked by assuming a moment of arbitrary magnitude at any one of the joints—for instance, at point A in Fig. 1. With the aid of the stiffness and carry-over factors corresponding to the chosen value of W, a routine moment distribution is carried out. If the calculator succeeds in "whittling down" the unbalanced moments to negligibly small quantities at all the joints, the procedure converges and by the convergence criterion the framework is stable. Next a larger value of W is selected and the $\frac{L}{j}$-values and ΣS-values are computed. If the procedure again converges, a still larger value of W is chosen until it is found that the unbalanced moments increase rather than decrease after a number of balancing operations. Then it is known that the framework is unstable under the chosen load W.

Theoretically, the lack of convergence can be caused either by actual instability of the framework or by the incompetence of the calculator. However, experience has shown that the average engineer conversant with the moment distribution method has no difficulty in converging the process when the applied load is 1% or 2% smaller than the theoretical critical load. Similarly, the divergence is very marked as soon as the applied load exceeds the critical value by 1% or 2%. Consequently, the buckling load can be bracketed very closely with little effort if the convergence criterion is used. It is immaterial to which joint the moment M is applied, but experience has shown that the convergence, or its absence, can be judged with the least effort if the moment is assumed to act at the joint to which the most highly compressed bars are attached. Details of the calculations are evident from the example presented in Appendix 2.

Proof of the Convergence Criterion.—The proof of the convergence criterion was published in various places during 1941.[5,6,7] It is based on properties of the total potential of the system which is the sum of the strain energy stored in the framework and the potential of the external loads. It can be shown that the analytical expression for the total potential of a stable framework has a minimum value with respect to the rotations of all the joints when the framework is in equilibrium. Each cycle of the moment distribution procedure consisting of balancing, distributing, and carry-over corresponds to a rotation of the joint balanced. The angle of rotation is of the magnitude required for the equilibrium of the subsystem comprising the joint and the bars attached to it. As in each step of the moment distribution procedure the far ends of the bars are considered rigidly fixed, the subsystem is always stable because of the requirement $\Sigma S > 0$. Consequently, the establishment of equilibrium in the subsystem decreases the total potential of the subsystem to its minimum (compatible with the requirement of no rotations at the far ends of the bars) and diminishes the value of the total potential of the entire framework.

It is known, therefore, that in each cycle of the balancing procedure the total potential of the system decreases and thus approaches the minimum corresponding to stable equilibrium. At the same time the moments at the ends of the bars approach the values that prevail when the system is in equilibrium. Experience with numerical examples has shown that the approach is quite rapid or, in other words, that the procedure is well convergent.

On the other hand, when the framework is unstable under the loading, equilibrium corresponds to either a maximum or a saddle point of the total potential of the system. Consequently, the moment distribution procedure whose every step diminishes the total potential cannot lead, as a rule, to the unstable state of equilibrium, and the end moments cannot approach values that correspond to equilibrium. After a limited number of initial cycles the difference between the instantaneous value of the total potential and the value corresponding to unstable equilibrium becomes greater in each cycle of the moment distribution process, and the unbalanced moments increase rather than decrease. Therefore, the moment distribution process is divergent.

Theoretical examples illustrating the different behavior patterns have been presented elsewhere,[7] and the changes in the values of the total potential during the individual steps of the moment distribution process were plotted in diagrams. It was also shown that for particular types of loading certain sequences in the balancing procedure may lead to equilibrium even though the system is unstable. In such cases, however, any change in the sequence causes a divergence of the process and thus the set of end moments obtained is not unique. In practical analysis such an occurrence is most unlikely.

Extension of the Original Theory.—As the convergence criterion is based on theorems well established in mechanics, an experimental verification of the criterion proper is not required. Nevertheless, eight specimens were constructed and tested in the aeronautical laboratories of the Polytechnic Institute of Brooklyn at Brooklyn, N. Y., mainly for the following reasons: In the basic theory each bar is replaced by a mathematical line segment, the center line of the bar, ending at the intersection points of the center lines, and the bending rigidity of the bar is considered constant. In reality the bars have finite cross-sectional dimensions, and they are attached to one another by rigid gusset plates. Consequently, in the vicinity of the joints the bars are stiffened by the gusset plates and by the other bars converging in the same joint, with the result that the bending rigidity of the ends is greater than that of the middle of the bar. The effect of this increase in bending rigidity is an increase in the buckling load of the framework which can be considerable when the gusset plates are large. The problem is amenable to a theoretical solution after suitable assumptions are made for the variation of the bending rigidity of the bar in the region where it is reinforced by the gusset plate. The validity of the assumptions can only be checked by experiments. Moreover, tests are required to obtain information regarding reduced modulus values when the stresses in some bars exceed the elastic limit of the material before the framework buckles.

Gusset Plate Effects.—Both bending and buckling tests showed conclusively that actual bars with gusset plates are considerably stiffer then their theoretical counterparts which are assumed to have a constant rigidity between their mathematical end points. The rather obvious next assumption is to consider the bending rigidity constant from gusset plate to gusset plate and infinitely large between the edge of the gusset plate and the mathematical end point of the bar. However, experiments indicate that this assumption makes the bars materially stiffer than they are in reality. Therefore, a further refinement in the assumption is necessary. Each bar is considered to have its actual constant bending rigidity between the gusset plates, and this rigidity is increased according to a hyperbolic law from the edge of the gusset plate to the mathematical end point of the bar where the rigidity becomes infinite. This assumption gave satisfactory agreement with bending and buckling tests when the gusset plates were not excessively large. When their length amounted to 16% or 20% of the length of the bar, the hyperbolic law gave too conservative results. In almost all cases the experimental points fell between the theoretical results calculated from the infinite gusset plate rigidity and the hyperbolic law assumptions.

Values of the stiffness factors have been computed[11] for both the infinite rigidity and the hyperbolic law assumptions for bars not subjected to axial loads. In Appendix 1, these calculations are extended to bars with compressive or tensile end loads. At the same time values of the carry-over factor are also determined. The numerical values computed are presented in a number of diagrams.

Inelastic Buckling.—Although the gusset plate problem can be considered as solved satisfactorily for practical purposes, efforts made to establish rules for the determination of effective modulus values, when the stresses in the framework exceed the elastic limit, could not be brought to a final conclusion. It was observed in the tests that the bars of the framework were subjected to considerable bending moments and were visibly curved before the framework buckled. Thus, bending stresses were superimposed upon the uniformly distributed compressive stresses in the bars.

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[11] "Transversely Loaded Framework Members," by N. J. Hoff, *Journal of the Royal Aeronautical Society*, Vol. XXXIX, 1935, p. 718.

A diagram was developed on the assumption of the constancy of the axial load which permits an easy determination of the reduced modulus when the extreme fiber strains are known. The difficulty lies in finding the extreme fiber strains. An attempt was made in this investigation to explain the bending moments in the bars by calculating the secondary stresses and evaluating the effects of initial curvature and eccentricity.[12] It was found possible to assume curvature and eccentricity values which yielded extreme fiber strains in satisfactory agreement with those measured in the different stages of loading, but no method of predicting such values in a new design could be found. Moreover, the assumption of a constant compressive force during the buckling process is open to question in view of the work reported by F. R. Shanley.[13,14]

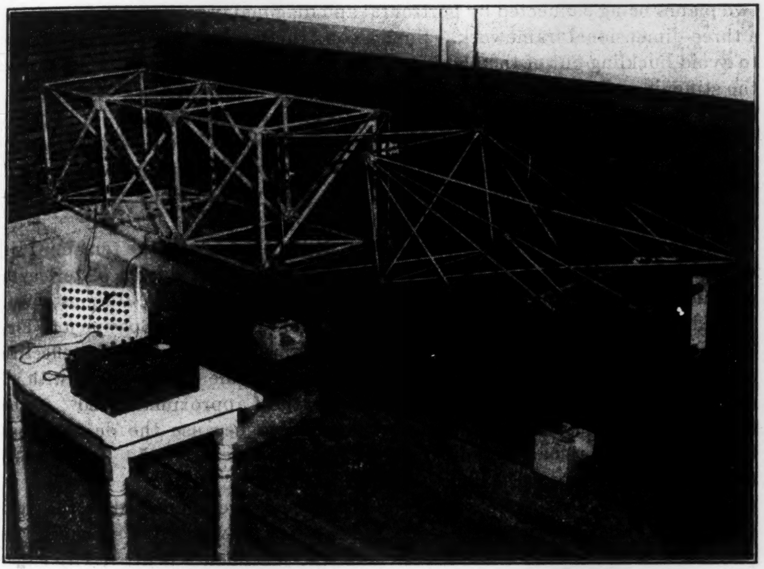

FIG. 2.—TEST SPECIMEN AND RIG

The final solution of the buckling problem of frameworks in the inelastic range cannot be expected until the basic problem of the inelastic buckling of columns is clarified completely. On the other hand, this clarification would be accelerated if more experimental data were obtained in tests on rigid-jointed frameworks.

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[12] "Buckling of Rigid-Jointed Frameworks," submitted by Dept. of Aeronautical Eng. and Applied Mechanics of the Polytechnic Inst. of Brooklyn, Brooklyn, N. Y., to the Office of Naval Research of the U. S. Navy under contract N6onr-263, Task Order I, Washington, D. C., February, 1948.

[13] "The Column Paradox," by F. R. Shanley, *Journal of the Aeronautical Sciences*, December, 1946, p. 678.

[14] "Inelastic Column Theory," by F. R. Shanley, *ibid.*, May, 1947, p. 261.

Description of Experiments and Comparison of Test Results with Theory

Test Specimens.—Eight framework specimens were tested in the course of this investigation. Of these all but one failed through buckling; of the seven that buckled two were designed to fail in the inelastic range and the other five buckled with stresses below the proportional limit.

A specimen and the test setup are shown in Fig. 2. Each framework was made of two identical vertical planes with a length-to-height ratio of 3:1, these two planes being connected by horizontal and diagonal braces so as to constitute a three-dimensional framework. Such a construction was considered necessary to avoid buckling out of the plane in the framework. However, in the analysis the stiffening effect of the braces between the two planes was neglected so that the frameworks were in effect considered to be two dimensional.

The configuration of the vertical planes of each framework is shown in Fig. 3. The frameworks were tested under the combined action of a vertical load and a bending moment applied in such a way as to cause tension in the upper chords and compression in the lower chords.

Specimens 1 and 2 were of welded steel construction, and the remaining six specimens were riveted of 24S-T aluminum alloy. The gusset plate length was measured from the center of the joint to the center of the outermost rivet. Rounded out values of these lengths are indicated in Fig. 3. Specimen 5 (Fig. 3(*d*)) was tested twice, first with the last rivets missing so that the effective gusset plate length was about 13% of the length of the bar and then with all rivets driven and thus with a gusset plate length of approximately 20% of the length of the bar. This procedure was possible because the deformations under the buckling load were fully elastic with specimens 1 to 5. Specimens 6 to 8 failed at stresses exceeding the elastic limit.

Test Rig and Test Procedure.—The specimens were connected to the rigid wall of the test rig which accommodated a fitting to which the specimen was pinned at its upper end. The lower end of the specimen was fitted with either a roller bearing or a bushing which bore against a smooth plate fastened to the rig. The specimens were loaded in cantilever fashion through a loading frame attached to the specimen itself by pins, so that no bending moment could be transmitted at the joint. The point of the application of the load was always at a distance from the rigid wall equal to twice the length of the specimen. The load was applied to the loading frame by a mechanical jack and lever arrangement, and was measured by a pair of SR-4 type A-1 "metalectric" strain gages cemented to opposite sides of a calibrated load link.

The strains in some of the members of the framework specimens were determined with the aid of SR-4 type A-1 or A-11 "metalectric" strain gages. Usually the gages were cemented at any one location in pairs, on opposite sides of the framework member. This arrangement allowed the measurement of the pure axial strain directly when the pairs of gages were connected in series, or of the bending moment when they were connected in parallel. For some of the specimens, however, individual strain measurements were recorded.

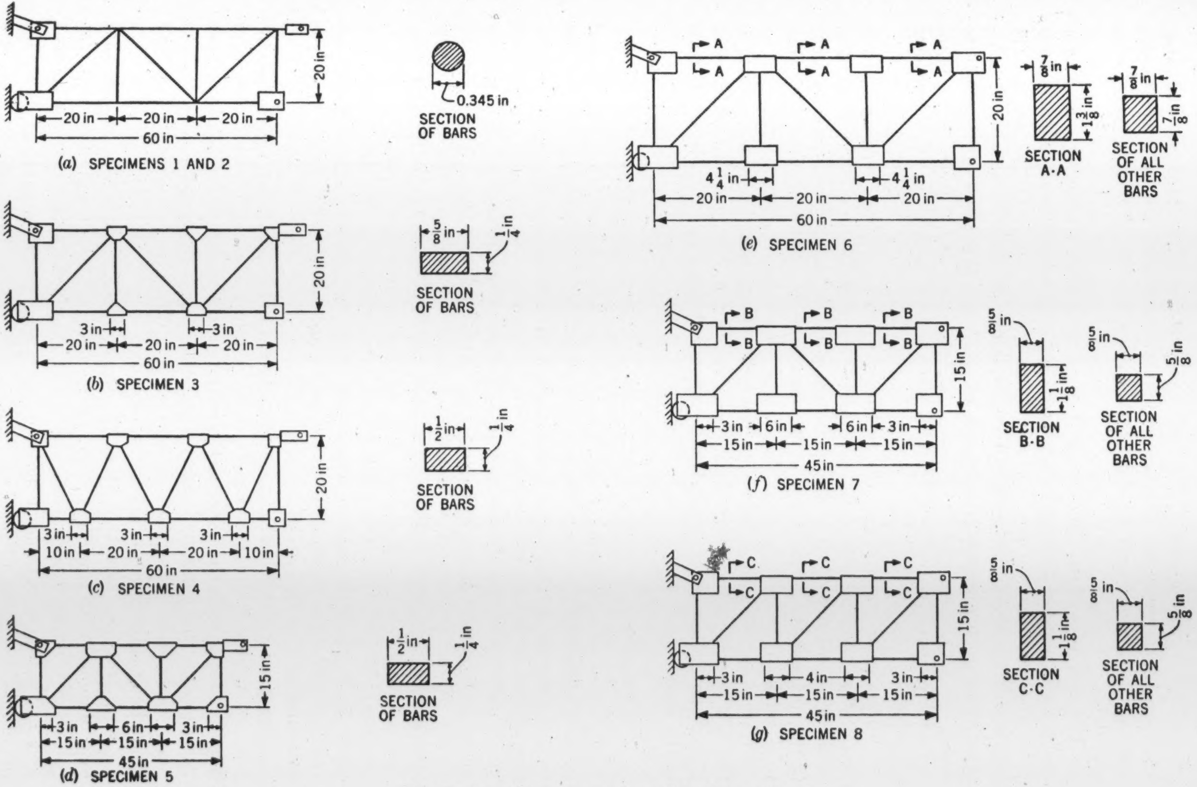
20 in
20 in
20 in
20 in
60 in
0.345 in
SECTION OF BARS
(a) SPECIMENS 1 AND 2
3 in
3 in
$\frac{5}{8}$ in
$\frac{1}{4}$ in
(b) SPECIMEN 3
3 in
10 in
$\frac{1}{2}$ in
(c) SPECIMEN 4
15 in
6 in
45 in
(d) SPECIMEN 5
A
$4\frac{1}{4}$ in
$\frac{7}{8}$ in
$1\frac{3}{8}$ in
SECTION A-A
SECTION OF ALL OTHER BARS
(e) SPECIMEN 6
B
$\frac{5}{8}$ in
$1\frac{1}{8}$ in
SECTION B-B
(f) SPECIMEN 7
C
4 in
SECTION C-C
(g) SPECIMEN 8

FIG. 3.—SCHEMATIC DRAWINGS OF TEST SPECIMENS

The strain was measured with an SR-4 portable electronic strain indicator. Switching from one strain gage to another was accomplished by a banana plug and jack arrangement. The load link was calibrated in a lever-type testing machine.

The test procedure adhered approximately to the following outline: For four or five values of the load below the critical value, force and bending moment data were recorded for the framework members equipped with gages. The load was then slowly increased to the buckling or critical value while simultaneous strain readings were taken on gages on the upper and lower side of the most highly compressed bar. Thus, the strains in this bar were known to the moment of buckling. They were needed in the calculation of the reduced modulus.

With specimens failing in the elastic or Euler range the buckling load was observed as the load at which a continued deflection of the point of application of the load was noted without an increase in the load. With specimens failing in the inelastic range the value of the load dropped suddenly when the framework buckled.

Care was taken that the shear and bending moment be evenly applied to the two panels of the framework. Equalization of the load was attained by placing shims under the roller bearings or by shifting the point of application of the load.

COMPARISON OF CALCULATED AND MEASURED BUCKLING LOADS

It was mentioned that in the calculations the actual framework was replaced by its two vertical plane trusses. Consequently, the stiffening effect of the braces connecting the two vertical planes of the actual framework was entirely neglected. It is believed that the buckling loads of the frameworks were not materially influenced by the stiffness of these braces because an effort was made to have the applied load equally distributed between the two vertical planes of the framework. The displacement patterns of the two vertical planes were always very similar, and thus the motion of the braces connecting the two planes consisted mainly of a rigid body motion with little twisting or bending. A check on the additional stiffness provided by the braces was made in the experiments in which the theory of gusset plate stiffness was verified. The assumption that the braces are not of major importance was substantiated by these tests.

The experimental buckling loads are presented in Table 1 together with the nominal values of the gusset plate size. Table 1 also contains the calculated buckling loads with the effect of the gusset plates neglected, with a hyperbolic variation of the bending rigidity in the gusset plate regions, and with an infinite moment of inertia in these regions. Where appropriate, the effective modulus of elasticity was taken into account.

With frameworks 1 to 5 (Fig. 3), which buckled elastically, the buckling load obtained with the effect of the gusset plate neglected is always lower than the corresponding experimental load. (It should be noted that specimens 1 and 2 (Fig. 3(*a*)) had gusset plates only on one bar at one joint.) When the hyperbolic variation of bending rigidity is assumed, the agreement between experimental and theoretical loads is materially improved, the average percentage error being 7.53% as compared to 33.03% obtained when the gusset plates were neglected. (All percentages are based on the experimental values.) The conclusion is reached that the assumption of a hyperbolic variation is not sufficiently accurate for bars with gusset plates whose length is of the order of magnitude of 20% of the length of the bar, but represents with a satisfactory degree of accuracy the behavior of bars with gusset plates extending over not more than about 13% of the length. Furthermore, with the exception of specimens 1 and 3 which showed a difference of 1.1% and 15.2% between theory and experiment, all values calculated from the hyperbolic variation assumption are slightly lower than the actual buckling load and are, therefore, conservative.

TABLE 1.—BUCKLING LOADS OF TEST SPECIMENS

Specimen No. (see Fig. 3)	Nominal gusset plate size[a]	COMPUTED BUCKLING LOADS				Experimental buckling loads	ERROR BASED ON EXPERIMENTAL VALUE (%)			
		Gussets neglected	Variable rigidity	Infinite rigidity	Reduced modulus[b]		Gussets neglected	Variable rigidity	Infinite rigidity	Reduced modulus[b]
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
1	12[c]	407	440	....	....	435	− 6.4	+ 1.1	....	....
2	12[c]	407	440	....	....	445	− 8.5	− 1.1	....	....
3	9	178	212	248	....	184	+15.2	− 3.2	+34.8	....
4	9	175	205	255	....	216	−18.0	− 5.1	+18.1	....
5	13	245	300	360	....	300	−18.3	0	+20.0	....
5	20	245	320	520	....	443	−44.7	−22.7	+17.3	....
6	10	10,520	....[d]	....[d]	....[d]	....	....	....	....	....
7	20	4,950	6,300	....	5,875	5,600	−11	+12.5	....	+4.9
8	15	5,750	....	....	5,875	5,420	....	....	....	+8.4

[a] Percentage of bar length. [b] Reduced modulus and variable rigidities. [c] Gusset plate only at roller bearing. [d] Did not fail by buckling.

As the upper limit for the stiffness is represented by the assumption of an infinite moment of inertia in the regions of the gusset plates, the buckling loads were next calculated on this basis. The results obtained were always substantially higher than the experimental values, as may be seen from Table 1. Consequently, the buckling loads calculated from the assumption of an infinite rigidity must be considered unsafe.

Specimen 6 (Fig. 3(*e*)) did not fail by buckling but by the shearing of some of the riveted connections. The stresses at failure in the most highly compressed member of specimen 7 were above the proportional limit of the material. For this reason a reduced modulus of elasticity E_r was used in the calculations. Since E_r depends on the strains at the two extreme fibers at the bar, care was taken in the experiment to measure the strains in the highly loaded bars at sufficiently close intervals to allow an extrapolation to the moment of failure. The strains were found to vary somewhat along the bars. The stiffness and carry-over factors were calculated with the aid of a single value for the modulus of elasticity for any particular bar, and thus it was necessary to replace the variable modulus distribution along the bar by a constant value. In obtaining this average modulus more weight was given to the values along the middle of the bar. Table 1 shows that the buckling load calculated with consideration of the effects of both the reduced modulus and the gusset plates is in satisfactory agreement with experiment.

Specimen 8 was the second framework designed to fail in the inelastic range. The critical load calculated for this specimen by the convergence criterion, neglecting the effects of the gusset plates and of the stresses above the proportional limit was 5,750 lb, applied at the end of the loading frame. It was not expected that the experimental buckling load would differ materially from this value; yet, when the load of 4,600 lb was reached in the test, one of the vertical bars of the specimen suddenly appeared to undergo a vertical translation and possibly also some stretching. The value of the applied load dropped and the zero positions of the strain gages were altered so much that no reliable strain readings could be obtained afterward. It was possible, however, to increase the applied load to a value of 5,420 lb, at which time the framework failed in the more usual displacement pattern involving only rotations of the joints. No further increase in the load was possible.

Because strain readings were not available near the maximum load, the effective modulus could not be calculated with any degree of accuracy. When the effective modulus corresponding to the average compressive stress was computed from the von Kármán expression and the gusset plate effect was neglected, the buckling load of the framework was found to be 5,400 lb by the convergence criterion. The buckling load was also calculated on the assumption of a hyperbolic increase of the stiffness in the gusset plate region. At the same time the modulus was taken equal to the tangent modulus corresponding to an average compressive stress calculated for a pin-jointed framework. The load obtained was approximately 5,875 lb and therefore did not differ widely from the experimental value. In view of the uncertainties involved in the choice of the effective modulus the agreement obtained between theory and experiment can be considered satisfactory provided the final experimental buckling load is considered as the correct one.

Joint Displacements.—In an attempt to obtain an explanation for the sudden change in the displacement pattern of specimen 8 at the lower load of 4,600 lb, the buckling load of this framework was also evaluated considering the displacements as well as the rotations of the joints. In the calculations the Hardy Cross method had to be replaced by the more general Southwell approach and the technique employed was that suggested by one of the authors (Mr. Boley[15]). When the only displacements taken into account were those of the joints at the ends of the vertical bar which underwent a translation, the value obtained for the load was 5,400 lb which was practically indistinguishable from that given by the Hardy Cross method. When the displacements of all the joints were considered, the buckling load was found to be between 5,340 lb and 5,400 lb, thus showing only a negligible change in the value of the buckling load. Both calculations including joint displacements were carried out on the basis of a value of the reduced modulus of elasticity calculated from the von Kármán formula. It was concluded that the unusual behavior of the framework was not due to buckling involving joint displacements, but was probably due to some inaccuracy in the construction or test setup. Unfortunately, because of lack of time this conclusion could not be checked experimentally by testing a second specimen of the same design.

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[15] "Numerical Methods for the Calculation of Elastic Instability," by Bruno A. Boley, *Journal of the Aeronautical Sciences*, June, 1947, p. 337.

A systematic, although somewhat limited, theoretical investigation of the effect of joint displacements on the instability of rigid-jointed frameworks led to the conclusion that this effect is negligible.

Stiffness Tests.—The correctness of the assumptions regarding the stiffness of the gusset plates was checked indirectly by comparing theoretical and experimental buckling loads. To obtain a more direct verification a moment was applied to one joint of a plane truss and the resulting rotations of this and neighboring joints were measured. The rotations were also calculated using the stiffness and carry-over factors determined in Appendix 1, and the calculated values were compared with the test results.

The experiments were conducted with framework specimen 5 after the buckling tests of the specimen had been completed. As the stresses attained at buckling were well below the proportional limit, the buckling tests caused no permanent deformation. Stiffness tests were run first on the complete three-dimensional framework; then the braces connecting the two vertical planes were removed and the tests were repeated on the resulting two-dimensional truss. The size of the gusset plates was varied between 13.3%, 16.5%, and 20% of the length of the bars by inserting or removing some of the rivets attaching the gusset plate to the bar.

The moment was applied by an arm fitted to the upright entering the joint to be tested. Dead weights were hung from the moment arm at a point 40 in. from the joint. The loads were varied from 1 lb to 10 lb in seven or eight approximately equal increments. The specimen was thus subjected to the combined action of a shear load and a moment, but the length of the moment arm was considered sufficient to make the effect of the shear load negligible. This assumption was permissible because the resistance of the framework to forces applied centrally at the joints was very much greater than the resistance to moments that tended to rotate the joints.

To measure the rotation of a joint a short channel section was mounted with two bolts and lock nuts approximately 1 in. from the joint in question. A long channel section supported at several points was attached to the rig independently of the framework. Two dial gages were fastened to the latter channel with their plungers resting on the short channel section attached to the specimen. The difference in the gage readings divided by the distance between the gages was taken as the angle of rotation of the joint in radians.

Figs. 4 and 5 are samples of comparisons between measured and calculated values of the stiffness. The following conclusions can be drawn from these figures as well as from other graphs not shown:

(1) All the experimental points are located between the curve corresponding to infinite gusset plate rigidity and that corresponding to the hyperbolic law.

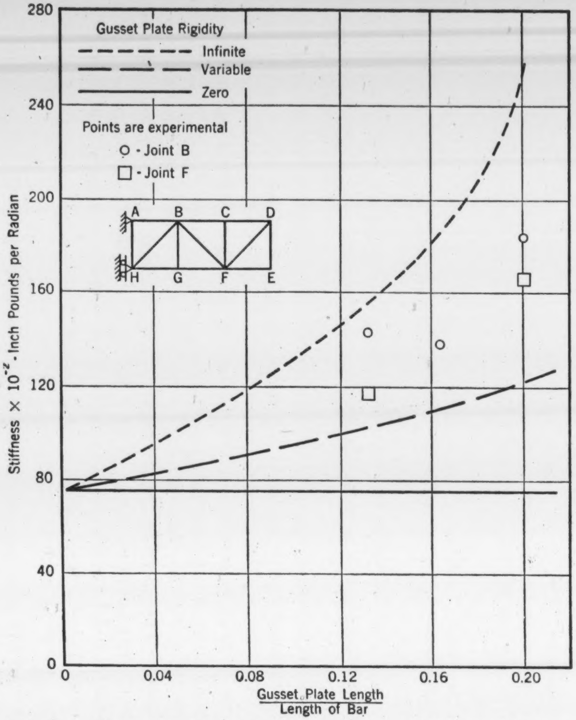

FIG. 4.—STIFFNESS OF JOINTS B AND F, SPECIMEN 5; NO AXIAL LOADS

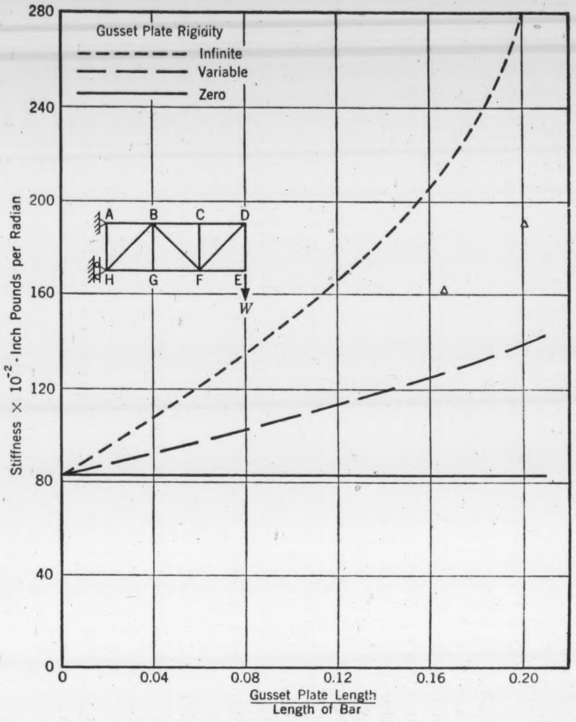

FIG. 5 —STIFFNESS OF JOINT B, SPECIMEN 5; BARS SUBJECTED TO AXIAL LOADS

(2) For small gusset plates the hyperbolic law represents a good approximation. For large gusset plates it gives a conservative estimate.

(3) The increase in stiffness due to gusset plate effect is considerable. With 20% gusset plates the actual stiffness is from two times to three times the value without gusset plates.

CONCLUSIONS

Tests on eight frameworks proved that the convergence criterion of the Hardy Cross moment distribution method can be used with confidence to predict the buckling loads of frameworks if the effect of the gusset plates on the stiffness and carry-over factors is taken into account. Values of these quantities were calculated on the basis of two different assumptions and the numerical results obtained are presented in eight graphs. It was found that the assumption of a hyperbolic law for the bending rigidity distribution generally yields conservative results for the buckling of trusses which are accurate enough for practical purposes when the gusset plate length divided by the total theoretical length of the truss member is not greater than 0.13. The assumption of infinite rigidity for the gusset plate regions leads to nonconservative results. The actual buckling load of a truss is bracketed, as a rule, by the values obtainable on the basis of these two assumptions. A numerical example is presented in Appendix 2 to show how a practical problem can be solved.

ACKNOWLEDGMENT

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APPENDIX 1. STIFFNESS AND CARRY-OVER FACTORS FOR BARS WITH GUSSET PLATES

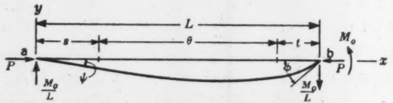
It has been customary to evaluate the stiffness and carry-over factors of bars on the basis of a constant bending rigidity throughout the length of the bar. When these factors were used to calculate the buckling load of the frameworks described in the experimental part of this report, the resulting theoretical buckling loads were lower than the experimental values. To eliminate this discrepancy the bending rigidity of a bar in a framework was assumed to increase hyperbolically toward the joint in the regions reinforced by gusset plates. The effect of such a variation on the stiffness of a bar with end moment, but no axial load, has been demonstrated elsewhere.[11] The theory is herein extended to bars subjected to both end moment and axial load.

As shown in Fig. 6(*a*), the bar is considered in three sections; the sections *s* and *t* are reinforced by gusset plates and the middle section has constant bending rigidity $E\,I$. If I is the moment of inertia of the middle section, the hyperbolic variation in the ends gives

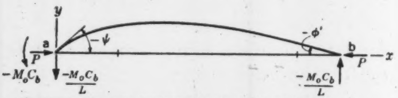
$$I_s = \frac{s\,I}{x} \qquad (\text{for } 0 \leqq x \leqq s) \ldots\ldots\ldots\ldots\ldots\ldots (2a)$$

and

$$I_t = \frac{t\,I}{L - x} \qquad (\text{for } L - t \leqq x \leqq L) \ldots\ldots\ldots\ldots\ldots (2b)$$



(*a*) BEAM-COLUMN MOMENT APPLIED AT POINT b



(*b*) BEAM-COLUMN MOMENT APPLIED AT POINT a

FIG. 6.—FORCES ON BARS WITH GUSSET PLATES

Hence, the following three moment-curvature equations can be written for the three sections if both ends of the bar are pin jointed:

$$\frac{E\,s\,I}{x}\,y''_s + P\,y_s = \frac{M_0\,x}{L} \qquad (\text{for } 0 \leqq x \leqq s) \ldots\ldots\ldots\ldots (3a)$$

$$E\,I\,y''_m + P\,y_m = \frac{M_0\,x}{L} \qquad (\text{for } s \leqq x \leqq L - t) \ldots\ldots\ldots (3b)$$

and

$$\frac{E\,I\,t}{L - x}\,y''_t + P\,y_t = \frac{M_0\,x}{L} \qquad (\text{for } L - t \leqq x \leqq L) \ldots\ldots\ldots (3c)$$

in which y'' is the second derivative of the deflection y with respect to x. The solutions to Eqs. 3 give the deflection y of the bar as

$$y_s = x^{1/2}\left[A_1\,J_{1/3}\left(\frac{2\,k\,x^{3/2}}{3\,s^{1/2}}\right) + B_1\,J_{-1/3}\left(\frac{2\,k\,x^{3/2}}{3\,s^{1/2}}\right)\right] + \alpha\,x \qquad (\text{for } 0 \leqq x \leqq s) \ldots (4a)$$

$$y_m = A_2 \sin k\,x + B_2 \cos k\,x + \alpha\,x \qquad (\text{for } s \leqq x \leqq L - t) \ldots . (4b)$$

and

$$y_t = (L - x)^{1/2}\left\{A_3\,J_{1/3}\left[\frac{2\,k(L - x)^{3/2}}{3\,t^{1/2}}\right] + B_3\,J_{-1/3}\left[\frac{2\,k(L - x)^{3/2}}{3\,t^{1/2}}\right]\right\} + \alpha\,x \qquad (\text{for } L - t \leqq x \leqq L) \ldots (4c)$$

in which $k^2 = \frac{P}{E\,I}$, $\alpha = \frac{M_0}{L\,P}$, $J_{1/3}$, and $J_{-1/3}$ are Bessel functions of order 1/3 and of order $-1/3$, respectively, and A_1, A_2, A_3, B_1, B_2, and B_3 are arbitrary constants. To insure continuity of deflection and slope along the bar, the following conditions must be satisfied:

$$\left.\begin{aligned} y_s(s) &= y_m(s); \qquad & y_m(L-t) &= y_t(L-t) \\ y'_s(s) &= y'_m(s); \qquad & y'_m(L-t) &= y'_t(L-t) \end{aligned}\right\} \qquad (5)$$

Since the ends of the bar can rotate but not displace, $y(0) = y(L) = 0$. These six conditions determine the six arbitrary constants as

$$\left.\begin{aligned} A_1 &= \frac{y_1 - \alpha s}{s^{1/2} J_{1/3}(z_s)} \\ A_2 &= (y_2 - \alpha L + \alpha t)\cos k s - (y_1 - \alpha s)\cos k(L-t) \\ A_3 &= \frac{r\left[y_2 - \alpha(L-t) + \alpha L\left(\dfrac{k}{3}\right)^{1/3} t^{-1/6}\,\Gamma(\tfrac{2}{3}) t^{1/2} J_{-1/3}(z_t)\right]}{t^{1/2} J_{1/3}(z_t)} \end{aligned}\right\} \qquad (6)$$

and

$$\left.\begin{aligned} B_1 &= 0 \\ B_2 &= -r\left[(y_2 - \alpha L + \alpha t)\sin k s + (y_1 - \alpha s)\sin k(L-t)\right] \\ B_3 &= -\alpha L\left(\frac{k}{3\,t^{1/2}}\right)^{1/3}\Gamma(\tfrac{2}{3}) \end{aligned}\right\} \qquad (7)$$

in which

$$y_1 - \alpha s = \alpha r L\left(\frac{k}{3\,t^{1/2}}\right)^{1/3}\Gamma(\tfrac{2}{3}) t^{1/2} J_{-1/3}(z_t)\,\frac{R(z_t)}{Q(z_s)\,Q(z_t) - r^2} \qquad (8a)$$

$$y_2 - \alpha(L-t) = \alpha L\left(\frac{k}{3\,t^{1/2}}\right)^{1/3}\Gamma(\tfrac{2}{3}) t^{1/2} J_{-1/3}(z_t)\,\frac{R(z_t)\,Q(z_s)}{Q(z_s)\,Q(z_t) - r^2} \qquad (8b)$$

In Eqs. 8,

$$R(z_t) = \frac{J_{2/3}(z_t)}{J_{-1/3}(z_t)} + \frac{J_{-2/3}(z_t)}{J_{1/3}(z_t)} \qquad (9a)$$

$$Q(z_t) = \frac{J_{-2/3}(z_t)}{J_{1/3}(z_t)} + \cot k\theta \qquad (9b)$$

$$Q(z_s) = \frac{J_{-2/3}(z_s)}{J_{1/3}(z_s)} + \cot k\theta \qquad (9c)$$

Also

$$r = \frac{1}{\sin k\theta} \qquad (10a)$$

$$z_t = \tfrac{2}{3} k t \qquad (10b)$$

$$z_s = \tfrac{2}{3} k s \qquad (10c)$$

and

$$\theta = L - s - t \qquad (10d)$$

From the foregoing solutions $y'(L)$ and $y'(0)$ are given as follows:

$$y'(L) = \alpha\,\lambda_1(s, t) \qquad (11a)$$

$$y'(0) = \alpha\,\lambda_2(s, t) \qquad (11b)$$

in which

$$\lambda_1(s, t) = 1 - \left[\frac{k\,L}{3} \left(\frac{L}{t} \right)^{1/2} \right]^{2/3} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})} \frac{J_{-1/3}(z_t)}{J_{1/3}(z_t)} \left[1 - \frac{R(z_t)\,Q(z_s)}{Q(z_s)\,Q(z_t) - r^2} \right] \quad \text{(12a)}$$

and

$$\lambda_2(s, t) = 1 - \left[\left(\frac{k\,L}{3} \right)^2 \left(\frac{L}{t} \right)^{1/2} \left(\frac{L}{s} \right)^{1/2} \right]^{1/3} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})}\, r \frac{J_{-1/3}(z_t)}{J_{1/3}(z_s)} \frac{R(z_t)}{Q(z_s)\,Q(z_t) - r^2} \quad \text{(12b)}$$

The stiffness coefficient p_b at end b of the bar in Fig. 6(a) is defined implicitly by

$$\phi = \frac{M_0\,L}{p_b\,E\,I} \quad \text{(13a)}$$

If the deflection is small, $\tan \phi \cong \phi$, and

$$y'_b(L) = \frac{M_0\,L}{p_b\,E\,I} \quad \text{(13b)}$$

Thus the stiffness coefficient at end b is

$$p_b = \frac{(k\,L)^2}{\lambda_1(s, t)} \quad \text{(14)}$$

and the carry-over factor is zero here since end a is pin jointed.

To be of use in a relaxation process such as the Hardy Cross moment distribution method, the stiffness and carry-over factors at end b of the bar must be determined with end a fixed. For this purpose a moment $(-\,M_0\,C_b)$ is applied at joint a (Fig. 6(b)), in which C_b is defined implicitly by the expression:

$$\psi = \frac{M_0\,C_b\,L}{p_a\,E\,I} \quad \text{(15)}$$

and p_a is the stiffness coefficient of joint a when joint b is pin jointed. Therefore,

$$C_b = \frac{\psi}{\phi} \frac{p_a}{p_b} \quad \text{(16a)}$$

If ϕ and ψ are small, the angles in Fig. 6(a) can be replaced so as to give

$$C_b = \frac{y'_b(0)}{y'_b(L)} \frac{p_a}{p_b} \quad \text{(16b)}$$

in which the subscript b in $y'_b(0)$ and $y'_b(L)$ indicates that M_0 is applied at joint b. With the notation of Fig. 6(b), the coefficient C_a is defined by the expression:

$$\phi' = \frac{M_0\,C_b\,C_a\,L}{p_b\,E\,I} = \frac{C_a\,p_a}{p_b}\,\psi \quad \text{(17)}$$

or

$$C_a = \frac{\phi'}{\psi} \frac{p_b}{p_a} \quad \text{(18a)}$$

If ϕ' and ψ are assumed to be small,

$$C_a = \frac{y'_a(0)}{y'_a(L)} \frac{p_b}{p_a} \quad \text{(18b)}$$

in which the subscript a in $y'_a(0)$ and $y'_a(L)$ indicates that the moment is applied at joint a, and the abscissas 0 and L indicate joints b and a, respectively. This system can be derived from the system in Fig. 6(a) by interchanging s and t, since in that figure both ends are pin jointed. Thus,

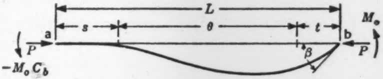


FIG. 7.—BEAM COLUMN WITH ONE END FIXED

$$y'_a(0) = \alpha\, \lambda_2(t, s) \qquad \text{and} \qquad y'_a(L) = \alpha\, \lambda_1(t, s) \quad \text{(19a)}$$

and, consequently,

$$p_a = \frac{M_0 L}{E\, I\, y'_a(L)} = \frac{(k\, L)^2}{\lambda_1(t, s)} \quad \text{(19b)}$$

By superimposing the loading of Fig. 6(b) on that of Fig. 6(a), the system of Fig. 7 is obtained in which

$$\beta = \phi - \phi' = \frac{M_0 L}{p_b\, E\, I} (1 - C_b\, C_a) = y'_b(L) \left[1 - \frac{y'_b(0)}{y'_b(L)} \frac{y'_a(0)}{y'_a(L)}\right] \quad \text{(20)}$$

or

$$\frac{\beta}{\alpha} = \lambda_1(s, t) \left[1 - \frac{\lambda_2(s, t)\, \lambda_2(t, s)}{\lambda_1(s, t)\, \lambda_1(t, s)}\right] \quad \text{(21)}$$

With the use of the definition expressed by Eq. 13a, the stiffness coefficient f at end b of the bar in Fig. 7 is defined by the equation:

$$\beta = \frac{M_0 L}{f\, E\, I} \quad \text{(22)}$$

so that

$$f = \frac{(k\, L)^2}{\frac{\beta}{\alpha}} \quad \text{(23)}$$

The carry-over factor C, defined as the moment induced at joint a by a unit moment at joint b (Fig. 7), is then

$$C = C_b = \frac{y'_b(0)}{y'_b(L)} \frac{p_a}{p_b} = \frac{\lambda_2(s, t)}{\lambda_1(s, t)} \quad \text{(24)}$$

In the case of equal gusset plates, Eqs. 20, 21, and 24 reduce to

$$\beta = y'(L) \left\{1 - \left[\frac{y'(0)}{y'(L)}\right]^2\right\} \quad \text{(25)}$$

or

$$\frac{\beta}{\alpha} = \lambda_1(s, s) \left\{1 - \left[\frac{\lambda_2(s, s)}{\lambda_1(s, s)}\right]^2\right\} \quad \text{(26)}$$

and

$$C = \frac{y'(0)}{y'(L)} = \frac{\lambda_2(s, s)}{\lambda_1(s, s)} \qquad (27)$$

since $p_a = p_b$, etc., due to symmetry.

The same procedure can be followed for a bar in tension with the load $(-P)$ in place of P. With this substitution, Eqs. 11 become

$$y'(L) = \alpha \, \bar{\lambda}_1(t, s) \qquad (28a)$$

and

$$y'(0) = \alpha \, \bar{\lambda}_2(t, s) \qquad (28b)$$

in which

$$\bar{\lambda}_1(t, s) = 1 - \left[\frac{k\,L}{3} \left(\frac{L}{t} \right)^{1/2} \right]^{2/3} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})} \frac{I_{-1/3}(z_t)}{I_{1/3}(z_t)} \left[1 - \bar{R} \, \frac{\bar{R}(z_t)\, \bar{Q}(z_s)}{\bar{Q}(z_s)\, \bar{Q}(z_t) - (\bar{r})^2} \right] \qquad (29a)$$

and

$$\bar{\lambda}_2(t, s) = 1 - \left[\frac{k\,L}{3} \left(\frac{L}{t} \right)^{1/2} \right]^{1/3} \left[\frac{k\,L}{3} \left(\frac{L}{s} \right)^{1/2} \right]^{1/3} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})}$$

$$\times \left[\frac{\frac{t}{L}}{\frac{s}{L}} \right]^{1/2} \frac{I_{-1/3}(z_t)}{I_{1/3}(z_s)} \, \bar{r} \, \frac{\bar{R}(z_t)}{\bar{Q}(z_t)\, \bar{Q}(z_s) - (\bar{r})^2} \qquad (29b)$$

Also,

$$\bar{R}(z_t) = \frac{I_{-2/3}(z_t)}{I_{1/3}(z_t)} - \frac{I_{2/3}(z_t)}{I_{-1/3}(z_t)} \qquad (30a)$$

$$\bar{Q}(z_t) = \frac{I_{-2/3}(z_t)}{I_{1/3}(z_t)} + \coth k\,\theta \qquad (30b)$$

$$\bar{Q}(z_s) = \frac{I_{-2/3}(z_s)}{I_{1/3}(z_s)} + \coth k\,\theta \qquad (30c)$$

$$z_t = \tfrac{2}{3}\, k\, t \qquad (30d)$$

$$z_s = \tfrac{2}{3}\, k\, s \qquad (30e)$$

$$\bar{r} = \frac{1}{\sinh k\,\theta} \qquad (30f)$$

and

$$\theta = L - s - t \qquad (30g)$$

In Eqs. 29 and 30, I is the usual notation for the modified Bessel function.

Calculations have been made and curves of stiffness and carry-over factors drawn for bars of equal gusset plates occupying 9%, 13.3%, 20%, and 30% of the length of the bar. The curves are presented in Figs. 8 and 9.

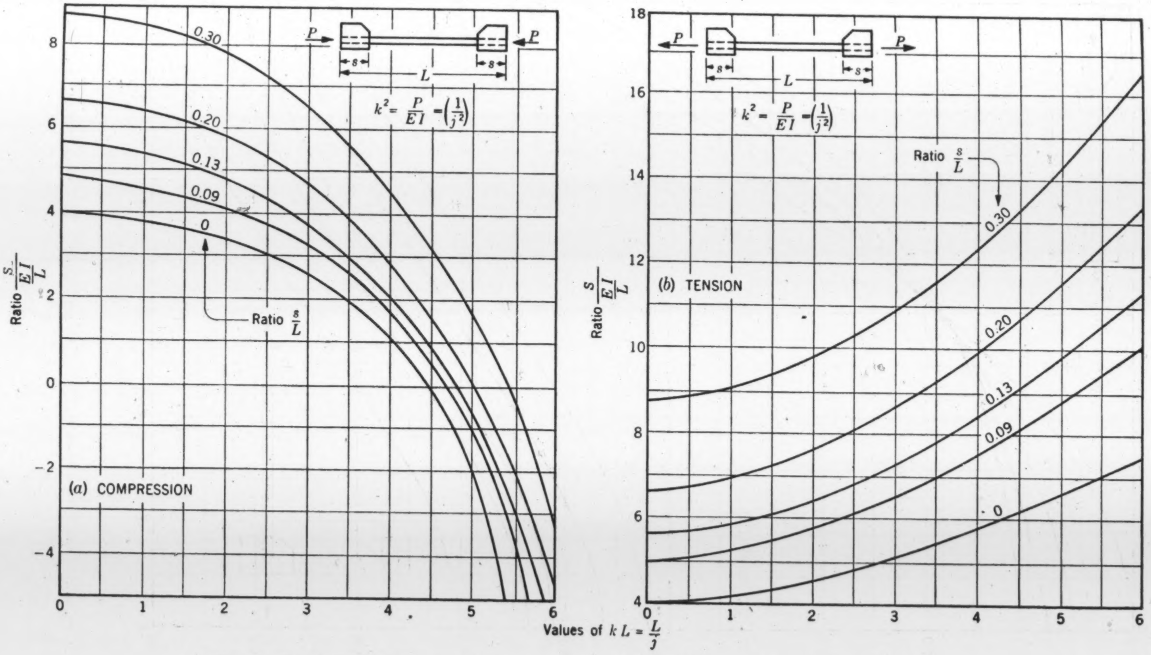


FIG. 8.—STIFFNESS FACTORS FOR BARS WITH GUSSET PLATES VARIABLE GUSSET PLATE RIGIDITY

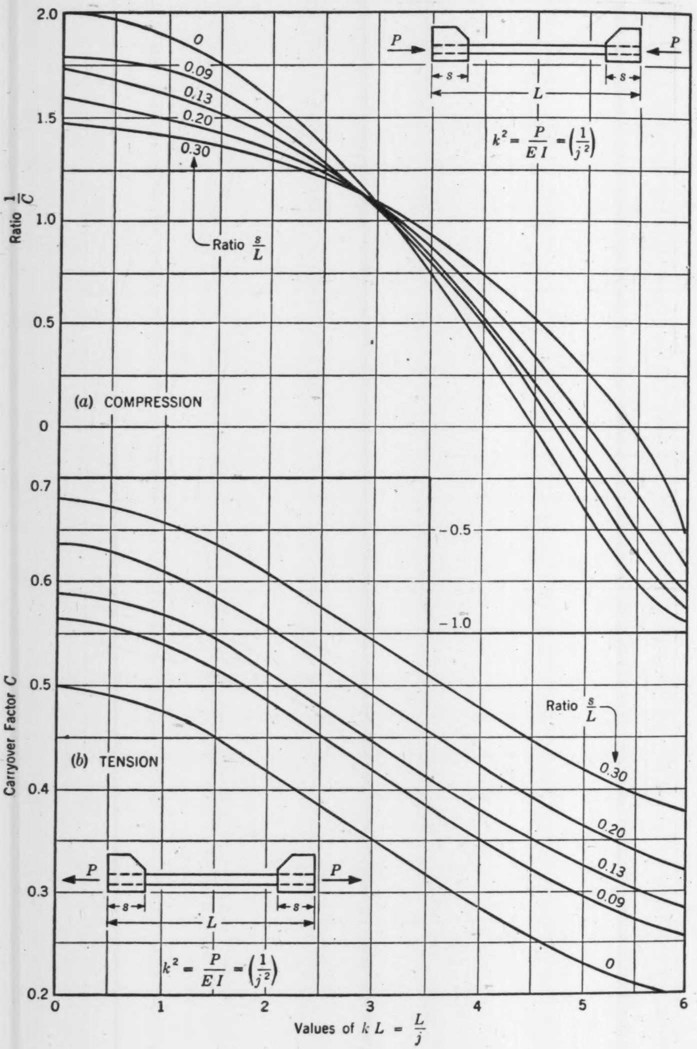


FIG. 9 —CARRY-OVER FACTORS FOR BARS WITH GUSSET PLATES; VARIABLE GUSSET PLATE RIGIDITY

For purposes of comparison it is of interest to calculate stiffness and carry-over factors based on the assumption of infinitely rigid gusset plates. The differential equations of bending, for the bar in Fig. 6(a), when the infinite moment of inertia extends over regions of lengths s and t at the left and right ends of the bar, respectively, are

$$y''_s = 0 \qquad (\text{for } 0 \leqq x \leqq s) \tag{31a}$$

$$E\,I\,y''_m + P\,y_m = \frac{M_0\,x}{L} \qquad (\text{for } s \leqq x \leqq L - t) \tag{31b}$$

and

$$y''_t = 0 \qquad (\text{for } L - t \leqq x \leqq L) \tag{31c}$$

in which y'' is the second derivative of the deflection y with respect to x. The solutions to these equations give the deflection y of the bar as

$$y_s = A_1\,x + B_1 \qquad (\text{for } 0 \leqq x \leqq s) \tag{32a}$$

$$y_m = A_2 \sin k\,x + B_2 \cos k\,x + \alpha\,x \qquad (\text{for } s \leqq x \leqq L - t) \tag{32b}$$

and

$$y_t = A_3\,x + B_3 \qquad (\text{for } L - t \leqq x \leqq L) \tag{32c}$$

in which $k^2 = \frac{P}{E\,I}$, $\alpha = \frac{M_0}{L\,P}$, and A_1, A_2, A_3, B_1, B_2, and B_3 are arbitrary constants. The same conditions on the deflection y and the slope y' as given in the preceding calculations may be used to determine these six arbitrary constants. The result is

$$\left.\begin{aligned}
A_1 &= y'(0) = \alpha - \frac{\alpha\,k\,L}{(1 - k^2\,s\,t)\sin k\,\theta + k(t + s)\cos k\,\theta} \\
A_2 &= \frac{1}{k \sin k\,\theta}\left[(A_1 - \alpha)\sin k(L - t) - (A_3 - \alpha)\sin k\,s\right] \\
A_3 &= y'(L) = \alpha - (\alpha\,k\,L)\,\frac{\cos k\,\theta - k\,s \sin k\,\theta}{(1 - k^2\,s\,t)\sin k\,\theta + k(t + s)\cos k\,\theta}
\end{aligned}\right\} \tag{33}$$

and

$$\left.\begin{aligned}
B_1 &= 0 \\
B_2 &= \frac{1}{k \sin k\,\theta}\left[(A_1 - \alpha)\cos k(L - t) - (A_3 - \alpha)\cos k\,s\right] \\
B_3 &= -\,A_3\,L
\end{aligned}\right\} \tag{34}$$

in which

$$\theta = L - s - t \tag{35}$$

A procedure entirely analogous to that employed for the calculation of stiffness and carry-over factors under the assumption of a hyperbolic variation of the moment of inertia in the regions of the bar reinforced by gusset plates leads, in this case, to the following expressions for the stiffness factor S and the carry-over factor C for a bar whose far end is fixed:

$$S = \frac{E\,I}{L}\,(k\,L)^2\,\frac{\alpha}{\beta} \tag{36a}$$

and

$$C = \frac{A_1}{\lambda(s, t)} \quad \text{(36b)}$$

in which

$$\beta = \lambda(s, t)\left[1 - \frac{A^2_1}{\lambda(s, t)\,\lambda(t, s)}\right] \quad \text{(37)}$$

For the case of a bar in compression,

$$\lambda(s, t) = A_3 \quad \text{(38)}$$

whereas, for a bar in tension,

$$\lambda(s, t) = \alpha - (\alpha\, k\, L)\,\frac{\cosh k\,\theta + k\, s \sinh k\,\theta}{(1 - k^2\, s\, t)\sinh k\,\theta + k(t + s)\cosh k\,\theta} \quad \text{(39)}$$

Numerical values of the stiffness and carry-over factors have been calculated for gusset plates of equal length at the two ends of the bar extending over 9%, 13.3%, and 20% of the length of the bar. The values are plotted in Figs. 10 and 11.

APPENDIX 2. NUMERICAL EXAMPLE

The buckling load will be calculated here for specimen 4 whose configuration and dimensions are given in Fig. 3(*c*). The gusset plates extend over 9% of the length of the members. The material used in the construction of the framework is 24S-T aluminum alloy. The first step is the assumption of a value for the load applied to the structure. In this case the load W was chosen as 100 lb acting at the end of the loading frame previously described, which corresponds to an applied bending moment of 6,000 in.-lb, and an applied shear force of 100 lb at the loaded end of the structure. This assumption leads to the forces given in Cols. 3 and 4, Table 2(*a*). The entries in Cols. 5 and 6 are calculated from the forces and the geometric and mechanical properties of the bar. The reciprocal of the carry-over factor is given for the compression bars in Col. 7 as read from the curves of Fig. 9(*a*). The values in Col. 8 are found from Fig. 8(*a*) for the compression members and from Fig. 8(*b*) for the tension members. The stiffness factors of Col. 10 are easily obtained from the values in Cols. 1 and 2. The corresponding distribution factors are given in Fig. 12(*a*).

Since the values of $k\,L\left(=\frac{L}{j}\right)$ are smaller than $2\,\pi$ for all the compression members, and greater than π for some of them, and because the sums ΣS of the stiffnesses are positive at every joint, the stability must be determined by the convergence criterion. First, a moment of 1,000 in.-lb is applied at joint J. The joint is balanced, and moments are carried over to the adjacent joints. The balancing procedure, shown in Fig. 12(*a*), is then continued in a straightforward manner, except that carry-over moments are accumulated at joint J and are never balanced out. When all the joints but J are balanced, the total accumulated moments at J are added. In this case, the sum is found to be a

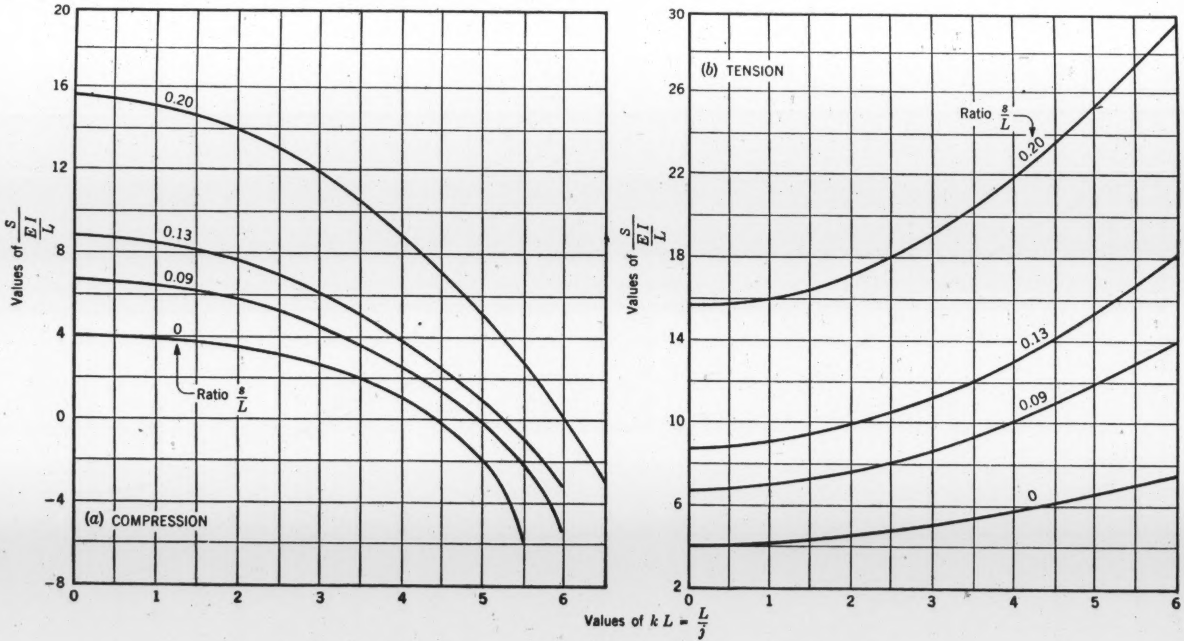


FIG. 10.—STIFFNESS FACTORS FOR BARS WITH GUSSET PLATES; INFINITE GUSSET PLATE RIGIDITY

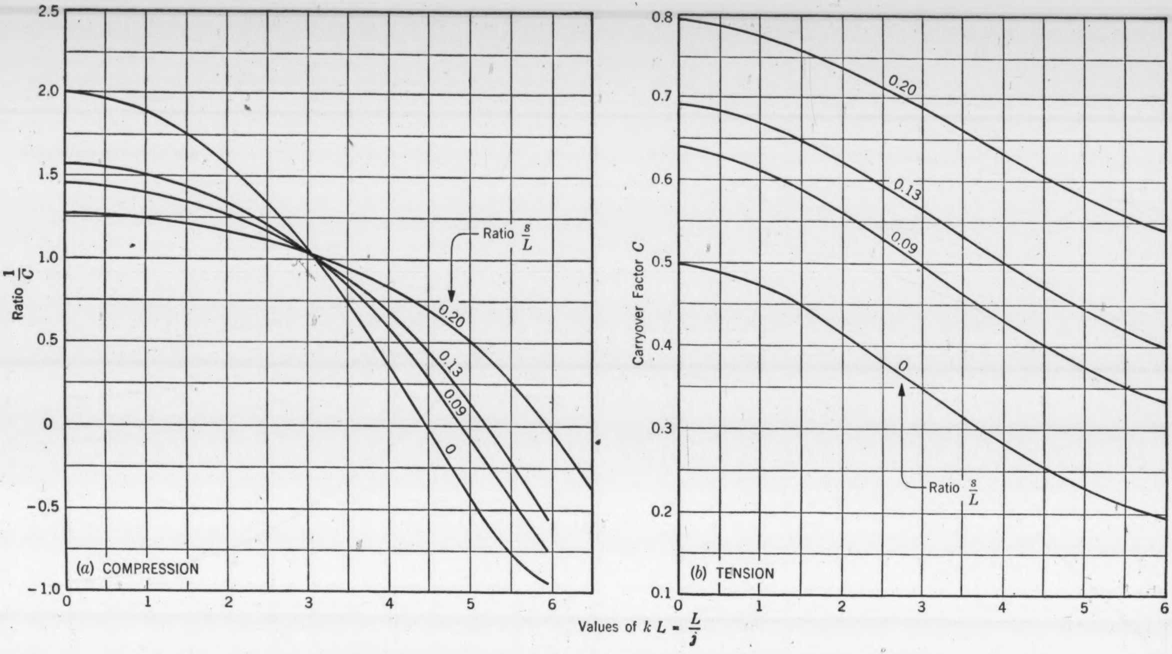
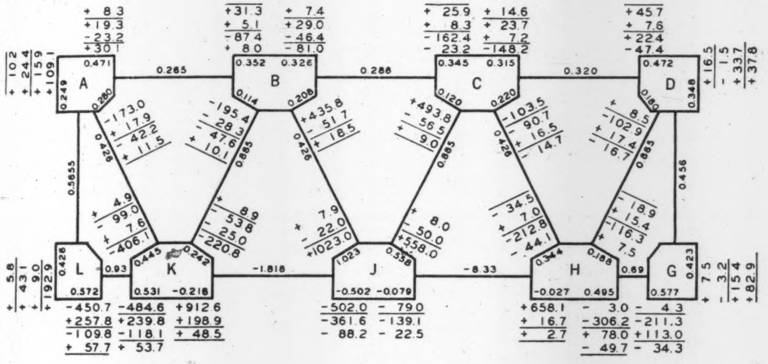
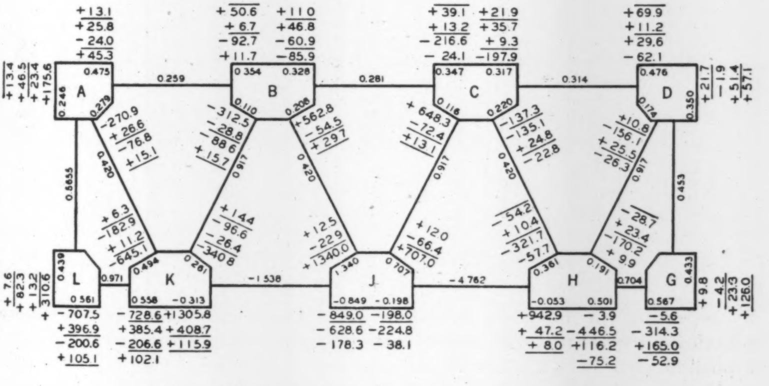


FIG. 11.—CARRY-OVER FACTORS FOR BARS WITH GUSSET PLATES; INFINITE GUSSET PLATE RIGIDITY

moment of 668 in.-lb. Since this is smaller than the originally applied 1,000 in.-lb, a repetition of the entire procedure would reduce the unbalance at J further—namely, to (668/1,000)668. Obviously, the unbalance can be diminished at will by undertaking the necessary number of operations, and thus the procedure is convergent. Consequently, by the convergence criterion the framework is stable.


A
B
C
D
L
K
J
H
G
0.471
0.249
0.280
0.265
0.352
0.326
0.114
0.208
0.288
0.345
0.315
0.120
0.220
0.320
0.472
0.348
0.180
0.5655
0.426
0.885
0.456
0.428
0.572
0.93
0.445
0.242
0.531
-0.218
-1.818
1.023
0.338
-0.502
-0.079
-8.33
0.344
0.188
-0.027
0.495
0.69
0.423
0.577
+ 8.3
+19.3
-23.2
+30.1
+31.3
+ 5.1
-87.4
+ 8.0
+ 7.4
+29.0
-46.4
-81.0
+ 25.9
+ 8.3
-162.4
- 23.2
+ 14.6
+ 23.7
+ 7.2
-148.2
+45.7
+ 7.6
+22.4
-47.4
+ 10.2
+ 24.4
+ 15.9
+109.1
+16.5
- 1.5
+33.7
+37.8
-173.0
+ 17.9
- 42.2
+ 11.5
-195.4
- 28.3
- 47.6
+ 10.1
+435.8
- 51.7
+ 18.5
+493.8
- 56.5
+ 9.0
-103.5
- 90.7
+ 16.5
- 14.7
+ 8.5
-102.9
+ 17.4
- 16.7
+ 4.9
- 99.0
+ 7.6
-406.1
+ 8.9
- 53.8
- 25.0
-220.8
+ 7.9
- 22.0
+1023.0
+ 8.0
- 50.0
+558.0
- 34.5
+ 7.0
-212.8
- 44.1
- 18.9
+ 15.4
-116.3
+ 7.5
+ 5.8
+ 43.1
+ 9.0
+192.9
- 450.7
+257.8
-109.8
+ 57.7
-484.6
+239.8
-118.1
+ 53.7
+912.6
+198.9
+ 48.5
-502.0
-361.6
- 88.2
- 79.0
-139.1
- 22.5
+658.1
+ 16.7
+ 2.7
- 3.0
-306.2
+ 78.0
- 49.7
- 4.3
-211.3
+113.0
- 34.3
+ 7.5
- 3.2
+15.4
+82.9

(a) APPLIED LOAD W = 100 LB.


A
B
C
D
L
K
J
H
G
0.475
0.246
0.279
0.259
0.354
0.328
0.110
0.208
0.281
0.347
0.317
0.116
0.220
0.314
0.476
0.350
0.174
0.5655
0.420
0.917
0.453
0.439
0.561
0.971
0.494
0.281
0.558
-0.313
-1.538
1.340
0.707
-0.849
-0.198
-4.762
0.361
0.191
-0.053
0.501
0.704
0.433
0.567
+13.1
+25.8
-24.0
+45.3
+50.6
+ 6.7
-92.7
+11.7
+11.0
+46.8
-60.9
-85.9
+ 39.1
+ 13.2
-216.6
- 24.1
+21.9
+35.7
+ 9.3
-197.9
+69.9
+11.2
+29.6
- 62.1
+13.4
+46.5
+23.4
+175.6
+21.7
- 1.9
+51.4
+57.1
-270.9
+ 26.6
- 76.8
+ 15.1
-312.5
- 28.8
- 88.6
+ 15.7
+562.8
- 54.5
+ 29.7
+ 648.3
- 72.4
+ 13.1
-137.3
-135.1
+ 24.8
- 22.8
+10.8
-156.1
+ 25.5
- 26.3
+ 6.3
-182.9
+ 11.2
-645.1
+ 14.4
- 96.6
- 26.4
-340.8
+ 12.5
- 22.9
+1340.0
+ 12.0
- 66.4
+707.0
- 54.2
+ 10.4
-321.7
- 57.7
- 28.7
+ 23.4
-170.2
+ 9.9
+ 7.6
+ 82.3
+ 13.2
+310.6
- 707.5
+396.9
- 200.6
+105.1
- 728.6
+385.4
- 206.6
+102.1
+1305.8
+408.7
+115.9
- 849.0
- 628.6
- 178.3
-198.0
-224.8
- 38.1
+942.9
+ 47.2
+ 8.0
- 3.9
-446.5
+116.2
-75.2
-5.6
- 314.3
+165.0
- 52.9
+9.8
-4.2
+23.3
+126.0

(b) APPLIED LOAD W = 105 LB.

FIG. 12.—MOMENT DISTRIBUTION OF BUCKLING LOAD CALCULATION FOR SPECIMEN 4, FIG. 3(*c*)

All the calculations must now be repeated for higher assumed loads until the load selected is sufficiently high to cause instability. Then the actual buckling load is bracketed between an upper limit and a lower limit. In this calculation an applied load of 105 lb was chosen which corresponds to a bending moment of 6,300 in.-lb plus a shear force of 105 lb at the loaded end of the structure. Table 2(*b*) is then set up like Table 2(*a*). The balancing procedure, similar to that in Fig. 12(*a*) is shown in Fig. 12(*b*). In this case the moments accumulated at joint J add up to 1,135 in.-lb; and, since this value is larger than the originally applied 1,000 in.-lb, the structure is unstable.

It may be concluded, therefore, that the buckling load of this framework is between 100 lb and 105 lb. The buckling load can be obtained more accurately by repeating the foregoing procedure for intermediate loads, but for most practical purposes it is sufficient to state that the buckling load is 102.5 lb with a possible error of ±2.5%.

TABLE 2.—COMPUTATION OF BUCKLING LOAD FOR SPECIMEN 4, FIG. 3(*c*) ($EI = 6{,}835.5$ LB-IN.^{2} FOR ALL BARS)

Bar (see Fig. 1) (1)	Length, L (in.) (2)	Load, P (lb) (3)	Type of load[a] (4)	$k = \sqrt{\frac{P}{EI}}$ (in.$^{-1}$) (5)	kL (6)	$\frac{1}{C}$[b] (7)	$\frac{S}{EI/L}$ (8)	C (9)	$\frac{20\,S}{EI}$ (10)
(*a*) APPLIED LOAD $W = 100$ LB									
AB	20.00	550.0	T	0.28366	5.67	....	9.63	0.265	9.63
BC	20.00	450.0	T	0.25658	5.13	....	8.92	0.286	8.92
CD	20.00	350.0	T	0.22628	4.53	....	8.15	0.320	8.15
LK	10.00	600.0	C	0.29627	2.96	1.075	3.40	0.930	6.80
KJ	20.00	500.0	C	0.27046	5.41	−0.550	−2.80	−1.818	−2.80
JH	20.00	400.0	C	0.24191	4.84	−0.120	−0.44	−8.330	−0.44
HG	10.00	300.0	C	0.20950	2.09	1.450	4.10	0.690	8.20
AL	20.00	0	....	0	0	....	5.078	0.5655	5.078
AK, BJ, CH	22.37	111.9	T	0.12795	2.86	....	6.38	0.426	5.70
KB, JC, HD	22.37	111.9	C	0.12795	2.86	1.130	3.48	0.885	3.11
DG	20.00	100.0	T	0.12095	2.42	....	6.02	0.456	6.02
(*b*) APPLIED LOAD $W = 105$ LB									
AB	20.00	577.5	T	0.29066	5.81	....	9.82	0.259	9.82
BC	20.00	472.5	T	0.26292	5.26	....	9.08	0.281	9.08
CD	20.00	367.5	T	0.23187	4 64	....	8.30	0.314	8.30
LK	10.00	630.0	C	0.30359	3.04	1.03	3.25	0.971	6.50
KJ	20.00	525.0	C	0.27714	5.54	−0.65	−3.65	−1.538	−3.65
JH	20.00	420.0	C	0.24788	4.96	−0.21	−0.85	−4.762	−0.85
HG	10.00	315.0	C	0.21467	2.15	1.42	4.00	0.704	8.00
AL	20.00	0	....	0	0	....	5.078	0.5655	5.078
AK, BJ, CH	22.37	117.5	T	0.13111	2.93	....	6.44	0.420	5.76
KB, JC, HD	22.37	117.5	C	0.13111	2.93	1.09	3.40	0.917	3.04
KG	20.00	105.0	T	0.12394	2.48	....	6.10	0.453	6.10

[a] T = "tension" and C = "compression." [b] Compression bars only.

It is of interest to compute the end fixity coefficient obtained for the most highly compressed member. As in bar KJ the average value of $k\,L = \frac{L}{j}$ is about 5.47 from Table 2; the end fixity coefficient is $(5.47/\pi)^2 = 3.02$.

DISCUSSION

Jacob Karol,[16] M. ASCE.—This interesting paper is significant theoretically for its presentation of stiffness and carry-over factors for bars with gusset plates, and practically for the experimental results whereby the theories could be checked and a sound method of design developed. The purpose of this discussion is to show that a consistent theory of gusset plate action can be derived which checks the test results, including specimen 5 with 20% gusset plates, with an accuracy of ± 6%.

The discrepancy noted by the authors between theory and experiment, especially for specimen 5 with 20% gussets, might possibly be explained on the assumption that secondary moments reduce the primary stresses in the bars, making the actual ($k\ L$)-values in the bars for a particular load on the truss less than the theoretical values. Since no data are given in the paper comparing theoretical and measured primary stresses, it is assumed that such differences are small enough to be neglected.

The two assumptions for gusset plate rigidity given by the authors are: (1) Infinite bending rigidity over the entire length of the gusset plate and (2) hyperbolic variation of the bending rigidity from that of the member at the edge of the gusset to infinity at the center of the truss joint. The second assumption sounds correct, yet the experimental results in almost all cases fall between the theoretical results calculated from both assumptions. An obvious third assumption, suggested by the writer, is that the gusset plate is infinitely rigid over only part of its length.

A study of Figs. 4 and 5 indicates that a theoretical effective gusset length of infinite rigidity for symmetrical members given by the expression:

$$s_e = s\left[1 - \left(\frac{L - 2\,s}{L}\right)^3\right] \qquad (40)$$

fits the experimental data quite well. It is worth noting that, if the exponent of the second term in the brackets is taken as infinity, the result is full effectivity as assumed by the authors.

Further study of Figs. 4 and 5 indicates that, for the hyperbolic law assumption, the increase in stiffness is a linear function of the nominal gusset plate length. Also, the slope of the tangent at the origin to the infinite rigidity stiffness curve is twice that for the curve based on the hyperbolic law. The expressions for the stiffness, based on the two assumptions, can then be given by the expressions: For variable rigidity—

$$S = S_o + K_1\left(\frac{s}{L}\right) \qquad (41a)$$

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[16] Kansas City, Mo.

and, for infinite rigidity—

$$S = S_o + 2\,K_1\left(\frac{s}{L}\right) + K_2\left(\frac{s}{L}\right)^4 \ldots\ldots\ldots\ldots\ldots (41b)$$

in which K_1 and K_2 are constants fitting the theoretical data.

From Table 1, the theoretical buckling loads for specimen 5, based on variable rigidity, indicate a linear increase in load with an increase in gusset plate length. Consequently it has been assumed that expressions for buckling load for specimen 5 analogous to Eqs. 41*a* and 41*b* may be written: For variable rigidity—

$$W = W_o + K_3\left(\frac{s}{L}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (42a)$$

and, for infinite rigidity—

$$W = W_o + 2\,K_3\left(\frac{s}{L}\right) + K_4\left(\frac{s}{L}\right)^4 \ldots\ldots\ldots\ldots\ldots (42b)$$

in which K_3 and K_4 are suitable constants. It is further assumed that Eqs. 41 and 42 apply, with proper constants, to specimens 3 and 4.

Specimens 1 and 2 will not be considered, since it is apparent that with a gusset plate at only one joint, the effect of the assumption of infinite rigidity would be to increase the buckling load only slightly more than the assumption of variable rigidity.

Several computed buckling loads in Table 1 for specimens 3, 4, and 5 were checked by the writer; hence it is assumed that all the computed loads are correct. The expressions for buckling load for gussets of infinite rigidity then are: For specimen 3—

$$W = 178 + 705{,}000\left(\frac{s}{L}\right)^4 \ldots\ldots\ldots\ldots\ldots\ldots (43a)$$

for specimen 4—

$$W = 175 + 667\left(\frac{s}{L}\right) + 300{,}000\left(\frac{s}{L}\right)^4 \ldots\ldots\ldots\ldots (43b)$$

and, for specimen 5—

$$W = 245 + 750\left(\frac{s}{L}\right) + 122{,}000\left(\frac{s}{L}\right)^4 \ldots\ldots\ldots\ldots (43c)$$

Results, using Eq. 40 to determine the effective gusset length and Eqs. 43 to find the buckling loads for these lengths, are shown in Table 3. The agreement between the modified theory and actual experiment is very good in all cases.

The effect of buckling beyond the proportional limit can be considered very simply with a good degree of accuracy. Neglecting, temporarily, the increase in section of the top chord, specimen 7 is similar to specimen 5. The buckling loads, for constant E with gussets considered, would then be proportional to the buckling loads with gussets neglected. Using 437 lb as the theoretical value for specimen 5 with 20% nominal gussets, $W_7 = 437 \times \frac{4{,}950}{245} = 8{,}830$ lb. The maximum compressive stress in the bottom chord is $\frac{P}{A} = \frac{2.5 \times 8{,}830}{0.391} = 56{,}500$ lb per sq in., and the apparent strain is $\frac{f}{E_r} = \frac{56{,}500}{10{,}500{,}000} = 0.00538$. From a curve of reduced modulus against stress, the stress for $\frac{f}{E_r} = 0.00538$ is 33,300 lb per sq in. and the buckling load is $\frac{33.3}{56.5} \times 8{,}830 = 5{,}200$ lb. The actual buckling load will be higher because both the top chord and the web members are stressed below the proportional limit, and their stiffness, relative to that of the last two panels of the bottom chord, is increased considerably. Buckling calculations based on modified gusset lengths, infinite rigidity, and reduced modulus resulted in a buckling load of 5,550 lb.

TABLE 3.—BUCKLING LOAD OF TEST SPECIMENS

Specimen No.	Gusset Plate Size (Percentage of Bar Length)		Computed Buckling Loads Based on Effective Gusset Lengths		Experimental buckling loads	Errors Based on Experimental Value (%)	
	Nominal	Effective	Infinite rigidity	Reduced modulus		Infinite rigidity	Reduced modulus
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
3	9	4	180	....	184	−2.2	....
4	9	4	203	....	216	−6.0	....
5	13	7.8	308	....	300	+2.7	....
5	20	15.7	437	....	443	−1.4	....
7	20	15.7	5,550[a]	5,200[b]	5,600	−0.9	−7.2
8	15	9.9	....	5,060[b]	5,420	....	−6.6

[a] Includes effect of reduced modulus, as described in the text. [b] Approximations based on specimen 5, as explained in the text.

If specimen 8 is considered similar to specimen 5, an approximate buckling load may be obtained as shown herein for specimen 7. The theoretical load for specimen 5 with 15% nominal gussets is 331 lb. Carrying through the calculations, the apparent strain is 0.00473 and the buckling load is 5,060 lb.

It is realized that the indicated percentage errors in Table 3 are correct only in so far as the assumption of similarity between (1) the shape of the curves for stiffness and (2) the shape of the curves for buckling load (expressed as functions of the gusset plate length) is valid. This assumption is based on the theoretical results for specimen 5. Its extension to the other specimens yields satisfactory agreement with experiment. The possibility of the general validity of the modified theory presented by the writer can perhaps be corroborated or refuted by the authors in their closing discussion.

Alfred S. Niles,[17] Assoc. M. ASCE.—The development of a practical procedure for taking account of the stiffening effect of gusset plates in the design of trusses is commendable. Prior to the presentation of this paper the theory of the stability of trusses with rigid joints had been fairly well developed; but there were few, if any, data to verify the theory, the computations were extremely tedious, and the actual stiffening effect of the joint construction was unknown.

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[17] Prof., Aeronautical Eng., Stanford Univ., Stanford University, Calif.

The writer was particularly interested in the methods used for determining the stiffness and carry-over factors for beam columns of nonuniform section. The graphical method presented in 1935 by D. Williams,[18] and the writer's adaptation of that method to numerical calculation,[19] although applicable to arbitrary variation in moment of inertia, are both methods of successive approximations, and too tedious for practical application. The authors are to be congratulated for having found a direct method for obtaining these factors and presenting curves which will be easy to use in practice.

The question arises, however, as to whether the choice of a hyperbolic variation in I was due solely to the fact that it was amenable to mathematical manipulation, or whether it had any rational basis. The writer believes the former to be the case, but would like to be certain. Even though the assumption has only an empirical justification, it so simplifies the procedure of practical analysis that the paper should prove a landmark in the development of the theory of truss action.

The use of the ratio "L/j" as a measure of the intensity of loading on a member deserves some comment. This ratio has been used for some years in aeronautical work, and has proved very convenient. Since $j^2 = E\,I/P$, $\pi\,j$ is the length of a pin-ended column for which P is the critical load. It might well be termed the "Euler length" of a section for a given axial load P, just as $\pi^2\,E\,I/L^2$ is called the "Euler load" of a section for a given pin-ended length L. Because of this relationship, the distance j has a physical significance which is lacking in its reciprocal $k = \sqrt{P/E\,I}$ which is so often used in the literature of elastic stability. The writer believes this advantage outweighs the apparently greater simplicity of the formulas when k is used, but grants that the difference may be slight to all but the typesetter.

It is to be noted that the method of this paper is not well adapted to the common civil engineering practice of designing to specified allowable stresses which must not be exceeded under the maximum expected load. Therefore, it will be more enthusiastically received by the followers of J.A. Van den Broek, M. ASCE, and his predecessors in the aeronautical industry who have been using the philosophy of "limit design" for at least thirty years. To them it is the long hoped for solution to a very vexatious problem.

THOMAS C. KAVANAGH,[20] ASSOC. M. ASCE.—The results of the painstaking investigation described by the authors constitute a valuable and gratifying verification of a long-accepted and used theory of stability. The aeronautical profession as a whole has made notable developments in the field of framework buckling, developments which have so unified the understanding of this problem that the civil engineering profession has only recently become aware of the pioneering work of some of its own men in this same field.

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[18] "A Successive Approximation Method of Solving the Continuous Beam Problem," by D. Williams, *Reports and Memoranda No. 1670*, Aeronautical Research Committee, London, 1935.

[19] "Airplane Structures," by Alfred S. Niles and Joseph S. Newell, John Wiley & Sons, Inc., New York, N. Y., 3d Ed., 1943, Vol. 2, pp. 132–140.

[20] Prof., Civ. Eng., The Pennsylvania State College, State College, Pa.

In a survey of the status of stability problems, Messrs. A. S. Niles and J. S. Newell indicated some time ago that "* * * the only type of complex structure for which there is available a reasonably satisfactory method of handling the problem of group stability is the rigid-jointed truss * * *".[21] The present experimental verification of this theory and the refinement of the theory to include gusset plate effects bring the problem nearer to final solution. Other aspects are currently under investigation by the Column Research Council of the Engineering Foundation and should eventually clarify civil engineering practice with reference to column design for end restraints.

The following itemized comments are offered with reference to the details and results of the tests:

1. The statement (under the heading "Comparison of Calculated and Measured Buckling Loads"), relative to the interaction of the two trusses, is very important and may need elaboration. If the experimental critical load of the double-truss frame is determined by the deflection of the applied load at the end of the rig, it would seem that this value would be more representative of the lower of the critical loads of the two component trusses of the frame, rather than an average of the two. It seems hardly to be expected that the workmanship of each of the two component trusses, as well as the uncontrollable effects of accidental eccentricities, and so forth, would be identical and that the two component trusses would fail at identical values. For example, did the irregularity of action noted for specimen 8 occur in both trusses simultaneously?

In other words, it would seem reasonable to expect the average critical load of the two trusses to be higher than the observed value. This possibility may be important because, although one of the major conclusions of this paper points to the loading calculated with the variable rigidity as giving closest agreement with the loading test results, the more direct approach followed in the stiffness tests cited in the paper points to a value between the variable and infinite rigidities. It is quite possible that the latter result is more nearly correct and that the direct loading test results are too low on an average basis, as indicated in the preceding paragraph.

2. Comparing the computed buckling loads for specimens 1 and 2 (Table 1), with gussets neglected versus gussets included, it appeared at first surprising that the presence of only one gusset plate in the entire truss would account for an increase of 8% in the buckling load. Closer scrutiny of the truss details, however, reveals that the "critical" (or most highly stressed) compression member is considerably stiffened at one end by this gusset plate. Engineers experienced in this type of stability analysis will recognize this phenomenon as further evidence that the truss is no stronger than its weakest part. Frequently a framework, well proportioned throughout with the exception of one compression member, will be found to have a relatively low value of buckling load; but a slight redesign of the weak member will immediately result in a startling increase in the over-all buckling strength.

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[21] "Airplane Structures," by A. S. Niles and J. S. Newell, John Wiley & Sons, Inc., New York, N. Y., 3d Ed., 1943, Vol. II, p. 312.

3. In view of the fact that tension-member stiffnesses increase very slowly with increased loading, whereas compression-member stiffnesses decline rather rapidly with increasing loading, it is often the practice to neglect the favorable effect of axial tension. It would appear from Fig. 8 that the error from neglecting axial tension would remain at about the same relative magnitude provided the gusset plate be included in the unloaded member stiffness, whereas the error might be increased considerably if the gusset plate effect were omitted entirely.

4. No mention is made of the possible effect of slip at the riveted joints, or of the elastic behavior of such joints, which have all been considered as rigid in the analyses. Unfortunately the make-up of the joints is not indicated; but, even if this detail were included, it is probable that insufficient experimental data would be available on this specific type of connection.

It is well known that beam connections in structural framing exhibit a rotational string constant or stiffness of the order of 1×10^8 in.-lb per radian at initial low moments, and that this stiffness may decline to one twenty fifth or one fiftieth of this value at high values of the applied moment. The effect of this condition (which civil engineers refer to as "semi-rigid framing") is essentially to reduce the restraints and to lower the critical buckling load from that which would obtain had the members been rigidly attached to one another. It is possible to extend the general theory to include the effect of such elastic joints. The writer has been able to demonstrate that, using typical initial values of the spring constant of the connection as found in ordinary structural work, a reduction in the critical load of a few percent is possible due to joint elasticity.

5. The technique of direct stiffness measurement described by the authors has much to commend it as a method of verifying the theory. The writer has employed a similar principle to single axially-loaded bars to check the effective modulus experimentally, the latter modulus usually being assumed in calculations as equal to the tangent modulus.

The authors' calculated values in Figs. 4 and 5 are presumably the stiffnesses S' of the joint with the far ends elastically restrained. The authors indicate that a comparison was made with experimental values obtained from the rotation of the joint "and neighboring joints." Inasmuch as the experimental $S' = M/\theta$, the writer would like to inquire as to the need for computing rotations in adjacent joints.

As a further matter of technique, one of the principal difficulties encountered in direct stiffness tests is the necessity for keeping the applied moment low (as the authors have done) in order not to introduce secondary yielding. It would be quite possible, however, that the maximum applied moment of 400 in.-lb (assumed equally divided between the five members at the joint, and resulting in a flexural stress of $\frac{M\,c}{I} = \frac{80 \times 0.125}{\frac{1}{12} \times 0.5 \times (0.25)^3} = 15{,}360$ lb per sq in.) could give stresses in the plastic range if combined with stresses due to axial loads. The effect of such yielding would probably be evident in a curvature of the (M-θ)-diagram for higher values of the moment.

The problem of the buckling of trusses is identical with that of the buckling of the individual columns of the truss, each considered as an elastically restrained strut. As such, it should be of interest to all civil engineers, who would be alert to the economic potentialities involved in the theoretical 4-to-1 ratio of areas required for the pin-ended column versus fully-restrained column in the elastic range.[22]

In so far as the design of compression members in trusses is concerned, it is a simple matter to demonstrate with the stability theory mentioned in this paper that, for fixed loadings and the customary factors of safety employed in structural work in steel, all compression members should be designed as pin ended. Tension members reach their yield point long before the compression members are ready to buckle, and become "plastic hinges," incapable of offering restraint.

GEORGE WINTER[23] AND OLIVER G. JULIAN,[24] MEMBERS, ASCE.—Among other things, this paper illustrates forcibly the interdependence of various branches of the engineering profession. It appears that the oldest branch of engineering—structural—can learn much from one of the youngest—aeronautical. This is probably because the latter is not steeped in tradition. It has been forced, by the necessity of weight saving, into the adoption of rational methods of design and analysis, largely verified by tests up to and including absolute failure.

It is the writers' understanding that the aeronautical engineer ordinarily considers the term "factor of safety" as the ratio of the loading that a structure is capable of sustaining to the imposed loading which tends to destroy the structure. "Factor of safety" so considered is a factor of overload and has real meaning. On the other hand, most structural codes and many texts are written in terms of working stress, disregarding the fact that the principle of superposition is not general. These codes leave one in an impenetrable fog as to the factor of overload, actual factor of safety, and probability of failure.

Written in terms of critical loads, this paper is refreshing, stimulating, and instructive. It is especially valuable in that it considers phenomena beyond the elastic range, investigates the influence of gusset plates mathematically, and demonstrates empirically that their effect cannot be neglected, nor can gussets reasonably be considered as infinitely rigid. It is shown that, in all cases up to and including that in which the gusset plate length at each end of the member segment is 13% of the corresponding bar length, the hyperbolic rule used gives results within a practicable degree of accuracy (5%). For longer gussets, more complicated and refined methods are in order if a close design is required. The error, in case the plates are 20% as long as the bar length, is of the order of 20% on the safe side.

The influence of riveted gusset plates on buckling loads is presently under experimental study at Cornell University, Ithaca, N. Y., as part of a project sponsored jointly by the Public Roads Administration (Department of Commerce) and the Column Research Council (Engineering Foundation). These tests differ from those reported in the paper in that they are made on full size members comprising standard structural shapes (angles and channels) in contrast to the authors' small-scale, rectangular members. Although only half these tests are completed at this writing (1951), evidence so far indicates a significant strengthening effect of these gusset plates, which, together with the authors' evidence, points to the practical importance of accounting for this effect.

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[22] "End Restraints on Truss Members," by Harold E. Wessman and Thomas C. Kavanagh, *Transactions*, ASCE, Vol. 115, 1950, p. 1135.

[23] Prof. and Head of Dept. of Structural Eng., Cornell Univ., Ithaca, N. Y.

[24] Chf. Structural Engr., Jackson & Moreland, Cons. Engrs., Boston, Mass.

The writers believe that the inelastic range should be entered with caution. This is especially true for structures subjected to oscillating loads. Fatigue phenomena must then be given careful consideration.

The energy proof of the convergence criterion by the authors is a valuable addition to the previous work of Benjamin Wiley James[8] and Eugene E. Lundquist,[10] Assoc. M. ASCE. The paper should be studied against the background of three sources of information in addition to those cited—a paper by Mr. Lundquist[25] (1938), a paper by M. R. Horne[26] (1947), and a bulletin published by the Cornell University Engineering Experiment Station[27] (1948). The research at Cornell University[28] extends the moment distribution method, with the effect of axial forces considered, to include frames subjected to sidesway. It also gives an alternative method[29] of end restraints for determining the critical loads of frames. This method, like those of Hardy Cross,[30] Hon. M. ASCE, and R. V. Southwell[31] is essentially one of successive joint relaxation, but is distinguished from the moment distribution method presented by Messrs. James, Lundquist, and Hoff inasmuch as it results in explicit information on the actual amounts of end restraints, effective length (between hypothetical pins), and critical load for each member. Since it allows one to single out the weaker members primarily responsible for instability, it appears to have distinct advantages.

It is realized, of course, that fixed end moments, as well as the stiffness and carry-over factors, are functions of the axial forces. Equations and a chart for determining fixed end moments considering the effect of axial forces are given in the Cornell University bulletin.[32]

The statement (under the heading "The Convergence Criterion: Calculation of the Buckling Load")—

> "When $\frac{L}{j}$ is greater than 2π in any one compression bar, the entire truss is unstable because the bar in question would be unstable even if its two ends were rigidly fixed."

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[25] "Principles of Moment Distribution Applied to Stability of Structural Members," by Eugene E. Lundquist, *Proceedings*, Fifth International Cong. for Applied Mechanics, Cambridge, Mass., 1938, p. 145.

[26] "A Moment Distribution Method for Rigid Frame Steel Structures Loaded Beyond the Yield Point," by M. R. Horne, Inst. of Welding, *Transactions* (*Welding Research Supplement*), London, England, Vol. 10, No. 4, August, 1947, p. 6.

[27] "Buckling of Trusses and Rigid Frames," by George Winter, P. T. Hsu, Benjamin Koo, and M. H. Loh, *Bulletin No. 36*, Cornell University Engineering Experiment Station, Ithaca, N. Y., April, 1948.

[28] *Ibid.*, pp. 19–21.

[29] *Ibid.*, pp. 24–26.

[30] "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, *Transactions*, ASCE, Vol. 96, 1932, p. 1.

[31] "Relaxation Methods in Engineering Science," by R. V. Southwell, *Oxford Engineering Science Series*, Oxford, England, Clarendon Press, 1940.

[32] "Buckling of Trusses and Rigid Frames," by George Winter, P. T. Hsu, Benjamin Koo, and M. H. Loh, *Bulletin No. 36*, Cornell University Engineering Experiment Station, Ithaca, N. Y., April, 1948, pp. 13–15, and p. 56, Graph I.

—warrants further explanation. It prompts the questions: May not redundant compression bars be unstable without the entire truss being unstable? May not the compression force in a redundant compression bar exceed four times (or any other value of) the Euler load for a pin-ended member without the entire truss being unstable?

Data as to the difference in action between trusses fabricated with bolted, riveted, and welded gusset plates would be most interesting. The tests presently being conducted (1951) at Cornell indicate that the effect of bolt or rivet slip is far from negligible.

It is hoped that the studies reported by Mr. Hoff and his associates will be extended to include freestanding frames and trusses which are not braced normal to their planes. It is also hoped that the studies will be extended to include cases involving nonprismatic (and noncylindrical) members.

The writers take this opportunity to congratulate the authors on the presentation of an excellent paper. It is at once cogent, clear, and concise, and the charts present the results in such a manner as to be of real value to practicing designers.

N. J. Hoff,[33] M. ASCE, Bruno A. Boley,[34] S. V. Nardo,[35] and Sara Kaufman.[36]—The willingness to read and to check the paper as well as to comment upon it are appreciated by the writers. They are glad to answer the questions raised and wish to submit some remarks regarding the comments made. Mr. Karol suggested that some of the discrepancies between theory and experiment noted in connection with specimens having very long gusset plates (specimen 5) might be caused by the secondary stresses. The authors have not carried out an analysis of the secondary stresses in the specimens listed in the paper. It appears reasonable to assume that the secondary stresses developed in a framework with gusset plates are greater than those developed in frameworks without gusset plates in approximately the ratio of the stiffness factors of the bars having gusset plates to the stiffness factors of bars without gusset plates. This assumption is based on the customary method of calculating secondary stresses, a method by which the displacements of the joints are determined from the extensions of the bars under the primary stresses calculated for pin-jointed frameworks. The secondary stresses for the framework are then obtained by means of the Hardy Cross moment distribution method, on the assumption that these joint displacements remain unchanged. The unbalanced reactions at the joints accompanying the bending moments in the bars necessitate corrections in the primary stresses. However these corrections are insignificant for frameworks of the types shown in Fig. 3, and thus recalculation of the joint displacements on the basis of the modified primary stresses is unnecessary. Consequently the approximate procedure described must be considered as a rapidly convergent one. When the gusset plate effect is taken into account, the displacements of the equivalent pin-jointed framework are calculated exactly as before. The secondary stresses are likely to be two to three times as high as in the framework without gusset plates because this is the ratio of the stiffness factors of the bars having gusset plates to those of the bars without gusset plates. Consequently the corrections in the primary stress values are also doubled or tripled. With frameworks of the types investigated, these corrections will still not exceed the order of magnitude of 1%. Therefore, the secondary stress effect cannot explain the much larger discrepancy observed.

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[33] Prof. of Aeronautical Eng., Polytechnic Inst. of Brooklyn, Brooklyn, N. Y.

[34] Asst. Prof. of Aeronautical Eng., Univ. of Ohio, Columbus, Ohio.

[35] Asst. Prof. of Aeronautical Eng., Polytechnic Inst. of Brooklyn, Brooklyn, N. Y.

[36] Mrs. Bruno A. Boley.

The assumption of an empirical gusset plate length that gives the same stiffness when it is considered as perfectly rigid as the actual elastic gusset plate is a useful one, because it eliminates the complex mathematical treatment which follows from the hyperbolic stiffness variation. However, the establishment of an empirical formula is advisable only if a wealth of experimental data is available. The authors did not feel that they had obtained a sufficient number of experimental points to warrant the suggestion of an empirical formula. They understand that work in this direction is now in progress at Cornell University, Ithaca, N. Y.

Even though the buckling loads calculated by Mr. Karol agree well with test results as indicated by Table 3, it is too optimistic to expect that the same agreement will be obtained with all future frameworks. His empirical formulas are based on far too scanty information. Moreover the formulas proposed for the buckling load are of little practical value because they give the buckling load of the framework having gusset plates as a function of the buckling load of a framework without gusset plates. It is no more trouble to determine the buckling load of a framework having gusset plates than to calculate the buckling load of a framework without them, as long as all the stiffness and carry-over factors can be obtained from graphs, tables, or formulas. Hence the practical engineer may just as well calculate the buckling load of a framework including the gusset plate effects.

Finally it should be mentioned that the modification of the buckling stress, used by the discusser to take into account the inelastic behavior of the material, is incorrect. If Young's modulus is replaced by the tangent modulus or the reduced modulus in the Euler formula, the buckling load or the buckling stress is obtained for the short column, This does not mean, however, that the buckling strain ϵ is given by the equation:

$$\epsilon_{cr} = \frac{\pi^2}{(L/\rho)^2} \quad \ldots\ldots (44)$$

in which ϵ_{cr} is the critical strain and ρ is the radius of gyration of the cross section. The correct buckling strain is obtained by dividing the buckling stress by the corresponding secant modulus. The difference betweeen the correct critical strain and the value given incorrectly by the above equation can be easily demonstrated if some nonlinear stress-strain relationship is assumed. For instance, when the stress σ is given by the formula

$$\sigma = E\,(\epsilon - K\,\epsilon^2) \quad \ldots\ldots (45)$$

the tangent modulus E_t is represented by

$$E_t = E\,(1 - 2\,K\,\epsilon) \quad \ldots\ldots (46)$$

If the tangent modulus stress

$$\sigma_{cr} = \frac{\pi^2 E_t}{(L/\rho)^2} \qquad (47)$$

is considered as the correct buckling stress, substitution of the tangent modulus value leads to

$$\sigma_{cr} = \frac{\pi^2 E}{(L/\rho)^2} (1 - 2 K \epsilon_{cr}) \qquad (48)$$

On the other hand, the stress-strain relationship assumed requires that

$$\sigma_{cr} = E (\epsilon_{cr} - K \epsilon^2_{cr}) \qquad (49)$$

and thus one obtains the following quadratic equation:

$$\epsilon^2_{cr} - \epsilon_{cr} \left[\frac{1}{K} + \frac{2 \pi^2}{(L/\rho)^2} \right] + \frac{\pi^2}{K (L/\rho)^2} = 0 \qquad (50)$$

The solution of the quadratic is

$$\epsilon_{cr} = \epsilon_E + \frac{1}{2 K} - \sqrt{\epsilon^2_E + \left(\frac{1}{2 K} \right)^2} \qquad (51)$$

in which ϵ_E represents the Euler strain:

$$\epsilon_E = \frac{\pi^2}{(L/\rho)^2} \qquad (52)$$

This equation clearly shows that the critical strain ϵ_{cr} differs from the Euler strain ϵ_E, which was assumed by Mr. Karol to be the critical strain for a short column.

Mr. Niles was perfectly correct in assuming that the hyperbolic variation in the stiffness was chosen for the analysis, purely because it leads to equations that can be solved in closed form. Any other formula representing a smooth transition between the bending rigidity of the bar proper and the infinite rigidity stipulated for the mathematical intersection point of the bars would have been equally acceptable on purely physical grounds. One additional reason for the selection of the hyperbolic law was the fact that it had been used successfully at an earlier date in the calculation of the deflections of framework members.[11]

Mr. Kavanagh raises a point of great interest in connection with the interaction of the two trusses. Because the tests have shown that the critical load of a perfectly elastic truss corresponds to a true neutral equilibrium (in the sense that large deflections can take place under a load that is constant for all practical purposes), it is safe to assume that the two trusses carried equal loads at buckling when the stresses did not exceed the elastic limit. This was the case with the majority of the specimens. On the other hand the inelastic specimens buckled catastrophically. It is obvious, therefore, that the buckling of the weaker or more highly loaded truss must have thrown additional loads upon the second truss and precipitated its buckling. The experimental buckling loads of the inelastic specimens must be assumed, therefore, to be lower than the true values, as Mr. Kavanagh has pointed out. However, the writers believe that the inaccuracy resulting from this condition must be a minor one because of the care taken in distributing the loads evenly between the two trusses. In any case the conclusions reached on the basis of the fully elastic specimens need not be modified. Mr. Kavanagh was right in assuming that the irregularity noted for specimen 8 occurred in only one of the trusses.

It was not the purpose of the paper to evaluate the economy of present-day structures but rather to provide a somewhat more refined tool for the analysis of the ultimate load of a framework. It is hoped that the availability of such a tool will ultimately lead to design specifications which will permit the best use to be made of the structural materials.

The writers were very glad to learn from the discussion of Mr. Winter and Mr. Julian that the strengthening effect of the gusset plates has been observed in current tests at Cornell University also. They are also willing to admit the inaccuracy of one of their statements in the paper. The sentence (under the heading "The Convergence Criterion: Calculation of the Buckling Load")

> "When $\frac{L}{j}$ is greater than 2π in any one compression bar, the entire truss is unstable because the bar in question would be unstable even if its two ends were rigidly fixed."

is correct only for statically determinate frameworks. Naturally a redundant member of a truss can be loaded beyond its buckling load without impairing the load-carrying capacity of the truss. On the other hand the convergence criterion is equally applicable to statically determinate and redundant frameworks.

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Paper No. 2455

REINFORCED CONCRETE SKEWED RIGID-FRAME AND ARCH BRIDGES

BY MAURICE BARRON,[1] M. ASCE

WITH DISCUSSION BY MESSRS. D. A. NETTLETON, ERNEST H. HARDER, EGIDIO O. DI GENOVA, KUANG-HAN CHU, HAROLD E. LEVENSON, WALTER L. SCHWARTZ, LEO SOS, LEIF ARUP, AND MAURICE BARRON.

SYNOPSIS

This paper presents a method of analysis and design by which the effects of skew on a barrel arch or rigid-frame bridge may be evaluated separately. The skew problem is thus resolved into a sort of secondary stress analysis in which the ordinary stresses for a rectangular structure are used as primary stresses. The procedure is extended so that final steel areas of reinforcement and unit stresses are determined as functions of the areas and unit stresses for the rectangular structure. Equations and transformations are derived for all applied loads and for volume changes. Criteria are presented for the effects of skew. For single-span or double-span structures with less then critical skew, no skew analysis is required. For structures with greater than critical skew, an independent adjustment is shown for analysis and design.

The practical significance of this development can better be appreciated if one considers the modern parkway or expressway as a sinuous ribbon of constant width. If the horizontal clearances are constant, every overpass on the parkway has the same square span. The single-span or the double-span structures may be so proportioned that they differ (structurally) only in width or skew, and sometimes in the depth of foundations. Therefore, one basic rectangular structure becomes the parent of a family of bridges.

Another feature of practical and theoretical significance is the fact that, for the rectangular structure, a designer may use any preferred method of analysis and design. If required, the independent secondary analysis for skew effects may be readily superimposed.

The results obtained by this method have been compared with the results obtained by the analysis of existing structures designed by current methods which involve the numerical solution of several simultaneous equations. The agreement is convincing evidence of the accuracy of the proposed methods.

NOTE.—Published in April, 1950, as *Proceedings-Separate No. 13.* Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Farkas & Barron, Cons. Engrs., New York, N. Y.

INTRODUCTION

Modern express highways and channelization of traffic have created problems in the economic planning and execution of highway projects. The high speed through highway includes a minimum of three lanes in each direction, and safety considerations have brought about the center mall separating opposing traffic. Thus, the typical cross section will have a reasonable shoulder on each side and two multiple-lane roadways separated by a center mall. The highway will cross intersecting highways usually at some skew, the right-angle intersection being a rare exception. Therefore, the structure separating the grades will usually be a skewed bridge. Maladjustment of alinement to avoid skewed structures is not tolerated in current practice. The double-span rigid-frame bridge has become a recognized type of structure admirably suited to the imposed conditions, the center mall being used to accommodate the center pier.

A single-span structure is used where no center mall is provided, at traffic interchanges where center piers are undesirable, or where opposing traffic is widely divided and two separate structures are indicated.

In 1924, J. Charles Rathbun, M. ASCE, presented a pioneering paper[2] on the theory of the skewed arch which demonstrated the derivation of the elastic equations for the fixed skewed arch ring. Later, at the request of Arthur G. Hayden, M. ASCE, Mr. Rathbun extended his theory to include the elastic equations for the hinged concrete rigid-frame bridge.

This extension was revised, adapted to office use, and first published by Mr. Hayden in 1931.[3] The numerical work required by this method is so difficult, even for single-span structures, that the skewed frame or arch was often abandoned in favor of more easily designed types. A worthy step toward a practical simplification for single-span skewed structures was incorporated in a paper[4] by Richard M. Hodges, in 1943.

The current theory is so complex and the amount of computation required for double-span skewed structures is so much greater than for single-span structures that the former are being used less and less. This is most unfortunate because the need for double-span structures is increasing as more parkways and freeways are built or projected.

Notation.—The letter symbols introduced in this paper are defined where they first appear, in the text or in the illustrations, and are assembled for reference alphabetically in the Appendix.

DERIVATION OF ELASTIC EQUATIONS; SINGLE SPAN

The conventional structural analysis of hinged single-span skew arches and rigid-frame bridges assumes a rectangular coordinate space system with a reaction force applied in the direction of each of the three mutually perpendicular axes, and a reaction moment about each of two of these three axes. The removal of the redundant reaction forces and redundant reaction moments leaves a statically determinate system. This system is usually called the base frame, transformed structure, or residual structure. For each redundant, an elastic equation may be written so that, simultaneously, the redundants satisfy the elastic requirements for the residual frame, and the remaining reactions independently satisfy the laws of statics.

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[2] "Analysis of the Stresses in the Ring of a Concrete Skew Arch," by J. Charles Rathbun, *Transactions*, ASCE, Vol. LXXXVII, 1924, p. 611.

[3] "The Rigid-Frame Bridge," by Arthur G. Hayden, John Wiley & Sons, Inc., New York, N. Y., 1931.

[4] "Simplified Analysis of Skewed Reinforced Concrete Frames and Arches," by Richard M. Hodges, *Transactions*, ASCE, Vol. 109, 1944, p. 913.

Any structural system that is statically indeterminate to the *n*th degree will have *n* redundants and the simultaneous solution of *n* elastic equations is required to determine the redundants. If the *n* redundants are removed and the structure is loaded, elastic deformations in the direction opposite to each redundant will occur. The redundants must produce equal and opposite deformations simultaneously in each of the *n* directions. When *n* is equal to 4, this condition may be written mathematically as follows:

$$\left.\begin{aligned} \delta_a &= 0 = A\,\delta_{aa} + B\,\delta_{ab} + C\,\delta_{ac} + D\,\delta_{ad} + \Delta_a \\ \delta_b &= 0 = A\,\delta_{ba} + B\,\delta_{bb} + C\,\delta_{bc} + D\,\delta_{bd} + \Delta_b \\ \delta_c &= 0 = A\,\delta_{ca} + B\,\delta_{cb} + C\,\delta_{cc} + D\,\delta_{cd} + \Delta_c \\ \delta_d &= 0 = A\,\delta_{da} + B\,\delta_{db} + C\,\delta_{dc} + D\,\delta_{dd} + \Delta_d \end{aligned}\right\} \qquad \text{(1)}$$

By transposition,

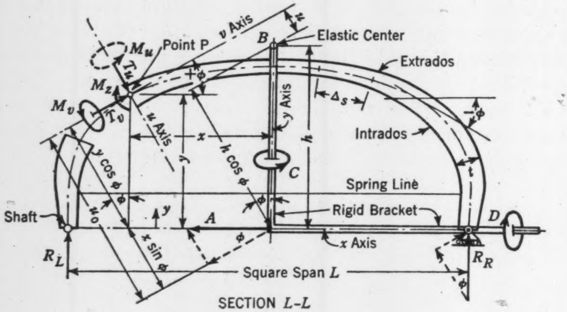
$$\left.\begin{aligned} A\,\delta_{aa} + B\,\delta_{ab} + C\,\delta_{ac} + D\,\delta_{ad} &= -\Delta_a \\ A\,\delta_{ba} + B\,\delta_{bb} + C\,\delta_{bc} + D\,\delta_{bd} &= -\Delta_b \\ A\,\delta_{ca} + B\,\delta_{cb} + C\,\delta_{cc} + D\,\delta_{cd} &= -\Delta_c \\ A\,\delta_{da} + B\,\delta_{db} + C\,\delta_{dc} + D\,\delta_{dd} &= -\Delta_d \end{aligned}\right\} \qquad \text{(2)}$$

In Eqs. 1 and 2: Symbols *A*, *B*, *C*, and *D* are the four redundants for a single-span, hinged skewed structure as shown in Fig. 1; δ_{aa}, δ_{bc}, δ_{ad}, etc., denote the deformation at the point of application and in the direction of the redundant indicated by the first subscript due to a unit load (or unit moment) acting at the point of application, and in the direction, of the redundant indicated by the second subscript; and Δ_a, Δ_b, Δ_c, and Δ_d denote the deformation of the residual structure at the point of application, and in the direction, of the redundants *A*, *B*, *C*, and *D*, due to loads, volume changes, or movements of the foundations.

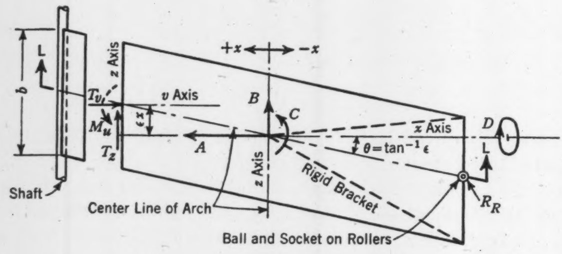
Fig. 1 shows a plan and square section of the residual structure for an elliptical skewed arch. The right support is assumed to be sustained by a frictionless ball and socket on frictionless rollers. A rigid (inelastic) bracket is attached to the residual structure above the ball and socket, and the bracket extends to the elastic center which is the point of application of the redundant *B*. Each redundant is positive in the direction shown. The redundant *A* is applied to the rigid bracket as shown in plan and section of Fig. 1. The redundant *D* is applied about the *x*-axis. The redundant *C* is applied about the *y*-axis, and the redundant *B* is applied to the bracket at the elastic center on the *y*-axis.

The left support of the residual structure is assumed to be a shaft that permits frictionless rotation about the *z*-axis. Section LL, Fig. 1, shows a cut section at point P. There are three mutually perpendicular axes, *u*, *v*, and *z*, at the cut section.

The moment stresses at the cut section are shown as M_z, M_v, and M_u, which are the moments about each of the mutually perpendicular axes at point P. There is also a normal thrust T_v, a cross shear T_z, and a radial shear T_u. The moment about the u-axes, M_u, is indicated by a dotted ellipse. Although this moment is quite large in value, the section over which it acts is the full width of the bridge, and the resulting stresses are very small. Therefore, the moment M_u is neglected. As customary, the radial shear T_u is also neglected, since it results in negligible stresses. Table 1 shows the unit moments m at any section of the residual structure for each unit redundant.


v Axis
u
Point P
B
Elastic Center
Extrados
M_u
T_u
M_z
$+\phi$
M_v
T_v
u Axis
x
y Axis
Δ_s
Intrados
y cos φ
h cos φ
y
h
C
t
φ
Spring Line
Rigid Bracket
Shaft
y
A
D
u_0
x sin φ
φ
x Axis
R_L
R_R
Square Span L
φ

SECTION L-L


L
b
z Axis
+x
−x
T_v
v Axis
B
C
εx
M_u
T_z
A
x Axis
D
θ = tan⁻¹ ε
L
z Axis
Rigid Bracket
Shaft
Center Line of Arch
R_R
Ball and Socket on Rollers

PLAN

FIG. 1

TABLE 1.—MOMENTS m FOR UNIT REDUNDANTS

Redundant	m_z	m_v
$A = +1$	$-y$	$+\epsilon x \sin \phi$
$B = +1$	0	$-u$
$C = +1$	0	$-\sin \phi$
$D = +1$	0	$+\cos \phi$

A complete understanding of the basic relations shown in Fig. 1 will remove the principal source of difficulty in the analysis of skewed structures. If the designer masters these basic relations and can derive the values shown in Table 1, he has mastered the major part of all that has been written on the analysis of this type of skewed structure.

The following equations are readily derived from Fig. 1 and from Table 1:

$$u_o = x \sin \phi + y \cos \phi \tag{3a}$$

$$u = u_o - h \cos \phi \tag{3b}$$

$$M_z = M_{oz} - A\, y \tag{4a}$$

$$M_v = M_{ov} + \epsilon\, A\, x \sin \phi - B\, u - C \sin \phi + D \cos \phi \tag{4b}$$

$$T_v = V \sin \phi + A \cos \phi \tag{5a}$$

and

$$T_z = B \tag{5b}$$

Since the single-span hinged skewed arch or rigid-frame bridge is a structure which is indeterminate in the fourth degree, Eqs. 1 and 2 will apply. The right-hand term in each Eq. 2 represents the deformation of the residual structure in the direction of each redundant because of the loads on the structure. These deformations may also be the result of a volume change (temperature change, rib shortening from axial stress, plastic yield, etc.), or from yielding of the foundations.

Unit Deflections.—Each term on the left side of Eqs. 2 consists of two parts, one a redundant and the other a unit deformation. The redundant cannot be made zero, but an analysis of each unit deformation will disclose the possibility of making it zero. In Eqs. 2, the unit deformations become coefficients for the redundants.

The equation for each unit deformation δ may be written directly from Table 1 using the Maxwell theorem of deflection. This theorem applies to both the rectangular elastic system, and to the torsional elastic system. The basic formulas are as follows:

For the rectangular elastic system—

$$\Delta_R = \int \frac{M_z\, m_z\, ds}{E_c\, I} = \sum \frac{M_z\, m_z\, \Delta s}{E_c\, I} \tag{6a}$$

for the torsional elastic system—

$$\Delta_T = \int \frac{M_v\, m_v\, ds}{E_s\, F} = \sum \frac{M_v\, m_v\, \Delta s}{E_s\, F} \tag{6b}$$

In Eqs. 6, M_z and M_v are the components of the moment acting on the surface cut by a radial plane through point P about the z-axis and the v-axis, respectively. Moment M_v is positive when it causes the obtuse angle of the section on which it is acting to deflect toward the center of curvature, and M_z is positive when it produces compression in the extrados fibers. Symbols m_z and m_v are the moments corresponding to M_z and M_v when a unit load is applied at a point m. The factor F can be computed[5] by the equation:

$$\frac{1}{F} = \frac{205\,(t^2 + b^2)}{180\, b^3\, t^3}\, \pi \tag{7a}$$

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[5] *Bulletin No. 3*, Faculty of Applied Science and Eng., School of Eng. Research, Univ. of Toronto, Toronto, Ont., Canada.

which, in turn, may be written

$$\frac{1}{F} = \frac{205}{180}\,\pi\left(\frac{1}{b^3\,t} + \frac{1}{b\,t^3}\right) \qquad \text{(7b)}$$

Since b is usually large compared to t, the term $1/(b^3\,t)$ may be neglected; therefore,

$$\frac{1}{F} = \frac{205\,\pi}{180\,b\,t^3} = \frac{3.578}{b\,t^3} = G \qquad \text{(7c)}$$

Also

$$\frac{1}{I} = \frac{12}{b\,t^3} = G_1 \qquad \text{(7d)}$$

Eq. 6b may be written

$$\Delta_T = r\int\frac{M_v\,m_v\,ds}{E_c\,I} = r\sum\frac{M_v\,m_v\,\Delta s}{E_c\,I} \qquad \text{(8)}$$

in which r is an elastic element defined as $\frac{E_c\,I}{E_s\,F} = \lambda\,\frac{3.578}{12} = 0.685$.

Table 2 results from substituting the values for Table 1 into Eqs. 6. It shows the formula used for the evaluation of the unit deformations, and also the locations of the sections that contribute the major part to the unit deformation. It should be noted that four unit deformations have similar double subscripts—δ_{aa}, δ_{bb}, δ_{cc}, δ_{dd}. The unit deformations with similar double subscripts are the sum of squares and always positive, and they are of major

TABLE 2.—UNIT DEFLECTIONS

Deflection (1)	M (2)	m (3)	Equation[a] (4)	Major contribution by sections at the: (5)	Remarks (6)
(a) UNIT DEFORMATION FOR THE RECTANGULAR ELASTIC EQUATION (EQ. 6a)					
$\delta_{aa,R}$	$-y$	$-y$	$\Sigma(y^2/I)$	Crown	....
(b) UNIT DEFORMATION FOR THE TORSIONAL ELASTIC SYSTEMS (EQ. 6b)					
$\delta_{aa,T}$	$+\epsilon\,x\sin\phi$	$+\epsilon\,x\sin\phi$	$+\epsilon^2\,r\,\Sigma(x^2\sin^2\phi/I)$	Knee	....
$\delta_{bb,T}$	$-u$	$-u$	$+r\,\Sigma(u^2/I)$	Knee	....
$\delta_{cc,T}$	$-\sin\phi$	$-\sin\phi$	$+r\,\Sigma(\sin^2\phi/I)$	Knee	....
$\delta_{dd,T}$	$+\cos\phi$	$+\cos\phi$	$+r\,\Sigma(\cos^2\phi/I)$	Crown	....
$\delta_{ab,T}$	$+\epsilon\,x\sin\phi$	$-u$	$-\epsilon\,r\,\Sigma(x\,u\sin\phi/I)$	Knee	$\delta_{ab} = \delta_{ba}$
$\delta_{ad,T}$	$+\epsilon\,x\sin\phi$	$+\cos\phi$	$+\epsilon\,r\,\Sigma(x\sin\phi\cos\phi/I)$	Knee to quarter point	$\delta_{ad} = \delta_{da}$
$\delta_{ac,T}$	$+\epsilon\,x\sin\phi$	$-\sin\phi$	$-\epsilon\,r\,\Sigma(x\sin^2\phi/I)$	....	$\delta_{ac} = \delta_{ca} = 0$
$\delta_{bc,T}$	$-u$	$-\sin\phi$	$+r\,\Sigma(u\sin\phi/I)$	....	$\delta_{bc} = \delta_{cb} = 0$
$\delta_{dc,T}$	$+\cos\phi$	$-\sin\phi$	$-r\,\Sigma(\sin\phi\cos\phi/I)$	....	$\delta_{dc} = \delta_{cd} = 0$
$\delta_{bd,T}$	$-u$	$+\cos\phi$	$-r\,\Sigma(u\cos\phi/I)$	....	$\delta_{bd} = \delta_{db} = 0$

[a] The equations in Col. 4, multiplied by $\Delta s/E_c$, result from substituting values from Table 1 in Eqs. 6.

importance. Deformation δ_{aa} is the only unit deformation to which both the rectangular and torsional elastic systems contribute a part, as shown in Table 2, and subsequently in Eq. 13*a*. The other unit deformations have dissimilar subscripts.

Two unit deformations with dissimilar subscripts, which differ only in that there is an inversion in the subscripts, are equal in accordance with Maxwell's theorem of reciprocal deflections. This statement is expressed mathematically as $\delta_{mn} = \delta_{nm}$ and several applications are shown in Table 2. Also, as shown in Table 2, eight of the unit deformations may be equated to zero; δ_{ac} $(= \delta_{ca})$, δ_{bc} $(= \delta_{cb})$, and δ_{dc} $(= \delta_{cd})$ are equal to 0 by symmetry. The last unit deformation in Table 2, deformation δ_{bd} $(= \delta_{db})$, is made equal to 0 by the proper location of the elastic center. This location is established by substituting Eqs. 3*b* into the equation for δ_{bd} (last item of Col. 4, Table 2), equating to zero, and solving for h; thus: $\delta_{bd} = 0 = \sum \frac{u \cos \phi \, \Delta s}{E_s F} = r \sum \frac{(u_o - h \cos \phi) \cos \phi \, \Delta s}{E_c I}$ $= r \sum \frac{u_o \cos \phi \, \Delta s}{E_c I} - h \, r \sum \frac{\cos^2 \phi \, \Delta s}{E_c I} = 0.$ Therefore,

$$h = \frac{\Sigma(u_o \cos \phi / I)}{\Sigma(\cos^2 \phi / I)} \qquad (9)$$

This elastic center always occurs a little above the axis of the structure and on the center line of symmetrical structures as shown in Fig. 1.

The elastic center, as used in this demonstration, is a particular point in space which is the end of a rigid bracket of infinite stiffness. This bracket is attached to the structure so that, by an appropriate transformation, the effects of stresses, which would normally result at the point of attachment, are exactly duplicated by another set of stresses at the elastic center. This second set of stresses is much more easily determined than the set at the point of attachment. Applying the redundant B at the elastic center, the deformation $B \, \delta_{bd} = 0$, which means that the angular rotation in the direction of the redundant D is equal to zero for all values of B. Similarly, $D \, \delta_{db} = 0$, which means that the deformation in the direction of the redundant B is equal to zero for all values of the redundant D.

SOLUTION OF ELASTIC EQUATIONS; SINGLE SPAN

Rewriting Eqs. 2 and dropping the zero terms as shown in Table 2,

$$A \, \delta_{aa} + B \, \delta_{ab} + D \, \delta_{ad} = - \Delta_a \qquad (10a)$$

$$A \, \delta_{ab} + B \, \delta_{bb} = - \Delta_b \qquad (10b)$$

$$C \, \delta_{cc} = - \Delta_c \qquad (10c)$$

$$A \, \delta_{ad} + D \, \delta_{dd} = - \Delta_d \qquad (10d)$$

From Eq. 10*b*,

$$B = - \frac{\delta_{ab}}{\delta_{bb}} A - \frac{1}{\delta_{bb}} \Delta_b \qquad (11a)$$

From Eq. 10*c*,

$$C = -\frac{1}{\delta_{cc}}\Delta_c \ldots\ldots\ldots\ldots\ldots\ldots (11b)$$

From Eq. 10*d*,

$$D = -\frac{\delta_{ad}}{\delta_{dd}}A - \frac{1}{\delta_{dd}}\Delta_d \ldots\ldots\ldots\ldots\ldots\ldots (11c)$$

Substituting Eqs. 11 into Eq. 10*a*,

$$A\left(\delta_{aa} - \frac{\delta^2_{ab}}{\delta_{bb}} - \frac{\delta^2_{ad}}{\delta_{dd}}\right) = -\Delta_a + \frac{\delta_{ab}}{\delta_{bb}}\Delta_b + \frac{\delta_{ad}}{\delta_{dd}}\Delta_d \ldots\ldots\ldots (12)$$

From Tables 1 and 2,

$$\delta_{aa} = \delta_{aa,R} + \delta_{aa,T} \ldots\ldots\ldots\ldots\ldots\ldots (13a)$$

and

$$\Delta_a = \Delta_{a,R} + \Delta_{a,T} \ldots\ldots\ldots\ldots\ldots\ldots (13b)$$

Let

$$\delta_{aa,T} - \frac{\delta^2_{ab}}{\delta_{bb}} - \frac{\delta^2_{ad}}{\delta_{dd}} = Q_1 \ldots\ldots\ldots\ldots\ldots\ldots (14a)$$

$$\frac{\delta_{ab}}{\delta_{bb}} = Q_2 \ldots\ldots\ldots\ldots\ldots\ldots (14b)$$

$$\frac{\delta_{ad}}{\epsilon\,\delta_{dd}} = Q_3 \ldots\ldots\ldots\ldots\ldots\ldots (14c)$$

$$\frac{1}{\delta_{aa,R}} = Q_4 \ldots\ldots\ldots\ldots\ldots\ldots (14d)$$

$$\frac{1}{\delta_{bb}} = Q_5 \ldots\ldots\ldots\ldots\ldots\ldots (14e)$$

$$\frac{1}{\delta_{cc}} = Q_6 \ldots\ldots\ldots\ldots\ldots\ldots (14f)$$

and

$$\frac{1}{\delta_{dd}} = Q_7 \ldots\ldots\ldots\ldots\ldots\ldots (14g)$$

At this point, where various factors enter into the solution, the order of a quantity should be mentioned and some explanation given for what is meant by order. In many of their derivations, mathematicians drop differentials of higher order without diminishing the value of the demonstration. Similarly, structural engineers neglect terms where their effect on the final answer is small and of no practical importance. Thus, in the usual analysis of right arches, and rigid frames, the deflections caused by shear, axial stress, and plastic flow are usually of a different order from the deflections produced by primary flexure; and therefore they are ordinarily neglected. If these factors of little importance have different algebraic signs, they are compensating and the order of their total may then change to one of less importance.

This analysis consists primarily of combining terms in a particular order so that certain terms and combinations of terms tend to cancel each other. The parts that do not cancel are easily determined as secondary effects but they are usually of an unimportant nature. Thus, it is shown that the effects of skew may be found directly from the stresses determined for the corresponding rectangular structure. In effect, the latter part of the torsional stress analysis, therefore, is a kind of secondary stress problem.

It has been found, as for example in Eq. 12, that the combination of terms which comprises Q_1 is of very little importance when compared with the value of $\delta_{aa,R}$. In the usual rigid frame with a skew of 50° the ratio of Q_1 to $\delta_{aa,R}$ is of the order 1:2,000. In Eq. 12 it is proper, therefore, to drop out Q_1 inasmuch as it will affect the coefficient of A by only 0.05%. Similarly, for rigid frames, Q_2 has been found to equal almost exactly $-\epsilon$. Even in the full centered arch the error in assuming that $Q_2 = -\epsilon$ is of little consequence. Table 3 shows the relative values of these and other constants for three rigid-frame bridges and two elliptical arches.

TABLE 3.—RELATIVE VALUES OF DESIGN CONSTANTS FOR SINGLE-SPAN BRIDGE STRUCTURES

Structure No. (1)	Square span (ft) (2)	SKEW		CONSTANTS		
		Angle θ (3)	ϵ (4)	Q_2 (5)	Q_5 (6)	$\frac{L^2}{4}\sum_{0}^{n}\frac{1}{I}$ (7)
			(a) RIGID-FRAME BRIDGES			
1	52	47° 50′	1.10	−1.11	1/23,740	1/23,685
2	75	41° 00′	0.8693	−0.8735	1/ 1,075	1/ 1,074
3	56	43° 17′ 20″	0.9420	−0.9521	1/ 1,362	1/ 1,329
			(b) ELLIPTICAL ARCH BRIDGES			
4	56.1	45°	1.000	−1.0424	....	....
5	57.3	36° 34′ 33″	0.742	−0.771	....	....

Therefore, without significant error,

$$Q_2 = \frac{\delta_{ab}}{\delta_{bb}} = -\epsilon \qquad (15)$$

Substituting Eqs. 13, 14, and 15 into Eqs. 11 and 12,

$$A = Q_4 \left(-\Delta_{a,R} - \Delta_{a,T} - \epsilon\,\Delta_b + \epsilon\,Q_3\,\Delta_d\right) \qquad (16a)$$

$$B = \epsilon\,A - Q_5\,\Delta_b \qquad (16b)$$

$$C = -\,Q_6\,\Delta_c \qquad (16c)$$

and

$$D = -\,\epsilon\,Q_3\,A - Q_7\,\Delta_d \qquad (16d)$$

Substituting Eqs. 3 into Eq. 4*b* the result is the following very useful equation:

$$M_v = M_{ov} - (B - \epsilon A)\, u - C \sin\phi + (D + \epsilon A\, h) \cos\phi - \epsilon A\, y \cos\phi \,..(17)$$

The structural significance of the foregoing mathematical treatment may be better understood by inspection of Fig. 2(*a*). The redundant A' is here shown as turned through the angle θ, and acts along the longitudinal axis of the structure. The redundant A' is really the resultant of A and ϵA with a slight excess or deficiency. Eq. 16*b* means that part of force B has been used thus to turn A. The foregoing treatment isolates the effect of each redundant on the torsional moment M_v. This isolation is necessary in order to evaluate the contribution of each redundant to the total stress. The writer believes that the algebraic solution is easier to understand than a presentation involving skewed redundants and skewed residual structures.

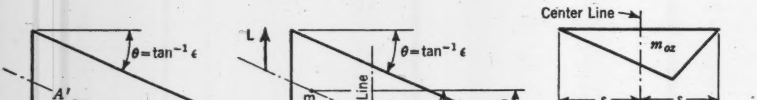


FIG. 2

Vertical Loads.—Fig. 2(*b*) shows a plan of a skewed structure with a unit vertical load applied at point A. The stresses at point m are required. Fig. 2(*d*) shows a square section L-L of the plan in Fig. 2(*b*). The distances from the center line to points A and m are indicated as a and x, respectively. Fig. 2(*c*) indicates the *m* moment diagram of the residual structure due to a unit load at point A, Fig. 2(*d*).

For a unit vertical load at point A,

$$m_{ov} = \frac{s + a}{2\,s}\,(s + x)\,\epsilon \cos\phi - (1 \cos\phi)\,\epsilon\,(a + x) \,.......(18a)$$

or

$$m_{ov} = \epsilon \cos\phi\; m_{oz} \,....................(18b)$$

and therefore, by superposition,

$$M_{ov} = \epsilon \cos\phi\; M_{oz} \,....................(19)$$

One of the most striking examples illustrating how terms combine or cancel out is shown when Eqs. 3, 4*a*, 16, and 19 are substituted into Eq. 17, thus: $M_v = M_{ov} + \epsilon A\, x \sin\phi - B\, u - C \sin\phi + D \cos\phi = \epsilon M_{oz} \cos\phi + \epsilon A\, x \sin\phi$

$$- C \sin \phi + D \cos \phi - (B - \epsilon A) u - \epsilon A (x \sin \phi + y \cos \phi - h \cos \phi); \text{ or}$$

$$M_v = \epsilon \cos \phi \, M_z + [- (B - \epsilon A) u - C \sin \phi + (D + \epsilon A h) \cos \phi] \quad (20)$$

For the vault of a rigid frame, in which $\sin \phi = 0$ and $\cos \phi = 1$, for applied loads the three terms within the bracket are of secondary order and their combination is even less significant. Therefore, for the vault—

$$M_v = \epsilon \cos \phi \, M_z \quad (21)$$

and, for the legs of a rigid frame, in which $\cos \phi = 0$, $\sin \phi = 1$, and $u = \frac{L}{2}$,

$$M_v = - \frac{(B - \epsilon A) L}{2} \quad (22a)$$

Eq. 22*a* may be evaluated from Eq. 16*b* as—

$$M_v = Q_5 \, \Delta_b \frac{L}{2} \quad (22b)$$

Eq. 21 indicates that influence lines for M_v are not required since they will be the influence lines for M_z to a different scale. For vertical loads, in addition to Eqs. 21 and 22,

$$A = - Q_4 \, \Delta_{a, R} \quad (23a)$$

$$(B - \epsilon A) = - Q_5 \, \Delta_b \quad (23b)$$

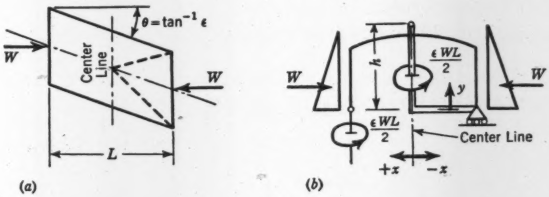
$$C = - Q_6 \, \Delta_c \quad (23c)$$

and

$$D + \epsilon Q_3 A = - Q_7 \, \Delta_d \quad (23d)$$

For symmetrical vertical loads $\Delta_c = 0$; therefore $C = 0$.

For rigid-frame bridges Δ_b is almost equal to zero; therefore $(B - \epsilon A)$ is almost equal to zero (less than 6% of ϵA). Eq. 23*a* is found from Eq. 16*a* in which the combination of terms $(- \Delta_{a, T} - \epsilon \Delta_b + Q_3 \Delta_d)$ is of secondary order for vertical loads and is therefore dropped out. The use of elastic weights and torsional elastic weights to determine $\Delta_{a, R}$, $\Delta_{a, T}$, Δ_b, Δ_c, and Δ_d for vertical loads is fully explained in the original paper filed for reference in the Engineering Societies Library and elsewhere.[6]



(*a*) (*b*)

FIG. 3

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[6] "The Rigid-Frame Bridge," by Arthur G. Hayden and Maurice Barron, John Wiley & Sons, Inc., New York, N. Y., 1950.

Earth Pressure; Both Sides.—Fig. 3 shows a residual structure loaded with earth pressure on both sides. The vertical reactions are zero and the structure is in static equilibrium due to the application of the two equilibrating moments $\epsilon W L/2$, one at the left support and the other at the center line on the rigid bracket. The equilibrating moment is divided into two equal parts, as shown, in order that the loaded residual structure may be symmetrical about the center axis. The moment M_{oz} may be determined by the shear increment method,[6] starting at the bottom of a leg. From the values of M_{oz} the value of $\Delta_{a,R}$ is found. This procedure is part of a standard rectangular analysis which is fully illustrated elsewhere.[6] For the torsional analysis,

$$M_{ov} = M_{oy} \sin \phi \tag{24a}$$

and

$$M_{oy} = \left(\frac{\epsilon W L}{2}\right) - \epsilon W \left(\frac{L}{2} - x\right) = + \epsilon W x \tag{24b}$$

Therefore, for the rectangular elastic system,

$$\Delta_{a,R} = - \Delta_s \Sigma M_{oz} \frac{y}{I E_c} \tag{25}$$

and, for the torsional elastic system,

$$\Delta_{a,T} = + \epsilon^2 \Delta_s W \sum \frac{x^2 \sin^2 \phi}{F E_s} = + W \delta_{aa,T} \tag{26a}$$

$$\Delta_b = - \epsilon \Delta_s W \sum \frac{u x \sin \phi}{F E_s} = + W \delta_{ab} \tag{26b}$$

$$\Delta_c = - \epsilon \Delta_s W \sum \frac{x \sin^2 \phi}{F E_s} = + W \delta_{ac} = 0 \tag{26c}$$

and

$$\Delta_d = + \epsilon \Delta_s W \sum \frac{x \sin \phi \cos \phi}{F E_s} = + W \delta_{ad} \tag{26d}$$

In Eqs. 26, F denotes a factor of torsion.

Substituting Eqs. 25 and 26 into Eqs. 16,

$$A = Q_4 \left[- \Delta_{a,R} - W \left(\delta_{aa,T} - \frac{\delta^2_{ab}}{\delta_{bb}} - \frac{\delta^2_{ad}}{\delta_{dd}} \right) \right] = Q_4 (- \Delta_{a,R} - W Q_1) = - Q_4 \Delta_{a,R} \tag{27a}$$

$$B = \epsilon A - W \frac{\delta_{ab}}{\delta_{bb}} = \epsilon A + \epsilon W = \epsilon (W + A) \tag{27b}$$

$$C = 0 \tag{27c}$$

and

$$D = - \epsilon Q_3 A - W Q_7 \delta_{ad} = - \epsilon Q_3 A - \epsilon Q_3 W = - Q_3 \epsilon (W + A) \tag{27d}$$

Substituting Eqs. 24 and 27 into Eq. 4*b*,

$$\epsilon W x \sin \phi + \epsilon A x \sin \phi - (W + A) u - 0 - \epsilon Q_3 (W + A) \cos \phi = M_v \tag{28}$$

Substituting Eqs. 3 into Eq. 28 and factoring,

$$M_v = \epsilon\,(W + A)\cos\phi\,(h - y - Q_3) \tag{29a}$$

Applying Eq. 5*a*, and remembering that $V = 0$ for balanced earth pressure, the torsional moment is then

$$M_v = \epsilon\,T_v\,(h - y - Q_3) \tag{29b}$$

Temperature Change.—Fig. 4 indicates a plan of the residual structure and also the deformed position of the residual structure due to a rise in temperature. Change in width is not important and therefore is not shown. Let c be the coefficient of temperature expansion and t be the temperature change in degrees Fahrenheit. Then, referring to Fig. 4, Δ_c, Δ_d, M_o, and C are each equal to zero, and

$$\Delta_a = -\,c\,t\,L \tag{30a}$$

and

$$\Delta_b = -\,\epsilon\,c\,t\,L \tag{30b}$$

Substituting the foregoing values into Eqs. 16,

$$A = Q_4\,(+\,c\,t\,L + \epsilon^2\,c\,t\,L) = Q_4\,(1 + \epsilon^2)\,c\,t\,L \tag{31a}$$

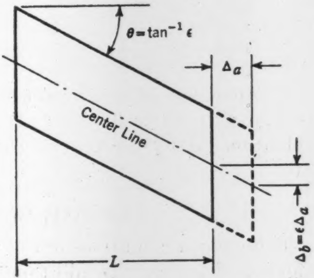


FIG. 4

$$B = \epsilon\,A + \epsilon\,Q_5\,c\,t\,L \tag{31b}$$

$$C = 0 \tag{31c}$$

and

$$D = -\,\epsilon\,Q_3\,A \tag{31d}$$

Substituting Eqs. 31 into Eq. 18*a*,

$$M_v = -\,\epsilon\,Q_5\,c\,t\,L\,u + \epsilon\,A\,(h - y - Q_3)\cos\phi \tag{32a}$$

Applying Eq. 32*a* to arches and rigid frames it has been found that, at points where the torsional moment M_v has any practical significance, the major contribution is from the term $-\,\epsilon\,Q_5\,c\,t\,L\,u$ and that the second term may be neglected. Therefore,

$$M_v = -\,\epsilon\,Q_5\,c\,t\,L\,u \tag{32b}$$

SUMMARY OF EQUATIONS; SINGLE SPAN

Inspection of Eqs. 21, 29*b*, and 32*b* shows that the tangent, ϵ, of the skew angle appears as a parameter (Table 4). Also these equations indicate that no torsional elastic analysis is required because, for any point, a rectangular stress, multiplied by a constant, yields the required torsional stress. For temperature change, a constant, $-\,\epsilon\,(Q_5\,c\,t\,L)$, multiplied by u gives the required torsional moment. Furthermore, Eqs. 27*b* and 31*b* show that the torsional cross shear T_z ($= B$) is susceptible to the same treatment. The equations necessary for a practical analysis of skewed structures are shown in Table 4.

TABLE 4.—EQUATIONS FOR THE ANALYSIS OF SINGLE-SPAN SKEWED STRUCTURES

Stress	Vertical Loads		Temperature Change		Balanced Earth Pressure	
	Equation	No.	Equation	No.	Equation	No.
M_z	$M_{oz} - A\,y$	4a	$\mp Q_4\,(1 + \epsilon^2)\,c\,t\,L\,y$	31a	$M_{oz} - A\,y$	4a
T_v	$V \sin\phi + A \cos\phi$	5a	$\pm Q_4\,(1 + \epsilon^2)\,c\,t\,L \cos\phi$	5a	$(W + A) \cos\phi$	5a
M_v/ϵ	$M_z \cos\phi$	21	$\mp Q_5\,c\,t\,L\,u$	32b	$T_v\,(h - y - Q_3)$	29b
T_z/ϵ	A	23b	$\pm A \pm Q_5\,c\,t\,L$	31b	$W + A$	27b

Comparisons of exact values and proposed approximations for Q_2 and Q_5 are shown in Table 3. Critical examination of the terms entering the exact evaluation justify the general conclusion that neglected terms are of a minor nature.

Analysis of Foundation Conditions

Foundation conditions are of particular importance for skewed rigid-frame bridges and arches. In addition to R_x and R_y the foundation design (see Fig. 5) must take into account the effects of R_z, M_x ($= R_y\,e'$), and M_y ($= R_x\,e$), in order to insure the structural action on which the analysis and design is predicated. The foundation restraints that are significant are shown in Table 5.

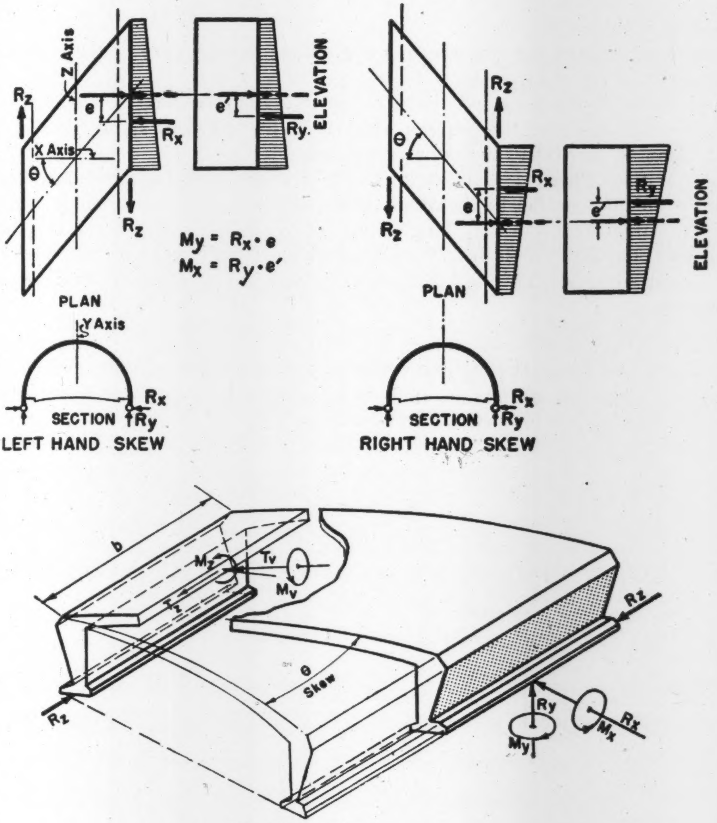
TABLE 5.—SIGNIFICANT FOUNDATION RESTRAINTS

Foundation restraint[a]	Dead load	Balanced earth pressure	Temperature change[b]
R_x	ΣH	ΣH	$(1 + \epsilon^2)\,\Sigma H$
R_y	Vertical reaction[c]	Vertical reaction[c]	Vertical reaction[c]
R_z	$\epsilon\,R_x$	$\epsilon\,(R_x - \Sigma W)$,	$\epsilon\left[R_x + \dfrac{6.7\,c\,t\,E_c}{(1 + \mu)\,s\,L\,\Sigma\frac{1}{I}}\right]$
M_x	$D + R_z\,y_c$[d]	$+ R_z\,y_c$	$+ R_z\,y_c$
M_y	Negligible	Negligible	$\epsilon\left[\dfrac{3.35\,c\,t\,E_c}{(1 + \mu)\,s\,\Sigma\frac{1}{I}}\right]$
M_z	0	0	0

[a] M_z assumed zero for a hinged structure. [b] See text for limits of $\Sigma 1/I$. [c] Symbol R_y denotes the vertical reaction for the transformed structure plus the vertical effect of the redundants. [d] See Eq. 23*d*.

The equations are for a unit foot of width which must be multiplied by b in order to obtain the total foundation restraints. Table 5 also shows the foundation restraints that can usually be neglected. (The quantity $\sum\frac{1}{I}$ denotes the summation for points in both end legs, and excludes values for the vault and for the center leg of the double-span frames.)

Values for live load may be determined by positioning the load for maximum M_z at the crown.


Z Axis
R_z
e
R_x
X Axis
θ
R_z
e'
R_y
ELEVATION
$M_y = R_x \cdot e$
$M_x = R_y \cdot e'$
PLAN
Y Axis
R_x
SECTION
R_y
LEFT HAND SKEW
R_z
θ
R_x
e
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R_y
e'
ELEVATION
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b
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Skew
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R_y
R_x
M_x
M_y
R_z

FIG. 5.—ANALYSIS OF FOUNDATION CONDITIONS

(a) Left-Hand Skew
(b) Right-Hand Skew
(c) Isometric Sketch

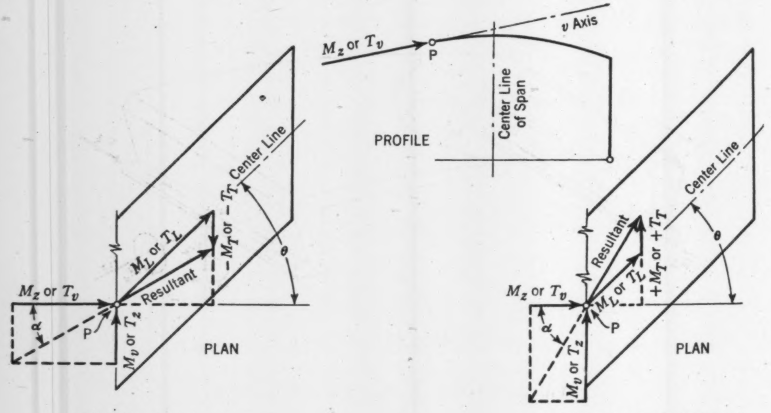
If the foundation material does not provide full restraint as assumed, it is possible to substitute the partial restraint, as a function of k (the soil modulus), into the equations. This will result in partial fixity about the x-axis and the y-axis but requires the assumption that k is a known and uniform value.

DESIGN OF LONGITUDINAL REINFORCEMENT

A simple process for the design of the longitudinal reinforcement is the result of an approach to the problem based on the following propositions:

1. The square-span moments (M_z) and thrusts (T_v) (that is, as computed on the span perpendicular to the abutments) are independent of the skew angle for all applied loads.
2. The square-span stresses for temperature change vary as the square of the secant of the skew angle.
3. Using required square-span areas of steel reinforcement, a simple "turning" process will give the required steel areas for the skew span—that is, for bars running parallel to the skew center line.
4. With a "turned" longitudinal steel pattern, only nominal transverse reinforcement, parallel to the abutments, will be required, except for bridges of very large skew or a very long span. For single-span structures, this limit will be about 50°. This transverse reinforcement will be amply provided for by 1-in.-round rods, 18 in. on centers, as usually shown in details for unskewed structures.

The foregoing propositions for single-span and multiple-span structures have been proved mathematically and verified by comparing the results of the exact and the simplified method of analysis as applied to a series of hypothetical rigid-frame bridges all having the same square span but different skews, to a maximum of 60°. In addition, the propositions have been verified by comparing the results of the simplified method with the results of exact analysis for a number of actual bridges. These proofs are not included in this paper but are filed for reference in the Engineering Societies Library.



Case 1. Angle $\alpha <$ Angle θ for Temperature Fall

Case 2. Angle $\alpha >$ Angle θ for Temperature Rise

FIG. 6.—TRANSFORMATIONS

Transformations.—Fig. 6 proposes two cases to demonstrate the "turning process" mentioned in proposition 3. In both cases, the following equations apply:

$$M_L = M_z \sec V \qquad (33a)$$

The angle V is the projection of the skew angle θ onto the plane tangent to the neutral axis at the point under consideration:

$$T_L = T_v \sec V \qquad (33b)$$

$$M_T = M_z \tan V - M_v \qquad (34a)$$

and

$$T_T = T_v \tan V - T_z \qquad (34b)$$

In general—

$$\tan V = \epsilon \cos \phi \qquad (35)$$

The reinforcement is found for bending and direct stress by using the equations instead of numerical values. Standard nomenclature used in concrete design is employed in the following demonstration: Referring to Fig. 7,

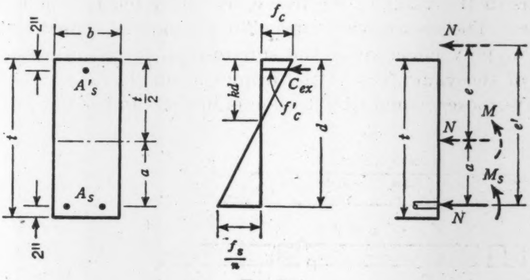


FIG. 7

$$e_L = \frac{M_L}{T_L} = \frac{M_z \sec V}{T_v \sec V} = \frac{M_z}{T_v} = e \qquad (36a)$$

and

$$e' = e + a \qquad (36b)$$

Then

$$M_{sL} = e'\,T_L = e'\,T_v \sec V = M_{sR} \sec V \qquad (37a)$$

Also

$$A_{sL} = \frac{M_{sL}}{f_s\,j\,d} - \frac{T_L}{f_s} = \left(\frac{e'\,T_v}{f_s\,j\,d} - \frac{T_v}{f_s}\right) \sec V = A_{sR} \sec V \qquad (37b)$$

In Eq. 37*b*, A_{sR} is the steel area (Fig. 8) before turning and the spacing of the bars must be measured parallel to the abutment. This is logical because A_{sR} must be increased to be equally effective in resisting M_{sR} when the bar is turned from its most effective direction. If the spacing perpendicular to the bars (that is, perpendicular to the face) is more desirable, either this spacing must be multiplied by $\cos \theta$, or the steel area must be multiplied by $\sec \theta$.

Therefore, for a pattern in which the main steel is parallel to the face of the bridge and in which the spacing is measured perpendicular to the bars,

$$A_{sL} = A_{sR} \sec V \sec \theta \ldots\ldots\ldots\ldots\ldots\ldots \text{(38)}$$

In the usual rigid frames $\sec V = \sec \theta$ for vault sections and $\sec V = 1$ for legs. Therefore, for vault sections—

$$A_{sL} = A_{sR} \sec^2 \theta = A_{sR} (1 + \epsilon^2) \ldots\ldots\ldots\ldots \text{(39}a\text{)}$$

for legs with spacing still measured perpendicular to the face—

$$A_{sL} = A_{sR} \sec \theta \ldots\ldots\ldots\ldots\ldots\ldots \text{(39}b\text{)}$$

and, for legs with spacing measured along the abutment—

$$A_{sL} = A_{sR} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(39}c\text{)}$$

Explanation of the "Turning Process" for Reinforcement.—The area of reinforcing bars in the vault of the frame, as calculated for the square span, is A_{sR} in Fig. 8. The area of the "turned" steel bars A_{sT} must be $A_{sR} \sec V$ in order to be equally effective in the square-span direction after turning. If the arching of the vault (top of the frame) is small, V may be assumed, for all practical purposes, equal to θ for points in the vault.

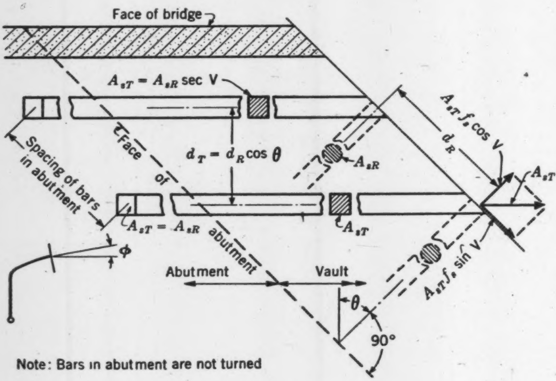

Face of bridge
$A_{sT} = A_{sR}$ sec V
$A_{sT} f_s \cos V$
d_R
Spacing of bars in abutment
℄ Face of abutment
$d_T = d_R \cos\theta$
A_{sR}
$A_{sT} f_s$
$A_{sT} = A_{sR}$
A_{sT}
ϕ
Abutment
Vault
$A_{sT} f_s \sin V$
θ
90°
Note: Bars in abutment are not turned

FIG. 8

As shown in Fig. 8, the plan spacing of the "turned" bars d_T, measured perpendicularly to the bars themselves, will be the spacing of the unturned bars multiplied by $\cos\theta$. The net result of the two multiplications is that the required steel area in the top of the frame of the skewed bridge is the required area calculated for the square span multiplied by $\sec V \sec \theta$. At the crown, $\sec V \sec \theta$ will be equal to $\sec^2 \theta$, which for all practical purposes may also be applied to the vault of rigid-frame bridges.

Turning the steel pattern parallel to the skew center line reduces its effectiveness in the square-span direction but increases its effectiveness in the transverse direction (parallel to the abutments), as seen from Fig. 8. The resolution of the stress in any bar into two components is indicated, one component being effective in resisting bending moment about the z-axis and the other being effective in resisting the cross shear T_z and the torsional moment M_v. This is analogous to the wedge-shaped beam in which one component of the stress in the bars resists bending and the other component resists shear.[7]

The effectiveness of the "turned" reinforcement in the transverse direction, combined with the effectiveness of the concrete in shear and torsion, and the effectiveness of the nominal transverse reinforcement, are sufficient to take care of all torsional effects resulting from skew, to a maximum skew angle of about 50° for the single-span structure.

The increase in steel areas contingent on the "turning" process does not apply to points in the vertical legs. For these points, the steel areas as calculated for the square span (Eq. 39c) are correct for the skew structure, with the spacing of bars as discussed.

EXTENSION TO DOUBLE-SPAN BRIDGES

A practical analysis results if propositions 1, 2, 3, and 4, which were developed for single-span structures, are applied to double-span skewed structures. To justify this application consider Fig. 9 which shows a double-span skewed rigid frame. This structure may be replaced by a single-span structure with a center post considered as a flexible bracket. A system of loads is applied to the end of this bracket and these loads are of such value that they exactly replace the reactions of the original structure. This transformation makes logical the application of the principles of single-span analysis to double-

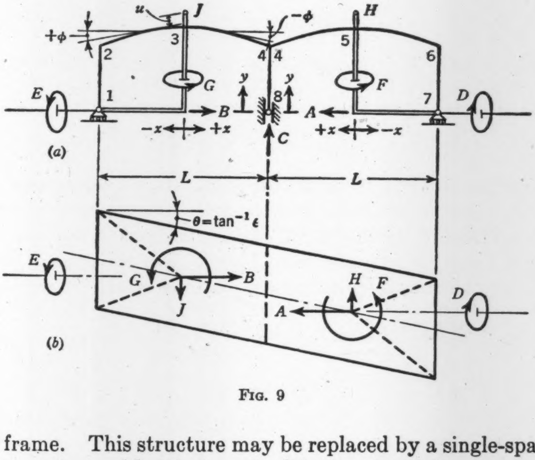


FIG. 9

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[7] "Concrete Engineers' Handbook," by George A. Hool and Nathan C. Johnson, McGraw-Hill Book Co., Inc., New York, N. Y., 1919, p. 314.

span skewed structures. The validity of this logical application has been checked by analyzing, independently, the rectangular elastic system for the rectangular redundants for a unit vertical influence load and by comparing the answers with those of the precise analysis. The precise analysis consisted of the solution of two sets of simultaneous equations. Each set contained six elastic equations, and the analysis is similar to the method proposed by Mr. Rathbun[8] in 1931, modified to apply to hinged foundations.

A simpler analysis of the double-span structure is indicated in the following demonstration. Fig. 9 shows the most advantageous residual structure because it is symmetrical.

TABLE 6.—MOMENT AT A GIVEN SECTION DUE TO EACH UNIT REDUNDANT

Section	$A = +1$		$B = +1$		$C = +1$		$D = +1$	$E = +1$	$F = +1$	$G = +1$	$H = +1$	$J = +1$
	m_z	m_v	m_z	m_v	m_z	m_v	m_v	m_v	m_v	m_v	m_v	m_v
1–2	0	0	$-y$	$-\epsilon x \sin \phi$	$-\frac{1}{2}\left(\frac{L}{2}+x\right)$	0	0	$+\cos \phi$	0	$+\sin \phi$	0	$-u$
2–3	0	0	$-y$	$-\epsilon x \sin \phi$	$-\frac{1}{2}\left(\frac{L}{2}+x\right)$	$m_z \epsilon \cos \phi$	0	$+\cos \phi$	0	$+\sin \phi$	0	$-u$
3–4	0	0	$-y$	$-\epsilon x \sin \phi$	$-\frac{1}{2}\left(\frac{L}{2}+x\right)$	$m_z \epsilon \cos \phi$	0	$+\cos \phi$	0	$+\sin \phi$	0	$-u$
4–8	$+y$	$-\epsilon x \sin \phi$	$-y$	$-\epsilon x \sin \phi$	0	0	$+\cos \phi$	$+\cos \phi$	$-\sin \phi$	$+\sin \phi$	$-u$	$-u$
4–5	$-y$	$-\epsilon x \sin \phi$	0	0	$-\frac{1}{2}\left(\frac{L}{2}+x\right)$	$m_z \epsilon \cos \phi$	$+\cos \phi$	0	$-\sin \phi$	0	$-u$	0
5–6	$-y$	$-\epsilon x \sin \phi$	0	0	$-\frac{1}{2}\left(\frac{L}{2}+x\right)$	$m_z \epsilon \cos \phi$	$+\cos \phi$	0	$-\sin \phi$	0	$-u$	0
6–7	$-y$	$-\epsilon x \sin \phi$	0	0	$-\frac{1}{2}\left(\frac{L}{2}+x\right)$	0	$+\cos \phi$	0	$-\sin \phi$	0	$-u$	0

Table 6, similar to Table 1, shows the moments at each section of the structure due to each unit redundant; and a tabulation similar to Table 2 may be easily developed from it, showing the unit deformations for each unit redundant. The location of the sections is shown in Fig. 9. For the general case, there are nine redundants and nine simultaneous equations, which are as follows:

$$\delta_a = 0 = A\,\delta_{aa} + B\,\delta_{ab} + C\,\delta_{ac} + D\,\delta_{ad} + E\,\delta_{ae} + F\,\delta_{af} + G\,\delta_{ag} + H\,\delta_{ah} + J\,\delta_{aj} + \Delta_a \quad \text{..(40a)}$$

$$\delta_b = 0 = A\,\delta_{ba} + B\,\delta_{bb} + C\,\delta_{bc} + D\,\delta_{bd} + E\,\delta_{be} + F\,\delta_{bf} + G\,\delta_{bg} + H\,\delta_{bh} + J\,\delta_{bj} + \Delta_b \quad \text{..(40b)}$$

$$\delta_c = 0 = A\,\delta_{ca} + B\,\delta_{cb} + C\,\delta_{cc} + D\,\delta_{cd} + E\,\delta_{ce} + F\,\delta_{cf} + G\,\delta_{cg} + H\,\delta_{ch} + J\,\delta_{cj} + \Delta_c \quad \text{..(40c)}$$

$$\delta_d = 0 = A\,\delta_{da} + B\,\delta_{db} + C\,\delta_{dc} + D\,\delta_{dd} + E\,\delta_{de} + F\,\delta_{df} + G\,\delta_{dg} + H\,\delta_{dh} + J\,\delta_{dj} + \Delta_d \quad \text{..(40d)}$$

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[8] "An Analysis of Multiple-Skew Arches on Elastic Piers," by J. Charles Rathbun, *Transactions*, ASCE, Vol. 98, 1933, p. 1.

$$\delta_e = 0 = A\,\delta_{ea} + B\,\delta_{eb} + C\,\delta_{ec} + D\,\delta_{ed} + E\,\delta_{ee} + F\,\delta_{ef} + G\,\delta_{eg} + H\,\delta_{eh} + J\,\delta_{ej} + \Delta_e \quad \text{..(40}e\text{)}$$

$$\delta_f = 0 = A\,\delta_{fa} + B\,\delta_{fb} + C\,\delta_{fc} + D\,\delta_{fd} + E\,\delta_{fe} + F\,\delta_{ff} + G\,\delta_{fg} + H\,\delta_{fh} + J\,\delta_{fj} + \Delta_f \quad \text{..(40}f\text{)}$$

$$\delta_g = 0 = A\,\delta_{ga} + B\,\delta_{gb} + C\,\delta_{gc} + D\,\delta_{gd} + E\,\delta_{ge} + F\,\delta_{gf} + G\,\delta_{gg} + H\,\delta_{gh} + J\,\delta_{gj} + \Delta_g \quad \text{..(40}g\text{)}$$

$$\delta_h = 0 = A\,\delta_{ha} + B\,\delta_{hb} + C\,\delta_{hc} + D\,\delta_{hd} + E\,\delta_{he} + F\,\delta_{hf} + G\,\delta_{hg} + H\,\delta_{hh} + J\,\delta_{hj} + \Delta_h \quad \text{..(40}h\text{)}$$

$$\delta_j = 0 = A\,\delta_{ja} + B\,\delta_{jb} + C\,\delta_{jc} + D\,\delta_{jd} + E\,\delta_{je} + F\,\delta_{jf} + G\,\delta_{jg} + H\,\delta_{jh} + J\,\delta_{jj} + \Delta_j \quad \text{..(40}i\text{)}$$

For symmetrical structures, the dead load, the balanced earth pressure load, and the temperature change effects are symmetrical. Only the applied live load can be unsymmetrical. It is obvious from Fig. 9 that for a symmetrical, double-span structure subjected to symmetrical loads the redundants are equal in pairs, as follows:

$$A = B \quad \text{.........................(41}a\text{)}$$

$$D = E \quad \text{.........................(41}b\text{)}$$

$$F = G \quad \text{.........................(41}c\text{)}$$

and

$$H = J \quad \text{.........................(41}d\text{)}$$

Fig. 10 shows the residual structure with the applied symmetrical redundants.

Eqs. 41 indicate, from symmetry, that the center post is subjected only to the vertical rectangular reaction C and that no torsional stresses exist for the center post sections. Thus, for symmetry, the nine elastic equations (Eqs. 40) are reduced to five elastic equations (Eqs. 42) written for five redundants, two being rectangular and three torsional.

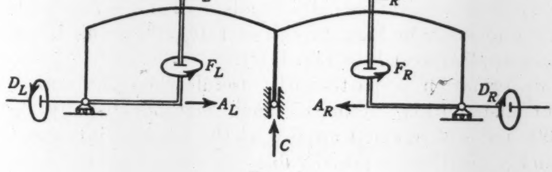

FIG. 10

If Eq. 40*a* is added to Eq. 40*b*, Eq. 40*d* to Eq. 40*e*, Eq. 40*f* to Eq. 40*g*, and Eq. 40*h* to Eq. 40*i*, five elastic equations may be written for symmetrical structures symmetrically loaded. If the conjugate redundants from Eqs. 41 are substituted into the five equations, and the conjugate deformations of

Table 7 are also substituted,

$$Q_{12} A + Q_{13} C + Q_{14} D + Q_{15} F + Q_{16} H = - \Delta_a \quad \text{.......(42a)}$$

$$Q_{13} A + Q_{17} C + Q_{18} D + Q_{19} F + Q_{20} H = - \Delta_c \quad \text{.......(42b)}$$

$$Q_{14} A + Q_{18} C + Q_{21} D + Q_{22} F + Q_{23} H = - \Delta_d \quad \text{.......(42c)}$$

$$Q_{15} A + Q_{19} C + Q_{22} D + Q_{24} F + Q_{25} H = - \Delta_f \quad \text{.......(42d)}$$

and

$$Q_{16} A + Q_{20} C + Q_{23} D + Q_{25} F + Q_{26} H = - \Delta_h \quad \text{.......(42e)}$$

In Eqs. 42 the conjugate deformations are written from the same formula and the sum of any two conjugate deformations may be found by including the new sections—that is, summing up within the new limits. Application of the logical propositions (substantiated by comparisons and tests) permits the separate determination of the rectangular and torsional redundants. This separation results in a practical solution whereas the simultaneous solution of the five equations results in a precise solution. The logical conclusion is that the same precision exists for the double span as exists for the single span.

If the redundant A and the redundant C are found independently from the rectangular elastic analysis, Eqs. 42 may be written as follows:

$$Q_{21} D + Q_{22} F + Q_{23} H = - \Delta_d - Q_{14} A - Q_{18} C \quad \text{.......(43a)}$$

$$Q_{22} D + Q_{24} F + Q_{25} H = - \Delta_f - Q_{15} A - Q_{19} C \quad \text{.......(43b)}$$

and

$$Q_{23} D + Q_{25} F + Q_{26} H = - \Delta_h - Q_{16} A - Q_{20} C \quad \text{.......(43c)}$$

Thus only three simultaneous equations are required to be solved for an exact analysis of the double-span symmetrical skewed structure with symmetrical loading. Of course, this analysis is not required unless the critical skew has been exceeded. It is estimated that more than 60% of all double-span, skewed frames and arches have less than critical skew.

Live load may be applied as an unsymmetrical load, but by a transformation it is possible to convert the unsymmetrical applied load into two elements which are symmetrical. Consider Fig. 11 which shows an unsymmetrical load P applied at point m. Fig. 11(*a*) shows a pair of loads $P/2$, applied symmetrically. Fig. 11(*b*) shows another pair of loads, $P/2$ and $-P/2$ applied as shown. Combining the loadings in Figs. 11(*a*) and 11(*b*), the result is exactly the unsymmetrically applied load P of Fig. 11(*c*).

This transformation is particularly useful when the analysis is one of the moment distribution methods. Such methods usually distribute 100 units or 1,000 units of moment applied at the center joint, and this distribution will quickly yield the required values for the moment, $P\,x$, applied to the center joint.

Numerous approximations have been uncovered by isolating and analyzing individual elements. Among the more useful approximations are the following, which eliminate considerable computations.

It has been observed that the element $(h - y - Q_3)$ in Eq. 29*b* may be approximated as $(y_c - y)$, in which y_c is the y-value for the crown. This

TABLE 7.—SCHEDULE OF CONJUGATE DEFORMATIONS

Eq. 40	COEFFICIENT		FORMULA ELEMENT NOS.:				Formula	CENTER LEG VALUES FOR ELEMENTS:					
	For:	No.	1	2	3	4		Formula	1	2	3	4	Σ
(a) SECTIONS 1-2-3-4-8 AND 8-4-5-6-7, FIG. 9(a), CONTRIBUTING TO TOTAL VALUE													
a, b	A	Q_{12}	$=\delta_{aa}$	$+\delta_{bb}$	$+\delta_{ab}$	$+\delta_{ba}$	$\pm\Sigma(G_1 y^2)$ $\pm\epsilon^2\Sigma(G x^2)\sin^2\phi$	$\pm\Sigma(G_1 y^2)$ $\epsilon^2\Sigma(G x^2)$	+ +	+ +	− −	− −	0 0
a, b c, c	C A	Q_{13}	δ_{ac}	δ_{bc}			$+\frac{1}{2}\Sigma\left[G_1 y\left(\frac{L}{2}+x\right)\right]$ $+\frac{\epsilon^2}{2}\Sigma\left[G x\left(\frac{L}{2}+x\right)\sin\phi\cos\phi\right]$	0 0	0 0	0 0			0 0
a, b d, e	D A	Q_{14}	δ_{ad}	δ_{bd}	δ_{ae}	δ_{be}	$-\epsilon\Sigma(G x\sin\phi\cos\phi)$	0	0	0	0	0	0
a, b f, g	F A	Q_{15}	δ_{af}	δ_{bf}	δ_{ag}	δ_{bg}	$\pm\epsilon\Sigma(G x\sin^2\phi)$	$\pm\epsilon\Sigma(G x)$	+	−	+	−	0
a, b h, j	H A	Q_{16}	δ_{ah}	δ_{bh}	δ_{aj}	δ_{bj}	$+\epsilon\Sigma(G u x\sin\phi)$	$\pm\epsilon\frac{L^2}{4}\Sigma G$	+	−	+	−	0
c, c	C	Q_{17}	δ_{cc}				$\frac{1}{4}\Sigma\left[G_1\left(\frac{L}{2}+x\right)^2\right]$ $+\frac{\epsilon^2}{4}\Sigma\left[G\left(\frac{L}{2}+x\right)^2\cos^2\phi\right]$	0 0	0 0				0 0
c, c d, e	D C	Q_{18}	δ_{cd}	δ_{ce}			$-\frac{\epsilon}{2}\Sigma\left[G\left(\frac{L}{2}+x\right)\cos^2\phi\right]$	0	0	0			0
c, c f, g	F C	Q_{19}	δ_{cf}	δ_{cg}			$\pm\frac{\epsilon}{2}\Sigma\left[G\left(\frac{L}{2}+x\right)\sin\phi\cos\phi\right]$	0	0	0			0
c, c h, j	H C	Q_{20}	δ_{ch}	δ_{cj}			$+\frac{\epsilon}{2}\Sigma\left[G u\left(\frac{L}{2}+x\right)\cos\phi\right]$	0	0	0			0
d, e	D	Q_{21}	δ_{dd}	δ_{ee}	δ_{ed}	δ_{de}	$+\Sigma(G\cos^2\phi)$	0	0	0	0	0	0
d, e f, g	F D	Q_{22}	δ_{df}	δ_{ef}	δ_{dg}	δ_{eg}	$\pm\Sigma(G\sin\phi\cos\phi)$	0	0	0	0	0	0
d, e h, j	H D	Q_{23}	δ_{dh}	δ_{eh}	δ_{dj}	δ_{ej}	$-\Sigma(G u\cos\phi)$	0	0	0	0	0	0
(b) SECTIONS 1-2-3-4-8 AND 8-4-5-6-7, FIG. 9(a), CONTRIBUTING TO TOTAL VALUE													
f, g	F	Q_{24}	δ_{ff}	δ_{gg}	δ_{gf}	δ_{fg}	$\pm\Sigma(G\sin^2\phi)$	$+\Sigma G$	+	+	+	+	+
f, g h, j	H F	Q_{25}	δ_{fh}	δ_{gh}	δ_{fj}	δ_{gj}	$+\Sigma(G u\sin\phi)$	$+\Sigma(G u)$	+	+	+	+	+
h, j	H	Q_{26}	δ_{hh}	δ_{jj}	δ_{jn}	δ_{hj}	$+\Sigma(G u^2)$	$+\Sigma(G u^2)$	+	+	+	+	+

eliminates the computation for h and Q_3. By inspection of Eq. 14e and Table 2, it is observed that the legs of a rigid frame contribute almost the entire value of Q_5 and therefore, with little error,

$$Q_5 = \frac{L^2}{4}\sum\frac{1}{F} = \frac{3.58\,L^2}{48}\sum\frac{1}{I} \quad\ldots\ldots\ldots\ldots\ldots\ldots (44)$$

(The constant multiplier is not shown in Eq. 44. The summation is for the legs only.)

This method of analysis may be extended to skewed structures of more than two spans. The method has given satisfactory results for a triple-span, skewed, rigid-frame parkway bridge.

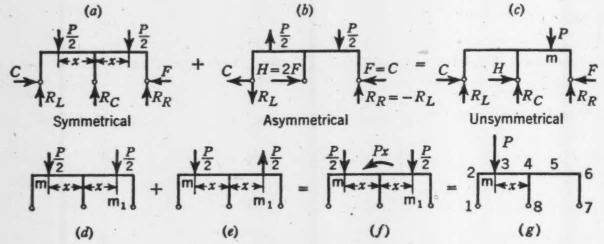
(a)
P/2
P/2
x
x
C
F
R_L
R_C
R_R
Symmetrical
+
(b)
P/2
P/2
C
H=2F
F=C
R_L
R_R=−R_L
=
(c)
P
m
C
H
F
R_L
R_C
R_R
Unsymmetrical
P/2
m
x
m_1
(d)
+
(e)
=
Px
(f)
=
P
2 3 4 5 6
1 8 7
(g)

FIG. 11

The free body diagram shown in Fig. 12 is the same for either single-span or multiple-span structures, and therefore any equation written for this free body is valid for all structures. As a result the stress equations and the design procedure for the single-span skewed structure may be applied to multiple-span skewed structures.

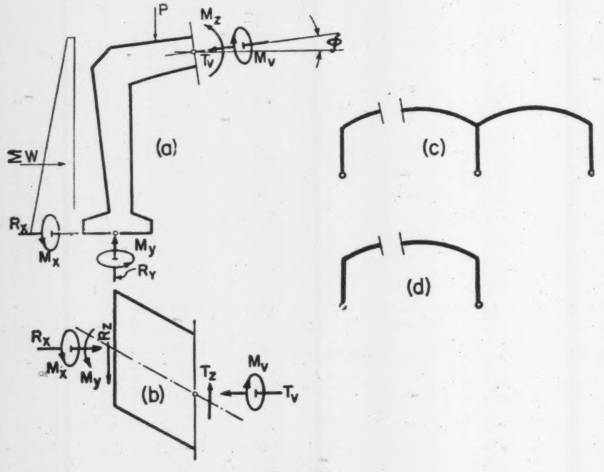
P
M_z
T_v
M_v
Φ
ΣW
(a)
R_x
M_x
M_y
R_Y
(c)
(d)
R_x
R_z
M_x
M_y
T_z
M_v
T_v
(b)

FIG. 12

TEST FOR LIMIT OF SKEW ANGLE WITH NOMINAL TRANSVERSE REINFORCEMENT

It has been stated that, if the longitudinal reinforcement is designed parallel to the face of the bridge, only nominal reinforcement in the transverse direction (parallel to the abutments) will be required. This statement will now be examined, and its limits will be determined by analyzing each term that enters into the design of the transverse reinforcement. The critical section is at the crown.

The design moments, M_T, and cross shears, T_T, are found from the transformations. For all the applied loads, M_T and T_T, for the crown section, are negligible. The stresses due to temperature change (see Table 8) are the only significant stresses. The values from Table 5 are used to determine the crown stresses M_T and T_T.

TABLE 8.—STRESSES FOR DESIGN OF TRANSVERSE REINFORCEMENT AT THE CROWN SECTION

Loading	Design moments, M_T	Cross shears,[a] T_T
Dead load	$\epsilon\, M_z - M_v$	$\epsilon\, T_v - T_z$
Live load plus impact	$\epsilon\, M_z - M_v$	$\epsilon\, T_v - T_z$
Temperature change	$\epsilon\, R_x\, y_c$	$\epsilon\, \dfrac{6.7\, c\, t\, E_c}{(1 + \mu)\, s\, L\, \Sigma 1/I}$

[a] See text for limits of $\Sigma 1/I$.

The next step is to find the unit stresses for the crown section due to M_T and T_T. The unit shearing stress v_t due to the torsion moment M_T may be found from the relation (see Fig. 13):

$$v_t = k\, \frac{M_T}{b\, t^2} \quad \text{(45)}$$

in which the constant k has been determined variously as shown in Fig. 13. For values of b/t greater than 6, the Saint Venant expression for k is more accurate than the Merriman value of 4.5. The latter is applicable only for values of b/t less than 2. In general, for barrel arches and rigid-frame bridges, the ratio of b/t is always greater than 16, and $k = 3$, in Eq. 45, is sufficiently accurate. The cross

TABLE 9.—DESIGN OF TRANSVERSE REINFORCEMENT (SEE FIG. 14)

Point	t (in.)	TORSION		DIRECT SHEAR		GEOMETRIC DATA							STEEL (SQ IN.)	
		M_T (kip-ft)	v_t	T_T (kips)	v_s	$v_t + v_s$	v	a_1	a_2	a_3	$a_3 + 2\,a_1$	$a_1 - 3$ (in.)	A_{s1}	A_{s2}
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
1	27	11.0	45	11.9	55	100	....	....	....	....	....	....	....	....
2	32	11.0	32	11.9	47	79	....	....	....	....	....	....	....	....
3	37	11.0	24	11.9	40	64	....	....	....	....	....	....	....	....
4	36.5	59.2	134	0.54	2	136	24	18.5	3.3	15.2	52.2	15.5	0.02	0.02
5	27	62.0	255	0.54	2	257	145	13.6	7.7	5.9	33.1	10.6	0.31	0.30
6	20	64.0	480	0.54	3	483	371	10.0	7.7	2.3	22.3	7.0	0.81	0.76
....[a]	15	9.0	120	1.63	14	134	22	8.4	1.4	7.0	23.8	4.0	0.016	0.008
11	24	64.2	338	0.54	3	341	229	12.1	8.1	4.0	28.2	9.1	0.51	0.50
12	34	63.0	163	0.54	2	165	53	17.2	5.5	11.7	46.1	14.2	0.08	0.08
13	48	60.8	79	0.54	1	80	....	....	....	....	....	....	....	....

[a] Center-line point for single span. Other points for double-span bridge.

shear T_T produces unit shearing stresses (v_s) which are distributed parabolically with a maximum of 1.5 times the average stress. The formulas and numerical values for several typical design sections are shown in Table 9. The allowable unit stress, v_c, in the concrete (see Fig. 14) is taken as 112 lb per sq in. which is the normal unit stress of 90 lb per sq in. increased by 25% (90 × 1.25) because temperature stress is included. The area A_{s1} is found by equating the moment of the shear wedge *a b c*, Fig. 14, about the neutral axis, to the moment of $A_{s1} f_s$. The area A_{s2} is found by equating the shear wedge to $A_{s2} f_s$. Referring to Fig. 14, the respective columns in Table 9 are computed by the following formulas: In Col. 3—

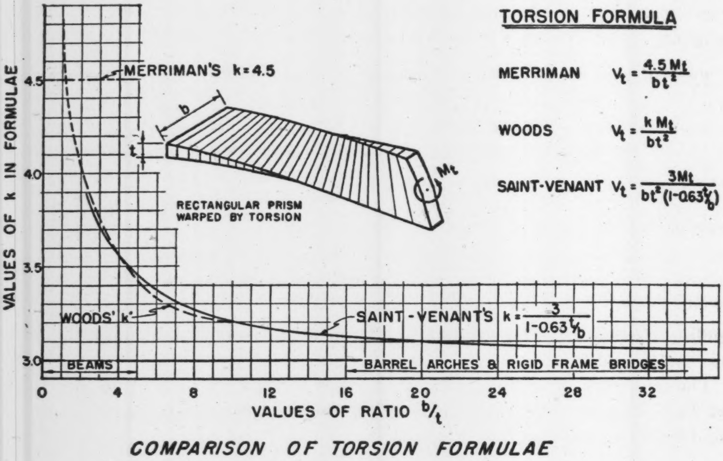

FIG. 13.—COMPARISON OF *k*-VALUES IN TORSION FORMULAS

$$v_t = \frac{3\,M_T}{12\,t^2} \qquad (46a)$$

In Col. 5—

$$v_s = \frac{1.5\,T_T}{12\,t} \qquad (46b)$$

In Col. 7—

$$v = v_t + v_s - v_c \qquad (46c)$$

In Col. 8—

$$a_1 = \frac{v_t + v_s}{2\,v_t}\,t \qquad (47a)$$

In Col. 9—

$$a_2 = \frac{v_t + v_s - v_c}{2\,v_t}\,t \qquad (47b)$$

In Col. 10—

$$a_3 = a_1 - a_2 \qquad (47c)$$

In Col. 13—

$$A_{s1} = \frac{2\,v\,a_2\,(a_3 + 2\,a_1)}{f_s\,(a_1 - 3\text{ in.})} \qquad (48a)$$

and, in Col. 14—

$$A_{s2} = \frac{6\,v\,a_2}{f_s} \qquad (48b)$$

In Eqs. 48, $f_s = 18{,}000 \times 1.25 = 22{,}500$ lb per sq in. The transverse steel area used is the larger of the two areas, and it is used in both faces.

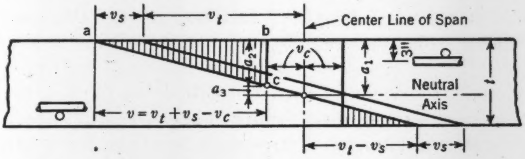

FIG. 14.—TRANSVERSE REINFORCEMENT IN A SKEWED STRUCTURE

Codes and specifications for bridges do not impose a limit for the maximum unit shear due to torsion, but the Bureau of Yards and Docks, United States Department of the Navy, has a commendable standard[9] which treats the subject of combined shears due to torsion and direct shear for beams in flat slab design. For flat slabs, beams at walls and openings, torsion shear plus direct shear, web reinforcement, and longitudinal bars specially anchored, the maximum allowable unit shearing stress for 3,000-lb concrete is 450 lb per sq in. With a 25% increase for the inclusion of temperature stresses, the maximum allowable unit shearing stress would be 450 × 1.25 = 562 lb per sq in. This maximum unit shearing stress is recommended for rectangular sections in the range of these bridges.

By substituting the maximum allowable shearing stress into the equations, the maximum skew for the bridge may be found. Other considerations lead to the tentative conclusion that the limit of skew as determined by the maximum unit shearing stress is close to the limit determined by nominal transverse reinforcement (0.50 sq in. per ft). A more complete demonstration and several numerical examples[6] have been filed for reference in the Engineering Societies Library.

CONCLUSIONS

It has been demonstrated that for all practical design purposes:

1. With few exceptions, the rectangular elastic system is independent of the torsional elastic system;

2. The rectangular redundants and the rectangular stresses for all applied loads are independent of the torsional elastic system, and independent of the skew angle;

3. Torsional redundants and torsional stresses for applied loads are a linear function of the tangent of the skew angle;

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[9] "Standards of Design for Concrete," *No. 3Yb*, Bureau of Yards and Docks, U. S. Navy Dept., Washington, D. C., 1929, Section 8–13d.

4. Conclusions 1, 2, and 3 are true for all the usual shapes of skewed, single-span arches or double-span arches and rigid-frame bridges, and also true for all applied loads regardless of the direction or the point of application of the load; and

5. The rectangular redundants due to temperature change vary as the $\sec^2 \theta$—that is, as $(1 + \epsilon^2)$;

In addition to conclusions 1 to 5 the following important conclusions may be written for the practical application of the method:

6. The torsional stresses may be written directly from the rectangular stresses by multiplying an appropriate rectangular stress by an appropriate geometric or elastic constant, thereby completely eliminating the torsional elastic analysis;

7. The torsional stresses may be written with ϵ as a parameter and thereby represent the torsional stresses for a family of skewed structures;

8. All rectangular and torsional stresses, other than those due to volume change or foundation yield, are independent of Poisson's ratio (that is, $E_c/E_s = \lambda$, since $\lambda = 2\,(1 + \mu)$);

9. All torsional stresses, other than those due to volume change or foundation settlement, are independent of the formula used to compute the value of F (or r), providing that F is a linear function of b, the width of the structure, and further providing that F is independent of the ratio of b/t within the applied range;

10. Any method that will determine the rectangular stresses satisfactorily may be used as the basis for determining the torsional stresses (such other methods include moment distribution and slope deflection);

11. Inasmuch as Δ_s is a parameter in the demonstration, the analysis is valid for all span lengths;

12. For practical office procedure no torsional analysis is required, and the longitudinal steel may be determined from the rectangular design using the "turning process"; and

13. For extreme skews, long spans, or multiple spans the shearing stresses due to torsion become important and the test for limitations should be applied.

ACKNOWLEDGMENTS

In the preparation of this paper the writer has received encouragement from Messrs. Hayden and Hodges, Leslie G. Holleran, G. D. Clarke, and E. H. Praeger, Members, ASCE, and the late Aubrey L. McCaffrey. In particular, the writer wishes to express his appreciation to Mr. Hayden without whose help and inspiration this work would never have been begun or completed.

APPENDIX. NOTATION

The following letter symbols, adopted for use in this paper and for the guidance of discussers, conform essentially with American Standard Letter Symbols for Structural Analysis (ASA—Z10.8—1949) prepared by a Sectional

Committee of the American Standards Association, with ASCE participation, and approved by the Association in 1949. In general, point P designates any spot on the neutral surface of an arch ring which is halfway between the spandrel walls. The neutral surface is assumed to be midway between the extrados and the intrados.

A subscript R identifies the main symbol with the rectangular elastic system and T, similarly, with the torsional elastic system or in transverse direction, as demonstrated in the text.

A = area; cross-sectional area as further defined by subscripts, s, denoting "steel reinforcement" and, c, denoting "concrete";

A = one of four redundants (A, B, C, and D) introduced in the design of a single-span structure as shown in Fig. 1 and one of nine redundants (A, B, C, D, E, F, G, H, and J) introduced in the design of a double-span skewed structure as shown in Fig. 8; A' = a resultant of redundant A and ϵA in Fig. 2(a); and subscripts L and R denote "left" and "right," respectively;

a = an x-distance measured from the center line of a beam to the point of application of a unit load (see Fig. 2(d));

a = design constants in Table 9, distinguished by numerical subscripts and defined by Eqs. 47 (see also Figs. 7 and 14);

B = a redundant design factor (see under A);

b = width of an arch ring measured parallel to the z-axis (on the skew) (Figs. 1, 5, and 13);

C = a redundant design factor (see under A);

c = coefficient of thermal expansion (Eqs. 30) = 0.0000065;

D = a redundant design factor (see under A);

d = effective depth of a concrete section (see Fig. 7);

E = modulus of elasticity, with subscripts c and s to identify "tension and compression" and "shear," respectively;

E = a redundant design factor (see under A);

e = eccentricity of reaction R_x; and e' = eccentricity of reaction R_y (Fig. 5 and Eqs. 36);

e = eccentricity of thrust, T_v, in Eq. 36a;

F = the factor of torsion (Eqs. 7);

F = a redundant design factor (see under A);

f = fiber stress in reinforced concrete design, the subscripts s and c identifying "steel" and "concrete," respectively;

G = a redundant design factor (see under A);

G = an elastic element defined as $1/F$; G_1 = an elastic element defined as $1/I$;

H = a redundant design factor (see under A);

h = the vertical distance from the x-axis to the elastic center at which the redundant B is applied;

I = moment of inertia of the area of the section cut by a radial plane through point P and parallel to the axis of the barrel (z-axis), about a horizontal line through point P and lying in the radial plane;

J = a redundant design factor (see under A);

j = proportion of effective depth;

k = soil modulus; soil compression constant—that is, the resisting force, in kips, exerted by the soil on a surface 1 ft square which is displaced normally a distance of 1 in.;

k = constant in torsion formula (Eq. 45) (see also Fig. 13);

L = span of the neutral surface measured on the square;

M = bending moment; M_v and M_z are the components of the moment acting on the surface cut by a radial plane through point P about the v-axis and the z-axis, respectively (M_v being positive when it causes the obtuse angle of the section on which it is acting to deflect toward the center of curvature, and M_z being positive when it produces compression in the extrados fibers); M_v is the torsional moment (M_u in Fig. 1 and M_x and M_y in Fig. 5);

m = moment resulting from a unit load applied at a point m; m_v, m_z, m_{ov}, and m_{oz} are the components of the moments corresponding to M_v, M_z, M_{ov}, and M_{oz} when a unit load is applied at point m;

n = a number, such as "indeterminate in the nth degree," or "n equations";

P = applied point load;

Q = substitution factor, with appropriate numerical subscript, function of δ (Eqs. 14);

R = resultant forces or reaction identified by appropriate subscripts x, y, z, etc., to a given axis;

r = an elastic element in Eq. 8;

Δs = the length of an elementary parallelopiped measured, on the square, along the neutral surface (see Fig. 1); and ds is the differential length of an elementary parallelopiped measured, on the square, along the neutral surface;

s = one half the span length, on the square, of an arch (Fig. 2);

T = resultant thrust on a section along an axis identified by appropriate subscript;

t = thickness (Figs. 1 and 13 and Eqs. 45 and 46);

t = temperature rise (+) or temperature fall (−), in degrees Fahrenheit (Eqs. 30);

u = the coordinate of the elastic center referred to the tangent and radial planes through point P and parallel to the plane $z = 0$, u being parallel to the radial plane (the sign is fixed by Eq. 3b);

u_o = a dimension defined as $x \sin \phi + y \cos \phi$ (Fig. 1);

V = projection of the skew angle θ onto the plane tangent to the neutral axis at the point under consideration;

V = vertical shear (Eq. 5a);

v = unit shearing stress with subscripts as shown in text (see also Table 7, Eq. 46c, and Fig. 14);

v = distances measured parallel to the v-axis;

W = total weight; load;

x = abscissa distance; coordinate with y and z;

y = ordinate (see under x);

z = coordinate (see under x);

α = slope of a reaction force (Fig. 6);

Δ = deformation; the deformation of the residual structure at the point of application of the redundants A, B, C, and D (identified by subscripts a, b, c, and d, respectively), and in the direction of the redundants A, B, C, and D, due to applied loads, volume changes, or yielding of foundations; subscripts L and T denote "longitudinal" and "transverse," respectively; subscripts R and T denote "rectangular" and "torsional," respectively;

δ = unit deformation; δ_{aa}, δ_{bc}, and δ_{ad} = the deformation at the point of application and in the direction of the redundant indicated by the first subscript, due to a unit load (or unit moment) acting at the point of application and in the direction of the redundant indicated by the second subscript;

ϵ = the tangent of the skew angle which is always positive;

θ = skew angle;

λ = modular ratio, $E_c/E_s = 2\,(1 + \mu)$; for concrete, $\lambda = 2.30$;

μ = Poisson's ratio; for concrete, $\mu = 0.15$; and

ϕ = the angle of the slope of a plane tangent to the neutral surface at point P.

DISCUSSION

D. A. Nettleton,[10] Assoc. M. ASCE.—An examination of Fig. 1, Eqs. 6, and Table 2 will show that the expressions for the indeterminate reactions are based on deformations from bending moments acting in planes perpendicular to the line of the abutments, and torsional moments acting in planes parallel to the line of the abutments. Also, on any plane cut through the structure parallel to the abutments, it is assumed that the moments (and therefore the deformations that determine the reactions) are the same on each unit of the width of the structure. The same assumptions as to bending and torsional moments were made by J. Charles Rathbun, M. ASCE, in setting up his equations,[2] published in 1924, and they have formed the basis for several papers since that date.

The writer questions the validity of these assumptions in regard to bending and torsional moments. In the vicinity of the sides of the deck, the bending moments must act in planes parallel to the sides and the torsional moments in planes perpendicular to the sides. For a rigid-frame bridge, narrow in relation to its span, this type of moment action applies fairly closely across the width of the deck. The writer has set up equations for the indeterminate reactions on the basis of this moment action. To compare the results obtained by the author's method (hereafter called Method A) and the writer's method (hereafter called Method B), the two have been applied to rigid-frame bridges with skew angles of 0°, 30°, and 60°. To simplify the calculations, the frame is assumed to be of constant thickness throughout the legs and deck, and the deck is assumed to be in a horizontal plane. The span, in a direction perpendicular to the line of the abutments is called L, and the height of the legs is taken as $0.25\,L$.

A comparison between the two methods is afforded by Table 10. The deck moments in Table 10(a) act in a plane parallel to the sides of the deck. For Method A, these values are obtained from moments acting in planes perpendicular to the line of the abutments by multiplying by the square of the secant of the skew angle. This corresponds to the author's "turning process" which he applies to the area of reinforcing required. "Turning" the moments instead of the required areas of reinforcing results in the same requirements for reinforcing, and is done here to facilitate a comparison between the two methods.

Method B gives larger knee moments for deck loading, and that, of course, results in smaller moments at the center of the span. In the case of temperature change, Method B gives very materially smaller knee moments, particularly in the deck. Deck moments by Method B are only 58% of the deck moments by Method A for a 30° skew, and 7% for a 60° skew. Neither method shows the variation in moment across the width of the section, and neither method yields any information as to the transverse moments.

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[10] Supervising Urban Engr., State Highway Dept., Dallas, Tex.

V. P. Jensen,[11] M. ASCE, has made an exact analysis for a skewed slab of constant section by the method of difference equations. A method such as this would show the stresses in a rigid-frame bridge but, to the writer's knowledge, no one has been successful in applying it to a rigid-frame bridge.

TABLE 10.—RELATIVE MOMENTS IN A FLAT-TOPPED FRAME OF CONSTANT THICKNESS

Quantity	Skew angle θ	Uniform Load of w Pounds per Square Foot		Temperature Rise of t°	
		Method A	Method B	Method A	Method B
		(a) Knee Moments			
Deck	0°	$-0.0714\ w\ L^2$	$-0.0714\ w\ L^2$	$-\ 3.42\ E\ I\ c\ t/L$	$-\ 3.42\ E\ I\ c\ t/L$
	30°	$-0.095\ w\ L^2$	$-0.102\ w\ L^2$	$-\ 6.05\ E\ I\ c\ t/L$	$-\ 3.5\ E\ I\ c\ t/L$
	60°	$-0.286\ w\ L^2$	$-0.330\ w\ L^2$	$-\ 54.8\ E\ I\ c\ t/L$	$-\ 3.81\ E\ I\ c\ t/L$
Leg	0°	$-0.0714\ w\ L^2$	$-0.0714\ w\ L^2$	$-\ 3.42\ E\ I\ c\ t/L$	$-\ 3.42\ E\ I\ c\ t/L$
	30°	$-0.0714\ w\ L^2$	$-0.0732\ w\ L^2$	$-\ 4.55\ E\ I\ c\ t/L$	$-\ 3.93\ E\ I\ c\ t/L$
	60°	$-0.0714\ w\ L^2$	$-0.0789\ w\ L^2$	$-\ 13.7\ E\ I\ c\ t/L$	$-\ 5.11\ E\ I\ c\ t/L$
		(b) Reactions at the Bottom of the Legs			
R_x	0°	$+0.286\ w\ L$	$+0.286\ w\ L$	$+\ 13.68\ E\ I\ c\ t/L^2$	$+13.68\ E\ I\ c\ t/L^2$
	30°	$+0.286\ w\ L$	$+0.293\ w\ L$	$+\ 18.2\ E\ I\ c\ t/L^2$	$+15.72\ E\ I\ c\ t/L^2$
	60°	$+0.286\ w\ L$	$+0.316\ w\ L$	$+\ 54.7\ E\ I\ c\ t/L^2$	$+20.4\ E\ I\ c\ t/L^2$
R_z	0°	0	0	0	0
	30°	$+0.165\ w\ L$	$+0.169\ w\ L$	$+\ 17.25\ E\ I\ c\ t/L^2$	$+15.81\ E\ I\ c\ t/L^2$
	60°	$+0.496\ w\ L$	$+0.548\ w\ L$	$+115\ E\ I\ c\ t/L^2$	$+55.7\ E\ I\ c\ t/L^2$
M_z	0°	0	0	0	0
	30°	$+0.0069\ w\ L^2$	$+0.0077\ w\ L^2$	$-\ 4.31\ E\ I\ c\ t/L$	$-\ 4.68\ E\ I\ c\ t/L$
	60°	$+0.0206\ w\ L^2$	$+0.0081\ w\ L^2$	$-\ 28.7\ E\ I\ c\ t/L$	$-14.67\ E\ I\ c\ t/L$
M_y	0°	0	0	0	0
	30°	0	0	$+\ 3.37\ E\ I\ c\ t/L$	$+\ 3.37\ E\ I\ c\ t/L$
	60°	0	0	$+\ 10.13\ E\ I\ c\ t/L$	$+10.13\ E\ I\ c\ t/L$

The design of the transverse reinforcement should be controlled by the transverse moments, and not by the torsional moments, because the torsional moments are generally so small that no transverse reinforcing is required. The analysis of skewed slabs by Mr. Jensen may be used as a rough guide to the amount of transverse reinforcement required in the deck. This analysis shows the following maximum transverse moments acting in a plane at right angles to the line of the skew span:

Skew	Transverse M
0°	$0.0085\ w\ l^2$
30°	$0.0148\ w\ l^2$
45°	$0.0193\ w\ l^2$
60°	$0.0241\ w\ l^2$

in which w is the uniform load per square foot, and l is the skew span.

The transverse moments in the central part of the span of a rigid-frame bridge may be obtained, roughly, by using these moment coefficients, taking l as the skew distance between points of zero moment for dead load and w as sum of the dead load and the average live load. In addition to this moment, there is a transverse moment,[12] due to concentrated wheel loads, equal to 0.3 P approximately, in which P is the concentrated load, in pounds, and the moment is in units of foot-pounds per foot. Since the transverse rods generally run parallel to the abutments instead of perpendicular to the skew span, this steel is not fully effective for transverse moments and, therefore, the required areas must be multiplied by the square of the secant of the skew angle. These transverse moments produce tension in the bottom of the slab, and therefore should be used to determine the transverse bottom steel. The transverse top steel may be made somewhat less.

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[11] "Analyses of Skew Slabs," by V. P. Jensen, *Engineering Experiment Station Bulletin Series No. 332*, Univ. of Illinois, Urbana, Ill., 1941, Tables 5 to 8, inclusive.

Ernest H. Harder,[13] M. ASCE.—Aside from the application of the author's method to double-span bridges, this paper is very much like that of Richard M. Hodges[4] except, perhaps, in the method of demonstration. Any attempt to simplify the lengthy exposition of the original paper by J. Charles Rathbun,[2] M. ASCE, is commendable. In the discussion of that paper, the writer demonstrated[14] that the solution of the reactions of any skewed arch bridge consisted of two parts—the solution of the reactions of a right arch and the solution of the effect of the skew on that right arch, provided that the x-axis is chosen perpendicular to the line of the abutments and the y-axis and z-axis are at right angles to the x-axis all through the crown of the arch.

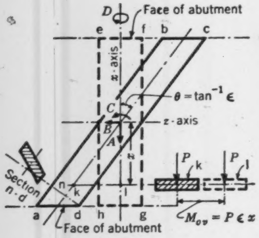

FIG. 15

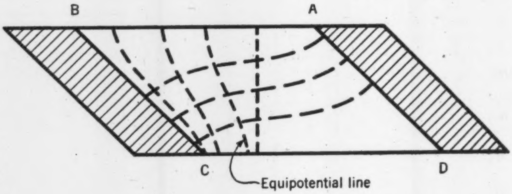

FIG. 16

It seems to the writer that it is time for a review of the problem that has been obscured by endless pages filled with integral signs, double subscript deltas, changes of notation, etc. Having selected, as a matter of convenience, the x-axis perpendicular to the face of the abutments and y-axis and z-axis at right angles to the x-axis at the center of the crown, the designer can erect about these axes a dummy arch (e-f-g-h, Fig. 15) which has the same curvature and thickness as the skew arch a-b-c-d when viewed in the direction of the z-axis. Then, whether the skew arch and the dummy arch are cut through the crown along the z-axis, or in any other manner, to create simple systems, any load such as P at point k may be transferred to the dummy arch at point l by an equal load P and a twisting moment $P \epsilon x$. Thus, the designer can operate on the dummy arch as on any right arch to obtain the thrusts, shears, and moments due to the vertical loads P and he can find the shears parallel to the z-axis, the torsions on any radial section, and the moments on the radial section which the author calls M_u. These shears and moments are in reality treated like secondary forces on the dummy arch. This is what the writer proposed in 1924 and it is the same as the author is proposing in 1950. In effect both methods are the same—merely the method of presentation is different. For purposes of illustration, the writer placed load P on the center line of the skew arch. A load can be placed anywhere on the skew arch and its secondary effects on the dummy arch are found in the same manner. It is also immaterial whether the ends of the arches are hinged or fixed.

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[12] *Public Roads*, March, 1930, p. 8, Fig. 6.

[13] Cons. Structural Engr., Newark, N. J.

[14] Discussion by Ernest H. Harder of "Analysis of the Stresses in the Ring of a Concrete Skew Arch," by J. Charles Rathbun, *Transactions*, ASCE, Vol. LXXXVII, 1924, p. 660.

By the usual methods of summation and using the areas and moments of inertia of the various sections of the dummy arch (all of which are still somewhat questionable), the several indeterminate reactions that hold the primary and secondary forces of the dummy arch in equilibrium can be found. Having found these reactions for the dummy arch, the designer is finished with it and any attempt to find moments, shears, thrusts, and torsions at section l of the dummy arch and then attempt to apply them at section k of the skew arch is wrong. That is what the author has done in Eq. 17, for instance. As the writer has stated elsewhere:[15]

> "Before these forces and torsions can be used to determine the stresses in the arch, T_x and T_y must be combined and the components of the resultant along the center line of the bridge at right angles to this axis must then be determined. This should also be done in connection with the bending moments and torsions so that the arch stresses can be found for sections at right angles to the faces of the arch."

So far, the only experimental substantiations of the theoretical methods are the tests made by the late George E. Beggs, M. ASCE, for the Special Committee on Concrete and Reinforced Concrete arches in 1926. In reviewing this matter in the final report of the committee neither the text nor the figures are too explicit regarding the direction of the coordinate axes. The figures, however, indicate that for both the experimental arch, and the theoretical one, the x-axis was parallel to the face of the skew arch whereas, in the past, all writers have chosen the x-axis perpendicular to the face of the abutment for theoretical computations. Since this writer is suggesting a review of the entire problem, perhaps this matter may be clarified.

In addition to these mistaken ideas of the general problem, there has been a tendency toward simplification. In some instances, it appears that the simplification has been over done, for example, neglecting M_u, which is a moment that causes the normal thrust T_v to depart more and more toward the obtuse corner of the arch. That there is a concentration of stress at the obtuse corner of the arch is well known. Such a concentration of stress was the cause of the failure of a skew arch bridge in Bendigo, Australia. Clyde T. Morris,[16] M. ASCE, clearly showed such stress concentration in the test of a model reinforced concrete arch having a span of 4 ft. Perhaps there is no great stress concentration in the obtuse corner of a rigid-frame span because of the flatness of the arch but then, Mr. Barron's paper, by its title, includes arch bridges. A word of caution is advisable.

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[15] Discussion by Ernest H. Harder of "Analysis of the Stresses in the Ring of a Concrete Skew Arch," by J. Charles Rathbun, *Transactions*, ASCE, Vol. LXXXVII, 1924, p. 672.

Writers seem to lay great stress on the computation of the indeterminate reactions, but not enough effort and thought have been given to the equally important problem of what to do with these reactions after they have been computed. Only one writer, Bernard L. Weiner,[17] M. ASCE, has gone into that subject extensively. Mr. Weiner has evolved a long, complicated, and highly technical exposition of the subject which renders it difficult to use for design purposes. A brief exposition of this subject is found in a later paper by Mr. Rathbun.[18] Mr. Rathbun follows the lines that this writer prefers—namely, transposing the reactions, as computed, into components parallel to the skew center line and at right angles thereto. For example, Mr. Rathbun has chosen to resolve T_x and T_y into a component T acting along the center line of the skew span and s, a shear, acting at right angles to the fascia. He stops there, however, and gives a method for distributing only the shear s to transverse reinforcing steel, which is similar to that method given in the paper by Mr. Barron. In the design of sections, Mr. Barron uses an oversimplified method which is also mentioned in the paper by Mr. Hodges. This method is called the "turning process" for reinforcement and involves a serious error. He virtually assumes that moment M_L and thrust T_L are acting on the diagonal section parallel to the abutments. After all, this diagonal plane is a section through the right dummy arch, and all that remains is to increase the computed steel areas by multiplying by sec V on the theory that the right section across the skew is shorter than the diagonal section. What about the last diagonal section at the face of the abutment? How can one justify the "turning process" there?

If one chooses to turn the entire coordinate axis system so that the x-axis falls along the center line of the skew arch, then the new components must act on the rather complex sections mentioned by the writer in his 1924 discussion,[19] and the last of such sections lies across the skew arch axis starting at the obtuse angle (Fig. 15). This step leaves a triangular area a-d-n in which the stresses cannot be found so readily. If the arch is very wide compared to its span length, there may be only one rectangular section at the crown—the rest is a triangular area. The writer has always felt that, for extremes of skew and width, the method proposed by Mr. Rathbun does not hold.

In the design of transverse reinforcement, as given by Mr. Barron and which is similar to that given by Mr. Rathbun in his 1924 paper (and also in that of Mr. Hodges), Mr. Barron, as well as Mr. Hodges, seems to make an error by finding the areas of steel needed for a length of arch of 1 in. On the other hand,

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[16] "Thrust of Skew Barrel Arch Measured on Laboratory Model," by Clyde T. Morris, *Engineering News-Record*, April 20, 1922, p. 638.

[17] "The Design of a Reinforced Concrete Skew Arch," by Bernard L. Weiner, *Transactions*, ASCE, Vol. 96, 1932, p. 1212.

[18] "An Analysis of Multiple-Skew Arches on Elastic Piers," by J. Charles Rathbun, *Transactions*, ASCE, Vol. 98, 1933, p. 12.

[19] Discussion by Ernest H. Harder of "Analysis of the Stresses in the Ring of a Concrete Skew Arch," by J. Charles Rathbun, *ibid.*, Vol. LXXXVII, 1924, p. 664, Fig. 14.

Mr. Rathbun gives his results for a length of 12 in. along the center line of the arch. The demonstrations offered by Mr. Barron in Eqs. 45 to 48 and Fig. 14 are based on a demonstration published in 1929.[9] To conform to that demonstration,[20] the shear values as given in Eq. 47*b* should be substituted for $(v-v')b$ in the usual formula for stirrup spacing.

The writer has no ready solution to offer for his criticism of this paper. What he wrote in 1924 he still affirms today. In reviewing the efforts of others, he believes nothing really new has been injected into the subject.

The only hope for any real advance would seem to lie in the realm of experimentation to ascertain the nature and pattern of stresses in the arch. Whether this could best be done by the electric analogy method or by the brittle lacquer technique is something for others to select. In 1903, the late W. C. Kernot,[21] M. ASCE, professor of engineering, University of Melbourne, Melbourne, Australia (who has been so frequently mentioned in connection with the failure of a skew arch bridge at Bendigo) produced a stress pattern by the electric analogy method. Although it is crude (Fig. 16) it is the first and only stress pattern which the writer has seen.

Mr. Kernot tried another experiment: He endeavored to construct and test an indiarubber model having parallel lines or circles of known diameter drawn upon it, the deformation of which would indicate the direction and magnitude of the stress. This model proved unmanageable so he used a sheet of rubber and suspended loads from it, to obtain elongations instead of compressions. The end results showed great stress at the obtuse corner of the arch. This idea, if carried out in a model of hard material, would suggest the brittle lacquer technique.

Egidio O. Di Genova,[22] M. ASCE.—The correlation of externally applied loads with internal stresses, as proposed in this paper, is quite ingenious. The fundamental concepts of the elastic theory, here applied, prove once more that, when a structural analyst masters the fundamentals of this wonderful art, he is always at ease in undertaking any problem pertinent to the safest and surest way of designing a structure. He acquires a well-founded judgment in deciding what is essential and what may be disregarded when analyzing the importance of certain induced stresses with respect to others.

Mr. Barron shows clearly how many forces can be disregarded, and the writer proposes still further approximations that can be made without impairing the essential safety of a structure. In a work entitled "Elastic Theory Applied to Solids," on file for reference at the Engineering Societies Library, the writer has treated similar subjects in considerable detail.

A reinforced concrete skewed structure can first be considered as a monolith fixed against excessive partial settlements. In a very general sense, the skewed arch can be compared with a square arch as follows: At any section of a square arch the total resulting moment is a function of the horizontal component of reaction X, the vertical component Y, the reaction moment M_R (if fixed), and the externally applied forces ΣP (for instance, vertical loads). The relative partial moments are, respectively, $X\,y$, $Y\left(\frac{L}{2}+x\right)$, M_R, and $\Sigma(P\,a_h)$, in which a_h represents the horizontal distance of each load at the left of the section under consideration.

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[20] "Standards of Design for Concrete," *No. 3Yb*, Bureau of Yards and Docks, U. S. Navy Dept., Washington, D. C., 1929, p. 64.

[21] "The Stresses on Skew Arches," by W. C. Kernot, *Engineering News*, June 11, 1903, p. 529.

[22] Water Supply Engr., Board of Water Supply, New York, N. Y.

Consider the skewed arch that has a square span L. The coordinates of the center of gravity of each section are x, y, and z, whereas in the square arch they were x and y. Consequently the partial moments at the center of gravity of each section of the skewed arch are: $X'\,y$ and $X'\left(\frac{L}{2}\tan\theta+z\right)$, $Y'\,x$ and $Y'\left(\frac{L}{2}\tan\theta+z\right)$, and $\Sigma\,(P\,a_h)+\Sigma\,(P\,a_h\tan\theta)$—$M_R$ being considered as an infinitesimal force at infinity which does not change.

It is obvious that Y' for the same width of bridge must be larger than Y, and so X' and M'_R are larger than X and M_R. It is already evident that a skewed bridge cannot be designed for an arch of span L, because the moments $X'\,y$, $Y'\,x$, and M_R are larger than the corresponding moments in the square arch.

Concentrating on the other partial moments relative to the skewed span—

$$M = X\left(\frac{L}{2}\tan\theta+z\right)\ldots\ldots\ldots\ldots\ldots\ldots(49)$$

—may be interpreted as proportional to M_u as defined in the paper in so far as the former acts horizontally whereas the latter is a component of the former, acting in the plane normal to the section under consideration. In the final analysis this moment acts (as the author states) on the full depth of the arch and consequently is not very important with respect to the unit stresses; the other component at an angle of 90° also generates shear on the full depth of the section and, for the same reason, may be considered unimportant. The parameter $Y'\left(\frac{L}{2}\tan\theta+z\right)$ can be partly compared to the M_v of the paper. Considered as its component in the plane of the section, it would generate shearing stresses on the full length of the section, whereas its component at an angle of 90° (normal to the section) would generate fiber stresses due to a moment for which the arm length of the resultant compression and tension is very nearly the depth of the bridge.

Effects similar to the foregoing result from the partial moments $\Sigma(P\,a_h\tan\theta)$ except that the corresponding stresses generally have opposite values because the former are derived from forces of reaction whereas the latter are derived from externally applied loads. In a skewed bridge, the total reaction has a component acting parallel to the z-axis, and the moment of reaction has a component acting in a plane parallel to the (y,z)-plane. Both these components would contribute to the total values of M_u and M_v as given in the paper and as interpreted by the writer.

Thus the writer would minimize the importance of the extra moments which the paper rightfully classifies as secondary stresses; but these moments do exist and may have an effect in increasing the maximum stresses on a skew span. Therefore, it is safe to add to any system of computations a certain margin of safety by increasing the resistance value of this type of arch. This margin of safety is supplied by increasing the span length as the angle θ of the skewed arch increases, for any value of θ.

For simplicity assume a skewed arch bridge with a constant cross section. If L represents the span perpendicular to the faces of the abutments and computations are applied to a strip of arch 1 ft wide and parallel to the faces of the bridge, this arch will have a skew span $L \sec \theta$, which is longer than the right span.

Referring to Fig. 4, the center line may be taken as the center line of a 1-ft rib containing the centers of gravity of the various finite or infinitesimal elastic elements into which this strip may be subdivided. This rib is referred to a system of coordinates similar to the one shown in the paper, the origin of which is on the z-axis, whereas the x-axis and the y-axis are in the vertical plane containing the centers of gravity of all the elastic elements of the rib.

For an arrangement such as this the values of y corresponding to the intersections of the intrados or extrados, with radiating planes (planes perpendicular to the line of the centers of gravity of the rib and consequently to the (x,y)-plane), are not constant—that is, these lines are curved. Consequently the normal sections that result in subdividing the rib into elastic elements of finite lengths Δs are of irregular shape; and they are not symmetrical with respect to the (x,y)-plane. However, the two elastic parts into which they are subdivided by this plane are equal. Furthermore, the trace of each elastic element on the (x,y)-plane represents a segment of arc similar to a segment of a square span of uniform thickness. In other words, the shape of the rib on the (x,y)-plane is the same as that of the developed intersection of the cylindrical vault with a plane parallel to the plane containing one of the faces of the bridge.

This conception of the rib is actually not in strict conformity with the theories of planar elastic arches; but in the realm of approximations, applying those theories to a rib only 1 ft deep, the theory is an admissible and practical possibility.

To be convinced of the reasonable nature of such an approximation one needs only to remember that the surface containing the central strata of fibers or the neutral strata, with respect to the moment stresses, is always parallel to the intrados and extrados (zones of maximum fiber stresses), and consequently the concepts of fiber stresses and of the work of deformation may be considered unaltered.

A larger degree of approximation accrues in the investigation of the plane of intersection of the vault with the abutments. End elements Δs are extremely irregular; that is, the extreme section of each of these end elastic elements is not radial but skewed. In the computation of the magnitude of the total reactions, the approximation is quite close to the actual values because a thorough analysis of any method generally used for computing the reactions of support of a square arch reveals that the formulas are derived from the concept of integrals or of summations. The influence of one term is not very important in obtaining the value of a force, a moment, or an elastic deformation, especially if mathematical expressions with the same degree of approximation are in the numerator as are in the denominator.

Uncertainties as to stress behavior at the support of this rib can be taken care of in a practical manner by increasing the elastic mass around that section so that it may be considered as a truly fixed support of the vault. In the writer's opinion, the failure to recognize the possible effect of such irregularity

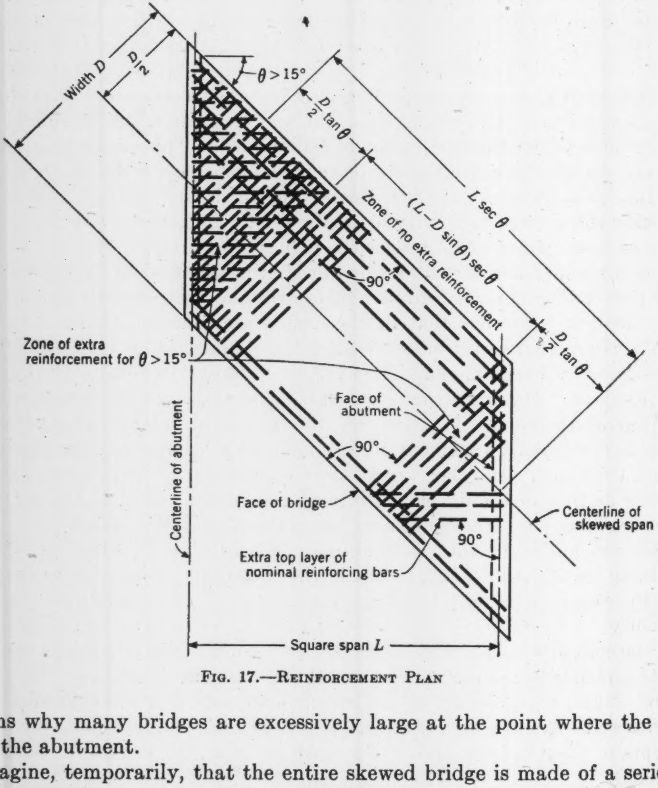
Width D
$\frac{D}{2}$
$\theta > 15°$
$\frac{D}{2} \tan\theta$
$L \sec\theta$
$(L - D \sin\theta) \sec\theta$
Zone of no extra reinforcement
90°
$\frac{D}{2} \tan\theta$
Zone of extra reinforcement for $\theta > 15°$
Face of abutment
90°
Centerline of abutment
Face of bridge
Centerline of skewed span
Extra top layer of nominal reinforcing bars
90°
Square span L

FIG. 17.—REINFORCEMENT PLAN

explains why many bridges are excessively large at the point where the arch meets the abutment.

Imagine, temporarily, that the entire skewed bridge is made of a series of ribs, and that the generatrix of the intrados and the extrados of each rib is a line perpendicular to the face of the bridge during the time that it is generating both the vault and the abutment. Such an assumption would not be objectionable to the stress analyst. The abutments would appear as a saw-toothed surface, as would the vault.

If the intrados is considered to be a cylindrical vault with the generatrix parallel to the z-axis (as in the paper), the analyst must always bear in mind that the maximum positive or negative moment—which happens to occur at a certain point near the region of one face of the bridge—will not be the same at a point of symmetry near the other face of the bridge and on the line normal to the longitudinal axis of the bridge from that point. This is true, even for symmetrical conditions of loading. Consequently, if a skewed bridge is designed on an assumed fictitious equivalent square span equal to $L \sec \theta$, the reinforcement along the entire bridge must be proportioned in accordance with the maximum moments resulting from the several conditions of loading and temperature variations.

It can be proved by comparative analysis that, for a small angle θ, the design based on the equivalent span $L \sec \theta$ is sufficiently satisfactory; but, for larger skews and for relatively large ratios L/D (Fig. 17), the equivalent span $L \sec \theta$ is very practical in that it creates an extra resistance of the vault at the skewed ends by increasing the nominal steel (by the same criteria as those used in determining the so-called nominal reinforcement for ordinary structures). This additional steel is shown in Fig. 17. It strengthens the top of the vault against the cantilever action of the skewed ends.

To summarize: A bridge, such as the one treated in this paper, can be designed as follows:

1. Analyze an equivalent square span of length $L \sec \theta$.
2. Proportion the main reinforcement along the entire vault in accordance with the maximum positive and negative moments, and locate it parallel to the faces of the bridge.
3. If the arch has vertical abutments and if there is no continuity of curvature, enlarge the mass of concrete in the region where the abutment and the arch intersect.
4. Place the nominal steel at an angle of 90° with the faces of the bridge.
5. For a skew angle θ greater than 15°, and for $L \geqq D \sin \theta$, provide an extra layer of nominal steel along the abutments at an angle of 90° with the face of the abutment of the bridge. Fig. 17 shows that, in passing from a square span to a skewed one, the limiting condition is $\theta = 0$, the main reinforcement is parallel to the face of the arch, and the nominal steel is at an angle of 90° to it. Therefore no extra reinforcement is required.

All the other details so masterfully presented in the paper will give the bridge analyst a very complete picture of this interesting subject.

Kuang-Han Chu,[23] Jun. ASCE.—Because it simplifies the analysis of skewed reinforced concrete rigid-frame and arch bridges, this is a highly valuable paper. However, there seems to be a need for some explanation for dropping certain terms. For instance, from Eq. 20, without dropping any term, the following equations are obtained: For the vault of a rigid frame, $\sin \theta = 0$ and $\cos \theta = 1$; therefore—

$$M_v = \epsilon \cos \theta \, M_z + [-(B - \epsilon A) \, u + (D + \epsilon A h)] \ldots\ldots (50)$$

For the legs of a rigid frame, $\cos \theta = 0$, $\sin \phi = 1$, and $u = \frac{L}{2}$; therefore—

$$M_v = \frac{-(B - \epsilon A)L}{2} - C \ldots\ldots\ldots\ldots\ldots\ldots (51)$$

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[23] Structural Designer, Ammann & Whitney, New York, N. Y.

It can be seen that Eq. 21 is obtained from Eq. 50 by dropping the term in the bracket and Eq. 22*a* is obtained from Eq. 51 by dropping the term C. The author states that the term $(B - \epsilon A)$ is almost equal to zero for rigid frames. For the vault the value of u is small, and the term $(B - \epsilon A) u$ may be dropped. Nevertheless, neither the term $(D + \epsilon A h)$ nor the term C is equal to zero; $C = 0$ only when the loads are symmetrical. In dropping these terms, some justification seems to be needed.

Since Mr. Barron touches the problem of design, another question may be worthy of notice. It is the validity of using the torsion formula for homogeneous uncracked sections in computing torsional stresses in reinforced concrete sections. The writer has not been able to find any authoritative theoretical or experimental treatment on this subject.

Harold E. Levenson,[24] **Jun. ASCE.**—The method developed by the author for the analysis and design of rigid-frame and arch bridges may be extended to those which are curved in plan or composed of forms other than a simple rectangle or rhombus. Such structures are suitable solutions to highway intersection problems where the alinement of the upper roadway is curved. A structure, such as shown in Fig. 18(*a*), has a minimum deck area for given span and side clearances. It should be observed that, if a bridge of a parallelogram plan were used, for the same alinement and clearances, the additional darkened area of Fig. 18(*b*) would be required. In addition to its economy, this type of structure produces a pleasing architectural effect with its smoothly curved parapets.

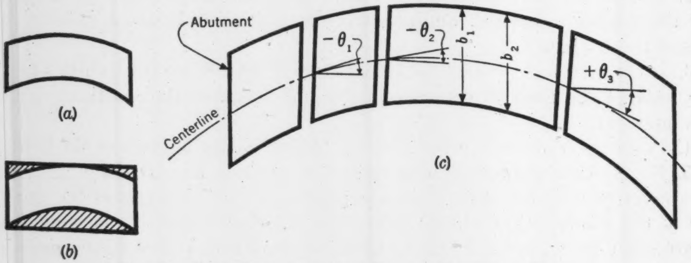

FIG. 18.—RIGID-FRAME OR ARCH BRIDGE, CURVED IN PLAN

For the purpose of analysis, the curved-in-plan structure may be cut with a series of sections parallel to the abutments as shown in Fig. 18(*c*). The normal at each section makes a skew angle θ with the center line. Thus a structure of "variable skew" is obtained. Also it should be noted that the width, b, is a variable when the roadway is of constant width.

Mr. Barron states that he has proved that the independence of the skew and rectangular effects for the cases of unsymmetrical structures include those of variable width. Accordingly, for a structure with a plan as in Fig. 18(*a*), using the author's simplified method of analysis, the equivalent rectangular structure, with the same span and with a width varying according to the width of the actual structure, is first analyzed. If the skew effects are to be determined, these may be determined by integration, either by the calculus or by the arithmetic summation of values for successive sections of length Δs.

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[24] Harold E. Levenson Co., Development Engrs., Brooklyn, N. Y.

The author has demonstrated that, for single-span structures of constant skew, the skew effects need not be considered in design when the skew angle is less than about 50°. For structures of variable skew of the type shown in Fig. 18(*a*), for radii of curvature of the alinement most commonly used in highway applications where the average skew is within the foregoing limit, the skew effects need not be considered in the design. The analysis should be based on the square span; the widths b will vary.

In the design phase of the work, it will be observed that, in addition to the stresses considered for the structure of constant skew, the curvature causes a lateral displacement of the line of action of redundant A from the centroids of the sections thus causing bending about a vertical axis at each section. However, for ordinary bridge sections the effect on the final design stress is less than 0.1%, due to the tremendous section modulus of the bridge for bending in this direction. Accordingly, this effect may ordinarily be disregarded.

The principle of the variable skew bridge may be extended to other cases.[25] In analyzing some of these structures, wedge-shaped segments may be taken, and the effects on the redundants may be evaluated in a way analogous to that used for structures of uniform skew.

The economy, simplicity of design, and architectural beauty of these new types of patented structures, especially the curved viaduct type, merit the attention of those responsible for the selection of highway bridge types, and currently several bridges with curved plans are being built. It is suggested that the testing of a series of curved-in-plan models to verify the theoretical considerations presented herein will prove a useful and fruitful investigation.

WALTER L. SCHWARTZ.[26]—The principal feature of this paper is the complete separation of the torsional effects from the rectangular effects, and the determination of the torsional effects, if required, as a product of a rectangular effect times an elastic or geometric constant.

This paper shows the care that must be exercised by researchers in comparing theoretical values with model tests data. Since the rectangular elastic system is shown to be independent of the torsional elastic system, it is not correct to conclude that the torsion theory is verified when the experimental data on test models agree with the rectangular results of the theoretical computations. Even a torsional analysis that is several hundreds per cent in error may yield correct results for the rectangular reactions and stresses. Test data previously correlated should be reviewed in this light.

From a practical standpoint the proof that a torsional analysis is usually of academic interest is a worthy step forward in the science of bridge building. The writer recalls an analogy from the author's personal correspondence which aptly describes this development:

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[25] "The Rigid-Frame Bridge," by Arthur G. Hayden and Maurice Barron, John Wiley & Sons, Inc., New York, N. Y., 3d Ed., 1950, p. 172.

[26] Brooklyn, N. Y.

> "Suppose, in the first application of the three equations of static equilibrium to the simple truss [a simple two-panel Warren truss] one additional secondary truss equation was written for each of the five joints and all eight equations were then solved simultaneously. The analysis and design would be correct and model tests would prove the validity of the method, but only experts would be able to design this simple structure."

The solution of numerous simultaneous equations for skewed structures involving rectangular and torsional redundants, although much more obscured, represents similar self-imposed complexity.

Leo Sos[27].—Much has been written concerning the inconsistencies in computations when simultaneous equations are solved numerically and small differences are encountered. In comparing the results of the author's method with the published analysis and design of a skewed rigid-frame bridge[28] several inconsistencies were found and small differences were suspected as being the cause. Therefore, the four simultaneous equations as published were recomputed to eight significant places and the results compared with published results which were computed only to four significant places.

TABLE 11.—Simultaneous Equations

No.	Equations	Vertical Unit Load at Point No.:			Earth pressure left	Temperature change
		6 R	8 R	10 R		
(1)	(2)	(3)	(4)	(5)	(6)	(7)
I	$+119.664\ R_x - 30.867\ R_z + 0.1341\ M_x - 0.5163\ M_y$	$= +19.72$	$= +37.57$	$= +48.15$	$= +13{,}039.0$	$= +1{,}215.0$
II	$+11.561\ R_x - 43.735\ R_z + 1.418\ M_x - 0.2477\ M_y$	$= -4.49$	$= -8.61$	$= -11.27$	$= -3{,}298.0$	$= -356.0$
III	$+0.0502\ R_x - 1.418\ R_z + 0.0664\ M_x + 0$	$= -0.219$	$= -0.424$	$= -0.538$	$= -146.9$	$= 0$
IV	$+0.1934\ R_x - 0.2477\ R_z + 0 \quad -0.0090\ M_y$	$= +0.0057$	$= +0.0078$	$= +0.0024$	$= -5.09$	$= 0$

TABLE 12.—Comparison of Final Values

No.	Redundant	Vertical Unit Load at Point No.:			Earth pressure left	Temperature change
		6 R	8 R	10 R		
(1)	(2)	(3)	(4)	(5)	(6)	(7)
	Moment M_y:					
1	Published =	−0.4581	+0.0658	+0.4296	+485.01	−1,477.8
2	Recomputed =	−0.49653720	+0.11116773	+0.39951280	+487.94191722	−1,475.33820977
	Moment M_x:					
3	Published =	−0.2218	−0.9908	−0.7040	+ 76.56	+1,466.0
4	Recomputed =	−0.19077984	−0.99691506	−0.67406015	+ 74.56316051	+1,463.19540918
	Reaction R_z					
5	Published =	+0.1515	+0.2660	+0.3642	+112.14	+ 69.2
6	Recomputed =	+0.15275882	+0.26542941	+0.36551424	+112.04037350	+ 69.22454545
	Reaction R_x					
7	Published =	+0.2021	+0.3838	+0.4988	+139.88	+ 20.0
8	Recomputed =	+0.20226988	+0.38402592	+0.49913904	+139.88560980	+ 20.00454282

Table 11 shows the four equations, and Table 12 compares the final values of the published and recomputed computation. Inconsistencies range almost to 100%.

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[27] New York, N. Y.

[28] "The Rigid-Frame Bridge," by Arthur G. Hayden, John Wiley & Sons, Inc., New York, N. Y., 2d Ed., 1940, p. 154.

Every skewed structure analyzed by numerical solution of simultaneous equations will manifest these inconsistencies unless carried to extreme accuracy. Unless the designer has great experience there is no way of determining beforehand the number of significant digits to carry.

The author's method of analysis eliminates this troublesome problem and can be used with much more confidence. There is less chance for error and a great saving in design time.

Leif Arup.[29]—Since Mr. Rathbun presented his theory of skew arch analysis in 1924,[2] various simplifications of the design method have been suggested. The value of these suggestions, of course, depends on the validity of the original assumptions. In the writer's opinion Mr. Barron's paper is the final word based on Mr. Rathbun's original theory, which is the basis of all current theories on skew arch analysis.

Mr. Barron has set up a timesaving method of analyzing the structures, which has enabled him to arrive at conclusions and simplifications quickly and, in a way, that makes the problems easy to visualize. By setting up such a "pilot design" and by stating in a clear and comprehensive manner which factors can be neglected and which simplifications are justified, the task of future designers of skew bridges should be lightened. The design can be based on the properties of a square span, by using the prescribed adjustments and limitations.

Mr. Barron's pilot design involves a vastly reduced amount of work but yields identical results with those obtained from Mr. Rathbun's theory. The general derivation and extension of the elastic loads to include torsion, and the use of these and the special redundant system shown in the paper were first applied by the writer in 1928 to the design of Bridge No. 1, County Road 68, White Plains, N. Y. A paper containing the methods and a sample design was checked by the late A. Ostenfeld, in Copenhagen, who issued a statement of approval (1928). This paper, however, was never published.

Mr. Barron's conclusions can be accepted if Mr. Rathbun's theory is accepted as valid. The writer, however, concurs with Alfred L. Parmé,[30] Assoc. M. ASCE, and J. Polivka,[31] M. ASCE, in their conclusions that the principles from which Mr. Rathbun derived his basic equations are theoretically incorrect. Two objections which affect this theory, and consequently all work based on it, are as follows:

1. The method is bound to the use of sections not normal to the axis of the member for computing deflections. This is in contradiction to the theory of beams, and causes an error as explained subsequently in Example 1.
2. The theory of beams is applied to the analysis of members, the width of which is not small compared with the span. This is not in accord with the definition of a beam. The consequences of this procedure will be demonstrated subsequently in Example 2.

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[29] Cons. Engr., New York, N. Y.

[30] Discussion by Alfred L. Parmé of "Simplified Analysis of Skewed Reinforced Concrete Frames and Arches," by Richard M. Hodges, *Transactions*, ASCE, Vol. 109, 1944, pp. 954–960.

[31] Discussion by J. Polivka of "Simplified Analysis of Skewed Reinforced Concrete Frames and Arches," by Richard M. Hodges, *ibid.*, pp. 967–968.

Example 1.—Consider a cantilever beam (see Fig. 19), for which the length l is large in comparison with the breadth b. The problem is to find the vertical deflection of the point of application of a load P. By the usual method of analyzing sections normal to the axis of the beam, the result will be

$$\Delta_N = \frac{P\,l^3}{3\,E\,I} \quad \text{(52)}$$

in which $I = \frac{b\,t^3}{12}$. By using the skew sections—that is, taking sections forming an angle θ with the normal to the axis, and by combining the flexural and torsional deformations in these sections (as is done in the paper)—it can be shown that the resultant deflection by this method is

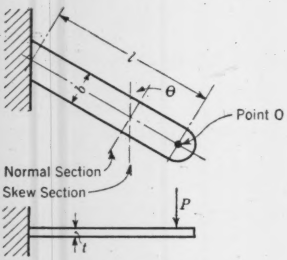

FIG. 19

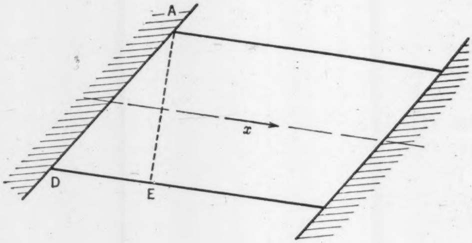

FIG. 20

$$\Delta_s = \frac{P\,l^3}{3\,E\,I} \times K_1 \quad \text{(53)}$$

in which

$$K_1 = \cos^2\theta\,[\cos^2\theta' + \sin^2\theta'\,\tfrac{3}{5}\,(1-\mu)] = f\,(\theta', \mu) \quad \text{(54)}$$

This means that the deflection is itself a function of the angle θ' and of Poisson's ratio, μ. How large is the difference in the results obtained by Eqs. 52 and 53?

Solving Eq. 54 with $\mu = 0.15$, a few K_1-values for selected skew angles are as follows:

θ'	K_1
0°	1.00
20°	0.84
40°	0.51
60°	0.19

In other words, for $\theta' = 60°$, Eq. 53 becomes

$$\Delta_s = \frac{P\,l^3}{3\,E\,I} \times 0.19 \quad \text{(55)}$$

which is about one fifth the value obtained by Eq. 52.

Since redundants in statically indeterminate structures depend on ratios between deflections rather than on the deflections themselves (except volume changes and foundation yield), the possibility might exist that K_1 enters as a constant factor in all deflections and thus cancels out. That this is not the case can be shown by investigating other deflections as, for example, the angular rotation ψ at point O around the z-axis, due to the load P. By using square sections the result will be

$$\psi = \frac{P\,l^2}{2\,E\,I} \cos\theta \qquad (56)$$

and by using the skew sections

$$\psi = \frac{P\,l^2}{2\,E\,I} \cos\theta \times K_2 \qquad (57)$$

in which

$$K_2 = \cos^2\theta \qquad (58)$$

Since $K_1 \neq K_2$ the ratio between deflections differs whether square or skew sections are used in the analysis. That such factors as K_1 and K_2 are not constants can also be seen from the fact that they are functions of θ'. The angle θ' is the angle between the sections and the line normal to the axis. In an arch this angle varies from point to point along the axis and, in a rigid frame, θ' closely equals the skew angle in the vault; it is O° in the abutments. Thus contributions to the deflections due to abutments are independent of Poisson's ratio μ and the skew angle θ; but contributions from the vault are functions of these quantities. Results obtained in this manner must obviously be arbitrary and incorrect.

Example 2.—Mr. Barron reaches the conclusion expressed in Eq. 21 which states that, for all vertical loads, $M_v = \epsilon\, M_z$ ($\cos\phi = 1$ for a flat topped frame). This means that the principal moments are always acting on planes normal to the faces of the bridge. Therefore, no torsional moments would be acting on these planes and consequently no torsional shears. Applying the theory of plates, the basic expression for torsional shears is

$$v_{xy} = -\frac{E}{1+\mu}\,\zeta\,\frac{\partial^2 w}{\partial x\,\partial y} \qquad (59)$$

in which μ is Poisson's ratio; ζ is the distance from the neutral plane; and w is the deflection at any point of the neutral plane, with the rectangular coordinates (x, y).

Choosing the x-axis parallel to the faces of the bridge, the torsional shear v_{xy} should be equal to zero. This is possible only if $\frac{\partial^2 w}{\partial x\,\partial y} = 0$. In the general case the deflection w is a function of both coordinates (x, y). That is, $w = f_1(x, y)$ and

$$\frac{\partial^2 w}{\partial x\;\partial y} = \frac{\partial\left(\frac{\partial w}{\partial x}\right)}{\partial y} \qquad (60)$$

in which $\frac{\partial w}{\partial x}$ is the slope of the deflected neutral plane in the direction of the x-axis, or parallel to the face of the bridge. Now $\frac{\partial\left(\frac{\partial w}{\partial x}\right)}{\partial y}$ can be equal to zero for all points only if $\frac{\partial w}{\partial x} = f_2(x)$—that is, only if $\frac{\partial w}{\partial x}$ is a function of x exclusively, which means that the slope of the deflection surface in the direction of the skew span is constant in a given section normal to the faces of the bridge. The inconsistency of this requirement becomes evident by considering a skew slab fixed at the supports as in Fig. 20. This case corresponds to a flat-topped, skewed, rigid frame, with infinitely stiff legs, for which Eq. 21 is supposed to be exact, as shown originally by Mr. Rathbun.[32]

It may now be seen from Fig. 20 that, at point A, as well as all along the fixed edge A–D, the slope $\frac{\partial w}{\partial x}$ is equal to zero. This makes $\frac{\partial w}{\partial x} = 0$ at all points inside the triangle ADE, which again means that this part of the structure does not deflect as the result of external vertical loading. That this conclusion is incorrect for wide structures seems obvious. It may be approximately true for very narrow slabs, where the width is negligible compared to the span—in other words, when the slabs are not slabs but beams.

From these considerations the writer concludes that Eq. 21, especially for wide structures, is not correct.

Since the elastic theory of beam analysis defines a beam as a member with two dimensions (depth and width) which are small compared with the third (span), it follows that this definition does not hold true for slabs, whether square or skew. It happens, however, that the method of analyzing beams satisfies the differential equation and all boundary conditions for a rectangular slab, supported on two opposite edges, provided that Poisson's ratio is assumed to be zero, which is a minor discrepancy. For skew slabs the boundary condition that the reaction along the free edges should be equal to zero is not satisfied, but gives antisymmetrically distributed reactions for a symmetrical loading. This can be seen by an inspection of the differential equation for the elastic slabs transformed into skew coordinates by Henry Favre.[33] Methods for approximating a solution that satisfies this condition are rather involved and cannot be given here, but the fact that this contradictory reaction is antisymmetrically distributed already indicates that its effect decreases from the free edge toward the center of the slab, and that the magnitudes of the moments near midspan are less affected than their directions. These conculsions are also verified by the many numerical examples solved by V. P. Jensen,[34] M. ASCE.

Since the writer finds that the moment trajectories for vertical loads do not follow lines parallel to the faces of the bridge (as Eq. 21 would indicate), but vary considerably in direction, the reinforcing steel for practical reasons cannot be placed to coincide with the moment trajectories. A method of designing the steel under these circumstances is needed. It has been indicated[34,35] that the practice of multiplying the area of the steel required in the direction of the moment trajectories by the sec^2 of the angle between these and the direction of the steel is not adequate. To explain this statement, consider the reinforcement designed in the direction of the moment trajectories (Fig. 21(*a*)), and for simplicity let this coincide with the direction of the square span. A small element (Fig. 21(*b*)) is cut out and it is assumed that a uniform tension p is acting along the two edges of the element which are normal to the main steel. The tension in each bar will then be $P = p\, d_R = A_{sR}\, f_s$ for which it is designed. Now the main bars placed in the direction of p are substituted by the same number of bars placed on the skew (Fig. 21(*c*)). These will form an angle θ with the direction of the tension p.

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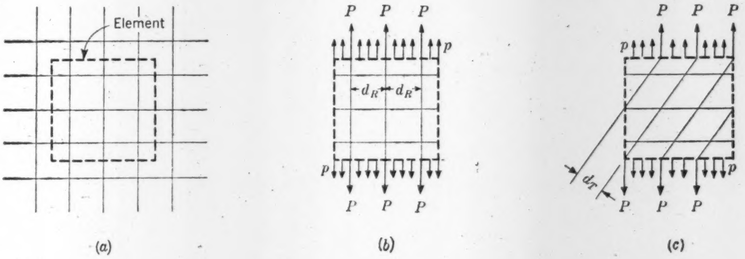
[32] "Analysis of the Stresses in the Ring of a Concrete Skew Arch," by J. Charles Rathbun, *Transactions*, ASCE, Vol. LXXXVII, 1924, p. 624, Eq. 16.

[33] "Contribution à l'étude des plaques obliques," by Henry Favre, *Schweizerische Bauzeitung*, Vol. 120, 1942, pp. 35–60.

[34] "Analyses of Skew Slabs," by V. P. Jensen, *Engineering Experiment Station Bulletin Series No. 332*, Univ. of Illinois, Urbana, Ill., 1941.

[35] "Laboratory Research on Concrete Bridge Floors," by F. E. Richart, *Transactions*, ASCE, Vol. 114 1949, p. 980.

It is now obvious that to produce the same component in the direction of p, the area must be increased sec-θ times and, since the spacing is multiplied by $\cos\theta$, the areas per unit width measured normal to the bars should be multiplied by $\sec^2\theta$. This procedure would be identical to the "turning process" described by Mr. Barron. However, the reinforcement in the element is not stable, and the main bars tend to rotate in counterclockwise direction for the given loading condition. This is prevented by stresses in the concrete, which again introduce additional tension in the steel. The direction of these concrete stresses can be found by determining the direction in which the maximum and minimum unit elongations of a fiber in the plane of the element occur, for an infinitesimal rotation of the main bars. These directions can be shown to coincide with the bisectors $\xi - \xi$ and $\eta - \eta$ of the angles at the intersection of the bars. Assuming that concrete can act in compression only, this force will act in the direction of $\xi - \xi$, Fig. 22. At the edge of the element the forces must be in equilibrium; P, acting as an external load on the element, must balance the steel tension T and the compressive force C in the concrete (see Fig. 23). By projecting on the axis, $\eta - \eta$, C is eliminated, and



(*a*) (*b*) (*c*)

FIG. 21

$$T \cos\left(45^\circ + \frac{\theta}{2}\right) - P \cos\left(45^\circ - \frac{\theta}{2}\right) = 0 \quad \text{.......... (61)}$$

which leads to the formula:

$$\frac{A_{sT}}{d_T} = \frac{A_{sR}}{d_R} \times \sec^2\theta\,(1 + \sin\theta) \quad \text{................ (62)}$$

If the moment trajectories fall between the square and the skew directions and they form an angle γ with the square direction, a similar consideration will give

$$\frac{A_{sT}}{d_T} = \frac{A_{sR}}{d_R} \times \sec^2\theta\,(1 + \sin\theta - \tan\gamma\sec\theta)\cos^2\gamma \ldots\ldots\ldots (63)$$

In Eqs. 61, 62, and 63 $\frac{A_{sT}}{d_T}$ is the required reinforcing steel area per unit width, measured normal to the direction of steel; $\frac{A_{sR}}{d_R}$ is the fictitious reinforcing steel area per unit width computed on the assumption that the direction of steel coincides with the moment trajectories; and γ is the angle between square span direction and moment trajectories. For skew bridges, the safest procedure seems to be to use $\gamma = 0$ (moments assumed to be acting on the square). A more exact analysis based on the theory of plates may permit reducing the required amount of steel when $\gamma > 0$.

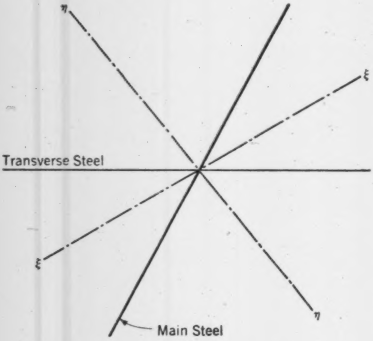


FIG. 22

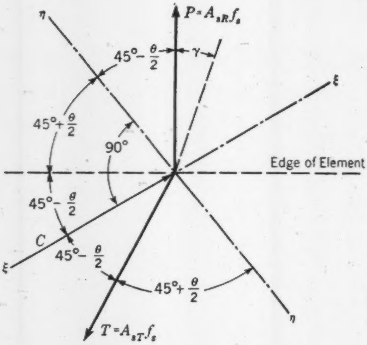


FIG. 23

Summary.—In conclusion, the writer wishes to state that, in his opinion, Mr. Barron has drawn all possible conclusions from Mr. Rathbun's work in an admirable manner. Since the introduction of the theory of plates and shells,[33,34,36] however, Mr. Rathbun's theory has become obsolete. Future investigations on skew arches should be based on modern research on plates and shells.

Acknowledgment.—The mathematical computations that support this discussion were made by Walter M. Zalite.

MAURICE BARRON,[37] M. ASCE.—The number of discussions and the constructive criticism they contain are encouraging to the writer. One type of discussion is lacking, however. The author hoped that engineers and designers would review their old designs based upon previously published methods[2,3,4,8]

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[36] "Die strenge Lösungdes, Problems der schiefen Platte," by P. Lardy, *Schweizerische Bauzeitung*, 1949, p. 207.

[37] Cons. Engr., Farkas & Barron, New York, N. Y.

and compare the results with those obtained by this practical office procedure. It may be assumed that several designers have made these comparisons and, finding them in agreement, have not reported the results. Some discussers have idealized the problem by application to very simple structures. Properly applied, the methods proposed in this paper yield the same results.

In any comparison of an analytical solution with test data (or with another analytical method) careful consideration must be given to the interpretation of results. If no correlation is shown, further tests must be applied to determine which is incorrect. Furthermore, even if exact correlation is shown, conclusions may still be incorrect because both results may be wrong to the same degree.

Mr. Nettleton compares his application (Method A) of the writer's method, and his application of his own method (Method B) to an idealized frame of uniform moment of inertia which consists of vertical legs and a horizontal vault with a uniform load applied. The results obtained by Method A are not in agreement with the results obtained for the same structure by the writer. The discrepancies are due to Mr. Nettleton's misinterpretation of the turning process. This process is not reversible. The statement by Mr. Nettleton—

> "* * * 'Turning' the moments instead of the required areas of reinforcing results in the same requirements for reinforcing, and is done here to facilitate a comparison between the two methods."

—contains a false premise. The reader is advised to think in terms of a homogeneous material, such as aluminum, rather than in terms of reinforced concrete. In the conventional design of reinforced concrete, unit shearing stresses in the concrete cannot be combined vectorially with the unit compressive stresses in the concrete. However, when a reinforcing bar is placed in a plane other than the one in which the bending moments and thrusts are computed, the tension (or compression) in the rod can be resolved into two components as shown in Fig. 8. It should be noted that turning the reinforcing does not imply that the concrete fibers are also turned. Turning the moments as Mr. Nettleton has done usually results in incorrect answers.

For example,[38] referring to a design section for the vault of a single-span rigid frame and using the notation of the paper, let: $M_z = 33.3$ kip-ft; $T_v = 11.2$ kips; $t = 25$ in.; $d = 23$ in.; $a = 10.5$ in.; $\theta = 47°50'$; $\epsilon = 1.1$; $\epsilon^2 = 1.21$; $\sec^2 \theta = (1 + \epsilon^2) = 2.21$; $f_s = 18$ kips per sq in.; $f_c = 1$ kip per sq in.; $n = 10$; and $K_b = 157.3$. Also: $e = \frac{M_z}{T_v} = \frac{33.3 \times 12}{11.2} = 35.6$ in.; $e' = e + a = 35.6 + 10.5 = 46.1$ in.; $M_s = e' T_v = 46.1 \times 11.2 = 516$ in.-kips; and $K = \frac{M_s}{12 \times d^2} = \frac{516}{12 \times 23^2} (\times 1{,}000) = 81$. Since K (= 81) is less than K_b (= 157.3), f_c is less than 1,000 lb per sq in. Consequently, $A_{sR} = \frac{M_s}{f_s j d} - \frac{T_v}{f_s} = \frac{516}{0.9 \times 18 \times 23} - \frac{11.2}{18} = 1.42 - 0.62 = 0.80$ sq in.

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[38] "The Rigid-Frame Bridge," by Arthur G. Hayden and Maurice Barron, John Wiley & Sons, Inc., New York, N. Y., 3d Ed., 1950, point 8, p. 159.

If the steel is turned into its most practical direction (parallel to the face), $A_{sT} = A_{sR}(1 + \epsilon^2) = 0.80 \times 2.21 = 1.76$ sq in. Turning the steel, however, does not change the unit compressive stress or the total compression. If the moments and thrusts are turned by a transformation,[4] K becomes (numerically) $81 \times 2.21 = 179.5$, which is greater than 157.3. Compression steel and additional tension steel is indicated. However, the transformation of moments cannot be justified because a unit shearing stress and a unit compressive stress have, in effect, been added vectorially. This example illustrates one important key to the design problem. All total stresses may be added vectorially but not all unit stresses. Mr. Nettleton's proposal for transverse reinforcement, using Mr. Jensen's formula,[11] does not include the effects of earth pressure or volume changes, which are so important in this type of structure. Furthermore, as far as the writer is aware, no exact analysis for skewed slabs by difference equations has been developed for variable moment of inertia, with edges stiffened by parapet walls, or with unrestricted deck curvature such as encountered in the structures under consideration. Mr. Nettleton contends that:

> "The design of the transverse reinforcement should be controlled by the transverse moments, and not by the torsional moments, because the torsional moments are generally so small that no transverse reinforcing is required."

It should be noted that the torsion moment, M_v, for the vault of a 45° skewed rigid-frame bridge is equal to M_z and for larger skews M_v is even greater than M_z. The reinforcement pattern should be designed to provide for all moments and shears due to all loads. Review of Mr. Nettleton's discussion, however, has revealed one interesting point in regard to the value of M_x for vertically applied uniform loads. Referring to Table 5 under dead load and to Eq. 23*d*: By evaluating the right-hand side of Eq. 23*d*, it can be shown that (at least, for rigid-frame bridges) D is equal to ϵ times the average of the two fixed end moments as usually determined. This statement is also valid for variable I and for any concentrated vertical load. Therefore, the evaluation of the foundation restraint M_x becomes very simple if tables[39,40,41] for fixed end moments are used.

From Mr. Harder's discussion it appears at times that he is in complete agreement with the proposed method and at other times diametrically opposed to it. Comparison of the results of his design with the results of this method reveals that both are in agreement. The easiest way to test the proposed method is to make these comparisons. To the writer's knowledge, no skewed structure was ever analyzed or designed using sections at right angles to the face. Hundreds of skew structures, designed and investigated by the writer, clearly show the necessity for a practical design procedure. No structure, carefully designed by qualified engineers using accepted published methods, has been endangered. The elastic theory, as applied to these structures, has been proved satisfactory by these many successful bridges, adequately performing

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[39] "Handbook of Frame Constants," Portland Cement Association, Chicago, Ill.

[40] "Beam Factors and Moment Coefficients for Unsymmetrical Members with Parabolic Haunches," Portland Cement Association, Chicago, Ill.

[41] "Continuous Concrete Bridges," Portland Cement Association, Chicago, Ill., 2d Ed., 1941.

their intended function. Failures and structures manifesting distress are clearly the result of inadequate design, detailing, or construction procedure. It is imperative that the structural action on which the design is predicated be realized. The foundations must fulfil their intended function, and the details must insure the structural action intended.

Mr. Harder, criticizing the design of the transverse reinforcement, states that an error is made in this paper, as well as in Mr. Hodges' paper.[4] This statement ignores the fact that all loadings and stresses are found for a 12-in. width of structure. Thus, the moments are given in kip-feet per foot of width and the shears in kips per foot of width. The resulting steel areas are for 12 in. and not 1 in. as Mr. Harder contends. This result is evident if the contention is checked against a numerical example. Mr. Harder's solution results in concrete sections containing as much as 50% steel reinforcing—an absurdity. The skew width of structure, *b*, is a linear parameter in each equation and in each step of the derivations. This fact is used to advantage, permitting the structure to be designed for a width of 1 ft. It also permits total effects to be determined by multiplying the values for a unit width by *b*, the skew width of the structure.

Mr. Harder requests clarification on several points. Concerning the relative importance of M_u, he is confusing M_u with M_y and M_x, which are always included in the method proposed in this paper. It should be noted that the moments M_x and M_y cause the major shift of R_y and R_x. Neglecting M_u in the analysis part is less significant than the omission of the deflection due to shear, and in the design part it has been shown several times that M_u causes insignificant stresses. Mr. Harder states that:

> "* * * Mr. Rathbun follows the lines that this writer [Mr. Harder] prefers—namely, transposing the reactions, as computed, into components parallel to the skew center line and at right angles thereto. * * * In the design of sections, Mr. Barron uses an oversimplified method which * * * virtually assumes that moment M_L and thrust T_L are acting on the diagonal section parallel to the abutments. * * * What about the last diagonal section at the face of the abutment? How can one justify the 'turning process' there?"

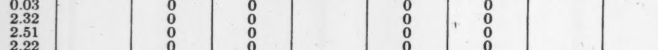
The method Mr. Harder favors and the proposed method yield the same results; but the writer determines the end product mathematically whereas Mr. Harder substitutes numerical values halfway through the operation of finding design stresses. Premature substitution of numerical values results in loss of clarity, a loss in accuracy, and a great loss in time.

Mr. Di Genova presents an approximate solution based on using the skewed span. In seeking simplification over a period of years, a solution similar to that of Mr. Di Genova was explored along with several other simplifications and approximate methods. A right-arch analysis using the skewed span was abandoned many years ago because it was found that this approximation was incorrect for stresses due to earth pressure, volume change, and live loads. It is fairly accurate for symmetrical vertical loads on symmetrical, single-span, rigid-frame bridges. For small skews the error is not serious. For larger skews the error is on the dangerous side when the right-arch analysis is applied to the vault of rigid frames. For the legs of rigid frames the error is on the safe side but results in wastefulness. An interesting point is raised by Mr. Di Genova in his discussion concerning reinforcing patterns. The pattern shown in Fig. 17 requires the cutting and placing of many short individual bars. Furthermore, when the vault is arched and the thickness varies, the difficulty in tying the intrados and extrados reinforcement together presents a problem because there is no uniformity in the depth of stirrup. This condition results from placing transverse steel perpendicular to the face of the bridge.

Several designers, including Mr. Chu, have found difficulty in properly expanding Eq. 20 and combining the terms. It should be carefully noted that each type of loading (vertical loads, horizontal loads, volume change) requires a different grouping for the vault and for the legs in order to separate primary from secondary effects.

Mr. Chu requests further explanation and justification for dropping out terms such as the terms within the bracket of Eq. 20. In order to show that $(B - \epsilon A)u$ may be neglected, Eq. 23*b* is used. Of course, for an idealized frame with flat vault, Eq. 23*b* reduces to zero because Δ_b is exactly zero (for

TABLE 13.—RECTANGULAR AND TORSIONAL ELASTIC WEIGHTS AND MOMENTS FOR UNIT VERTICAL INFLUENCE LOAD[a] FOR A SINGLE-SPAN SKEWED BRIDGE

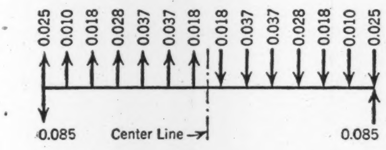


	(*a*) DEFORMATION Δ_a (RECTANGULAR) Elastic Weight $= y/I$			(*b*) DEFORMATION Δ_a (TORSIONAL) Elastic Weight $= \epsilon r x \sin \phi \cos \phi / I$			(*c*) DEFORMATION Δ_b Elastic Weight $= \epsilon r u \cos \phi / I$		
Point (1)	Elastic weight (2)	Shear (3)	Elastic moment (4)	Elastic weight (5)	Shear (6)	Elastic moment (7)	Elastic weight (8)	Shear (9)	Elastic moment (10)
1	0.03		0	0		0	0		
2	2.32		0	0		0	0		
3	2.51		0	0		0	0		
4	2.22		0	0		0	0		
5	2.63		0	0.64		0	+0.57		
		215.2			1.54			0.57	
6	4.05		215.2	0.21		1.54	+0.09		0.57
		211.1			1.33			0.66	
7	9.10		426.3	0.32		2.87	+0.12		1.23
		202.0			1.01			0.78	
8	19.20		628.4	0.39		3.88	+0.10		2.01
		182.8			0.62			0.88	
9	35.80		811.2	0.37		4.50	−0.04		2.89
		147.0			0.25			0.84	
10	60.30		958.2	0.22		4.75	−0.30		3.73
		86.7			0.03			0.54	
11	86.70		1,044.9	0.03		4.78	−0.54		4.27
$\frac{1}{2}\Sigma$	224.90			2.18			0		

[a] Constants (ϵ, r, Δ_s, Δ_x, etc.) which are common factors are not included in the numerical values.

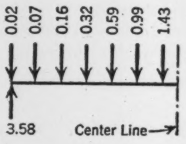
TABLE 13.—(CONTINUED[a])

(*d*) DEFORMATION Δ_c



Elastic Weight $= \epsilon\, r \sin\phi \cos\phi / I$

(*e*) DEFORMATION Δ_d



Elastic Weight $= \epsilon\, r \cos^2\phi / I$

Point (11)	Elastic weight (12)	Arm (13)	Static moment at center line (14)	Shear (15)	Elastic moment (16)	Elastic weight (17)	Shear (18)	Elastic moment (19)
1	0		1.10/13 = 0.085		0	0		0
2	0				0	0		0
3	0				0	0		0
4	0				0	0		0
5L 5R	−0.025 +0.025	13	0.32		0	0.02		0
				−0.060			3.56	
6L 6R	−0.010 +0.010	11	0.11		−0.060	0.07		3.56
				−0.050			3.49	
7L 7R	−0.018 +0.018	9	0.16		−0.110	0.16		7.05
				−0.032			3.33	
8L 8R	−0.028 +0.028	7	0.20		−0.142	0.32		10.38
				−0.004			3.01	
9L 9R	−0.037 +0.037	5	0.18		−0.146	0.59		13.39
				+0.033			2.42	
10L 10R	−0.037 +0.037	3	0.11		−0.113	0.99		15.81
				+0.070			1.43	
11L 11R	−0.018 +0.018	1	0.02		−0.043	1.43		17.24
				+0.088				
Σ			1.10			3.58		

vertical loads). Where the vault has a rise,[6] such as the usual rigid-frame bridge and in arches, Δ_b is not zero and u for the vault is not zero. Using torsional elastic weights it has been shown (unabridged paper filed for reference in Engineering Societies Library) that, even for the full center arch, Δ_b is relatively small, and the resulting correction ($= B - \epsilon A$) in the cross shear may be neglected. Examination of the elastic loads comparing y/I with $\epsilon\, r\, u \cos\phi/I$ and the elastic moments (see Table 13), indicates the relative value of ϵA and $(B - \epsilon A)$.

For symmetrical vertical loads on symmetrical structures (skewed rigid frames and arches), C is equal to zero, because Δ_c is equal to zero. Dissymmetry of load and usually dissymmetry of structure can be evaluated as secondary effects using the method shown in Table 13.

The following demonstration is not an exhaustive proof (except for the idealized frame), but it can be used to show that $(D + \epsilon\, A\, h) \cos\phi$ is never more than of secondary order. For an idealized frame, this term is zero for the legs ($\cos\phi = 0$) and equals $(D + \epsilon\, A\, h)$ for the vault ($\cos\phi = 1$). Using Eqs. 23*a* and 23*d*:

$$(D + \epsilon\, A\, h) = -\,\epsilon\, Q_3\, A - Q_7\, \Delta_d + \epsilon\, A\, h \tag{64}$$

and for an idealized frame the following expressions result from substitution of values in Table 2: Eq. 14*c* becomes—

$$Q_3 = 0 \tag{65a}$$

Eq. 14*g* becomes—

$$Q_7 = \frac{1}{\delta_{dd}} = \frac{1}{r \sum \frac{\cos^2\phi}{I}} = \frac{1}{r \sum_0^L \frac{1}{I}} \tag{65b}$$

Eq. 9 becomes—

$$h = \frac{\sum u_0 \frac{\cos\phi}{I}}{\sum \frac{\cos^2\phi}{I}} = y \tag{65c}$$

Eqs. 6*a* and 23*a* yield—

$$A = -\,Q_4\, \Delta_{aR} = \frac{\sum \frac{M_{0z}\, y}{I}}{\sum \frac{y^2}{I}} \tag{65d}$$

and Eqs. 6*b*, 8, and 19 yield—

$$\Delta_d = r \sum \left(\epsilon \cos\phi\, M_{0z} \frac{\cos\phi}{I} \right) \tag{65e}$$

Substituting Eqs. 65 into Eq. 64:

$$(D + \epsilon\, A\, h) = 0 - \frac{\epsilon\, r \sum M_{0z} \frac{\cos^2\phi}{I}}{r \sum_0^L \frac{1}{I}} + \epsilon \frac{\sum \frac{M_{0z}\, h}{I}}{\sum \frac{y^2}{I}} h = -\frac{\epsilon \sum \frac{M_{0z}}{I}}{\sum_0^L \frac{1}{I}}$$

$$+\,\epsilon \frac{\sum \frac{M_{0z}\, h^2}{I}}{\sum \frac{y^2}{I}} = \epsilon \left[\frac{\sum \frac{M_{0z}}{I}}{\sum \frac{y^2}{h^2\, I}} - \frac{\sum \frac{M_{0z}}{I}}{\sum_0^L \frac{1}{I}} \right] \tag{66}$$

The numerators of the two fractions in Eq. 66 are identical. For an idealized frame the denominators are almost identical. For several frames and for two full-centered arches, the computations show that $(D + \epsilon\, A\, h)$ is of secondary order.

Therefore, since the three terms in the brackets of Eq. 20 are of secondary order, their combination is of even less order. This has been confirmed by actual application to several structures.

Table 13 shows the conjugate structure (for a single-span, skewed bridge) loaded with elastic weights which are used to determine the redundant reactions resulting from a unit vertical influence load. Table 13(*a*) shows how to find the moments due to the rectangular elastic weights; and Table 13(*b*) represents the same conjugate structure loaded with torsional elastic weights which are used to evaluate ($B - \epsilon A$). By the law of superposition, these effects may be added together for the exact analysis. Comparison of the moments, however, shows that the additional effects due to torsion are negligible. It should be further observed that the torsional effects in Eq. 12 are compensating to a certain extent because their influence is reflected on both sides of the equation and to the same degree. Table 13(*c*) shows how to determine Δ_c for unit vertical influence load. Eq. 11*b* (or Eq. 16*c*) yields the redundant C from Δ_c and an elastic constant Q_6 (Eq. 14*f*). Table 13(*d*) shows the same procedure for Δ_d. A critical examination of each term supplies the best way to evaluate the slight errors introduced by dropping terms. Emphasis is placed on the fact that Δ_{aT}, Δ_b, Δ_c, and Δ_d, for each loading, may be used individually to evaluate the terms considered as secondary effects. Also, each major effect may be written as an elastic or geometric constant times a rectangular stress (conclusion 6).

Mr. Chu would have clarified this problem more had he made a numerical comparison of the proposed methods with actual designs in his office. The torsion formula as used in this paper and in many previous papers[2,4,42] has been justified by numerous tests.[43,44,16] Until additional research justifies a more refined formula, the writer recommends the torsion formula used (see Fig. 13). This recommendation is justified by observations at the University of Toronto[5] (Toronto, Ont., Canada), and the recorded experience of Walter C. Boyer[45] and Gordon B. Fisher,[42] Juniors, ASCE. Several other references are available.[46] Many universities have studied and tested skewed structures, and the results may be found in theses filed at such university libraries as The Johns Hopkins University[42,45] (Baltimore, Md.), Rensselaer Polytechnic Institute (Troy, N. Y.), Massachusetts Institute of Technology (Cambridge, Mass.), and the University of Utah (Salt Lake City, Utah).

Mr. Levinson's contribution broaches several very useful and fascinating subjects. The proposed analysis shows that the rectangular system is independent of the torsional elastic system and that each equation in the analysis may be written with a function of the skew angle as a parameter. It then becomes obvious that one rectangular structure may be considered as the parent of an entire family of bridges, each bridge having a different skew. A typical cross section becomes a generating element. Furthermore, if properly manipulated, this typical element will generate a curved-in-plan structure, the elements of which have family characteristics, easily analyzed and designed.

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[42] "An Experimental Study of the Reactions of a Two-Span Skewed Rigid Frame Bridge," by Gordon B. Fisher, thesis presented to The Johns Hopkins University, at Baltimore, Md., in partial fulfilment of the requirements for a graduate school degree in 1948.

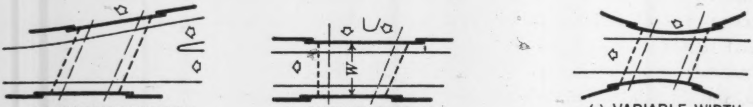
[43] "Skew Arch Reactions Measured by Reciprocal Method," by George E. Beggs, *Engineering News-Record*, July 21, 1927, pp. 106–107.

[44] "The Use of Models in the Solution of Indeterminate Structures," by George E. Beggs, *Journal of the Franklin Institute*, March, 1927, p. 375.

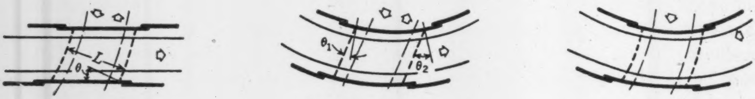
[45] "Effect of Skew Angle on Rigid-Frame Reactions," by Walter C. Boyer, *Proceedings-Separate No. 32*, ASCE, September, 1950.

[46] "Practical Design of Solid-Barrel Reinforced Concrete Skew Structures," by Bernard L. Weiner, *Proceedings-Separate No. 39*, ASCE, October, 1950.

The horizontal and vertical geometry required for these new structures is much simpler than that required for the fitting of a parallelogram structure to a curved roadway. When the upper roadway is curved, the faces of the bridge may be concentric, resulting in a uniform width of upper sidewalk. When the lower roadway is curved, the horizontal clearance is made constant by providing abutments or legs concentric with the curvature. Figs. 24 and 25 show application of these principles and other possible combinations. In a manner similar to that shown by Mr. Levinson, Fig. 25(e) shows how the skew angle varies from θ_1 at the left abutment to θ_2 at the right abutment. It is readily seen that each element has a different skew angle.

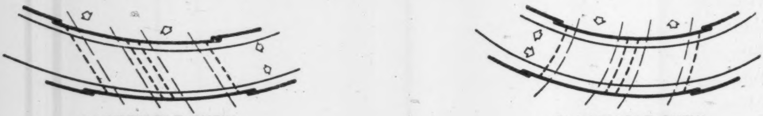


(a) VARIABLE WIDTH (b) VARIABLE SPAN (c) VARIABLE WIDTH AND CURVED FACES

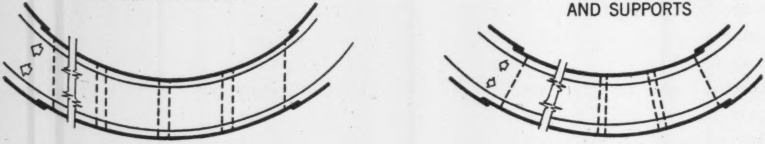


(d) VARIABLE SKEW AND CURVED SUPPORTS (e) VARIABLE SKEW AND CURVED FACES (f) VARIABLE SKEW WITH CURVED FACES AND SUPPORTS

FIG. 24.—REINFORCED CONCRETE, SINGLE-SPAN, BARREL ARCHES AND RIGID-FRAME BRIDGES



(a) VARIABLE SKEW AND CURVED FACES (b) VARIABLE SKEW WITH CURVED FACES AND SUPPORTS



(c) VARIABLE SKEW VIADUCT WITH PARALLEL SUPPORTS (d) VARIABLE SKEW VIADUCT WITH RADIAL SUPPORTS

FIG. 25.—REINFORCED CONCRETE, MULTIPLE-SPAN BARREL ARCHES AND RIGID-FRAME BRIDGES

The curved viaduct, consisting of a series of solid barrel rigid frames, may extend for an unlimited distance as a sequence of curved-in-plan and straight multispan rigid frames. This will result in an economical and aesthetic structure using a minimum of vertical depth. As much as 18 in. may be saved in the height of construction from the lower roadway surface to the upper roadway surface. The approach grades are much easier, and no separate wearing course or fill is required for these structures. Proper superelevation may be incorporated in the structural vault. In Fig. 18(*b*) Mr. Levinson shows the deck area that can be saved by using curved-in-plan construction. This saving amounts to as much as 30% on some actual designs for currently proposed projects. Properly designed forms, although costing a little more in dollars per square foot, also result in a saving because of the reduction in the total form area. These variable skew structures develop some peculiar stresses, especially in the foundations. The theory has been extended to evaluate all the effects of variable skew, and a few tests indicated that the theoretical extension is satisfactory. It would be a great advantage if additional model tests were made using curved-in-plan shapes to supplement the extension of the theory in the paper to these curved-in-plan structures.

Mr. Schwartz's discussion is short but to the point. The solution of simultaneous equations is tedious work, and the results are usually unsatisfactory. There are numerous examples to substantiate this statement, including the discussion by Mr. Sos, who has taken a design[3] that has been checked and rechecked numerous times and has been the subject of much discussion. The inconsistencies mentioned in Mr. Sos' discussion are not unusual. An examination of the equations and the step-by-step solution shows how these errors enter the analysis in such an unobtrusive manner that usually they cannot be detected.

Mr. Arup's discussion is gratifying to the writer in many ways, although it questions the validity of the basic assumptions. It is true that the paper is based on the elastic theory starting with the same assumptions as Mr. Rathbun's original theory. The questions raised by Mr. Arup as to the validity of these assumptions have been raised before[30,31] and the writer can present no better arguments than those written by Messrs. Hodges[46] and Rathbun.[47]

Concerning Fig. 19, it should be observed that for large skews (say 60°) the span at right angles is very short, the area in contact with the support is very wide, and the ratio of width to span is not comparable for beams and bridges. For these conditions it is questionable whether Eq. 52 applies.

To the writer's knowledge, the theory of shells and plates has not been sufficiently developed to apply to skewed rigid frames and arch bridges. Several solutions to the skew plate problem have been published[11,49,33] and suggestions[50,51] have been made to extend the methods to skewed rigid frames and to skewed arch bridges. The solution by finite differences[51] involves the numerical solution of several simultaneous equations. As so ably demonstrated by Mr. Sos, the numerical solution of simultaneous equations for skewed rigid frames often gives erroneous answers. The boundary conditions in skewed rigid frames and arch bridges are much more complicated than in the idealized structures to which these other solutions have been successfully applied. Until these other theories have been perfected and substantiated by test data, the application of the proposed method, which is based on the elastic theory and which has been substantiated by comparisons with numerous test data, is recommended. Several engineers have suggested[4,14,49,53,46] that an adequate design procedure might be developed by various lines of attack. All these were explored at great length and all were found inadequate; none yielded a practical design and yet permitted the evaluation of dropped terms; none showed how the limits of approximations could be determined; and none placed a limit on the combined unit shearing stress in the concrete. Exception must be taken to the part of Mr. Arup's concluding statement which maintains that:

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[47] "Simplified Analysis of Skewed Reinforced Concrete Frames and Arches," by Richard M. Hodges, *Transactions*, ASCE, Vol. 109, 1944, p. 969.

[48] *Ibid.*, p. 960.

[49] "Structural Skew Plates," by F. L. Ehasz, *Transactions*, ASCE, Vol. 111, 1946, p. 11.

[50] Discussion by Julius Sigman of "Structural Skew Plates," by F. L. Ehasz, *Transactions*, ASCE, Vol. 111, 1946, p. 1042.

[51] "Structural Skew Plates," by F. L. Ehasz, *Ibid.*, p. 1042.

> "Since the introduction of the theory of plates and shells,[33,34,36] however, Mr. Rathbun's theory has become obsolete."

A thorough canvass has not yielded a single skew rigid-frame or arch bridge designed by the theory used for plates and shells.

Exception must also be taken to Mr. Arup's two objections and the two examples he uses to justify them. The presentation appears to be a logical demonstration, but a close examination of the initial assumptions indicates the fallacy. If the width of structure in Fig. 19 and Fig. 20 is increased tenfold it becomes evident that Mr. Arup's formulas do not apply. Fig. 21(*c*) repeats an error frequently made in that only the force P is shown, omitting the component at right angles. If Fig. 21(*c*) is redrawn with both forces shown it will be evident that the main bars do not tend to rotate counter clockwise as Mr. Arup contends. If the element shown in Fig. 21(*b*) is subjected to a cross shear, the argument Mr. Arup presents for Fig. 21(*c*) should be applied to Fig. 21(*b*). In addition the effect of the cross steel must also be considered.

The successful application of this theory to hundreds of actual structures is final proof that the elastic theory is valid and may be applied as a practical office procedure for the design of reinforced concrete skewed rigid-frame and arch bridges.

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[52] "Design of a Reinforced Concrete Skew Arch," by Bernard L. Weiner, *Transactions*, ASCE, Vol. 96, 1932, p. 1286.

[53] "Solution of Difficult Structural Problems by Finite Differences," by Alfred Parmé, *Journal of the American Concrete Inst.*, November, 1950, p. 249.

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Paper No. 2456

PLASTICITY OF METALS—MATHEMATICAL THEORY AND STRUCTURAL APPLICATIONS

By D. C. Drucker,[1] Assoc. M. ASCE

With Discussion by R. W. Steed, Jacob Karol, S. K. Ghaswala, and D. C. Drucker.

Synopsis

A brief analysis is given of the philosophy and practice of engineering design. The importance of plasticity in most elastic design is emphasized and the use of elementary plastic analysis is illustrated. Applications to beam and truss frameworks (limit design and "shakedown") and structural stability are discussed. The great difference between plasticity and nonlinear elasticity is shown by simple examples. A short description is given of the development of mathematical theories of plasticity and the experimental results on which they are based. The necessity for additional correlated experimental and analytical investigation is made apparent, and the author's opinion of the initial direction this work should take is stated.

With the exception of the section which demonstrates that some members may unload when all loads applied to a structure increase in ratio, the paper presents a specialized survey and interpretation of existing information only.

Elastic Design and Plasticity

The usual civil engineering design apparently assumes completely elastic behavior. Working stresses are well below the elastic limit; internal stresses and accompanying deformations are computed from formulas such as

$$\sigma = \frac{M\,c}{I} \qquad (1a)$$

and

$$\Delta = \frac{P\,L}{A\,E} \qquad (1b)$$

—which are based on Hooke's law.

Note.—Published in August, 1950, as *Proceedings-Separate No. 27*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Associate Prof., Eng., Brown Univ., Providence, R. I.

Elastic behavior seems to be postulated for main members and also for the welded or riveted connections that join them. Actually, however, as is well known, the usual specifications are based on the explicit or implicit requirement of high ductility. For example, the assumption of uniform distribution of load among rivets in a tension group is obviously in error in the elastic range where the end rivets take far more stress than their share. Stress concentrations produced by abrupt changes in section as from flange to web (shear) or by cover plates, rivet holes, cutouts of various kinds, welded connections for fixed-end beams, stiffeners, or bearing plates—in fact, just about all the real details of the distribution of stress—are ignored. Ductility or plastic deformation has the effect of smoothing out all these irregularities for static loading so that they can safely be forgotten. However, this elimination of the high spots does not happen until appreciable plastic deformation takes place; marked stress concentrations are usually present at working loads. Mechanical engineers especially have been forced to allow for them in design where there is danger of fatigue failure.

In some cases, plastic action is used directly to produce desired stresses or to keep stresses within the elastic range. Boiler tubes are expanded plastically by interior rolls to produce positive pressure between them and the plates through which they pass; cold rolled metal is used instead of hot rolled metal; shot peening is employed to induce residual surface compression and thus to reduce the danger of fatigue; and springs are pre-set.

Thus, closer examination shows that there is often little validity in the concept of purely elastic design. Plasticity is almost always important. This statement does not mean that elasticity is useless except where fatigue is important. On the contrary, practically all experience on actual structures and experimental data on structures, models, and components have been correlated with nominal (or elastic) stress calculations. Specifications are based on this accumulation of knowledge so that conventional designs for conventional structures must of necessity prove to be satisfactory.

For this very reason, if designs must be pushed to the limit to save material, or if the structure contains novel features, the usual specification may not provide an adequate guide. The resulting factor of safety may be excessive or too small. An experienced and extremely capable engineer may be able to extrapolate his knowledge correctly, but it is clear that a more formal and precise approach is desirable. The first question to be decided is what the structure is to be designed against; certainly not against elastic breakdown at any point, for the result would be a very uneconomical design which would not make proper use of the known value of ductility. Clearly there should be an adequate factor of safety against complete failure; but of much more importance when structural steel or aluminum is used is the factor of safety against the excessive deformation which generally occurs long before failure, sometimes of course in the elastic range as in ceiling beams. A simple tension member will certainly deform unduly in most applications when its yield strength is exceeded, but a beam or shaft usually does not deform too much when the extreme fiber yields. Similarly, the yielding of one member of a statically indeterminate truss or of one cross section of a statically indeterminate beam does not necessarily cause excessive deflection of the structure.

SIMPLE PLASTIC DESIGN

A number of interesting and technically important conclusions can be drawn from the consideration of a material termed ideally plastic, one which does not work harden but which yields and continues to flow under constant stress. Structural steel seems to be in this category at the beginning of the plastic range, but this phenomenon is illusory. Initial yielding is inhomogeneous; no part of the material has a strain given by a point in the practically flat part, C, of the stress-strain diagram (Fig. 1). Such a point denotes only the effect of averaging over the gage length some material which is elastic (point A) and some material which has slipped appreciably (point B). However, regardless of the physical validity of such a theory of ideal plasticity for a particular metal, it is simple and often indicates the answer for work-hardening metals.

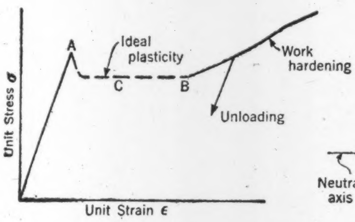
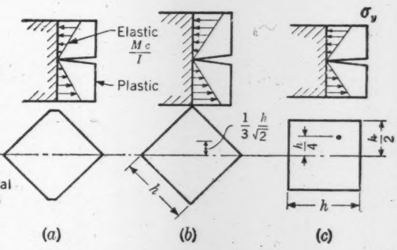


FIG. 1.—STRESS-STRAIN DIAGRAM FOR STRUCTURAL STEEL



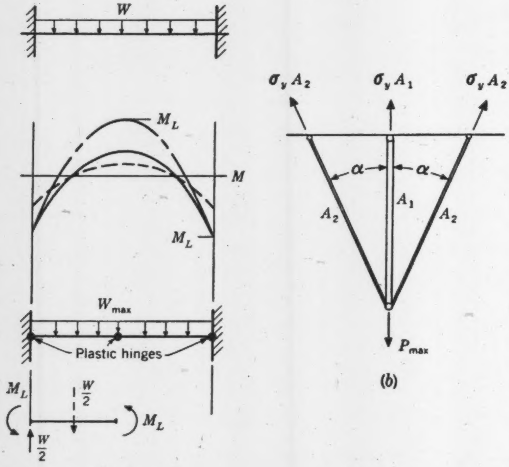
(a) (b) (c)

FIG. 2.—COMPARISON OF ELASTIC AND PLASTIC PROPERTIES
Elastic: (a) Better Than (b); for Given Moment, Maximum Stress for (b) Is $\sqrt{2}$ Times That for (c). Ideally Plastic: (b) Better Than (a); Moment Capacity of (c) Only 6% More Than (b)

Some differences between elastic and plastic design show up strongly in the pure bending of a beam of square cross section and of a beam of modified square cross section (Fig. 2). The maximum stress, σ, produced by a pure moment M in an elastic beam is $M\,c/I$ in which c is the distance from the neutral axis to the extreme fiber and I is the moment of inertia about the neutral axis. The triangular tip of Fig. 2(*b*) contributes more to c than to I so that, for a given moment, the maximum stress in section (*b*) is higher than that in section (*a*). A much greater difference is found between sections (*b*) and (*c*) because I is the same for both but the values of c are in the ratio $\sqrt{2}$ to 1. In the (ideal) plastic range the maximum resisting moment for symmetric sections is twice the product of the yield stress, σ_y, the area between the neutral axis and one extreme fiber, and the distance of the centroid of the area from the neutral axis. The ratio for Fig. 2(*b*) to Fig. 2(*c*) is just the ratio of the centroidal distances or $3\sqrt{2/4} = 1.06$—a much smaller difference than in the elastic case. Now section (*b*) appears preferable to section (*a*).

LIMIT DESIGN

The advantage of considering the plastic region lies in the economy of design that becomes possible. It has long been advocated for both simple cases and for statically indeterminate structures by J. A. Van den Broek, M. ASCE, under the designation, limit design.[2] Coupled with the assumption of ideal plasticity, the design is not only more efficient but the procedure is considerably simplified. The fixed-end beam under uniform load, Fig. 3(*a*), is a good example. Successive moment diagrams are shown for increasing values of load. In the elastic range and for an I-beam, almost until the end moments reach their limiting value M_L, the points of inflection remain the same and the end moment is numerically twice the center moment. Further increase in load after M_L is reached cannot increase the end moments because of the assumption of ideal plasticity. The center moment continues to rise until it too reaches M_L if the beam is of constant cross section. No additional load can then be supported. There is obviously no need to follow the process to calculate $W_{\max}$. The diagram showing three plastic hinges, points at which $M = M_L$ or $-M_L$, enables direct computation.



(*a*) (*b*)

FIG. 3.—LIMIT DESIGN—IDEAL PLASTICITY ASSUMED

Similarly, in the case of the three tension bars, Fig. 3(*b*), there is no need to solve the statically indeterminate problem. The limit load is directly

$$P_{\max} = \sigma_y A_1 + 2\,\sigma_y A_2 \cos\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which σ_y is the flow stress. In the first case (and usually in the second), the limit load will be considerably larger than the load that causes plastic deformation to start.

As must be expected, the reduction of complicated problems to static determinacy involves a sacrifice. The deformation of the structure does not have to be considered explicity. Conversely, the deformation is not calculated and there is always the most vital question of whether or not it is excessive and therefore the unsettled doubt exists as to the validity and the applicability of the assumption of ideal plasticity.

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[2] "Theory of Limit Design," by J. A. Van den Broek, John Wiley & Sons, Inc., New York, N. Y., 1948.

Alexander Hrennikoff, Assoc. M. ASCE, and others[3,4] have studied this question by taking the actual stress-strain curve into account. Once this is done, and the details of plastic action are examined more closely, other serious limitations appear. No mention has been made of the shear in the plastic beam of Fig. 3(*a*). Neglecting it would not be serious if the beam were rectangular in cross section because of its necessarily small magnitude. On the contrary, in I-beams, or more especially plate girders, the web is designed for shear. It is relatively thin, and the shear stress value may be high, so high that plastic deformation will begin at the web-flange junction rather than at the extreme fiber (see Fig. 4).

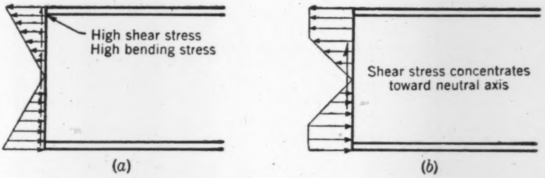

FIG. 4.—BENDING AND SHEAR

Furthermore, as plastic flow continues, the shear stress at the neutral axis increases and may well cause complete yielding of the section before the conventional limit load is reached. The answers to the foregoing questions require a much more complete theory of plasticity. In the subsequent discussion, it will be seen that, at present, the answers to many fundamental problems are still not known.

Most of these same unanswered queries do not apply to statically indeterminate trusses or to pin-connected frameworks. Each member is subjected to simple tension or compression and ideal plasticity is probably a good enough assumption. A limit design for a given loading would be a satisfactory answer. The straightforward approach is to apply the loading slowly as given, determine which member yields first, increase the loading, and find the next member to yield, continuing the process until enough members have yielded to cause failure. This process may well be tedious as it involves at least one, and possibly many, solutions of statically indeterminate elastic problems. A short cut should be found. In some cases it will be apparent which bars will yield to cause failure and the laborious calculations can be replaced by a simple statics computation. In many cases, however, there will be a few reasonable alternatives and some set procedure is required for deciding which of the statics computations is correct. H. J. Greenberg has proved the statement of S. M.

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[3] "Theory of Inelastic Bending with Reference to Limit Design," by Alexander Hrennikoff, *Transactions*, ASCE, Vol. 113, 1948, pp. 213–247.

[4] "Ueber das Verhalten statisch unbestimmter Konstruktionen aus Stahl nach Ueberschreitung der Elastizitaetsgrenze," by W. Prager, *Bauingenieur*, Vol. 14, 1933, pp. 65–67.

Feinberg[5] that any body (and therefore a structure in particular) will support the maximum load that it possibly can. Failure will not occur if any stable system of forces exists in the bars that will satisfy the equations of equilibrium. This is an extension of the dictum by N. C. Kist.[6]

William Prager and P. S. Symonds, at Brown University in Providence, R. I., have re-examined the basic principles that must apply to structures. In addition to limit design they explored the interesting problem of "shakedown," which arises when a structure such as a railway bridge is subjected to loads that change in magnitude between fixed limits. Each panel point load is assumed to vary independently of all other loads in any manner between a low value and a high value any number of times. The problem is whether or not, if plastic deformation does occur at some loading, eventually the structure will adjust itself and never again yield anywhere in tension or compression. Again, all the answers to this very practical problem are not yet known but an interesting and useful geometrical method is being applied.[7] It may be illustrated by reference to the simple indeterminate structure of Fig. 3(*b*). Assume that the load P alters in magnitude, and possibly changes in sign, in any manner between $+P_{\max}$ (tension) and $P_{\min}$ (compression). Calling the force in the central bar F and the force in each of the outer bars F_o, Fig. 5, the equation of equilibrium is

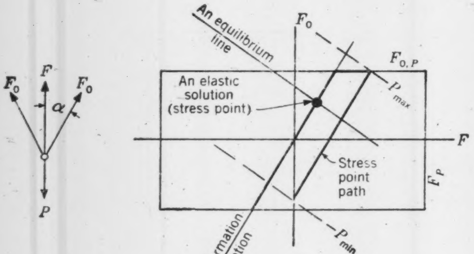

FIG. 5.—SHAKEDOWN

$$P = F + 2\,F_o \cos\alpha \quad \text{(3)}$$

In the elastic range, the second equation is determined by the deformation of the structure as

$$\frac{F\,L}{A\,E} = \frac{\dfrac{F_o\,L_o}{A_o\,E_o}}{\cos\alpha} \quad \text{(4a)}$$

or

$$F_o = \left(\frac{A_o\,E_o\,L}{A\,E\,L_o}\cos\alpha\right)F \quad \text{(4b)}$$

A coordinate system with axes F and F_o shows the behavior of the structure. The deformation equation plots as a straight line through the origin, with slope

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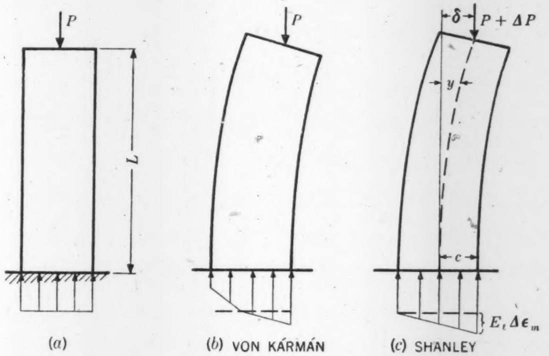
[5] "The Principle of Limiting Stress," by S. M. Feinberg, *Prikladnaya Mathematika i Mekhanika*, Vol. 12, 1948, pp. 63–68.

[6] "Theory of Limit Design," by J. A. Van den Broek, John Wiley & Sons, Inc., New York, N. Y., 1948, p. 39.

[7] "Problem Types in the Theory of Perfectly Plastic Materials," by W. Prager, *Journal of the Aeronautical Sciences*, Vol. 15, 1948, pp. 337–341.

$\frac{A_o\, E_o\, L}{A\; E\; L_o} \cos \alpha$. The equation of equilibrium for a given value of P plots as a straight line with a negative slope $\frac{1}{2 \cos \alpha}$. The stress point or intersection of the two lines gives the solution F, F_o. As the tensile force P increases in magnitude, the equilibrium line does not change in slope but simply moves parallel to itself, and the stress point moves straight out. This peaceful state of affairs is rudely interrupted by the onset of plastic action: F cannot exceed F_P and F_o cannot exceed $F_{o,P}$ on the assumption of ideal plasticity. These limits plus those for compression form a yield rectangle and the stress point cannot go outside it. As the load increases, the stress point moves out along the deformation line, hits the yield line, and moves along it. Unloading, or reducing the load, brings the stress point back along a line parallel to the original deformation line. Whether or not the system will "shake down" is clear from the picture. It will shake down for values of $P_{\max}$ and $P_{\min}$ as shown; it would not for the values $P_{\min} = - P_{\max}$.



(a) (b) VON KÁRMÁN (c) SHANLEY

FIG. 6.—COLUMN INSTABILITY (PLASTIC)

In actual practice the structure will have several bars each with its own value of force F_n and the coordinates chosen are modified; $\frac{F_n}{\sqrt{2\, A_n\, E/L_n}}$ would be used instead of F_n. The advantage is that the deformation line and the equilibrium line become perpendicular to each other in the elastic range. The distance from the origin to the stress point, $\frac{F^2{}_1}{2\, A_1\, E/L_1} + \frac{F^2{}_1}{2\, A_2\, E/L_2} + \cdots$, is the elastic strain energy that must be a minimum for a given set of loads.

The same method may be used for the analysis of rigid frames and continuous beams on the limit design type of assumption.

STRUCTURAL STABILITY

For many years it was felt that the problem of stability of columns in the plastic work-hardening range was clearly understood in principle despite the fact that predicted loads were often appreciably higher then those found experimentally. The method of analysis used was the same as that for elastic columns. In the latter case, the critical load is the load at which the equilibrium is neutral. Within the limitations of ordinary beam theory, at this load, no force is required to displace the column from its straight configuration; and, if displaced, the column will remain in the displaced position. The same procedure applied to the work-hardening range leads to formulas in which a reduced modulus occurs, a combination of elastic and plastic moduli.

As F. R. Shanley[8] demonstrated in 1947, this method of attack dces not take into account all possibilities, and so ignores the actual one. Consider a column of length L fixed at the bottom, free at the top (half of a hinged-end column) loaded centrally by a force P (see Fig. 6(*a*)). Instead of asking the question—"What happens if the top of the column is displaced at a given value of P (Fig. 6(*b*))?"— it is necessary to follow the loading process and to consider the possibility of deflection occurring as the load is increased (Fig. 6(*c*)). In the latter case there need not be unloading of any point in the column, and equilibrium may be maintained in the displaced position without a horizontal force applied to the column. The necessary conditions for such an occurrence for a small increment in load ΔP are that

$$\Delta P = 0.5\, E_t\, \Delta\epsilon_m\, A \qquad (5)$$

in which E_t is the plastic (tangent) modulus and $\Delta\epsilon_m$ is the maximum increment in strain. The column is here assumed symmetric in cross section. Moment equilibrium must also be satisfied so that

$$P\,\delta = \frac{0.5\, E_t\, \Delta\epsilon_m\, I}{c} \qquad (6a)$$

in which δ is the lateral deformation of the top of the member. In terms of the radius of curvature ρ,

$$P\,\delta = \frac{E_t\, I}{\rho_{\max}} \qquad (6b)$$

A similar equation must be satisfied at each cross section:

$$P\,(\delta - y) = \frac{E_t\, I}{\rho} \qquad (7)$$

Eq. 7 is the same as the equation for elastic stability except that the tangent modulus appears instead of the elastic modulus. Therefore, when

$$P = \frac{\pi^2\, E_t\, I}{4\, L^2} \qquad (8)$$

deflection can occur without application of horizontal forces providing P is increased by ΔP or more. However, if the load ΔP is added and the column is prevented from bending, $P + \Delta P$ does not make the column unstable. Horizontal motion without horizontal force requires still another increment in load. This type of effect is instability only in the sense that two equilibrium positions are possible. Both are stable; positive work must be done to displace the column from either position. In practice, however, a column will not be found perfectly straight at loads greater than those given by substitution of the tangent modulus in the elastic buckling formulas. Nevertheless, large deflection (apparent buckling) may not occur until the load is appreciably above this initial value; but it will occur below the von Kármán value. Mr. Shanley's concept reopens the entire field of plastic buckling; all previous solutions must be re-examined. Many basic principles are still to be established.

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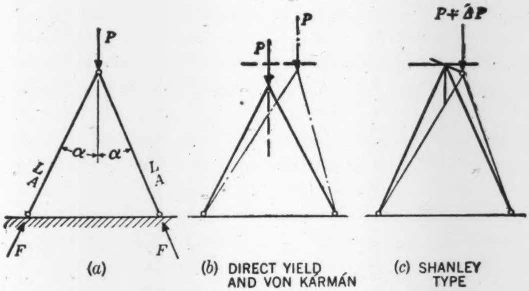
[8] "Inelastic Column Theory," by F. R. Shanley, *Journal of the Aeronautical Sciences*, Vol. 14, 1947, pp. 261–268.



(a) (b) DIRECT YIELD AND VON KÁRMÁN (c) SHANLEY TYPE

FIG. 7.—STRUCTURAL INSTABILITY (PLASTIC)

Another simple application to frameworks (Fig. 7) shows the differences between the old and new viewpoints. As the load P is increased, the bars become plastic but they work harden and will continue to carry additional load without failing. Failure may occur in several ways. The ultimate load is determined by the angle α and the shape of the stress-strain curve. When side motion of the apex is prevented, the system will collapse vertically at a load:

$$P = \frac{2\,E_t\,A\cos^3\alpha}{\sin^2\alpha} \qquad (9a)$$

No additional force is required to produce downward motion of the point of application at P. However, if

$$P = \frac{2\,E_t\,A\sin^2\alpha}{\cos\alpha} \qquad (9b)$$

which has a much smaller value if α is less than 45°, no external horizontal force is needed to cause horizontal displacement of the apex (Shanley effect). However, an increment in P will be required as in the case of the column. This increment is not unique, it has its least value for the case in which one bar takes on no additional load but does not unload (Fig. 7(c)). To obtain the equivalent of a von Kármán instability load (Fig. 7(b)), it is necessary to write the condition that P remains constant as lateral motion takes place without horizontal force. One bar will be found to unload and the value of P required is larger than that when P is permitted to increase. Another possible mode of failure—the Shanley type of buckling of the bars themselves—should also be considered.

MATHEMATICAL THEORIES OF PLASTICITY

The mathematical work required becomes more elaborate in the more complicated problems and is really formidable for general cases. It would be convenient to ignore them and avoid trouble. Unfortunately, determination of when deflections or stresses are excessive almost invariably involves plastic behavior. Beams or girders often fail by local web or over-all plastic buckling; and, as indicated in Fig. 4, the state of stress is not simple. At present, tests are relied on to supply the answers and no result can be as satisfactory as a careful test on the same structure or part. Tests are difficult, expensive, and time consuming, however. Extrapolation of results without an adequate theory is often misleading and tests covering all important cases are a physical impossibility.

Therefore, the next best procedure should be adopted: Develop as simple a theory as possible which will be adequate for the problems on hand. Its adequacy can be assured only by frequent experimental checks. It is by no means obvious which factors should be included and which omitted, and the final answer is still to be obtained. For example, time effects are customarily assumed to be ignored in structural and machine applications; but, when concrete is involved and the time is long, this assumption would not be reasonable.

Neglecting the time factor, it follows that at each stage in the plastic deformation some function of stress governs the continuing of the plastic action. In geometric language, yield or loading surfaces exist in stress space.[9,10] At any stage in the plastic deformation, if the point representing the state of stress moves inside the surface, elastic action only occurs. For a material that has been work hardened, if plastic action is to result, the stress must change so that the point representing it moves outside. Fig. 8 shows a few simple criteria of yield or loading for two-dimensional stress, the octahedral shearing stress, the maximum shearing stress, and the more general function of shearing stress $f(J_2, J_3)$ (J_2 and J_3 being the second and third invariants of the stress devia-

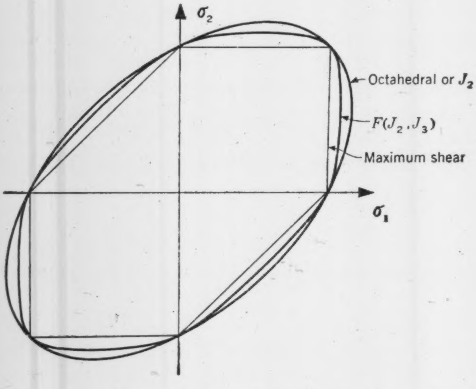


FIG. 8.—SEVERAL YIELD CRITERIA

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[9] "Recent Developments in the Mathematical Theory of Plasticity," by William Prager, *Journal of Applied Physics*, March, 1949, pp. 235–241.

[10] "The Significance of the Criterion for Additional Plastic Deformation of Metals," by D. C. Drucker, *Journal of Colloid Science* (rheology issue), Vol. 4, 1949, pp. 299–311.

tion[11] and J_2 being proportional to the square of the octahedral shearing stress). The yield surface, of course, is a yield curve in two dimensions. When there is no work hardening (ideal plasticity), if the stress point reaches the yield curve, flow occurs without limit for homogeneous states of stress. In work-hardened material, the stress point does move outside and the yield or the loading surface moves with it. A common but oversimple assumption is that the same variables still control. A larger value of the loading function is now required to continue plastic deformation—for example, a greater octahedral shearing stress equal to the largest one previously applied.

The concept of a loading function or of the variables that control plastic deformation is one facet of a mathematical theory of plasticity. Another is the actual form of the stress-strain relation. The simplest idea, and the one which has been used most extensively, is that stress and strain are uniquely related as long as loading continues (deformation theory). The stress-strain relation is then identical with the one for nonlinear elasticity except that on unloading a linear elastic recovery is assumed. One example of such equations for work-hardening material has the following typical expression for loading:

$$\epsilon_x = \frac{1}{C\,(\tau_o)}\,[\sigma_x - \nu'\,(\sigma_y + \sigma_z)] \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

in which $C\,(\tau_o)$ is a plastic modulus, a function of the octahedral shearing stress τ_o, and ν' denotes a plastic Poisson's ratio, often taken as 0.5. For unloading,

$$\Delta\epsilon_x = \frac{1}{E}\,[\Delta\sigma_x - \nu\,(\Delta\sigma_y + \Delta\sigma_z)] \ldots\ldots\ldots\ldots\ldots (11)$$

in which ν is the elastic Poisson ratio and the symbols Δ indicate changes in stress, σ, and in strain, ϵ.

The solution of general problems by this most elementary deformation theory is very difficult and discourages attempts to improve theory. It is, therefore, a most unpleasant fact that any deformation theory generally has validity only in the rare and special case in which all stresses remain fixed in direction and merely increase in ratio. This particular case is precisely the one most commonly used in obtaining test data. A thin-walled tube is loaded by a combination of direct pull, internal pressure, and, sometimes, torsion—increased in ratio. Therefore, some confusion has developed and some improper applications of the deformation theory have been made. The inconsistency lies in the incompatibility of a quasi nonlinear elasticity and a yield or a loading function when all paths of loading are considered.

An incremental theory is required instead. There cannot be a unique relation between stress and strain and only one relation for the increment in strain in terms of the existing state and history and the changes in stress. The simplest example of this type of theory has as a typical expression—

$$d\epsilon_x = \frac{1}{E}\,[d\sigma_x - \nu\,(d\sigma_y + d\sigma_z)] + \frac{1}{C_1\,(\tau_o)}\,[\sigma_x - 0.5\,(\sigma_y + \sigma_z)]\,d\tau_o \ldots (12)$$

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[11] "Strain Hardening Under Combined Stresses," by W. Prager, *Journal of Applied Physics*, Vol. 16, 1945, pp. 837–840.

—the sum of an elastic and plastic strain increment. Consistency with an octahedral stress loading surface is assured because, if the octahedral shearing stress does not change, $d\tau_o = 0$, the increment in strain is purely elastic. Actually much more elaborate expressions are required to fit just a few of the observed important facts: The Bauschinger and allied effects and the Mohr circle comparison[11,12] introduced by W. Lode. In tensor notation, using the summation convention, the most general form of incremental theory is

$$d\epsilon_{ij} = d\epsilon_{ij,e} + G \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{kl}} d\sigma_{kl} \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

in which f (the loading function) and G are functions of stress, strain, and the history of stress and strain and $d\epsilon_{ij,e}$ is the increment in the elastic or recoverable strain.[9,10] The loading function determines the incremental stress-strain law.

THE PATH OF LOADING

Another reason for the breakdown of the deformation type of theory, and the philosophy behind it, is illustrated simply by the familiar three-wire problem of elementary strength of materials (Fig. 9). This example shows the danger of wishful thinking. The deformation type of theory is applicable when the stress at each point increases in ratio; and, therefore, the hope was expressed that, if all loads applied to a body increase in ratio, the stresses will do likewise.[13] The three-wire problem proves this hope to be vain. Many other simple counterexamples can be found. Not only do the stresses not necessarily go up in ratio but unloading may (and will) often take place locally.

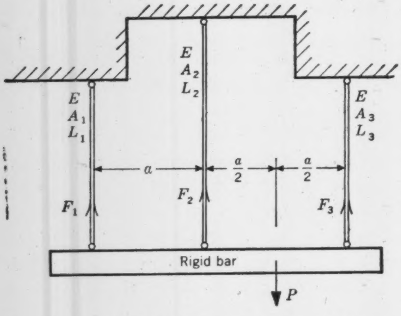

FIG. 9.—LOADING MAY PRODUCE LOCAL UNLOADING

It seems preferable to describe the solution in words first: In the elastic range, the load P applied to the rigid bar of Fig. 9 is divided by the three wires in accordance with their stiffness (spring constant) and the conditions of equilibrium. Assume the center wire to be strong but very flexible; imagine it temporarily, for this step only, as a rubber band. The outer two wires will both be stressed in tension. Despite the smaller load carried by the one on the left, this wire may yield first because its area is smaller. As the load P increases further, the force in the center and right-hand wires will increase; and, if the areas are in proper ratio, the right-hand wire will yield next. On continued loading, the central wire, which is elastic, will act as a pivot, the rigid bar will rotate clockwise, and the left-hand wire will unload.

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[12] "The Relation of Experiments to Mathematical Theories of Plasticity," by D. C. Drucker, *Journal of Applied Mechanics*, Vol. 16, 1949, pp. 349–357.

[13] "The Theory of Elastic-Plastic Deformation and Its Applications," by A. A. Ilyushin, *Izvestia*, Akademii Nauk U.S.S.R., Otdelenie Tekhnicheskikh Nauk, 1948, pp. 769–788.

The preceding description applies equally well to ideal plasticity or to the usual work-hardening stress-strain curves. Depending on geometry, and the type of these curves, the system will perform various tricks. For example, if the wires are really work-hardening bars capable of taking compression, the left-hand one may be made to yield in tension, unload, load in compression, yield, and unload again as the load P increases steadily.

The mathematical proof of unloading produced by loading merely requires writing the equations of equilibrium and of deformation. The geometric representation appears as Fig. 10. At all times, for equilibrium,

$$F_1 + F_2 + F_3 = P \qquad (14a)$$

and

$$3\,F_1 + F_2 - F_3 = 0 \qquad (14b)$$

In the elastic range,

$$\frac{F_1 L_1}{A_1 E} + \frac{F_3 L_3}{A_3 E} = 2\,\frac{F_2 L_2}{A_2 E} \qquad (15)$$

so that

$$F_1 = \frac{P\left(\dfrac{2\,L_2}{A_2 E} - \dfrac{L_3}{A_3 E}\right)}{2\left(\dfrac{L_1}{A_1 E} + \dfrac{4\,L_2}{A_2 E} + \dfrac{L_3}{A_3 E}\right)} \qquad (16)$$

will be tensile if $\dfrac{2\,L_2}{A_2 E} > \dfrac{L_3}{A_3 E}$. For sufficiently small values of A_1, F_1/A_1 will exceed the yield stress σ_y first (point A, Fig. 10); and, with the assumption of ideal plasticity, $F_1 = \sigma_y A_1$ then replaces the elastic deformation condition. As P increases, F_2 and F_3 increase equally and A_3 may be chosen to make bar 3 yield next point B (Fig. 10). At this stage, two bars have yielded but collapse does not occur because both have yielded in tension and the central bar is still elastic. The third equation is now $F_3 = \sigma_y A_3$ and direct calculation shows that F_1 must decrease as P increases further from point B to point C (Fig. 10).

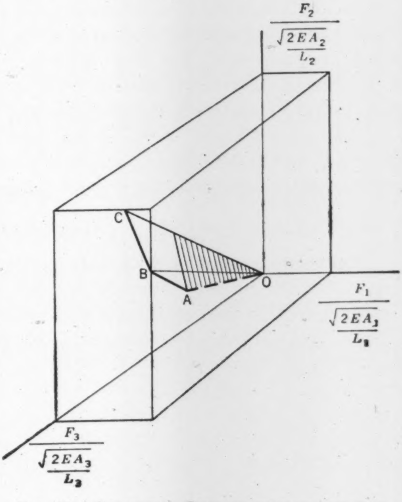
$\frac{F_2}{\sqrt{\frac{2EA_2}{L_2}}}$
C
B
O
A
$\frac{F_1}{\sqrt{\frac{2EA_1}{L_1}}}$
$\frac{F_3}{\sqrt{\frac{2EA_3}{L_3}}}$

FIG. 10.—GEOMETRIC REPRESENTATION FOR THE THREE-WIRE PROBLEM

If a reasonable extension of the Saint Venant principle in plasticity is accepted (an unreasonable one being easy to make), the case of bending plus tension, of which the three-wire problem is really a particular case, exhibits this same phenomenon. The case of torsion shows that the stress pattern alters considerably from the elastic to the plastic state as loading proceeds, although unloading does not occur.

The combination of the probable change in the pattern of stress and the possibility of unloading occurring locally makes it imperative that the history of loading be followed in detail in calculating the final state of stress and strain in a body that has been deformed plastically. Because of mathematical complexity, few problems have been solved completely; but work is in progress in many countries. Probably the most complicated problem solved so far is that of an infinite plate with a circular hole subject to unequal biaxial tension at infinity in such ratio that the entire boundary of the hole is plastic. The solution gives the stresses everywhere, and also the boundary of the plastic region under the assumptions of plane strain and ideal plasticity.[14]

CONCLUSION

If practical answers are to be found to many engineering problems of importance and the factor of safety is to be put on a firm calculable basis, a large number of complicated plasticity problems will have to be solved. The great expenditure of effort that will be required makes it desirable to use as simple a theory of plasticity as is permissible for each problem. Much more experimental information is necessary than exists at present. Work has already been started and should continue on an expanded scale along the fruitful path of investigating tubes under internal pressure, tension, and torsion where the ratios are purposely varied during the test to check particular questions. Strong efforts should be made to determine by experimental methods, in several fundamental cases, the importance of the Bauschinger and allied effects, time effects, and the deviation from similarity of the Mohr circles for stress and strain. Little should be taken for granted and all reasonable sounding but unproved statements should be viewed with skepticism. Opinions on what is important and what is not, such as those presented in this paper, may be useful guides but may also be misleading.

ACKNOWLEDGMENT

Acknowledgment is gratefully made to the Office of Naval Research and the Bureau of Ships, United States Department of the Navy, joint sponsors of research in this field at Brown University under contract N7onr-358.

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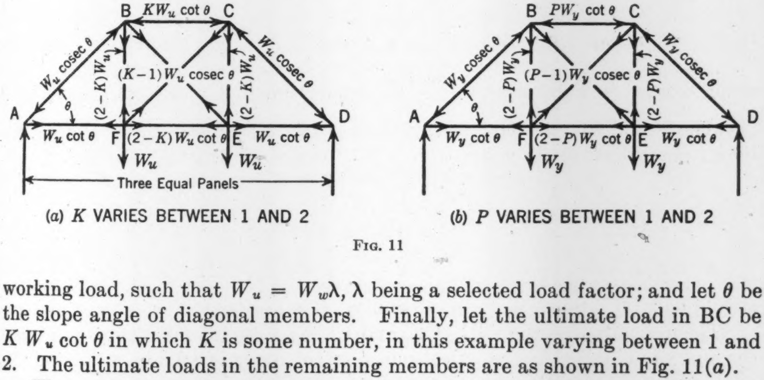
[14] "Plane Elastic-Plastic Problem, Plastic Regions Around Circular Holes in Plates and Beams," by L. A. Galin, *Prikladnaya Mathematika i Mekhanika*, Vol. 10, 1946, pp. 365–386.

DISCUSSION

R. W. **Steed**[15].—For some time the writer has been engaged in an investigation into the design of those triangulated steel structures which rely, primarily, for their resistance to loading on longitudinal forces. Each member is subjected to simple tension or compression. Ideal plasticity has been assumed as suggested in the paper.

It is not sufficient to choose an arbitrary load distribution for multiredundant structures under ultimate loading, unless the mechanism of failure is known, or unless recourse is made to the solution of statically indeterminate elastic equations. There is the possibility of compression members buckling before the ultimate load is reached. Insufficient experimental knowledge is available concerning the behavior of buckled compression members, as further load is added to a structure, for this premature buckling to be permissible. In the case of some structures there is also the possibility of local unloading after loading, even while all loads applied to a structure increase in ratio.

Triangulated structures, designed on the ultimate theory, having one or two symmetrically loaded, internal or external redundancies, can be analyzed as follows: Referring to Fig. 11: let W_u be the ultimate design load; let W_w be the working load, such that $W_u = W_w\lambda$, λ being a selected load factor; and let θ be the slope angle of diagonal members. Finally, let the ultimate load in BC be $K\,W_u \cot\theta$ in which K is some number, in this example varying between 1 and 2. The ultimate loads in the remaining members are as shown in Fig. 11(a).


B
$KW_u \cot\theta$
C
$W_u \operatorname{cosec}\theta$
$(2-K)W_u$
$(K-1)W_u \operatorname{cosec}\theta$
$(2-K)W_u$
$W_u \operatorname{cosec}\theta$
A
θ
D
$W_u \cot\theta$
F
$(2-K)W_u \cot\theta$
E
$W_u \cot\theta$
W_u
W_u
Three Equal Panels
B
$PW_y \cot\theta$
C
$W_y \operatorname{cosec}\theta$
$(2-P)W_y$
$(P-1)W_y \operatorname{cosec}\theta$
$(2-P)W_y$
$W_y \operatorname{cosec}\theta$
A
θ
D
$W_y \cot\theta$
F
$(2-P)W_y \cot\theta$
E
$W_y \cot\theta$
W_y
W_y

(a) K VARIES BETWEEN 1 AND 2

(b) P VARIES BETWEEN 1 AND 2

Fig. 11

The areas of members BF, CE, EF, BE, and CF are determined by dividing the ultimate loads in the members by σ_y, the yield stress of the material. As member BC is in compression, the buckling stress will be less than σ_y and can be denoted by σ_y/N, in which N is some number. The area of member BC, then, will be $K\,W_u\,N \cot\theta/\sigma_y$.

As both of the applied loads are increased incrementally, all members are in an elastic state until at some applied load W_y one member yields or buckles. Let the load in member BC now be $P\,W_y \cot\theta$, and the loads in the other members become as shown in Fig. 11(b). If the symbol Q be made to represent the ratio $\frac{\text{load capacity of a member}}{\text{elastic distribution for the member}}$, the following relations may be written:

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[15] Lecturer in Civ. Eng., Univ. of London, King's College, London, England.

$$Q_{BC} = \frac{W_u K}{P} \dots\dots (17a)$$

$$Q_{BF} = Q_{CE} = Q_{EF} = W_u \frac{2 - K}{2 - P} \dots\dots (17b)$$

and

$$Q_{CF} = Q_{BE} = W_u \frac{K - 1}{P - 1} \dots\dots (17c)$$

If P is greater than K, Eqs. 17*a* and 17*c* are less than W_u, and Eq. 17*b* is greater than W_u. Members BE and CF will then be the first to yield, as $\frac{K-1}{P-1} < \frac{K}{P}$. If P is less than K, Eqs. 17*a* and 17*c* are greater than W_u, and Eq. 17*b* is less than W_u. Under the latter conditions members BF, CE, and EF will be the first to yield.

From a strain energy analysis of the structure it can be shown that, assuming no lack of fit of the members:

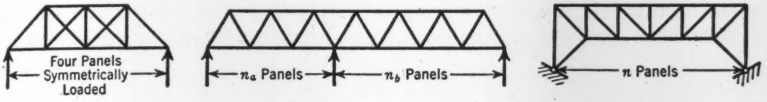
$$-2\frac{2-P}{2-K} - \cot^2\theta\frac{2-P}{2-K} + 2\operatorname{cosec}^2\theta\frac{P-1}{K-1} + \frac{\cot^2\theta\, P}{K\, N} = 0 \dots (18)$$

If $P = K$ in Eq. 18, $-2 - \cot^2\theta + 2\operatorname{cosec}^2\theta + \frac{\cot^2\theta}{N}$ should equal 0. In other words, $-1 + \operatorname{cosec}^2\theta + \frac{\cot^2\theta}{N}$ should equal 0. For all values of θ between 0° and 90° (the left hand side being always positive), this is an impossibility, regardless of the value of N. If P is greater than K, the quantity—$- < 2 - < \cot^2\theta + > 2\operatorname{cosec}^2\theta + \frac{> \cot^2\theta}{N}$—should equal 0.

The positive value of the left hand side is now even greater than before. If P is less than K the quantity—$- > 2 - > \cot^2\theta + < 2\operatorname{cosec}^2\theta + < \frac{\cot^2\theta}{N}$—should equal 0. Since this is a possibility, P is less than K.

Theoretically, members BF, CE, and EF, as a group, will always yield first for any value of θ. For practical purposes the areas of members BF and CE could be made slightly greater than their theoretical areas so that the only member to be in a plastic state as the load is increased from W_y to W_u would be member EF. With the application of the full design load W_u, all members would become plastic, and the structure would fail. It is appreciated that the theoretical areas here used would be adjusted to practical sizes but this would tend only to increase the ultimate failing load and thus the load safety factor for the structure. Using a similar argument, it can be shown that, whatever the value of α for a structure such as that shown in Fig. 3(*b*), designed by ultimate theory, the middle member will always yield before either of the other two members.

The writer has applied this method successfully to investigate many well known types of structures with one, or two symmetrically loaded, internal or external redundancies, and designed by ultimate theory. Structures investigated include those shown in Fig. 12.



(a) (b) (c)

FIG. 12

It will be appreciated that, although symmetrical loading has been used in the foregoing example, the method can be used for any design loading, either symmetrical or asymmetrical, on a frame having one redundancy.

By using generalized expressions, the writer has extended the method shown in the example to investigate several well known types of structures having multiple redundancies and designed by ultimate theory.

JACOB KAROL,[16] M. ASCE.—In his comprehensive review of the theory of plasticity of metals, Mr. Drucker has made many cautious generalizations which are difficult to appreciate. Emphasizing the need for a general theory of plasticity, he disregards solutions that are possible using existing theory. In particular cases, it appears that the naive approach of the practical engineer to stress problems in general is as consonant with reality as the "more formal and precise approach" which the author states is desirable.

Two problems discussed in detail by the author will be commented upon: (1) The limit design of I-beams and (2) the structural stability of columns.

Some concern is expressed by Mr. Drucker regarding the failure of others[3,4] to consider the effect of shear stress in the limit design of I-beams. Two generalizations in the paper can be used to show that this concern may be unwarranted. First, it is stated (under the heading, "Limit Design") "* * * that any body (and therefore a structure in particular) will support the maximum load that it possibly can." This statement seems axiomatic to the writer. Furthermore, under the heading, "The Path of Loading," the author states that "* * * unloading may (and will) often take place locally."

The important fact not considered by the author is that the web of an I-beam carries practically all the shear and that the flanges carry practically all the moment. Hence, it seems reasonable to assume that in the plastic range, if necessary (and one must repeat the "if necessary"), the shear will be resisted on a part of the web at the shear yield stress, and the moment will be resisted by the remainder of the section at the tensile (and compressive) yield stress. Failure of the I-beam, fixed-ended and uniformly loaded, will be assumed to take place when the outer fiber stress at the center section of the beam has just reached the tensile yield stress and when the end sections have yielded in shear, moment, or in a combination of shear and moment. The compression flanges are considered to be supported against lateral buckling. Since the failure load is the load at which the center section just begins to yield, one can calculate the deflections of the beam as indicated by Alexander Hrennikoff.[3]

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[16] Design Engr., Kansas City, Mo.

The load carried by a fixed-ended I-beam is given by the expression

$$W = 2\,\tau_y\,A_w\,\frac{y}{d} \qquad \text{(19)}$$

in which

$$\frac{y}{d} = \frac{4\,\sqrt{3}\,(S + 2\,Q)}{A_w\,d\,l/d}\left[1 - \frac{A_w\,d}{8\,Q}\left(\frac{y}{d}\right)^2\right] \qquad \text{(20)}$$

In addition to the symbols in the paper, in Eqs. 19 and 20 A_f = area of one flange; A_w = area of the web = $t_w\,d$; d = depth of beam; l = span length of beam; Q = static moment of a cross section; S = section modulus of cross section; t_w = web thickness; y = height of web stressed in shear; and τ_y = shear yield stress = $\sigma_y/\sqrt{3}$.

Since the ratio y/d cannot exceed unity, for low values of l/d the capacity of the beam is limited by its strength in shear. As a matter of interest, the formula for load, neglecting the effect of shear on the moment capacity, will also be given:

$$W = \frac{8\,\sigma_y}{l/d}\left(\frac{S + 2\,Q}{d}\right) \qquad \text{(21)}$$

The graph for an 18I54.7 (the same section used by Mr. Hrennikoff[3]) is shown as curves I and II in Fig. 13, abscissas representing the ratio l/d and ordinates representing $W/W_{\max}$, in which $W_{\max}$ is the yield capacity of the beam in shear.

The beginning of plastic action is reached at the critical section in one of three ways: (1) For short spans, when the shear stress at the neutral axis reaches the yield point; (2) for intermediate spans, when the apparent stress at the junction of the web and the flange (using the distortion-energy theory of failure) reaches the tensile yield point; and (3) for long spans, when the extreme fiber stress in bending reaches the tensile yield point. The corresponding expressions for load are: Short spans—

$$W = 2\,\tau_y\,A_w\,\frac{S}{2\,Q} \qquad \text{(22a)}$$

intermediate spans—

$$W = \frac{2\,\tau_y\,A_w\,S}{A_f\,d\left[1 + \frac{1}{3}\left(\frac{A_w\,l}{6\,A_f\,d}\right)^2\right]^{\frac{1}{2}}} \qquad \text{(22b)}$$

long spans—

$$W = \frac{12\,\sigma_y\,S/d}{l/d} \qquad \text{(22c)}$$

The loads as ratios of W_{max} are also shown in Fig. 13 as curve III. Plastic failure takes place in the region between curve II and curve III. The simple theory resulting in curve I seems to be a good approximation despite its lack of formality and precision.

The author's general discussion of structural stability is commendably concise and clear. One should add the important point that, in the range of slenderness ratios used in design, the difference between the buckling loads computed by the reduced modulus theory and that computed by the tangent modulus theory is small indeed. The limits thus set by theory are all the assurance the designer needs of the validity of his design procedure. Such procedure neglects the specific modes of failure which the author states (paragraph containing Eqs. 9*a* and 9*b*) indicate "* * * the differences between the old and new viewpoints." It will now be shown that the new viewpoint is not significant in practical design.

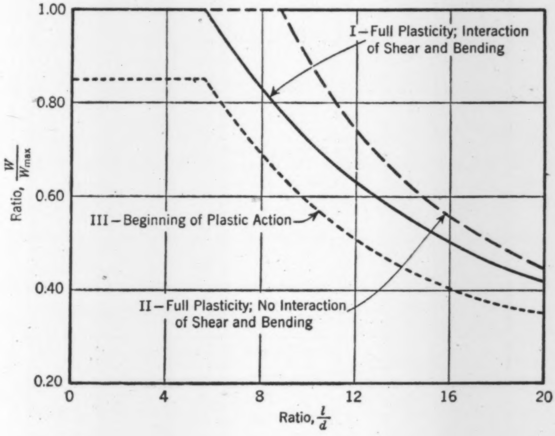

FIG. 13.—FAILURE LOADS FOR 18I54.7

The apparent strain in the inclined bars in Fig. 7 at which collapse will take place may be written directly from Eqs. 9*a* and 9*b*. For no side motion of the apex:

$$\frac{F}{A\,E_t} = \cot^2 \alpha \ldots\ldots\ldots\ldots\ldots\ldots (23a)$$

For side motion of the apex:

$$\frac{F}{A\,E_t} = \tan^2 \alpha \ldots\ldots\ldots\ldots\ldots\ldots (23b)$$

The third mode of failure (the Shanley mode of buckling of the bars themselves) is given by the expression:

$$\frac{F}{A\,E_t} = \frac{\pi^2}{(L/r)^2} \ldots\ldots\ldots\ldots\ldots\ldots (23c)$$

The zone of application of each mode of buckling is shown in Fig. 14, using the slenderness ratio as abscissa and the angle α as ordinate. It is seen that the ordinary tangent modulus theory is critical for determining the buckling load for any practical combination of slenderness ratio and frame angle, and is thus the most important, although the author's parenthetical mention of it might indicate otherwise.

This discussion is not intended to depreciate any efforts, experimental or theoretical, which may be made toward the development of a comprehensive theory of plasticity. It merely emphasizes the fact that upper and lower limits for the solution of some apparently complicated problems can be obtained very simply by using existing theory. For the theoretician as well as for the practical engineer, it is important to insist that lack of a general theory does not invalidate a particular fact.

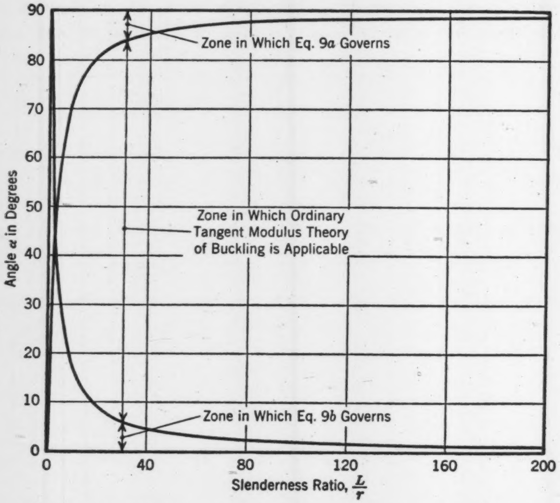

FIG. 14.—APPLICABILITY OF THEORIES OF STRUCTURAL INSTABILITY

S. K. GHASWALA,[17] Assoc. M. ASCE.—An interesting elementary account of the theory of plasticity is presented and should be studied by all concerned with the subject.

The design of engineering structures is based on the solid foundation of the theory of elasticity, whose empirical derivation dates from the seventeenth century. The classic observations of Galileo in 1638 on the fracture of loaded beams[18] and Robert Hooke's experiments some years later on the resilience of springs[19] suggested physical hypotheses capable of mathematical formulation.

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[17] Chartered Engr., Fort, Bombay, India.

[18] "Dialogues Concerning Two New Sciences" (English Translation), by Galileo Galilei, Northwestern Univ., Evanston and Chicago, Ill., 1939.

[19] "Lectures De Potentia Restitutiva," by Robert Hooke, published in "Early Science in Oxford," by R. T. Gunther, Vol. 8 (printed for author), Oxford, 1931.

As the years passed, the theory of elasticity continued to develop into an exact science and became part of mechanics and applied mathematics through the work of such pioneers as M. Navier, S. D. Poisson, A. L. Cauchy, M. B. de Saint-Venant, James and Daniel Bernoulli, Lord Kelvin, K. Pearson, P. G. Tait, and T. Young.[20,21] Today, for most of the simple design problems met by engineers, the scope and reliability of this theory has been amply demonstrated by the availability of exact results. During the past few years, however, intensive study has been directed towards evaluating the exact strength of composite and sandwich materials[22] and towards gaining a wider knowledge of elastically imperfect materials, of mechanical properties of ductile materials, of the crystal structure of solids, and of the behavior of metals during various fabricating processes. These efforts have led to a more satisfactory understanding of the importance of the plastic nature of solids and have necessitated the formulation of concepts not included in the theory of elasticity as propounded by its pioneers.

The behavior of material under stress so great that plastic action is set up is of interest in many spheres. Thus, in the bending, punching, and stretching of cold material, stresses beyond the yield point of the material develop; in the phenomenon of accidental overload in car couplers, draft rigging, and side frames of railroad rolling stock, plasticity becomes the governing factor in design. In problems concerning the stability of slender columns, stress concentration, and energy absorption, the theory of elasticity is no longer valid, and the theory of plasticity comes into play. Among problems of the latter class are structures built for air raid protection[23] and the elements of a torpedo protection system in which the ship's hull functions under explosive loading.[24]

In describing the theories of plasticity the author should have introduced the two concepts of plastic flow and plastic deformation more clearly so as to enable an uninitiated structural engineer to grasp the subject. The theory of plastic flow is based on the relation between stress and the rate of plastic strain, while the theory of plastic deformation is based on the relation between stress and plastic strain. The plastic flow theory is derived from the work of Saint-Venant, M. Levy, R. von Mises, L. Prandtl, T. Y. Lin, Assoc. M. ASCE, E. Reuss, and W. Prager, M. ASCE, and the plastic deformation theory is based on the work by A. Nádai, H. Hencky, and A. A. Ilyushin. The mathematical aspect of the subject has been ably treated by R. Hill.[25] In fact, from the standpoint of mathematics, the theory of plasticity forms one of the most active fields of research in the nonlinear mechanics of deformable media. The earliest statements of the concept of plastic design were made by N. C. Kist[26] and M. Gruning[27]; applications of the method to the design of transmission towers by

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[20] "A History of the Elasticity and of Strength of Materials," by Isaac Todhunter and Karl Pearson, Cambridge Univ. Press, Cambridge, England, Vol. I, 1886.

[21] "A Treatise on the Mathematical Theory of Elasticity," by A. E. H. Love, Cambridge Univ. Press, Cambridge, England, 4th Ed., 1927.

[22] "Elements of Sandwich Construction," by S. K. Ghaswala, *Journal*, Inst. of Engrs., India, Vol. 31, No. 1, Sept., 1950, pp. 47–62.

[23] "The Civil Engineer in War," by J. F. Baker, Inst. of Civ. Engrs., London, England, Vol. 3, 1948.

[24] "Plastic Deformation, Principles and Theories," by L. N. Kachanov, N. M. Beliaev, A. A. Ilyushin, W. Mostow, and A. N. Gleyzal, Mapleton House, New York, N. Y., 1948.

[25] "The Mathematical Theory of Plasticity," by R. Hill, Oxford Univ. Press, London, England, 1950.

[26] "Die Zähigkeit des Materials als Grundlage für die Berechnung von Brücken, Hochbauten und ähnlichen Konstruktionen aus Flusseisen," by N. C. Kist, *Der Eisenbau*, Leipzig, Vol. 11, 1920, pp. 425–428.

[27] "Die Tragfähigkeit statisch unbestimmter Tragwerke aus Stahl bei beliebig häufig wiederholter Belastung," by M. Grüning, Julius Springer, Berlin, 1926.

C. M. Goodrich in 1910 are described by J. A. Van den Broek,[28] M. ASCE, and a general review of work in this field prior to 1931 was given by the late Friedrich Bleich,[29] M. ASCE. One of the most important projects undertaken in England in recent times in the application of the theory of plasticity to structural design problems is due to the efforts of J. F. Baker, Assoc. M. ASCE, and his colleagues at Cambridge University, Cambridge, England.[30,31,32,33] It is regretted that the author has not made any specific or detailed mention of this work as it would have added much to the value of the paper. The so-called plastic method of design evolved by Mr. Baker is of particular application to welded structures, such as portal frames, where its use in lieu of the elastic design method leads to a more economical design, simplification of design calculations, and, most important, places the actual method of calculation on a more rational basis by maintaining a uniform safety factor in all members. In addition to English and American research, a flourishing school of research in this field has been established in Russia, mainly through the work of Mr. Ilyushin, L. N. Kachanov, N. M. Beliaev, and others.[24]

The subject of plasticity is closely linked with "anelasticity," a word coined by Clarence Zener of the newly established Institute for the Study of Metals at the University of Chicago (Chicago, Ill.).[34] Anelasticity denotes that property of solids by which stress and strain are not single-valued functions of one another in that low stress range in which no permanent set occurs and in which the relation of stress to strain is still linear. All manifestations of anelasticity, such as creep, hysteresis, internal friction, stress relaxation, and variation of modulus with frequency of measurements are capable of being interpreted in a better manner by this specialized study. The writer would like to see some discussion of this topic, which he believes will exert a profound influence in the future on the thought and practice of structural engineering.[35]

D. C. Drucker,[36] Assoc. M. ASCE.—It has been made painfully clear by the discussion that the paper attempted to cover too broad a phase of plasticity in too brief a manner. The objective was to indicate some of the directions of special interest to the author in which the field has been developing. There was no intention of criticizing accepted design procedures in the vast majority of cases that have been checked repeatedly by practical experience and laboratory tests. Simple illustrations were employed to introduce a few fundamental concepts to those unfamiliar with the theory. Clearly, the well-informed discussers are not in this category. The necessity for more advanced theory may not become apparent to the experienced engineer untill more elaborate problems are investigated.

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[28] "Theory of Limit Design," by J. A. Van den Broek, John Wiley & Sons, Inc., New York, N. Y., 1948, p. 64.

[29] "Stahlhochbauten, ihre Theorie, Berechnung und Bauliche Gestaltung," by Friedrich Bleich, Julius Springer, Berlin, 1932.

[30] "The Calculation of Collapse Loads for Framed Structures," by B. G. Neal and P. S. Symonds, *Journal of the Institution of Civil Engineers*, London, England, No. 1 (1950–51), November, 1950, pp. 21–40.

[31] "The Design of Steel Frames," by J. F. Baker, *The Structural Engineer*, London, England, Vol. 27, October, 1949, pp. 397–431.

[32] "A Review of Recent Investigations into the Behaviour of Steel Frames in the Plastic Range," by J. F. Baker, *Journal of the Institution of Civil Engineers*, London, England, No. 3 (1948–49), January, 1949, pp. 188–240.

[33] "Theory of Plasticity—Elements of Simple Theory," by J. W. Roderick, *Philosophical Magazine and Journal of Science*, London, England, 7th Series, Vol. 39, July, 1948, pp. 529–539.

[34] "Elasticity and Anelasticity of Metals," by Clarence Zener, Univ. of Chicago Press, Chicago, Ill., 1948.

[35] "Atomic Structure of Engineering Metals," by S. K. Ghaswala, *Science and Culture*, Calcutta, India, February, 1947, pp. 388–395.

[36] Prof. of Eng., Brown Univ., Providence, R. I.

However, a number of statements are made in the discussion indicating that, in a few cases, the inadequacy of past techniques and sometimes "engineering intuition" is not fully appreciated. Mr. Ghaswala implies that the design of engineering structures should be based on the "solid foundation of the theory of elasticity," except for structures which are intended to function in the plastic range. Elastic theory is certainly of great value but, as stated in the paper, design factors of safety have meaning only when consideration is given to what happens under overload—generally when plastic action takes place. In fact, the elastic theory, instead of being invalid, is of greatest value in problems of stress concentration when fatigue may occur and in problems of the buckling of slender members.

Mr. Ghaswala is critical of the definitions of deformation and of incremental (or flow) theory. To a considerable extent definitions are arbitrary, of course, but Mr. Ghaswala's definitions are far too restrictive. The facts are that deformation theory, which is basically incorrect, postulates (as stated in the paper under the heading "Mathematical Theories of Plasticity") "* * * that stress and strain are uniquely related as long as loading continues.* * *" Incremental theory, which is self-consistent, gives the "* * * increment in strain in terms of the existing state (stress, strain) and (strain) history and the changes in stress." These theories are described in considerable detail elsewhere.[9,10,12]

The author shares Mr. Ghaswala's high regard for the work of Mr. Baker and his colleagues at Cambridge University. He enjoyed a pleasant association with Bernard Neal and with Jacques Heyman, each of whom spent a year at Brown University, Providence, R. I., and returned to Cambridge. Close liaison between the Brown University group under Mr. Prager and the group under Mr. Baker has been materially aided by the two years spent by Paul Symonds in Cambridge.

There is complete agreement with Mr. Karol's opening remarks that a naive approach in particular cases is as good as a more formal one. The difficulty lies in deciding which cases are simple and which are not. It was one of the intentions of the paper to show possible stumbling blocks and caution signs. As an example, shear stress in the web of an I-beam is ignored in many limit design analyses today, and rightly so. However, Fig. 13 shows more clearly than the words of the paper that proper consideration of shear is important. The general question of placing upper and lower limits on the answer in more difficult examples is described in later papers.[37,38,39]

Mr. Karol has made a very able analysis of the buckling of the simple frame in Fig. 7. Fig. 14 demonstrates admirably that individual buckling of the bars usually governs. But, like Eq. 9*b*, it also shows that no matter how small the

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[37] "Limit Design of Beams and Frames," by H. J. Greenberg and W. Prager, *Proceedings-Separate No. 59*, ASCE, Vol. 77, February, 1951.

[38] "The Safety Factor of an Elastic-Plastic Body in Plane Strain," by D. C. Drucker, H. J. Greenberg, and W. Prager, Paper No. 51-A-3, given at annual meeting, American Society of Mechanical Engineers, Nov., 1951.

[39] "Extended Limit Design Theorems for Structures and Continuous Media," by D. C. Drucker, H. J. Greenberg, and W. Prager, to appear in the *Quarterly of Applied Mathematics* (publication pending).

l/r ratio is made, buckling is still important. This is another point that might be overlooked. Here again the illustration in the paper was to acquaint civil engineers with this problem, not to derive design rules for particular structures. Also, even though Shanley's concept most often leads to loads that are close to the reduced modulus loads, it does not always do so. Furthermore, his physical idea is the proper one and will certainly be employed in the future.

Mr. Steed is entirely correct in his statement that simple limit design of pin-connected trusses is not permissible. Buckling must be taken into account; in the case of structural steel it is known that after plastic buckling the load drops as the member is shortened. The history of the stress in each member must be considered in some way, or in some cases it may be possible to prestress the truss to avoid premature buckling. It would appear, however, that the technique described by Mr. Steed becomes very involved for a highly indeterminate structure if a design for minimum weight is desired. If a combination of Mr. Steed's method and an upper and lower bound procedure could be found, it would appear to have real promise for truly complicated framed structures.

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SUCCESSIVE APPROXIMATIONS FOR BEAMS ON AN ELASTIC FOUNDATION

BY E. P. POPOV,[1] ASSOC. M. ASCE

WITH DISCUSSION BY MESSRS. RALPH E. FADUM; J. V. DU PLESSIS; W. E. HANSON; JACOB KAROL; E. E. DE BEER AND D. KRSMANOVITCH; AND E. P. POPOV.

SYNPOSIS

A method resembling the well-known moment-area procedure is presented for the solution of beams on an elastic foundation. Essentially it is an extension, adaptation, and systematization of the famed Vianello-Stodola procedures as are very briefly mentioned by August Föppl[2] and Miklós Hetényi.[3] Because of the inherent power of these procedures, reasonably accurate results are obtained simply and quickly. The difficult mathematics that occur in the usual analytical solutions are completely avoided.

INTRODUCTION

The supporting media, such as soil, as well as a load-carrying beam or footing that rests on it, usually display a certain amount of elastic behavior; yet, for simplicity, it is customary in most cases to consider the footing to be infinitely rigid. The design is based on the assumption of uniform soil pressure distribution if the centroid of the footing coincides with the resultant of the applied forces. Similar simplified treatment is used for eccentric loads. Such an assumption becomes less and less accurate as the footing or the load-carrying beam becomes relatively more flexible. For example, in the case of a railroad rail supported by ties, the ordinary solution would lead to erroneous results and cannot be used. There is an important class of problems in which flexibility of the beam on an elastic foundation must be considered.

If the supporting medium is considered as a true elastic continuum, the problem becomes unusually difficult and only a few[4] rigorous mathematical

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[1] Associate Prof. of Civ. Eng., Univ. of California, Berkeley, Calif.

[2] "Vorlesungen über Technische Mechanik," by August Föppl, B. G. Teubner, Leipzig, 1900, Vol. III, p. 264.

[3] "Beams on Elastic Foundation," by Miklós Hetényi, The Univ. of Michigan Press, Ann Arbor Mich., 1946, p. 99.

[4] *Ibid.*, pp. 197–214.

solutions exist. However, if the reaction forces are assumed to be proportional to the deflection of the beam at that point, the problem becomes greatly simplified; yet, the solution of the simplified problem appears to yield results that are in reasonable agreement with the existing exact ones. In the case of a foundation problem, rather complex soil phenomena are involved and an extreme mathematical refinement in the solution of a practical problem seems unwarranted. Moreover, in many problems, such as those occurring at the stiffening rings in thin shells, grillage beams, etc., the foregoing assumption holds rigorously.

The idealized foundation assumed here may be thought of as one composed of an infinite number of independent springs. The elastic constant of these "springs" represents the foundation modulus. These springs can take both tension and compression forces. This assumption was first introduced[5] by Eduard Winkler. Many subsequent investigators used this assumption, perfected the mathematical aspect of the problem, and, in general, found good agreement between the experimental and analytical results. The earlier works of Hermann Zimmermann[6] and Keiichi Hayashi[7] are notable. A method developed by A. N. Krilov[8] is rather convenient and has been introduced into a textbook.[9,10] A book by Mr. Hetényi[3,11] discusses the same problem in a most complete and thorough manner. In this work, in addition to tables and complete solutions of many special cases, some series solutions are used which greatly facilitate the utility of the method for practical problems.

However, the foregoing procedures are thought by some to be difficult as they involve lengthy mathematical expressions. To overcome this difficulty a method by Zusse Levinton,[12] M. ASCE, was developed in which the entire procedure is mainly reduced to the solution of simultaneous equations. The determination of coefficients and constants in many cases may be routinized. This method involves some legitimate approximations from the engineering point of view. It becomes quite cumbersome, however, as the complexity of the problem increases.

To avoid the tediousness of a mathematical solution, ingenious electrical arrangements have been developed by A. L. Goflin[13] for solving the necessary differential equations. The method appears to be rapid, it can provide for the variable moment of inertia of the beam, but it requires special equipment and techniques.

To overcome some of the difficulties mentioned in this brief review of the available methods an approximate procedure is developed in this paper. The method proposed utilizes the great power of graphical integration. In many practical problems the moment of inertia of a beam as well as the foundation modulus are variable quantities along the length of the beam. Such variation may be discontinuous. Even the original mathematical formulation for the moment of inertia and the foundation modulus may require some approximations. Thereafter the solution may become very complex or entirely untractable. Although it lacks extreme accuracy, the method proposed overcomes these complexities. Essentially the problems may be solved with nearly the same ease whether they are simple or complex in the sense of mathematical formulation. The steps used in the solution of the problem are all very simple and are well known to engineers.

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[5] "Die Lehre von der Elastizitaet und Festigkeit mit Besonderer Rücksicht auf ihre Anwendung in der Technik," by Eduard Winkler, Dominicus, Prag, 1867, p. 182.

[6] "Die Berechnung des Eisenbahnoberbaues," by Hermann Zimmermann, W. Ernst and Sohn, Berlin, 2d Ed., 1930.

[7] "Theorie des Trägers auf Elastischer Unterlage und ihre Anwendung auf den Tiefbau," by Keiichi Hayashi, Julius Springer, Berlin, 1921.

[8] "Analysis of Beams on Elastic Foundations," by A. N. Krilov, Akademyia Nauk, Leningrad, 2d Ed., 1931.

[9] "Strength of Materials," by M. M. Filonenko-Borodich, *Stroiisdat*, Moscow, 1940, pp. 315–337.

[10] *Ibid*, pp. 540–551.

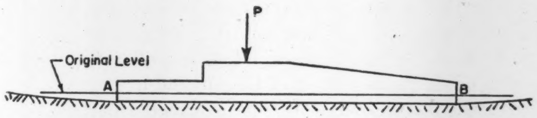
[11] "Strength of Materials," by S. Timoshenko, D. Van Nostrand Co., Inc., New York, N. Y., 2d Ed., 1941, Pt. II, p. 1.

[12] "Elastic Foundations Analyzed by the Method of Redundant Reactions," by Zusse Levinton, *Transactions*, ASCE, Vol. 114, 1949, pp. 40–52.

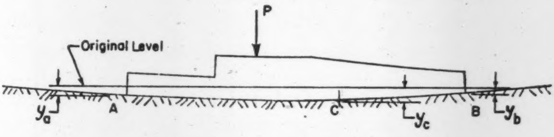
[13] "An Electric Model of an Elastically Supported Beam," by A. L. Goflin, *Bulletin*, Akademyia Nauk, Moscow, December, 1946, pp. 1743–1751.

General Procedure

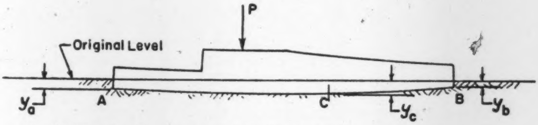
The usual assumption of a rigid beam on an elastic foundation is shown in Fig. 1(*a*), the true situation for a flexible beam is shown in Fig. 1(*b*), and in an idealized elastic foundation only the "springs" directly under the beam are loaded and deflected as shown in Fig. 1(*c*). The beam, being flexible, assumes some such curve as AB shown exaggerated in Fig. 1(*d*). Since the foundation is elastic, at point A the force exerted by the spring on the beam is $k_a\, y_a$, in which



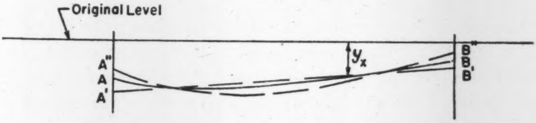
(a) Rigid Beam on an Elastic Foundation.



(b) Elastic Beam on an Elastic Foundation.



(c) Elastic Beam on an Idealized Elastic Foundation.



(d) Exaggerated Deflections.

Fig. 1.—General Case of a Beam on an Elastic Foundation

k_a is the foundation modulus at point A and y_a is the deflection of the beam at point A. Similarly, at point B such a force is $k_b\, y_b$, and at point C it is $k_c\, y_c$. Thus, knowing the foundation moduli, the soil pressure diagram can be easily constructed in a manner corresponding to a deflected beam. Statics considerations require that the resultant of the foundation pressure be equal and opposite to the forces applied to the beam, including its own weight. In this method this requirement must be kept in mind constantly as it is used over and over again to adjust the solutions obtained.

Initially the shape of the elastic beam is not known and must first be assumed perfectly rigid. This is the very assumption used in ordinary design and its accuracy for the final solution has been questioned herein. Under this assumption a symmetrically loaded beam will settle an equal amount everywhere if the foundation constant is uniform or variable, and symmetrical in relation to the beam. If the loading, or the foundation modulus, is unsymmetrical, the "rigid" beam will rotate as well as settle. The amount of this angular movement can be computed easily. Thus (thinking for a moment in terms of stresses, which can be converted into deflections once the foundation modulus is known), when there is rotation, the stresses are $s = P/A \pm M\, c/I$ instead of $s = P/A$ of the former case. The symbols in these relations have the usual meaning: P is the resultant of the applied loads; A is simply the area under the footing if k is uniform, or the transformed area (see Example 3, subsequently) if k is variable; c is the distance from the neutral axis to the part of the foundation investigated; I is the moment of inertia of the footing contact area (transformed area if k is variable) about its neutral axis; and M is the "rotating" moment, which is equal to P multiplied by the distance from the centroid of the footing area. In this manner, regardless of the type of loading or the foundation moduli, the pressure distribution under a rigid beam may be determined; and, knowing the pressure distribution, the corresponding deflections can be determined, if desired.

The result of such calculations could be represented by a straight line A′ B′ in Fig. 1(*d*), and the corresponding pressure distribution under the same beam would be $k_x\, y_x$. The subscript x in this relation designates the particular modulus and the particular deflection at a distance x from the end of the beam. Then the beam is considered flexible and its elastic curve is determined under the influence of the actual applied loads and the $k_x\, y_x$-loading. This is done by plotting the usual $\frac{M}{E\, I}$-diagram, subdividing it into a convenient number of segments, and graphically or arithmetically determining the elastic curve by the conjugate beam method. (In the expression $M/(E\, I)$, the symbol M denotes a bending moment at a section, and I is the moment of inertia of the corresponding cross-sectional area.) These deflections variously increase and relieve the foundation pressures, upsetting the static balance. By "locking" the beam in the curved shape just determined the results are adjusted to bring them into static balance—that is, the resultant of the foundation pressure under the deflected beam is made to be the equilibrant of the applied forces. This procedure is similar to the original treatment given a rigid beam, except that a constant force caused by the deflection of the beam enters into the calculations.

Such results are again shown diagrammatically in Fig. 1(*d*) by the curved line A″ B″. The shape of the curve A″B″ is the elastic curve previously determined by the conjugate beam method.

Curve A″ B″ exaggerates the deflection of the elastic curve as it was based on the loading that would occur on an infinitely rigid beam. In the case illustrated, the ends of the beam actually would "escape" such heavy loading by deflecting as a result of the flexibility of the beam. These deflections, and hence the corresponding pressures, "overshoot" the true values. The correct answer lies somewhere between the line A′ B′ and the curve A″ B″. Usually the best estimate that can be made at the end of this cycle is that the true elastic curve is the average between A′B′ and A″B″. For the more rigid members the answer is closer to A″B″, but for flexible members, it is closer to A′B′. In the latter case, if A″B″ is taken as the approximation for subsequent work, the solution may not converge. Thus for uniformity and to avoid the danger of working with a nonconvergent solution, it is recommended that the average of A′B′ and A″B″ be used.

After averaging the ordinates of curves A′B′ and A″B″, the first cycle of the solution is complete. The corresponding foundation pressures are $k_x y$, in which y is the average ordinate of the two curves at a point x. In order to improve the accuracy of the solution, the pressure distribution corresponding to the foregoing average deflection is then used. This pressure distribution, together with the actual loads, is used to obtain a new $\frac{M}{EI}$-diagram, and, proceeding as before, a new elastic curve is obtained. Then, it is adjusted for static compatibility. A beam "locked" into the curved shape of the new elastic curve is used in such calculations. The average of the initial elastic curve of this cycle with the new adjusted elastic curve is the desired answer for the two cycles.

It has been found that a solution made on the foregoing basis converges very rapidly. Seldom are more than two cycles required.

To illustrate the details of this method three examples follow, in which some pertinent points are discussed. Cases of variable moment of inertia, variable foundation moduli, and the utility of the transformed areas for a given case are shown. To facilitate comparison with other methods, data for Examples 2 and 3 are from published works to be cited. However, the type of problem in Example 1, which is unusually simple to solve by the method discussed, is normally avoided, as it presents some mathematical difficulties.

EXAMPLE 1

Consider a continuous concrete footing having a cross section (Fig. 2(*a*)) with a line load of 150 kips per ft (k/ft). The elastic modulus of concrete is taken as $E = 432{,}000$ kips per sq ft, and the foundation modulus is assumed to be $k_o = 300$ kips per cu ft. In this example the soil pressure distribution is found for the foregoing conditions. The weight of the beam is neglected.

Assuming the footing to be rigid, the pressure distribution is $s = 150/30 = 5$ kips per sq ft (k/ft²), and the uniform deflection is $\Delta = 5/300 = 0.0167$ ft. Considering a strip of the footing of unit width, the bending moment diagram is

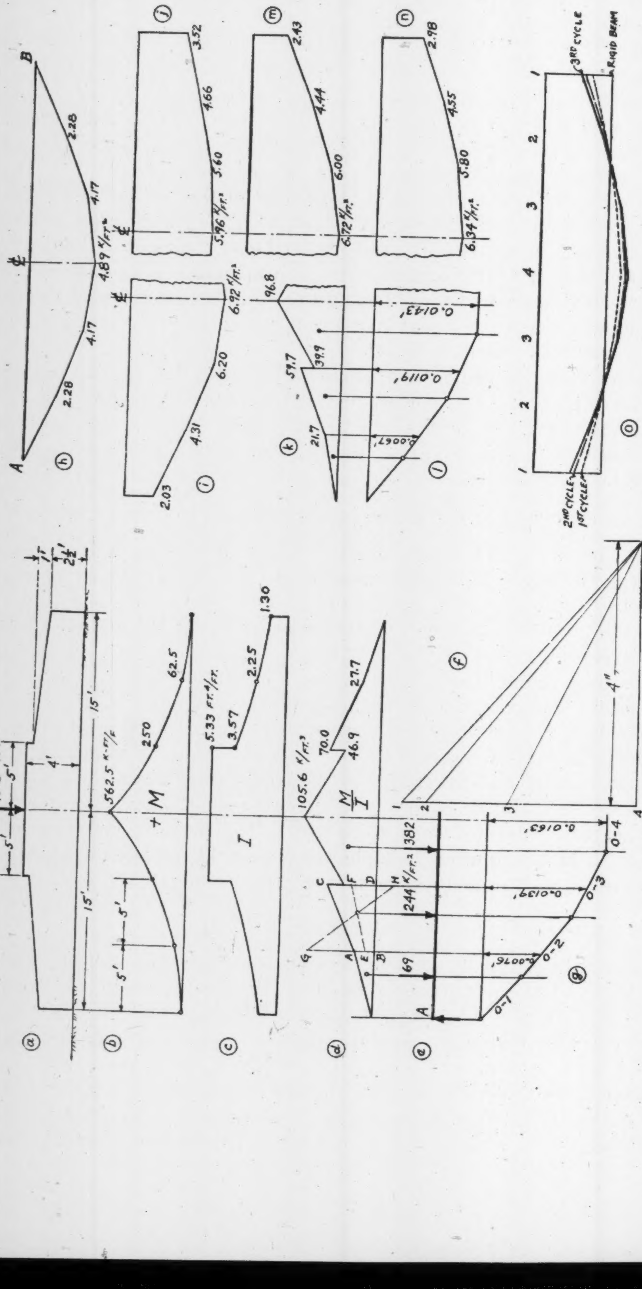

(a)
150 K/FT.
5'
5'
4'
1'
2½'
15'
15'
(b)
562.5 K-FT/FT
+M
250
62.5
5'
5'
(c)
I
5.33 FT.⁴/FT.
3.57
2.25
1.30
(d)
105.6 K/FT.³
M/I
70.0
46.9
27.7
G
A
E
B
C
F
D
H
(e)
A
69
244 K/FT.²
382
(f)
1
2
3
4
O
4"
(g)
0-1
0-2
0-3
0-4
0.0076'
0.0139'
0.0163'
(h)
A
B
℄
2.28
4.17
4.89 K/FT.²
4.17
2.28
(i)
2.03
4.31
6.20
6.92 K/FT.²
(j)
5.96 K/FT.²
5.60
4.66
3.52
(k)
21.7
57.7
39.9
96.8
(l)
0.0067'
0.0119'
0.0143'
(m)
6.72 K/FT.²
6.00
4.44
2.43
(n)
6.34 K/FT.²
5.80
4.55
2.98
(o)
1
2
3
4
3
2
1
2ND CYCLE
1ST CYCLE
3RD CYCLE
RIGID BEAM

FIG. 2.—EXAMPLE 1, SYMMETRICAL LOADING

computed for values of 5 kips per ft up, and 150 kips per ft down (Fig. 2(*b*)). In Fig. 2(*c*), a plot of the moment of inertia of the cross-sectional area is shown; this was computed at 5-ft intervals using the relation $I = \frac{1}{12} b\, d^3$. For example, at the center $I = \frac{1}{12} \times 1 \times 4^3 = 5.33$ ft^4 per ft, and directly below this, in Fig. 2(*d*), the $\frac{M}{I}$-diagram is shown. Only four values are computed and the actual curve is approximated by a series of straight lines. Only one half of the diagram is shown as this problem deals with complete symmetry. The M/I-diagram gives the elastic loading for the conjugate beam. The distributed loadings are concentrated at their respective centers of gravity. It will be recalled that such centers for triangles are located at a distance of one third from the base. From a trapezoid, one way of finding the center of gravity is to bisect distances such as AB and CD (Fig. 2(*d*)) and also to lay them off on the opposite sides as shown. This procedure yields points E, F, H, and G. The intersection of lines GH and EF gives the desired point.

To compute the elastic weights on the conjugate beam (Fig. 2(*e*)) simply add the areas of the $\frac{M}{I}$-diagram for a particular segment. For example, $\frac{1}{2} \times 5 \times (27.7 + 70.0) = 244$ kips per sq ft. Then the elastic weights are plotted vertically to a scale in Fig. 2(*f*), a pole O is selected, and lines parallel to the resulting rays are drawn in Fig. 2(*g*). The funicular polygon (Fig. 2(*g*)), to a certain scale, represents the deflected beam subjected to the assumed loading. In order to obtain actual deflections the vertical distances in inches are multiplied by a scale factor. This scale factor is obtained by multiplying the scale of the horizontal beam distances by the scale of the vertical measurements in the force polygon, by the pole distance in inches, and, in the present example, also by dividing by the elastic modulus of the beam. Thus, the factor in the original solution was $\frac{5 \times 200 \times 4}{432{,}000} = 0.00925$ ft per in. Several deflection ordinates from a straight line passing through the end points of the beam are recorded in Fig. 2(*g*). The pole distance may be adjusted to make the scale factor more convenient.

The foundation modulus is $k_o = 300$ kips per cu ft. The deflections, in feet (Fig. 2(*g*)), correspond to the pressures (Fig. 2(*h*)) which are obtained by the relation $k_o\, y$, provided that, for awhile, the ends are assumed to be precisely at the surface. Then the beam is "locked" with the curvature shown and the entire curve is depressed uniformly into the foundation to obtain a static balance. Summing the foundation pressures already developed when the ends A and B are at the surface, $P = 2 \times \frac{1}{2} \times 5\,(2 \times 2.28 + 2 \times 4.17 + 4.89) = 89$ kips. Hence, $150 - 89 = 61$ kips remains to be uniformly distributed; that is, $s = \frac{61}{30} = 2.03$ kips per sq ft. Adding this value to the ordinates of Fig. 2(*h*) the results are as shown in Fig. 2(*i*). Averaging this with the initially assumed uniform pressure distribution of 5 kips per sq ft yields the results of one cycle of the semigraphical solution. This is shown in Fig. 2(*j*).

To improve the accuracy of the results, another cycle may be made. Soil pressure loading shown in Fig. 2(*j*), together with the actual loads, are used in the next approximation. The $\frac{M}{I}$ -diagram is shown in Fig. 2(*k*), and the elastic curve, in Fig. 2(*l*); the accompanying force polygon is not shown. After adjusting the results for static compatibility that accommodate the curve (Fig. 2(*l*)), the results shown in Fig. 2(*m*) are obtained. Averaging these results with those in Fig. 2(*j*) gives the results of the second cycle (Fig. 2(*n*)).

Two cycles are believed to give results that are sufficiently accurate for practical purposes. If more accuracy is desired, larger diagrams and more segments must be used, and, perhaps, a few more cycles of work must be performed. In a comparison of the results, the foregoing two cycles and one additional cycle are shown in Fig. 2(*o*) and Table 1.

TABLE 1.—REFINEMENT POSSIBLE BY EXTRA CYCLES, EXAMPLE 1 (PRESSURE INTENSITY IN KIPS PER SQUARE FOOT)

Point	Rigid beam	First cycle	Second cycle	Third cycle
1	5.0	3.5	3.0	2.7
2	5.0	4.7	4.6	4.5
3	5.0	5.6	5.8	5.9
4	5.0	6.0	6.3	6.5

EXAMPLE 2

A "weightless" beam, 10 in. by 8 in., with the loading shown in Fig. 3(*a*) is resting on an elastic foundation. The modulus of the foundation $k_o = 200$ lb per cu in. The elastic modulus of the beam $E = 1.5 \times 10^6$ lb per sq in. (psi), and its moment of inertia $I = 426.7$ in.^{4} Pressure distribution under this beam is determined at this point. (Mr. Hetényi has solved this problem mathematically.[14] Data for Example 2 were selected from his work.)

The resultant of the applied loads is found to be located 7.5 in. off center as shown in Fig. 3(*b*). This indicates that, in addition to the uniform settlement, angular displacement takes place. The contact area of the beam with the foundation is 1,200 sq in. and the moment of inertia of this area $I = \frac{1}{12} \times 10 \times 120^3 = 1{,}440{,}000$ in.^{4} Hence, it is concluded that, if the beam were rigid, the pressure at end A would be −11.23 lb per sq in., and at end B, −5.11 lb per sq in., since $s_{A \text{ or } B} = \frac{P}{A} \pm \frac{M\,c}{I} = -\frac{9{,}800}{1{,}200} \mp \frac{9{,}800 \times 7.5 \times 60}{1{,}440{,}000} = -8.17 \mp 3.06$. This pressure variation is linear as shown in Fig. 3(*c*). As usual, minus signs indicate compression.

With actual applied loads as in Fig. 3(*a*) and pressure distribution as in Fig. 3(*c*), the bending moment diagram was calculated. It is approximated with the series of straight lines shown in Fig. 3(*d*). After computing the elastic weights, plotting them to a suitable scale, and selecting a pole (just as in the previous example), the elastic line shown in Fig. 3(*e*) is obtained. The deflections from the line passing through the ends are noted. Lack of symmetry in this case required that the entire curve be drawn in. The scale factor had

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[14] "Beams on Elastic Foundation," by Miklós Hetényi, The Univ. of Michigan Press, Ann Arbor, Mich., 1946, p. 47.

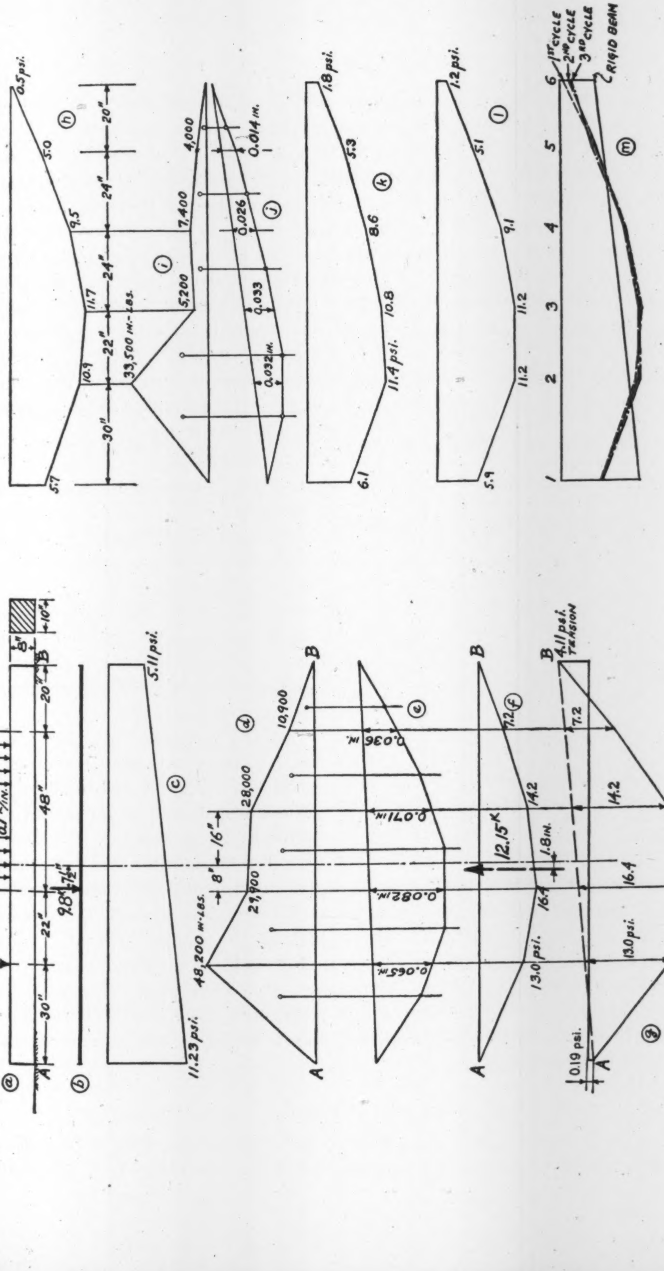
ⓐ
5^K
0.1^K/IN.
30"
22"
48"
20"
8"
10"
9.8^K
7½"
A
B
℄
ⓑ
ⓒ
11.23 psi.
5.11 psi.
ⓓ
48,200 IN-LBS.
21,900
28,000
10,900
8"
16"
ⓔ
0.065 IN.
0.082 IN.
0.071 IN.
0.036 IN.
ⓕ
12.15^K
1.8 IN.
13.0 psi.
16.4
14.2
7.2
ⓖ
0.19 psi.
4.11 psi.
TENSION
ⓗ
5.7
10.9
11.7
9.5
5.0
0.5 psi.
ⓘ
33,500 IN.-LBS.
5,200
7,400
4,000
ⓙ
0.032 IN.
0.033
0.026
0.014 IN.
ⓚ
6.1
11.4 psi.
10.8
8.6
5.3
1.8 psi.
ⓛ
5.9
11.2
11.2
9.1
5.1
1.2 psi.
ⓜ
1
2
3
4
5
6
1ST CYCLE
2ND CYCLE
3RD CYCLE
RIGID BEAM

Fig. 3.—Example 2, Unsymmetrical Loading

to be divided by the moment of inertia of the beam, as bending moment rather than the $\frac{M}{I}$ -diagram was used.

By "locking" the beam in the deflected shape just found, ends A and B are moved so that they are at the surface. The pressures developed in the foundation by the curved beam in this position are $k_o y$. These are shown in Fig. 3(*f*). The resultant of these pressures is 12.15 kips (12.15^k) upward, 1.8 in. to the left of the center line. These quantities were obtained by subdividing the pressure area into triangles and rectangles and by taking their moments about one end of the beam. This resultant is in excess of the applied loads which total 9.8 kips; therefore, it is necessary to apply an upward pull of 2.35 kips to the "locked" curved beam. Moreover, the pressure resultant develops a clockwise resisting moment of only 21.8 kip-in., whereas, to keep the applied forces in balance, 73.5 kip-in. of moment is required. To develop the foundation resistance for the difference of 51.7 kip-in., an additional rotation to the curved beam (as a unit), of this magnitude, must be applied in the counterclockwise direction. Hence, from the known relations these pressure increments at the ends are: $s_A = \frac{P}{A} \pm \frac{M\,c}{I} = + \frac{2{,}350}{1{,}200} - \frac{51{,}700 \times 60}{1{,}440{,}000} = -0.19$ lb per sq in.; and, similarly, $s_B = -4.11$ lb per sq in. This pressure distribution varies linearly from point A to point B. Adding it algebraically to the pressure distribution shown in Fig. 3(*f*), the distribution shown in Fig. 3(*g*) is obtained. This diagram is statically compatible with the applied loads. Its resultant is equal and opposite to the 9.8-kip resultant of the applied loads.

TABLE 2.—REFINEMENT POSSIBLE BY EXTRA CYCLES, EXAMPLE 2 (PRESSURE INTENSITY IN POUNDS PER SQUARE INCH)

Point	Rigid beam	First cycle	Second cycle	Third cycle	Exact
1	11.23	5.7	5.9	5.9	6.07
2	9.70	10.9	11.2	11.3	...
3	8.58	11.7	11.2	11.4	...
Center[a]	8.17	11.0	10.5	10.7	10.39
4	7.36	9.5	9.1	9.2	...
5	6.14	5.0	5.1	4.8	...
6	5.11	0.5	1.2	1.3	1.26

[a] Pressures at the center are obtained by interpolation.

The average of the pressure distribution at the corresponding points in Figs. 3(*c*) and 3(*g*) yields the desired results for one cycle, as shown in Fig. 3(*h*).

For the second cycle, the pressure distribution in Fig. 3(*h*), together with the applied loads, gives an approximate moment diagram and the new elastic curve in Figs. 3(*i*) and 3(*j*), respectively. By "locking" this curved beam and adjusting the results for static compatibility, the pressure distribution shown in Fig. 3(*k*) is obtained. The average of the values in Figs. 3(*h*) and 3(*k*) represents two cycles (see Fig. 3(*l*)).

For comparison, the results of one additional cycle and the results of a mathematical solution,[14] are given in Table 2 and Fig. 3(*m*). The agreement appears to be satisfactory.

EXAMPLE 3

Consider a "weightless" beam, 8 in. by 6 in. (Fig. 4(*a*)), with two equal loads of 100 lb each at the ends. The constants for the beam are: $E = 2.5 \times 10^6$ lb per sq in. and $I = 144$ in.[4] The foundation modulus is assumed to vary linearly from $k_{oA} = 87.5$ lb per cu in. at end A to $k_{oB} = 12.5$ lb per cu in. at end B. Mathematical solutions for the deflection line and pressure distribution for these data are available.[15] Herein it is solved by the semigraphical method.

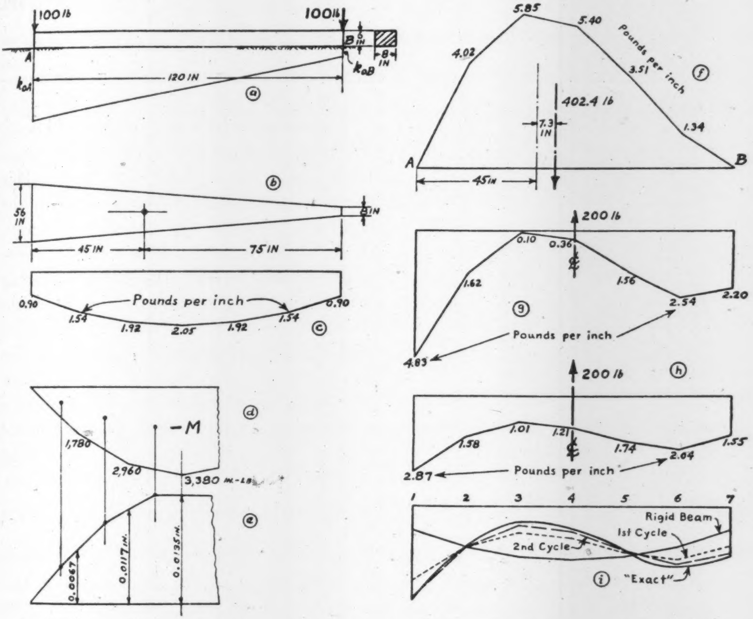
100 lb
100 lb
A
B
5 IN
8 IN
k_{oA}
k_{oB}
120 IN
(a)
56 IN
8 IN
45 IN
75 IN
(b)
0.90
1.54
1.92
2.05
1.92
1.54
0.90
Pounds per inch
(c)
1,780
2,960
3,380 IN.-LB
−M
(d)
0.0067
0.0117 IN.
0.0135 IN.
(e)
5.85
5.40
4.02
3.51
1.34
Pounds per inch
402.4 lb
7.3 IN
45 IN
A
B
(f)
200 lb
0.10
0.36
1.62
1.56
2.54
2.20
4.83
Pounds per inch
(g)
200 lb
(h)
1.01
1.21
1.58
1.74
2.04
1.55
2.87
Pounds per inch
1
2
3
4
5
6
7
Rigid Beam
1st Cycle
2nd Cycle
"Exact"
(i)

FIG. 4.—EXAMPLE 3, VARIABLE FOUNDATION MODULUS (CRITICAL VALUES AT 20 IN. APART)

This problem differs from the previous cases as the foundation modulus varies along the length of the beam. The stiffer foundation at end A offers more resistance to the load. However, precisely the same amount of resistance would be obtained with a uniform foundation modulus but with the width of the beam increased or decreased in the ratio of the foundation modulii. In the example considered, if k_{oB} is designated the uniform base modulus, the width of the beam at end A becomes $\frac{87.5}{12.5} \times 8 = 56$ in. As the foundation modulus, in this case, varies linearly, the equivalent area over the region where k_{oB} is the modulus is a trapezoid as shown in Fig. 4(*b*). Of course, any other modulus may be used as a base. The transformed area of the contact surface is 3,840 sq in.; its centroid is 45 in. from end A; and the moment of inertia of this area about its centroidal axis is $I = 3{,}744{,}000$ in.[4]

The loading is symmetrical in this case but the center of the foundation pressure resistance is not. The remaining parts of the problem are similar to Example 2. The pressure distribution for a rigid beam at point B is: $s_B = \frac{P}{A} \pm \frac{M\,c}{I} = -\frac{200}{3{,}840} - \frac{200 \times 15 \times 75}{3{,}744{,}000} = -0.1122$ lb per sq in., or $8 \times s_B = -0.90$ lb per in. of beam. At the center line, $s_C = -\frac{200}{3{,}840} - \frac{200 \times 15 \times 15}{3{,}744{,}000} = -0.0641$ lb per sq in. on the fictitious beam, or $32 \times s_C = -2.05$ lb per in. on the actual rigid beam, 32 in. being the width of the transformed area at the center. In this manner pressure distribution on the rigid beam has been computed at 20-in. intervals. Results are plotted in Fig. 4(*c*). This pressure distribution, together with the two loads, yields the moment diagram shown in Fig. 4(*d*), and an elastic curve shown in Fig. 4(*e*). As before, placing ends A and B at the surface of the foundation, the pressure distribution caused by the deflected beam is as shown in Fig. 4(*f*). The deflections in Fig. 4(*e*) are multiplied by the corresponding variable foundation modulii. The resultant of this pressure, together with the two end loads, gives a total force of 602.4 lb acting 9.8 in. to the right of the centroidal axis. This is adjusted to be statically compatible, using the transformed contact area: $s_B = -\frac{602.4}{3{,}840} - \frac{602.4 \times 9.8 \times 75}{3{,}744{,}000} = -0.275$ lb per sq in., or $-8 \times 0.275 = -2.20$ lb per in. of actual beam, etc. These results are shown in Fig. 4(*g*). Averaging this curve with the initial distribution shown in Fig. 4(*c*), the results for one cycle are as shown in Fig. 4(*h*). Results for two cycles together with the analytical values[15] are given in Table 3 and Fig. 4(*i*). Again, the agreement found may be considered entirely satisfactory.

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[15] "Beams on Elastic Foundation," by Miklós Hetényi, The Univ. of Michigan Press, Ann Arbor, Mich., 1946, p. 111.

TABLE 3.—Refinement Possible by Extra Cycles, Example 3 (Pressure Intensity in Pounds per Inch)

Point	Rigid beam	First cycle	Second cycle	Exact
1	0.90	2.87	3.54	3.62
2	1.54	1.58	1.57	..
3	1.92	1.01	0.73	0.57
4	2.05	1.21	0.95	..
5	1.92	1.74	1.70	1.65
6	1.54	2.04	2.22	..
7	0.90	1.55	1.79	1.89

Conclusions

The paper, with the illustrative examples, treats most of the important points that are encountered in practice in connection with beams on elastic foundation. Rather complex problems with variable moments of inertia of the beam and variable foundation moduli can be solved with relative ease.

In the examples, the writer assumed consistently that the foundation can develop tension as well as compression stresses. This detail is of no importance during the performance of calculations. However, the end results must be interpreted with caution. If it is a soil problem, no tension is possible. The part of the beam that is experiencing only an upward pressure distribution must be statically compatible. Likewise, it must be borne in mind that, since examples involving soils were used, the foundation modulus is not a simple quantity. Reaction does not depend only on a deflection at a point; the adjoining areas also affect this value. To provide for this influence of adjoining areas, however, is one of the merits of the suggested procedure (k_o may be compensated in the light of this knowledge) and the problem remains comparatively simple.

With minor modifications, the method discussed is well adapted to other similar problems. Its utility for flexible floating structures is particularly notable.

The development of the paper followed the semigraphical sequence; but the method is similar to the well-known moment-area procedure, and, if desired, those who abhor graphics, may set up tables and perform all the necessary calculations accordingly. An attempt was made in this paper to illustrate the power and simplicity of the graphical method. All procedures are conventional. It should require little time to master the over-all scheme.

The method is not proposed for use in situations in which ready analytical solutions are available. Unfortunately, these cover only the simple cases.

DISCUSSION

Ralph E. Fadum,[16] Assoc. M. ASCE.—The author has proposed a method of analysis that can be used as an aid in designing a beam supported on an "elastic" (yielding) foundation. The solution to this problem, by successive approximations, is simple and involves procedures that are well known to structural engineers. It is applicable to beams whose moment of inertia (I) and modulus of elasticity (E) either are constant or vary with the length of the beam. Furthermore, the method is applicable irrespective of the relation, which is assumed to exist, between the settlement of the loaded soil surface and the corresponding soil reaction on the beam.

Mr. Popov is to be commended especially for setting forth clearly a procedure involving simple operations that has a wide field of application. The writer proposes to comment on the assumed relations between settlement of a loaded soil surface and the soil reaction; he also will discuss how the method involving successive approximations, as used by the author in the solution of the beam problem, can be applied to several other common structural problems involving yielding supports.

Relation Between Soil Reaction and Settlement of Soil Surface.—In the three examples, given to illustrate the method, it has been assumed that the relation between the settlement of the soil surface and the corresponding intensity of the soil reaction is a linear function expressed by the equation $p = k\,y$, in which p is the intensity of pressure; k is the so-called modulus of subgrade reaction expressed in units of pressure per unit of settlement; and y is the settlement of the soil surface. According to this expression, the beam is assumed to receive the same type of support that it would receive were it resting on springs that change length in accordance with Hooke's law. As the author has pointed out, this expression would also describe precisely the relation between the pressure exerted on the bottom of a floating structure and the depth of submergence of the bottom on which the pressure is exerted. In this instance, k would equal the unit weight of the liquid in which the structure floated. It is also to be noted that in the case of a pile cap supported by a group of piles, which are capable of resisting axial load only, the relation between the load, P, supported by a given pile, and its change of length y, is $P = \frac{E\,A}{L}\,y$, in which E is the modulus of elasticity; A is the average cross-sectional area; and L is the effective length of the pile. In the analysis of this case by the author's method, it can be seen that k is equivalent to $E\,A/L$.

The late H. M. Westergaard,[17] M. ASCE, has presented a convenient method of determining the distribution of load to each of a group of battered piles supporting a rigid pile cap. The author's method provides a means of extending Mr. Westergaard's analysis to take into consideration the flexibility of the pile cap.

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[16] Prof. and Head, Dept. of Civ. Eng., North Carolina State College, Raleigh, N. C.

[17] "The Resistance of a Group of Piles," by H. M. Westergaard, *Journal*, Western Soc. of Engrs., December, 1917, pp. 704–713.

It is known that, in the case of granular soils, the resistance to deformation varies with confining pressure. This property can easily be demonstrated by placing dry sand in a rubber bag. By evacuating the interior of the bag, atmospheric pressure is brought to bear on the sand. It will be noted in this simple experiment that, as the air is evacuated and the confining pressure acting on the sand is thereby increased, the stiffness of the mass of sand increases. When the sand is subjected to full atmospheric pressure, it has the consistency of a hard solid. In accordance with this demonstration, it would appear reasonable to assume that, in the case of a beam supported by a granular (gravel, sand, or sand and gravel) foundation, the modulus of subgrade reaction should be assumed to increase in value from the ends of the beam toward the center. Since the confining pressure acting on the granular foundation increases in this direction, the value of the modulus of subgrade reaction should be assumed also to increase in this direction. According to Mr. Popov, variations in the value of the modulus of subgrade reaction, k, can be taken into consideration by the transformed area concept. Thus, his method is well suited to the analysis of beams supported by granular foundations.

Analysis of Rigid Frames Supported by Yielding Foundations.—It may be noted that the general method of analysis for a beam supported by an elastic foundation, as suggested by the author, is also applicable to the case of a rigid frame supported by a yielding foundation. A procedure for analyzing this problem was proposed by the writer (at a meeting of the Soil Mechanics and Foundations Division, during the Annual Meeting of the Society in New York, N. Y., on January 19, 1945), as follows:

1. Determine the footing loads assuming no distortion of the frame due to foundation settlement.
2. Compute[18,19] the settlement of the individual footings using the loads found in step 1. It should be noted that, unless the distance between the footings is large in comparison to the thickness of the supporting soil strata, it would not be reasonable to assume that the settlement of the soil support is only a function of the load supported by the footing immediately above the point of settlement. In general, the settlement of a given footing, as determined from a settlement analysis, is a function of the load on the footing in question as well as the loads acting on all neighboring footings. The greater the thickness of the compressible soil strata is with respect to the distance between footings, the more the settlement of a given footing is influenced by the loads on neighboring footings.
3. Determine the change in footing loads resulting from the distortion caused by the differential settlements found in step 2. Note that the greater the absolute stiffness of the frame, the greater will be the change in footing loads resulting from a given magnitude of differential settlement. If the frame is perfectly flexible, of course, there will be no shift in footing loads accompanying differential settling.

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[18] "Fundamentals of Soil Mechanics," by D. W. Taylor, John Wiley & Sons, Inc., New York, N. Y., 1948, pp. 268-288.

[19] "Soil Mechanics in Engineering Practice," by Karl Terzaghi and Ralph B. Peck, John Wiley & Sons, Inc, New York, N. Y., 1948, pp. 56-74.

4. If the change in footing loads due to the distortion caused by differential settlements is significant, recompute the settlements using the corrected footing loads.

5. If the settlements as found in step 4 are materially different from those found in step 2, determine the change in footing loads resulting from the distortion of the frame caused by the differential settlements computed in step 4.

6. Repeat steps 4 and 5 successively until changes in footing loads are not significant.

Since the settlements as found in steps 2, 4, etc., cannot be determined with any refined degree of accuracy, one would not be justified, in most cases, in extending the analysis beyond step 3.

Analysis of Rigid Frames Supported by a Continuous Beam on an Elastic Foundation.—No difficulty is encountered in extending the method of successive approximations to the case of a frame supported by a continuous beam which is in turn supported by an "elastic" (yielding) foundation. The procedure for analyzing this case might be itemized as follows:

a. Determine the column reactions assuming no distortion of the frame;

b. Using the loads thus found in step *a*, determine the soil reactions acting on the beam by the author's method;

c. Determine the change in column reactions due to the distortion of the beam found in step *b*; and

d. Using the corrected column reactions, repeat step *b*, etc.

J. V. DU PLESSIS,[20] JUN. ASCE.—This clear paper demonstrates a delightfully simple method of designing foundations on the basis of the more rational assumption of an elastic soil than that of a rigid foundation. The necessary calculations can perhaps be minimized by making a closer first assumption as to the correct soil reaction than that indicated by the rigid foundation calculation leading to line A′B′ in Fig. 1(*d*). With this latter value determined, the ordinates may be adjusted from the analyst's experience, bearing in mind statical necessities only. Using this amended distribution of soil reaction, it is possible to determine, as before, the relevant moments, slopes, and deflections for the foundation beam. The closeness of this derived deflection curve to the one assumed—or the analyst's experience—will indicate any necessity for further iteration.

In the interest of economizing the analyst's time, a safety factor or load factor should provide for such contingencies as the excess strength introduced in detailing, the erroneous assumptions presented in an analysis, and slight discrepancies in measurement.

Bearing safety or load factors in mind, the rapidity of the convergence of the author's examples indicates that, with experience, further investigation by the analyst will be unnecessary. The once-amended, assumed, soil-reaction diagram will be adequate for determining the moments for which the foundation is to be proportioned.

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[20] Research Asst., Univ. Eng. Laboratories, Bristol, England.

W. E. Hanson,[21] Assoc. M. ASCE.—A wide adoption of Mr. Popov's method for the analysis of many structural problems can be expected in the future. The simplicity and soundness of the structural analysis cannot be disputed if the basic assumptions are satisfied. It is somewhat unfortunate that the examples deal with the type of problem for which these basic assumptions are least applicable. It is for this reason that the writer wishes to present some of the problems involved in surmounting the complexities of the basic assumptions as applied to flexible foundations on soil. On the other hand, much could be written concerning the advantages and exactness of the method for the analysis of other problems such as the distribution of load from continuous slabs to supporting joists.

The actual final distributions of contact pressure probably are not far from those obtained by the author if the flexible foundations are located on a deep deposit of uniform sand with several feet of surcharge above the base. Also, similar pressure distributions would be expected if the foundations were located on the surface of sand which possessed considerable cohesion. However, if the same foundations rest on clay, quite different pressure distributions would exist.

To explain the reasoning that leads to the preceding statements, one must forget for a moment all the theory that follows the injection of a fictitious foundation modulus into the discussion of the problem. First, and of primary importance, is the influence of soil type on the distribution of contact soil pressure beneath rigid footings. Secondly, the changes in the pressure distribution resulting from the flexibility of the footing must be visualized. Fig. 5 shows the qualitative influences of soil type and flexibility. Such contact pressure distributions as those shown in Fig. 5 have been substantiated experimentally by Oscar Faber[22] and discussed by A. E. Cummings,[23] M. ASCE. Mr. Faber's tests, conducted with a somewhat flexible bearing plate, illustrate clearly the difference in behavior between sands and clays with the test plate at the surface and at a depth below the surface.

The fictitious foundation modulus (sometimes referred to as the coefficient of subgrade reaction) has been ignored intentionally in the preceding remarks. The writer believes that this is the only approach to the problem which demonstrates explicitly the complexity of the relation between the stress-strain characteristics of the subsoil, the flexibility of the footing, and the contact pressure on the base.

Presumably, the variation in the foundation modulus used in Example 3 is an attempt to account for the effects of the variation of the density or compressibility of the soil. The foundation area is transformed into one which agrees with the assigned variation in k_o. Perhaps the author has in mind a similar transformation for obtaining the initial pressure variation under a rigid footing on clay. Such a method would appear feasible if it were not impossible to determine the magnitude and variation in the foundation modulus across the footing. This will be borne out by the following discussion of the method commonly used to determine the foundation modulus.

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[21] Associate Prof. of Civ. Eng., Univ. of Illinois, Urbana, Ill.

[22] "Pressure Distribution Under Bases and Stability of Foundations," by Oscar Faber, *The Structural Engineer*, London, March, 1933, p. 116.

[23] "Lectures on Foundation Engineering," by A. E. Cummings, *Circular Series No. 60*, Univ. of Illinois, Eng. Experiment Station, Urbana, Ill., p. 89.

After much discussion of the limitations of the fictitious foundation modulus, Karl Terzaghi, Hon. M. ASCE, and Ralph B. Peck, M. ASCE,[24] state

> "Therefore, the evaluation of K_s (foundation modulus) involves many uncertainties, and the customary procedure for determining K_s on the basis of small-scale load tests is subject to all the limitations of the load-test method * * *."

The notorious habit of nature of laying down soil deposits in a haphazard fashion presents the possibility of obtaining soil conditions similar to those shown in Fig. 6. If a footing 30 ft wide, such as that used in Example 1, were placed above such a variation, the effects of the dense or stiff pockets would be quite pronounced; yet, no load test within the boundaries of the footing at the elevation of the base would show the influence of the dense or stiff material. Furthermore, the results of a second load test may differ considerably from the first when conducted nearby on what appears to be uniform soil. This fact has been emphasized by Woodland G. Shockley,[25] Assoc. M. ASCE. The location of the water table also deserves consideration if the results of load tests are to be applied successfully to the design of a footing.

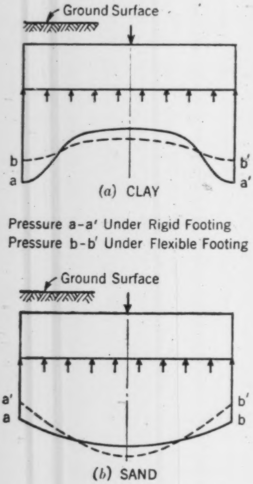
Ground Surface
b
a
b'
a'
(a) CLAY
Pressure a-a' Under Rigid Footing
Pressure b-b' Under Flexible Footing
Ground Surface
a'
a
b'
b
(b) SAND

FIG. 5.—EFFECTS OF SOIL TYPE AND FOOTING FLEXIBILITY ON CONTACT PRESSURE DISTRIBUTION BENEATH A FOOTING

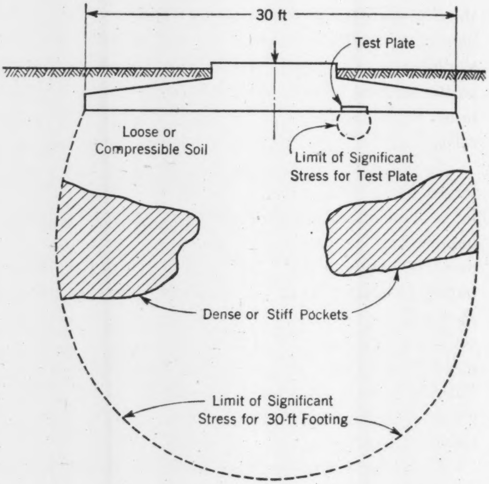
30 ft
Test Plate
Loose or Compressible Soil
Limit of Significant Stress for Test Plate
Dense or Stiff Pockets
Limit of Significant Stress for 30-ft Footing

FIG. 6.—VARIABLE SOIL CONDITIONS AND LIMITS OF SIGNIFICANT STRESS FOR FOOTING AND FOR TEST PLATE

Despite the limitations listed in the preceding paragraph, let it be assumed that the results of the load tests are accepted. Then, by proper computations involving the shape and dimensions of the prototype, as well as such uncertainties as the weight of surcharge and Poisson's ratio, the foundation modulus for the footing is obtained. The pertinent question for footings on clay still remains unanswered—"How can a single value of k_o, thus obtained, be adjusted to give contact pressures which agree with the variation shown in Fig. 5(*a*)?" Suddenly, the investigator realizes that the contact pressure distribution beneath the somewhat flexible bearing plate used for the load test must have been quite complex. Mr. Faber[22] proved this fact.

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[24] "Soil Mechanics in Engineering Practice," by Karl Terzaghi and Ralph B. Peck, John Wiley & Sons, Inc., New York, N. Y., 1948, p. 216.

[25] "Field Tests Determine Foundation Modulus for Bayou Bodcau Dam," by Woodland G. Shockley, *Civil Engineering*, July, 1950, p. 24.

In view of the complexities and uncertainties involved and the usual lack of thorough subsurface exploration and good load-test data, can the analytical refinement given in the paper be justified for common flexible footings? For certain soil conditions, the flexible footing in Example 1 could conceivably have shears and moments equal to, or even slightly greater than, those produced by a uniform pressure distribution. Unless extensive boring data are available which indicate reliably that uniform soil exists throughout the soil mass subject to significant stress from the footing, the writer prefers the use of a uniform or planar distribution for the design of common footings even though they may possess considerable flexibility. Mr. Cummings[23] has presented some comparisons between moments resulting from different extreme variations in contact pressure. These comparisons, coupled with a knowledge of the complexities involved, make the uniform or planar pressure distribution sufficiently justifiable for most design purposes.

Mr. Popov has presented a commendable paper from the standpoint of structural analysis. His method is easily applied to many engineering problems approachable through the requirement that the deflections of a structure must be compatible with (1) the elastic properties of the supports and (2) the flexibility of the structure. The tools of analysis used by the author, such as moment area and successive approximations, are what might properly be termed the everyday language of the structural designer.

JACOB KAROL,[26] M. ASCE.—The point of particular interest in this excellent paper is the use of the rigid beam pressure distribution as a basis for starting the analysis of beams on an elastic foundation. By a slight variation in the subsequent procedure, it will be shown that only two steps are required to determine the final pressures on the foundation.

The proposed procedure of analysis is as follows:

1. Calculate the rigid beam pressures from

$$s = \frac{P}{A} \pm \frac{M\,c}{I} \qquad \text{(1)}$$

Determine the deflections of the beam for the applied load and the foundation pressures.

2. Calculate the pressures due to the deflections in step 1. Find the total pressure, P_1, and the eccentricity, e_1, if any. Add rigid beam pressures to balance the load P_1 and the moment, if any. Determine the deflections of the beam for this balanced load.

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[26] Kansas City, Mo.

3. Calculate the pressures due to the deflections in step 2. Find the total pressure, P_2.

4. Determine the ratio:

$$r = \frac{P_1}{P_1 + P_2} \qquad (2)$$

The final pressures are found by adding r times the balanced pressures in step 2 to the rigid beam pressures in step 1.

The theory underlying the suggested procedure may be simply stated. For step 1, the balanced rigid beam pressures yield a deflection curve, y_1, resulting in an unbalanced pressure, P_1. For step 2, the balanced deflected beam pressures yield a deflection curve y_2, resulting in an unbalanced pressure P_2. Assuming that the two deflection curves are similar in shape, for compatibility of deflections and pressures,

$$P_2 r = P_1 (1 - r) \qquad (3)$$

which reduces to Eq. 2. Hence the pressures at any point are the sum of the rigid beam pressures in step 1 and r times the balanced pressures in step 2.

The proposed method will now be applied to Example 1. The parts of the author's calculations used subsequently are assumed to be correct.

a. The deflections for step 1 are shown in Fig. 2(*g*).

b. The pressures for step 1 are shown in Fig. 2(*h*). The total pressure, P_1, is 88.95 kips. The balancing rigid beam pressure is $-$ 88.95/30 = $-$ 2.965 kips per sq ft. The balanced pressures are shown in Table 4(*a*).

c. The deflections for step 2 are shown in Table 4(*a*). The total pressure, P_2, is 22.95 kips.

d. The ratio, r, is 88.95/111.9 = 0.793, and the final pressures are shown in Table 4(*a*). The author's results are also shown in the table for comparison.

To demonstrate that the suggested procedure is applicable to an unsymmetrical case, Example 3 was also solved. The values of P_1 and P_2 are 402.4 lb and 135.6 lb, respectively, hence the ratio, r, is 0.748. The results are tabulated in Table 4(*b*).

A study of the method indicates that it is similar to a trigonometric series solution where only one term of the series is used. The only difference is the use of a calculated deflection curve instead of the single sine wave. The accuracy obtained is excellent for beams whose values of λ are less than 3.0, where the parameter λ is given by the expression:

$$\lambda = L \sqrt[4]{\frac{k}{4 E I}} \qquad (4)$$

For higher values of λ, the pressures are in error because the final deflection curve, even for a concentrated load applied at the center of a beam, has reverse curvature near the ends. It is more important to note, however, that the maximum moment for a beam with a concentrated load at the center was practically correct for a value of λ = 6, although the pressure distribution was considerably in error. These considerations seem to justify the conclusion that the approximate procedure is sufficiently accurate for all practical purposes.

In discussing his general procedure, the author cautions against the danger of a nonconvergent solution. It would be interesting to know if his method will work for beams more flexible than he has considered. The writer's procedure, however, involves no convergence difficulties for any degree of flexibility, and gives an answer consistent with the assumption that the final deflection curve is of the same shape as that due to the applied load and the balancing rigid beam pressures.

TABLE 4.—SIMPLIFIED PROCEDURE APPLIED TO THE EXAMPLES OF THE PAPER

Steps	Points:						
	1	2	3	4	5	6	7
(*a*) EXAMPLE 1 (SEE FIG. 2)							
Step 1.—Deflections, in Feet:							
y_1	0	0.0076	0.0139	0.0163			
Step 2.—Pressures, in Kips per Square Foot:[a]							
Unit pressures, 300 y_1	0	2.280	4.170	4.890			
Balancing pressures, 88.95/30	−2.965	−2.965	−2.965	−2.965			
Net pressures	−2.965	−0.685	1.205	1.925			
Step 3.—Deflections, in Feet:[a] y_2							
Hence, $P_2 = 1{,}500\ \Sigma y_2 = 22.95$ kips	0	0.0021	0.0036	0.0039			
Step 4.—Final Pressures, in Kips per Square Foot:[b]							
Rigid beam pressures	5.00	5.00	5.00	5.00			
$r(= 0.793)$ times net pressures (step 2)	−2.35	−0.54	0.96	1.53			
Final pressures	2.65	4.46	5.96	6.53			
Author's results, Example 1	2.7	4.5	5.9	6.5			
(*b*) EXAMPLE 3 (SEE FIG. 4)							
Step 1.—Deflections, in Inches (Fig. 4(*e*)):							
y_1	0	0.0067	0.0117	0.0135	0.0117	0.0067	0
Step 2.—Pressures, in Pounds per Inch:[a]							
Unit pressures (Fig. 4(*f*))	0	−4.02	−5.85	−5.40	−3.51	−1.34	0
Balancing pressures[c]	3.88	4.08	4.03	3.76	3.18	2.37	1.32
Net pressures	3.88	0.06	−1.82	−1.64	−0.33	1.03	1.32
Step 3.—Deflections, in Inches:[a] y_2							
Hence, $P_2 = 20\ \Sigma(k\ y_2) = 135.6$ lb	0	0.0023	0.0041	0.0046	0.0037	0.0020	0
Step 4.—Final Pressures, in Pounds per Inch:[d]							
Rigid beam pressures	0.90	1.54	1.92	2.05	1.92	1.54	0.90
$r(= 0.748)$ times net pressures (step 2)	2.90	0.04	−1.36	−1.23	−0.25	0.77	0.99
Final pressures	3.80	1.58	0.56	0.82	1.67	2.31	1.89
Author's results (Example 3)	3.54	1.57	0.73	0.95	1.70	2.22	1.79
Exact results	3.62		0.57		1.65		1.89

[a] Computed from the results of the preceding step.

[b] r (by Eq. 2) $= \frac{88.95}{111.9} = 0.793$.

[c] With $P_1 = 402.4$ lb; and $e_1 = 7.35$ in.

[d] r (by Eq. 2) $= \frac{402.4}{538.0} = 0.748$.

E. E. DE BEER[27] AND D. KRSMANOVITCH[28].—A valuable contribution to the solution of problems concerning beams resting on soil has been presented by

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[27] Ingénieur en Chef, Directeur des Ponts et Chaussées, and Directeur de Inst. Géotechnique de l'État, Ghent, Belgium.

[28] Visiting Engr., Inst. Géotechnique de l'État, Ghent, Belgium, and Lecturer, Univ. of Serajevo, Yugoslavia.

Mr. Popov. Indeed, his semigraphical method for finding the distribution of the reactions under a beam with eccentric loading is based on a clever technique, which consists in taking the mean between the elastic deflections found by successive steps. This should be recorded as a personal achievement for Mr. Popov, because it decreases the computation work considerably.

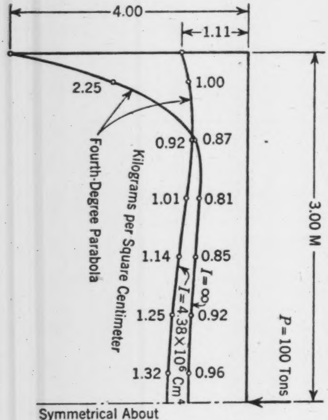

FIG. 7

The mathematical deductions of the paper are based on the "modulus of soil reaction *k*." This modulus does not correspond to a physical property of the real soil layers, as has been shown by one of the writers (Mr. De Beer)[29] for the case of centric loads. The distribution of the reactions obtained by the *k*-method can differ widely from the exact solution.

Making use of a semigraphical method similar to that described by Mr. Popov, one of the writers (Mr. Krsmanovitch) has made some calculations[30] in an effort to find a more exact distribution of the reactions under an eccentrically loaded beam resting on a soil with a constant modulus of elasticity, E_s. The work was reported in metric units for which typical conversion equivalents are presented in Table 5. However, for comparison purposes, the pressure distribution for an axially loaded beam is shown in Fig. 7 for a beam in which the width $b = 1.50$ m and the length, $l = 6.00$ m. The soil has a constant modulus of elasticity, E_s, of 1,000 kg per sq cm. Curves are presented for two values of the moment of inertia, *I*, both curves being in the form of fourth-degree parabolas. The imposed external load is a 100-ton (metric) force applied at the center of the beam.

TABLE 5.—CONVERSION OF TYPICAL METRIC QUANTITIES CITED IN THIS DISCUSSION

Symbol (1)	Definition (2)	Metric units (3)	English units (4)
E	Modulus of elasticity................	....	....
E_b	Value of *E* for a beam..............	200,000 kg per sq cm	182,858 tons per sq ft
E_s	Value of *E* for a soil..............	1,000 kg per sq cm	91,429 tons per sq ft
I	Rectangular moment of inertia......	4.38×10^6 cm^4	105,230 in.^{4}
b	Width of a beam section............	1.5 m	4.92 ft
l	Length of a beam section...........	6.00 m	19.68 ft
k	Reaction modulus..................	6.0 kg per cu cm	167.2 tons per cu ft

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[29] "Calcul de poutres reposant sur le sol," by E. E. De Beer, *Annales des Travaux Publics de Belgique,* June, 1948, p. 393; August, 1948, p. 525; October, 1948, p. 653; and December, 1948, p. 721.

[30] "Calcul de poutres reposant sur le sol—cas des sollicitations multiples et excentrées," by D. Krsmanovitch, *Annales des Travaux Publics de Belgique* (publication pending), 1951.

If the load is moved off center a distance $\frac{l}{12}$, the distribution for three different values of I is shown in Fig. 8. Before the results obtained for constant E_s can be compared with the k-method proposed by Mr. Popov, it is necessary to define k, since it has no true physical significance in relation to the soil. This may be done by writing[31]

$$k = \frac{E_s}{1.33\,(b^2\,l)^{\frac{1}{3}}} \quad \ldots . (5)$$

in which k is the modulus of reaction. Substituting the foregoing values of E_s, b, and l, Eq. 5 yields $k = 6.0$ kg per cu cm (167.2 tons per cu ft). Applying Mr. Popov's computation, with the same loading conditions, the corresponding distribution curves are indicated by the dashed lines in Fig. 8.

A direct comparison between the maximum moment, M, obtained by the "E_s-method" and the maximum moment, M', by Mr. Popov's method is afforded by the data in Table 6. Also shown, are the values of the unit reactions p and p' and the percentage differences (e), in results by both methods, at the point of application of load P and at ends A and B. The values corresponding to a trapezoidal distribution of the soil reactions are also included in the table. Examination of illustrations and Table 6 reveals that the k-method yields values of maximum moments that are consistently too low—especially in the case of stiff beams.

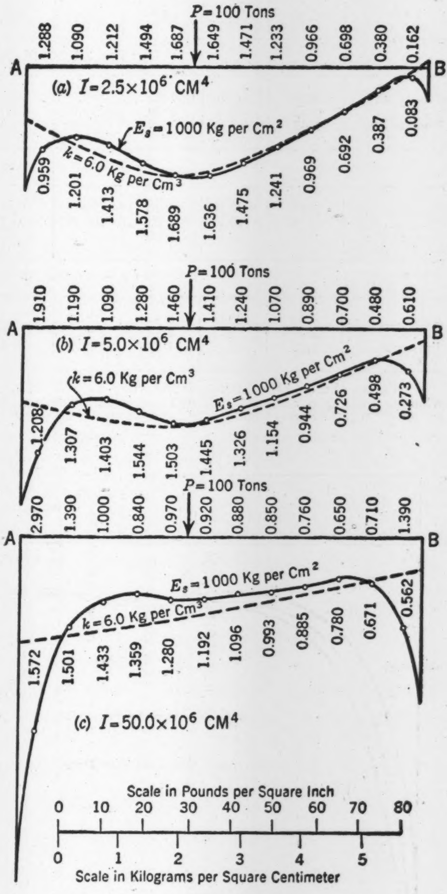

FIG. 8

To study the influence of eccentricity of loading, Fig. 9 shows the distribution curves for three positions, superposed. Even when the load is located

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[31] "Calcul de poutres reposant sur le sol," by E. E. De Beer, *Annales des Travaux Publics de Belgique*, October, 1948, p. 670.

TABLE 6.—COMPARISON OF MAXIMUM MOMENTS BY THREE METHODS

Method[a]	AT END A		AT LOAD P		AT END B		M, M', M'' (ton-m)	e_M (%)	Difference (%)
	p, p', p''	e_p	p, p', p''	e_p	p, p', p''	e_p			
(1)	(2)[b]	(3)[c]	(4)[b]	(5)[c]	(6)[b]	(7)[c]	(8)	(9)[c]	(10)
			(a) $I = 2.5 \times 10^6$ cm^4 =						
E_s..........	2.00	100.0	1.69	100.0	0.50	100.0	59.13	100.0	....
k...........	0.85	42.5	1.69	100.0	−0.07	<0	57.22	96.8	−3.2
Trapezoidal..	1.67	83.5	1.19	70.5	0.55	110.0	71.32	120.8	+20.8
			(b) $I = 5.0 \times 10^6$ cm^4 =						
E_s..........	2.85	100.0	1.46	100.0	1.15	100.0	67.96	100.0	....
k...........	1.16	40.7	1.54	105.5	0.17	14.8	62.38	91.8	−8.20
Trapezoidal..	1.67	58.6	1.19	81.5	0.55	47.9	71.32	105.0	+5.0
			(c) $I = 50.0 \times 10^6$ cm^4 =						
E_s..........	6.00	100.0	0.97	100.0	2.50	100.0	84.40	100.0	....
k...........	1.07	17.8	1.14	117.5	1.06	42.5	69.87	82.7	−17.3
Trapezoidal..	1.67	27.8	1.19	122.7	0.55	22.0	71.32	84.5	−15.5

[a] Values obtained by the E_s-method when E_s = 1,000 kg per sq cm. Values obtained by the k-method when k = 6.0 kg per cu cm. [b] Unit reaction in kilograms per square centimeter (kg per sq cm = 14.2234 lb per sq in.). [c] Percentage of the value obtained by the E_s-method.

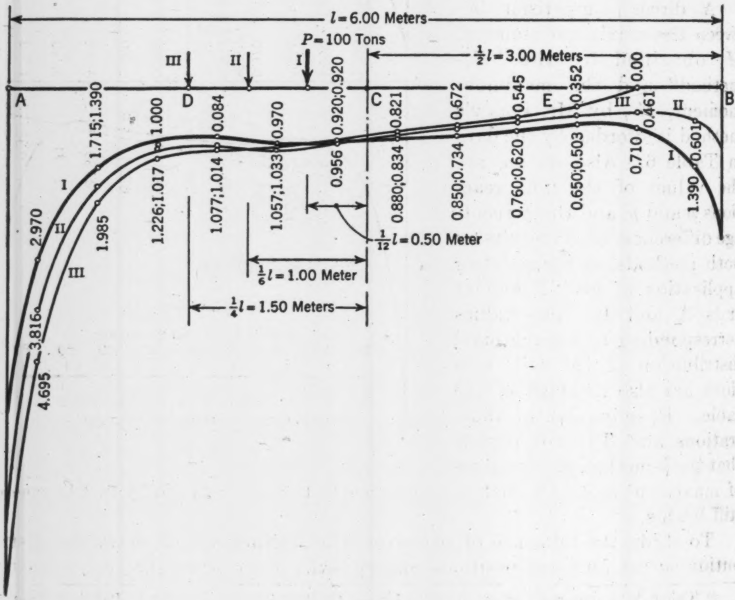

FIG. 9

one sixth of the span length off center, the reaction at the far end B is different from zero. Different distributions would be obtained by the *k*-method. Computations based on the modulus of elasticity E_s are more complicated and require slightly more time than the *k*-method proposed in the paper; but diagrams can be prepared by applying the method systematically to a selected number of cases, such that intermediate cases can be solved by interpolation.

E. P. Popov,[32] Assoc. M. ASCE.—The unanimously favorable comments on this paper are highly gratifying. Such an attitude on the part of experts is encouraging and indicates that the proposed method may prove useful to the profession. The writer thanks every discusser for adding much of value to the contents of this paper.

Originally the foundation modulus was not discussed in detail. On the other hand, some structural designers may not have a complete picture of the complexities involved in selecting this quantity. Mr. Hanson's discussion of the pitfalls which designers should guard against when using the proposed method treats this subject admirably. He is correct in stating:

> "* * * the variation in the foundation modulus used in Example 3 is an attempt to account for the effects of the variation of the density or compressibility of the soil."

An appropriate variation of the foundation modulus must be made depending on the nature of the material below the footing. The writer agrees with Mr. Hanson that the usual assumption of uniform pressure distribution for the design of "common" footings is acceptable. However, for footings possessing considerable flexibility, the usual basis of design may be unsafe.

The necessity for varying the foundation modulus in some cases is clearly brought out by the discussions of Messrs. Fadum, De Beer, and Krsmanovitch. For this purpose the latter two discussers cite some important work done on this problem in Europe. However, the comparison of Mr. De Beer's accurate solution with the proposed method needs further clarification. In the method proposed by the writer, the foundation modulus may be varied according to the principles of soil mechanics, tests, and judgment. Hence by using the transformed area of the footing as suggested in Example 3, results identical to those obtained by the "E_s-method" will be obtained.

The broad applicability of the proposed method to mathematically related problems was recognized by Messrs. Hanson and Fadum as being perhaps the most important use of this method. With minor modifications the proposed method may be applied to the analysis of interconnected girders of a bridge, pipe lines or tubes stiffened with collars, shells, and domes. At the University of California work has been done to obtain solutions of such problems by successive approximations. In these problems the "foundation modulus" may be definitely determined[33] and is not subject to the same objections that are inherent in the foundation problems. An interesting illustration of an approximation procedure to be followed in a difficult problem is given by Mr. Fadum.

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[32] Associate Prof. of Civ. Eng., Univ. of Calif., Berkeley, Calif.

[33] "Beams on Elastic Foundation," by Miklós Hetényi, The Univ. of Mich. Press, Ann Arbor, Mich., 1946, pp. 35, 36, 100, 119, and 163.

Improvement of the proposed method is the subject of the discussions by Messrs. du Plessis and Karol. Mr. du Plessis correctly states that with experience in the use of the method, the analyst's time may be economized materially by making a closer assumption than the average of the beam's deflections at the beginning and at the end of a cycle. For stiff beams the actual shape of the deflected beam approximates more closely the shape of the deflected beam subjected to the assumed loading than does the straight beam initially assumed. The inverse is true for flexible beams. The averaging process suggested in the paper is an attempt to reduce the design procedure to a routine, and it does not necessarily furnish the most rapid solution. In a strict sense, the process of averaging deflections at the beginning and end of a cycle is applicable only to short and medium length beams. Such beams, according to the classification proposed by Mr. Hetényi,[34] are such that $\lambda < \pi$, in which λ is as defined by Eq. 4. When the length of a beam is such that λ is greater than π, much work may be required to converge the solution by a trial and error process. However, in such cases the process may be simplified greatly by following Mr. Karol's suggestion.

A remarkable acceleration toward a convergent solution may be obtained by using the modified procedure developed by Mr. Karol. Using the ratio r defined in Eq. 2, immediate convergence of a solution is obtained. In difficult problems the use of this ratio eliminates the necessity of using many cycles to obtain a solution. In an average problem the convergence to a solution is accomplished by a process equivalent to only two cycles, although one additional cycle may be desirable as a final check. The foregoing procedure is somewhat analogous to the "one cycle distribution" in the Hardy Cross method of frame analysis, except that the Karol procedure is very simple to apply.

A procedure analogous to the one given by Mr. Karol has also been developed independently by V. A. Plumb,[35] Jun. M. ASCE. By successively integrating an assumed algebraic function based on the deflection of a rigid beam, the "convergence factor" r at any ith cycle in terms of Mr. Karol's notations may be stated as

$$r_i = \frac{P_i}{P_i + P_{(i+1)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

For $i = 1$ this reverts to Eq. 2. Great accuracy may be obtained rapidly for flexible beams by using this generalized expression for the convergence factor. For flexible beams, instead of seeking a solution at the end of two cycles, three or more cycles may have to be employed. By using the convergence factors, the danger of a nonconvergent solution is eliminated, and reverse curvature of the deflected beam, if such exists, may be determined. Mr. Plumb's analysis shows that the ratio P_{i+1}/P_i is $c\,\lambda^4$, in which c is practically a constant after a number of cycles.

In conclusion the writer wishes again to thank the discussers for their contributions which complement the paper so well.

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[34] *Ibid.*, p. 46.

[35] "Approximate Methods of Analysis of Beams and Circular Plates on an Elastic Foundation," by V. A. Plumb, to be submitted in partial fulfilment for a degree of Master of Science in Civ. Eng. at the Univ. of Calif., Berkeley, Calif., June, 1951.

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ECONOMIC EFFECTS OF RESERVOIR SEDIMENTATION

By William E. Corfitzen,[1] M. ASCE

With Discussion by Messrs. T. Blench, Martin Maevers, and William E. Corfitzen.

Synopsis

This paper is the result of several months of study by members of the Subcommittee on Economic Effects of Reservoir Sedimentation. The subcommittee is convinced that the problem has many facets, and an ultimate satisfactory solution can be obtained only after each of these facets has been recognized, defined, and properly coordinated with others which make up the entire problem. This paper seeks to separate the elements of a problem which are of paramount importance wherever large expenditures are proposed for the construction of dams. It discusses bases for estimating sediment loads and rates of sedimentation, briefly discusses the effects of such sedimentation on uses, such as irrigation, power, municipal water supply, flood control, and navigation, and concludes with brief considerations to be observed in computing the cost of sedimentation in any given reservoir.

Elements of the Problem

More storage reservoirs probably have been created since 1925 than in any period of history. Investigations lead engineers to believe that sedimentation will limit the usefulness of most of them to less than 200 years. With dam sites limited, and construction costs advancing, planning engineers realize that a serious economic problem exists, particularly where large dams have been (or are to be) constructed, for municipal water supply, irrigation, or flood control.

In estimating the economic effects of sedimentation in any reservoir, few, if any, generalizations can be applied. Each reservoir must be analyzed in the light of the particular data applicable to that reservoir. First in the order of analysis is the anticipated rate of sedimentation. The best basis for such an estimate would be long-term records of suspended-load and bed-load measure-

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Note.—Published in August, 1950, as *Proceedings-Separate No. 30.* Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Reclamation Specialist, E. C. A., Rome, Italy.

ments made at, or near, the reservoir site. If such records are not available, sampling station should be established at, or near, the site and the results correlated with those of other long-term records elsewhere on the same stream. Frequent inability to find any sampling stations serves to emphasize the urgent need for the immediate establishment of a coordinated network of stream-gaging and sediment-gaging stations on all major streams. With basic daily data the sediment load, W_d, in tons per day, may be estimated by the formula:

$$W_d = \frac{86{,}400 \times 62.4\, p\, Q}{2{,}000} \qquad \text{(1)}$$

in which Q is the daily mean flow, in cubic feet per second, and p is the percentage of sediment (by weight) expressed as a decimal. The conversion constants in Eq. 1 are 86,400 sec per day, 62.4 lb per cu ft, and 2,000 lb per ton.

Since daily load records are desirable for many sedimentation studies, they should be converted to acre-feet per year for economic studies of reservoirs. The weight of an acre-foot of sediment in place will vary with the physiographic forms and general features of surface relief that characterize different watersheds or different regions of the same watershed. The weight of deposited sediment varies from 40 lb per cu ft to more than 100 lb per cu ft. The laws of compaction as applied to reservoirs are unknown and present another field where research is urgently needed. At present the best an engineer can do is to obtain several samples of deposited material and determine the weight per cubic foot, γ_s. An estimate of the weight of an acre-foot of sediment in place, W_a, can then be obtained from the relation:

$$W_a = \frac{43{,}560\, \gamma_s}{2{,}000} = 21.78\, \gamma_s \qquad \text{(2)}$$

in which W_a is the weight of an acre-foot of sediment in place, in tons; and γ_s is the weight of a cubic foot of sediment, in pounds. The conversion constants are 43,560 cu ft per acre-ft and 2,000 lb per ton. The average annual sediment influx to a reservoir S, in acre-feet, will then be obtained from the relation:

$$S = \frac{\sum_{n=1}^{n} W_d}{\frac{n}{365}\, 21.78\, \gamma_s} = \frac{0.17 \sum_{n=1}^{n} W_d}{n\, \gamma_s} \qquad \text{(3)}$$

in which n is the number of days on which samples were taken. The conversion constant is 365 days per year.

At this point the engineer will have to determine how much of the load will pass through the reservoir and how much will be deposited in it. Depending on the size and the operating conditions of the proposed reservoir, some part of the sediment load may be transported through the basin in the form of a density current and may be discharged over the spillway or through the outlet works. If the basic data from which the estimate of sediment inflow was made did not contain bed-load data, an estimate of that load should be made now. Also of value for future use would be the determination of the sources of the sediment retained in the basin for, as the economic studies proceed, they may indicate the desirability of undertaking remedial measures at certain upstream points to prolong the life of the proposed reservoir. At present the lack of adequate stream-flow data and of quantitative evaluation of the influence of erosion-control measures makes such a determination very difficult.

The economic life of a reservoir will depend on a number of factors, an important one being the nature of the use or uses of the impounded water, together with the relationship between such uses and the accumulating silt. Other factors involved in estimating the economic life of a reservoir are:

1. The established use pattern of the reservoir and the economic life without further controls;
2. Possible controls, their costs, their effectiveness, and their economic justification; and
3. The cost of a reservoir to replace the existing reservoir. Pertinent considerations are irrigation, power, municipal water supply, flood control, navigation, and sediment retention.

Irrigation.—Before the construction of a dam, the unregulated flow of the river provided water for the irrigation of certain lands. Complete sedimentation of the reservoir and a return to this condition might not affect the water supply. However, the loss of reservoir capacity allocated to irrigating these lands, in excess of natural flow conditions, represents a very definite monetary deficit. Although the value of the water is dependent on such uncertain factors as extent, character, and market conditions of future crops, current practice is to place a reasonable value on irrigation water depending on the locality. With such information and with the irrigation area dependent on storage known, the annual revenue from irrigation can readily be computed. From the rate of sedimentation, the engineer can estimate the number of years hence when the irrigation pool will begin to be reduced and the ultimate time when irrigation storage no longer will be possible. This is not the only loss, however. Studies have indicated that one farm family on an irrigated area supports four additional families on that and surrounding areas. Consequently, as the irrigation pool is depleted, there would naturally follow a decrease in agricultural production with an attendant decline in income to the farm family and to each dependent family. The land values of agricultural and near-by urban properties probably would decline. Such reductions in income affect the taxes that people pay and lead to the inescapable conclusion that the problem is one of national importance.

During the years when storage is being lost there exists the danger that farmers, who have made substantial investments in their lands, will plant crops beyond the availability of water to sustain growth. Consequent crop failures could be tremendous and might conceivably lead to relocating populations.

Power.—When a reservoir is put into operation, definite estimates can be made of the annual income to be derived from firm power and dump power. If sedimentation affects the minimum regulated outflow, the effect may be to decrease firm power and to increase dump power, with great loss of revenue. However, if the rate of sedimentation is known, such revenue losses may be evaluated not only at the cost of power at the plant in question but also at the cost of replacement power which, in some instances, may be much higher.

As in the case of irrigation, there will be individuals and industries who will make investments on the assumption that power always will be available. Any decline in power not replaced from some other source will result in loss of revenue to individuals and industries and may even result in the physical relocation of industries.

Municipal Water Supply.—In addition to the actual revenue produced through the sale of water for domestic uses there are intangible benefits inherent in maintaining a continuous water supply. Inability to depend on the reservoir supply may lead to private pumping plants requiring huge expenditures. Such plants may lower the ground-water level to their own detriment and that of agriculturalists. Continuous release from a reservoir may have real value in improving the quality of river water by diluting alkali salts and in improving sanitary conditions by diluting industrial wastes and sewage.

Flood Control.—If an allocation has been made to flood control, and if sediment deposits prevent that use, valuable property below the dam may be subjected to recurrent and increasing flood damages. Besides actual damages such as those to crops, animals, buildings, and human life, a definite drop in the value of unprotected lands will follow.

Navigation.—Where reservoirs impound water for navigation, definite releases must be made at certain times of the year to provide water in channels that may be hundreds of miles below the reservoir. The value of such water lies in having it in the proper channel when required for navigation needs. In studying this use, trends in river traffic would be considered as well as attendant benefits of the reservoir—such as more stable ground water and dilution of pollution.

Sediment Retention.—The actual retention of sediment in a reservoir may have a monetary value worthy of consideration in an economic analysis. The sediment load in the stream below the dam will be decreased over a period of time as the channel adjusts itself to the new conditions. Consequently, sediments which formerly clogged canals and were deposited on fields, but are now retained in the reservoir, should no longer cause expensive maintenance. Sediment deposition in the reservoir, particularly during periods of high water, may lower treatment costs of municipal supplies diverted from the reservoir or from the river below the dam, and should lower maintenance costs on pumping equipment. There is another side to this "picture," however: The deposition of sediments in the reservoir and the subsequent release of clear water will generally cause a scour or degradation of the river bed below the dam. The removal of bed deposits by the clear water may lower the river below existing intake structures, and the material removed may be deposited elsewhere on the lower reaches of the stream. Both of these phenomena may be undesirable and costly. Under certain conditions the removal of sediment leads to an increase of algae with resulting higher costs for the treatment of public water supplies.

In addition to the foregoing considerations, it may be desirable to consider the effect of sediment on fish and wild life, recreation, scenery, value of properties bordering the reservoir, and possibly other uses applicable to a specific reservoir.

CONSIDERATIONS INVOLVED IN COMPUTATIONS OF COST

In arriving at the total cost of sedimentation of any given reservoir, certain items should be excluded if they relate to features that can be amortized. For example, it may be found that power plants, transmission lines, canal structures, and similar features which are subject to depreciation or obsolescence will be replaced one or more times during the life of the reservoir. Such structures generally are not affected by sedimentation but suffer losses from ordinary wear and tear, and from obsolescence. However, losses in features with remaining useful life but not subject to salvage, at the time they are rendered useless by sedimentation of the reservoir, would be a chargeable cost.

In regard to interest to be applied in computing justifiable expenditures to protect existing works or to provide for their replacement, two schools of thought exist. One group believes that interest (especially compound interest) should be eliminated when making computations of the costs and the benefits of conservation projects. The basic theory appears to be that the conservation of natural resources can be justified at any cost. On the other hand, there are those who hold the majority view that capital is a commodity. Capital is a commodity that is bought and sold, and the price paid for that commodity is the rate of interest. If government funds are available or must be raised by taxation, interest is not paid to any creditor, but the element of interest is pertinent nevertheless, because the government which has the money (or the citizen who pays the taxes) could lend that money at some rate of interest. The rate will depend on economic conditions existing at any given time but for most public expenditures the logical rate appears to be the current rate on long-term offerings of the United States Treasury Department.

The actual cost of reservoir sedimentation will vary depending on whether the basis adopted is the construction cost of the dam and the reservoir that are being rendered useless, or whether the basis is the replacement cost if another structure had to be built at the same or some other site. It is doubtful if engineers are ever justified in using the original construction cost in estimating sedimentation damage. If construction costs remained resonably constant, it might be allowable to use the original construction cost less any salvage of the work. For example, this approach might be adopted in arriving at sedimentation losses if the problem were to determine how much can justifiably be spent now, or over a period of years, on upstream erosion-control measures to prevent the present reservoir from becoming useless. However, some engineering economists believe that engineering construction costs will be 60% to 75% above the 1939 level for the 1950–1960 decade and that they probably will not recede to less than from 40% to 50% above 1939 costs for many years to come. On the other hand, if actual replacement storage is to be found, it must be borne in mind that the best reservoir site is the one usually selected for the first structure because that, generally, is the most economical. Almost invariably, adding to the height of an existing dam or planning the construction of a new dam at a new site will involve costs considerably in excess of those of the original structure. If replacement is necessary, the costs used should be those of the new structure; or, as an alternative to replacement costs, the future service value of the reservoir may be an acceptable measure of its worth.

SUMMARY

In the ultimate computation of damages occasioned by sedimentation, any one of three standard methods may be used: (1) The annual cost method; (2) the present worth method; or (3) the capitalized cost method which is a variation of the present worth approach. The annual cost method is more generally used by engineers, and is probably more readily understood by business men who are accustomed to thinking in terms of annual expenditures. For large reservoirs, however, where annual utility losses are not uniform and comparisons are to be made involving conditions with and without conservation-control measures, it is doubtful if the method has any advantage over the present worth or the capitalized cost methods. Present worth is defined as the conversion of a money-time series into an equivalent single payment at some particular date. For convenience, the present date is generally used. Capitalized cost refers to that sum of money, the future interest on which will provide for restoring the structure in question at specified intervals indefinitely.

DISCUSSION

T. Blench,[2] M. ASCE.—Experience in India makes the writer see this paper not as a study in economics but as a statement of some of the unforeseen, but foreseeable, consequences of interfering with sediment-bearing rivers of regime type. In India, British engineers began interfering with such rivers at the end of the nineteenth century. As they built nothing higher than diversion dams, they learned about river regime at small cost. From experience with canal systems (after some forty years of expensive fighting against nature) they established the basic laws of regime behavior. After about fifty years, these engineers conducted the world's largest quantitative investigation of waterlogging and alkali formation but only when, in the area investigated, land was turning alkaline at the rate of about 25,000 acres annually. Not until 1947—on a motion by the writer—did the Central Board of Irrigation (Government of India) Research Committee resolve that soil conservationists should be associated on equal terms with engineers as soon as a project was contemplated on a sediment-bearing river. Apparently the acquisition of vital knowledge, so simple in retrospect, is a slow and expensive process.

There seems no reason in these days why expert knowledge should not be used to prevent projects from being conceived in ignorance of possible ill effects. The Soil Conservation Service, United States Department of Agriculture, has published information on the effects of soil mismanagement and sedimentation damage, and that agency knows the practical ways of attacking the trouble at its source. The technique of measuring sediment load in streams has been developed mainly in the United States and is satisfactory. Agricultural experts in Canada and the United States are fully acquainted with the physics of the alkali problem, and experience with alkali formation on irrigation projects in India is extensive and practical. Hydrologists know the principles of making soil-water inventories.

In India, regime theory has been in use since the late nineteen twenties to estimate and explain the behavior of regime-type rivers after interference. Publications have appeared at intervals in Europe and India; yet with all this information in existence the author has to draw engineers' attention to: (1) The aggradation that experts know must occur upstream; (2) the degradation that must occur downstream; and (3) the consequences of reservoirs becoming dwindling assets and perhaps irreplaceable assets. With justice, he might have added items such as published misapprehensions about the unlikelihood of alkali formation, uncontrolled withdrawals from the water table, danger of waterlogging in some areas, and water-table depletion in others due to control of river discharges.

The author is to be congratulated on clarifying some of the important points that should be considered at the inception of a project.

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[2] Cons. Engr.; Associate Prof., Civ. Eng., Univ. of Alberta, Edmonton, Alta., Canada.

MARTIN MAEVERS[3].—An interesting summary of the study made by the Subcommittee on Economic Effects of Reservoir Sedimentation has been presented.

The committee is concerned with the short life of the large number of reservoirs which have been created since 1925 and those which are to be constructed for water supply, irrigation, power, and flood control in the future. The author describes the losses to individuals and society that occur from sedimentation of a reservoir: reduced farm income, loss to families with derivative income from irrigation farming, decline of land values, crop failures, loss of undepreciated investment, loss of taxes, and many more.

The question may well be asked, "Are these really losses in the true sense of the word?" Losses can be caused only by forces either unforeseen or forces whose occurrence is a matter of probability. However, sedimentation of reservoirs is a known factor, and its occurrence can be calculated within the limits of our ability to determine the sedimentation rate.

If these so-called losses were to be avoided when planning a particular reservoir with a given sedimentation rate, allowance would have to be made for: (1) A declining income stream from the land; (2) amortization of farm and industrial investment during the life of the reservoir; (3) a slow emigration of population and capital (amortization funds) as the services of the reservoir decline.

If the benefits from the services of the reservoir under these conditions are greater than the cost, the construction of the dam would be justified from a short-run point of view.

It is doubtful, however, that people who approve expenditures for large irrigation, power, or flood-control projects have such a short-run view in mind. It is equally doubtful that the planning engineers think in those terms, or they would not talk about "damages" from sedimentation of reservoirs.

With the development of any plan for dam construction, therefore, it must be decided in advance whether the economy created or protected by the services of the reservoir should be temporary or permanent.

What are the considerations that enter into the cost of furnishing a permanent service?

They are: (1) The probable rate of sedimentation. The source of sedimentation must be known, whether it can be reduced, by what means and by how much, and whether such reductions are temporary or permanent. A primary consideration, therefore, must be the condition of the watershed and the cost of reducing erosion to a point where long-time operation of the planned reservoir is possible; (2) the cost of the dam and appurtenant works; (3) the cost of maintenance of reservoir space for sediment which cannot be controlled upstream; (4) possible damage caused downstream of the dam because of its construction. This would include channel deterioration due to shifting of sediment from below the dam to downstream areas not threatened now by flood damage. It would include damage to land adjoining the river downstream from the dam, caused by lowering or raising of the water table and beach-erosion damage due to reducing the normal supply of sand to the sea.

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[3] Agricultural Economist, California Forest and Range Experiment Station, Berkeley, Calif.

All of these costs enter into the expenditures for constructing a dam. All of them are calculable factors. And all of these costs must be repaid from the services which the reservoir furnishes.

Objection is often raised to this approach in that it requires projection of events too far into the future, and "Who can look into the seeds of time and say which grain will grow and which will not?" Actually, it is not required to look into the distant future; but a complete plan which includes all factors within the present knowledge of things must be laid out. For instance, it is known that a reservoir will silt up and that the rate of sedimentation can be estimated. Provision will have to be made for substitute space, or for cleaning of the reservoir, or for building upstream barriers, or for taking the upstream watershed out of use. It can be estimated when such measures have to be undertaken and what their cost will be. This has to be included in a complete plan. Such a plan must be tentative in some of its phases, because some of the proposed measures will not require installation until many years have passed, when the thoughts of today will have given way to new ideas, developments, and inventions. To avoid taking all the benefits for our present generation and burdening the future with the enormous cost of either providing debris control or abandoning the project, a complete plan must be worked out.

Some of the project plans, like watershed land improvement and channel stabilization in areas above the proposed dam or dams, have to be undertaken first. Cover improvements should be undertaken well in advance of construction because they require time to become effective. The cost of these measures should be counted at compound interest to the day the services from the reservoirs become available. And the cost of those measures that are to be constructed well in the future must be discounted to the date the reservoir services become available. To be comparable all costs should be expressed in terms of deflated dollars.

Against these costs must be balanced the benefits. It is important that the benefits be expressed in terms of net returns. Only then is it possible to judge whether or not the investments are beneficial to society. A misinvestment today diverts funds not only from more profitable present investment opportunities but also reduces the availability of funds for investment at a future day. Misinvestments of today depress the future potential standard of living of the nation.

All secondary benefits, sometimes called "contributions to the national income," must be excluded when comparing benefits with cost of a specific project. The first and essential point in the benefit-cost analysis is to find out whether the investments made in the project pay for themselves at a rate comparable to investments made in the open market. Secondary benefits accompany any large investments. They, therefore, only enter into consideration when it must be decided which of two investment opportunities with equal primary benefits to choose.

The planning and analysis of river basin developments should lead to recommendations of two types of projects. The first represents construction that should be termed temporary. It would generally be exploitative. It would allow not only for repayment of the cost of the dams but also complete amortization of investments in the development area served by the reservoir. The second would allow a permanent development which would add throughout time to the net national income.

Present planning and analysis is mostly made on the basis of repayment or amortization of the dam itself. But this evades the main consideration, which is the amortization or permanency of the economy developed by the reservoir services. It is only when this is not taken into consideration that reservoir sedimentation can be called "damages." After having built the dams and justified their construction by proving that the services they render would pay for the structures over a given period of years, it is realized that reservoirs fill up with sediment, and the question arises as to what to do with the economy created because of them. The economy having been created, remedial measures must be designed to perpetuate the services of the reservoirs.

The result of evaluating remedial measures for existing dams may lead to these conclusions: It may be found that a plan for perpetuating the services of a reservoir will render enough benefits to justify its execution or that the cost of a remedial plan is less than the cost to liquidate the economy created by the original dam.

In the first case the plan provides a true benefit to society; in the second case it minimizes the loss to society.

William E. Corfitzen,[4] M. ASCE.—The number of publications available on this subject has been emphasized by Mr. Blench, who states that with all the information available, engineers' attention must be drawn to certain fundamental aspects of the problem. This was particularly interesting as the same thought was expressed in at least three of the discussions presented in 1949 by the ASCE Hydraulics Division in the symposium of the Joint Committee on Design and Operation of Multiple Purpose Reservoirs.[5]

Considering the earlier basic studies, such as those of the British engineers in India and the more recently published data on the subject (which were probably collected more scientifically), it seems likely to the writer that available technical information is still of greater interest to the research man than to the practicing engineer. Perhaps there yet remains a zone where more printed information is required in simplified form relating to the application of what the research men have found.

It must be recognized, however, that there can be no clear-cut approach to the solution of the sedimentation problems of any given stream. So many factors are involved, and such an infinite combination of those factors exists, that each stream presents a special problem that must be studied in detail. In this connection the writer recalls the sedimentation studies conducted on the lower Colorado River prior to the construction of Imperial Dam and desilting works. Sedimentation was a new field to most of the engineers working on the problem, and it was their hope that the early studies of the British engineers in India would provide a ready answer. However, application of the formulas developed for conditions there did not furnish results which checked observations of conditions found in the field. It was realized soon, however, that the Indian studies were of infinite help in pointing the way toward research of local significance and in helping the investigators to avoid many pitfalls. Mr. Blench has hit a keynote with which the writer agrees completely—that soil conservationists should be associated on equal terms with engineers as soon as a project is contemplated on a sediment-bearing river. In their studies of natural phenomena engineers have become so highly specialized and involved in the complexities of their own fields that they often neglect to coordinate these jobs as fully as they should be coordinated. There really is no reason why this should happen but the fact remains that it does. It behooves the engineer associated with such problems to make certain that he has called upon all experts whose services may have some bearing on the solution of the sedimentation problem.

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[4] Reclamation Specialist, ECA, U. S. Embassy, Rome, Italy.

[5] "Multiple-Purpose Reservoirs, A Symposium," *Transactions*, ASCE, Vol. 115, 1950, p. 789.

Mr. Maevers has presented some interesting thoughts that should be kept in mind during the planning stages of reservoirs. It is clear that, although economists and engineers may use a different language, they intend to express the same concept. In his professional contacts with Europe, the Middle East, and Africa, the writer has become acutely aware of the need for watershed control. In many of the so-called "depressed areas" of the world, plans are being made for economic development, either through local resources or with United States assistance under the "Point IV Program."[6]

Regardless of their location, engineers are generally eager to approach their problems realistically and to understand the limitations of any proposal. Unfortunately, there are many outside the engineering fold who think of irrigation and power development as an easy cure-all for the economic ills of these depressed areas. As mentioned by Mr. Blench, there is danger that many projects will be proposed and constructed, involving millions of dollars, without reference to the large amount of technical data that are already available on this subject. Also deplorable is the fact that little or nothing is being done, in many of the depressed areas, to collect basic data, such as rainfall, temperature, wind velocity, and soils data, which would be useful in solving the problems that engineers know must be solved before engineering structures can be constructed safely and economically. Both the discussions serve to focus attention on the value and need of watershed control. Soil conservationists, engineers, and foresters must be brought closer together during the formulation of plans for these large reservoirs if they are to be constructed to render maximum service adequately and economically.

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[6] *Public Law 535, Section 401*, 81st Congress, June 5, 1950.

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Paper No. 2459

DESIGN CURVES FOR FOOTINGS ON SOIL

By Winfield A. McCracken,[1] Assoc. M. ASCE

With Discussion by Messrs. C. Y. Teng, Phil M. Ferguson, and Winfield A. McCracken.

Synopsis

Many design tables for footings have been published. However, where such tables are comprehensive, they become cumbersome, and, because of this cumbersomeness, are difficult to use.

A study of the Code of the American Concrete Institute (ACI 318-47) shows that the depth-width ratio and the reinforcing are controlled (*a*) in footings of uniform thickness by bond, and (*b*) in sloped, or stepped footings by bond or shear, or both. It also shows that the depth-width ratio for plain concrete footings is controlled by tension on a section at the face of the column.

The method presented herein is more flexible and less cumbersome than any method involving tables, and it shows the relation between the code requirements clearly. For reinforced footings, only three sets of curves containing seven variables need to be used. For plain concrete footings, only one set of curves with three variables is required.

Following a derivation of equations, the curves are explained and an example of their use is given to make the method of procedure clear. This section is followed by a brief explanation of the adaptation of the curves to rectangular, one-way, and continuous footings.

With the exceptions, as explained under the heading, "Derivation of Equations for Curves," the ACI code is used throughout.

Derivation of Equations for Curves

Notation.—The symbols in this paper, defined where they first appear (by illustration or in the text), conform essentially to Letter Symbols for Structural Analysis (ASA–Z10.8–1949) prepared by the American Standards Association, with ASCE participation, and approved by the Association in 1949. For

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Note.—Published in September, 1950, as *Proceedings-Separate No. 35*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Cons. Engr., Houston, Tex.

reinforced footings, seven variables are involved in the three sets of curves (Figs. 1, 2, and 3)—namely, $\frac{P}{(f_c)'}$, $\frac{d}{B}$, q, Σo, y, $\frac{v_c}{(f_c)'}$, and $f_s \frac{A_s}{B\,P}$. Their significance, where not conforming to the standard, is made clear by Fig. 4.

Shear (*Fig. 1*).—The critical section for shear is defined by the code as a section parallel to, and at a distance d from, the face of the column, d being

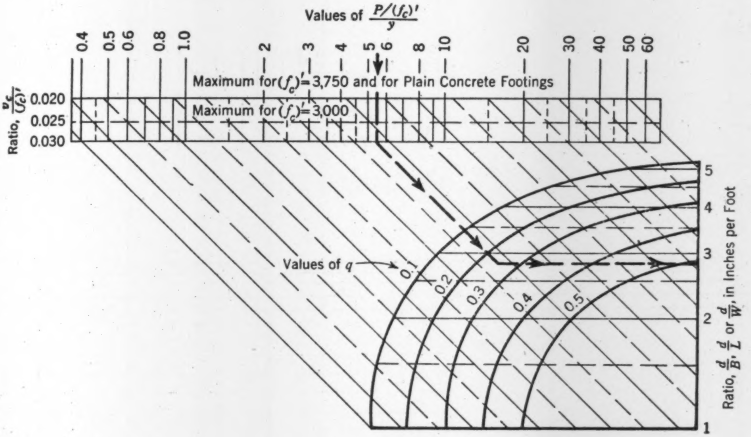

FIG. 1.—VALUES OF $\frac{P/(f_c)'}{y}$, $\frac{v_c}{(f_c)'}$, AND $\frac{d}{B}$

the effective depth of the footing at the base of the column, in inches. The external shear carried by this section is equal to the load on an area bounded by the section, 45° diagonals from the corners of the column, and the included edges of the footing. From Fig. 4, the length (l) of the section is

$$l = q\,b + 2\,d = b\left(q + \frac{2\,d}{b}\right) \dots\dots\dots\dots\dots\dots (1)$$

in which b is the width of a square footing, in inches, and q is the ratio of the width of the column to the width of the footing. The external shear carried is

$$V = \frac{P\,b^2\left[1 - \left(q + \frac{2\,d}{b}\right)^2\right]}{144 \times 4} \dots\dots\dots\dots\dots\dots (2)$$

in which V is the total shear on the section, in pounds per inch of width, and P is the load on the footing (soil bearing minus the weight of the footing), in pounds per square foot.

Let v_c be the unit shear on the section, in pounds per square inch; y be the ratio of the effective depth at the shear section to the effective depth at the column; and $(f_c)'$ be the compressive strength of concrete at twenty-eight days, in pounds per square inch. The standard concrete symbol j is the ratio of the distance between the resultant compressive stress and the resultant tensile stress, to the distance from the outer compressive fiber to the resultant tensile stress. Then,

$$v_c = \frac{P b^2 \left[1 - \left(q + \frac{2\,d}{b}\right)^2\right]}{576\, b \left(q + \frac{2\,d}{b}\right) j\, y\, d} \quad \text{.................... (3)}$$

and

$$\frac{v_c}{(f_c)'} = \frac{0.002 \left[1 - \left(q + \frac{2\,d}{b}\right)^2\right]}{\left(q + \frac{2\,d}{b}\right)\frac{d}{b}} \left[\frac{\frac{P}{(f_c)'}}{y}\right] \quad \text{.............. (4a)}$$

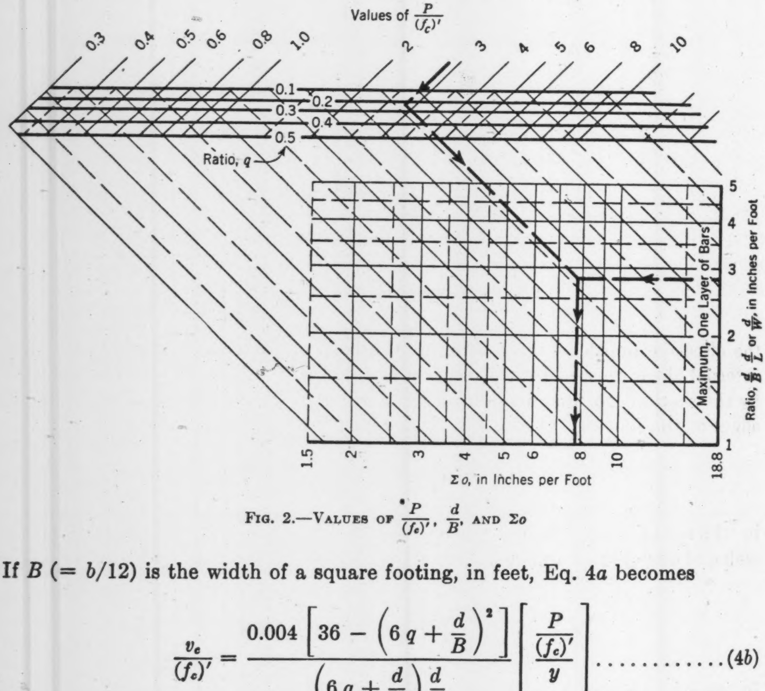

FIG. 2.—VALUES OF $\frac{P}{(f_c)'}$, $\frac{d}{B}$, AND Σo

If B ($= b/12$) is the width of a square footing, in feet, Eq. 4a becomes

$$\frac{v_c}{(f_c)'} = \frac{0.004 \left[36 - \left(6\,q + \frac{d}{B}\right)^2\right]}{\left(6\,q + \frac{d}{B}\right)\frac{d}{B}} \left[\frac{\frac{P}{(f_c)'}}{y}\right] \quad \text{............ (4b)}$$

Bond (*Fig. 2*).—The critical section for bond is specified as the same as that for bending moment and is at the face of the column. Its length is the full width of the footing. The code gives the permissible bond stress in a two-way footing as 0.056 $(f_c)'$, with a maximum of 200 lb per sq in. It also specifies that not less than 85% of the external shear carried by the section shall be considered in computing bond. Since 85% of 0.056 $(f_c)'$ for 3,750-lb concrete is only 210 lb per sq in., and since no reduction in the external shear need be made,[2] the 85% allowance and the 200-lb maximum have been ignored in the equation for bond. Thus, the external shear per inch of width is

$$V = \frac{P\,b\,(1-q)}{144 \times 2} \quad \text{.........................(5)}$$

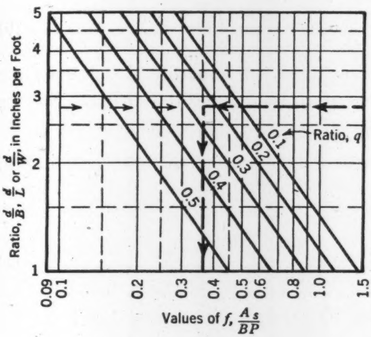

FIG. 3.—VALUES OF $\frac{d}{B}$ AND $f_s \frac{A_s}{B\,P}$

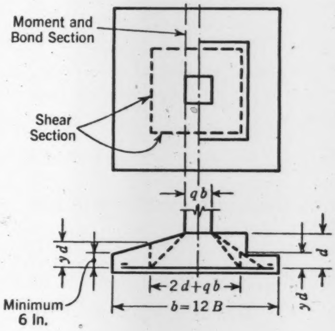

FIG. 4

The sum of the perimeters of the reinforcing bars, in inches per inch in width of footing, is

$$\Sigma o = \frac{V}{u\,j\,d} = \frac{0.071\,\dfrac{P\,(1-q)}{(f_c)'}}{\dfrac{d}{b}} \quad \text{...................(6)}$$

in which u is the allowable bond stress. To convert to inches per foot in width of footing, substitute B for b in Eq. 6.

Bending Moment (*Fig. 3*).—The moment M per inch in width is

$$M = \frac{P\,b^2\,(1-q)^2}{144 \times 8} = f_s\,A_s\,j\,d \quad \text{..................(7)}$$

in which f_s is the tensile stress in the reinforcing steel and A_s is the section area of steel per inch width of footing. Using an average value of $j = 0.833$,

$$\frac{f_s\,A_s}{b\,P} = 0.00104\,\frac{(1-q)^2}{\dfrac{d}{b}} \quad \text{...................(8a)}$$

If A_s is the steel area per foot, and $b = 12\,B$,

$$\frac{f_s\,A_s}{B\,P} = 1.80\,\frac{(1-q)^2}{\dfrac{d}{B}} \quad \text{...................(8b)}$$

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[2] "Reinforced Concrete Wall and Column Footings," by Frank E. Richart, *Proceedings*, American Concrete Institute, Vol. 45, 1948–1949, p. 97, title 45-6a, and p. 237, title 45-6b.

Since $f_s A_s$ varies inversely with j, the tension $f_s A_s$ for other values of j may be found by multiplying the values from Fig. 3 by $0.833/j$.

As in the case of bond, considering recent tests,[2] the 85% allowance has been ignored in Eqs. 8.

Plain Concrete Footings (Fig. 5).—The permissible tension in the concrete is limited to $0.03\,(f_c)'$ on a section at the face of the column. The moment per inch of width is

$$M = \frac{P\,b^2\,(1-q)^2}{144 \times 8} = f_c\,Z_c = \frac{0.03\,(f_c)'\,d^2}{6} \quad \ldots\ldots\ldots\ldots (9)$$

in which f_c is the allowable compressive stress in concrete, in pounds per square inch, and Z_c is the section modulus of a plain concrete footing. Solving for d/b,

$$\frac{d}{b} = 0.417\,(1-q)\left[\frac{P}{(f_c)'}\right]^{0.5} \quad \ldots\ldots\ldots\ldots (10a)$$

With the width expressed in feet,

$$\frac{d}{B} = 4.99\,(1-q)\left[\frac{P}{(f_c)'}\right]^{0.5} \quad \ldots\ldots\ldots\ldots (10b)$$

Explanation of Curves and Examples of Their Use

Fig. 1 is a set of curves that contains the variables $\frac{P/(f_c)'}{y}$, q, $v_c/(f_c)'$, and d/B, and gives the value of d/B when the other three are known. Fig. 2 contains the variables $P/(f_c)'$, q, d/B, and Σo, and gives the value of Σo when the other three are known. Fig. 3 contains the variables d/B, q, and $\frac{f_s\,A_s}{B\,P}$, and gives the value of the latter when the others are known. For plain concrete footings, Fig. 5 is a set of curves based on tension in the concrete, containing the variables d/B, q, and $P/(f_c)'$, which gives the value of d/B when the other two are known. In designing footings, the total superimposed load, the allowable soil bearing pressure, the column size ($q\,b$), and the concrete strength $(f_c)'$ are specified, or known. The width of the footing (B) is determined by the superimposed load and the allowable bearing; therefore, the problem is to determine the most satisfactory proportions for d/B, y, and A_s to fit the requirements. As the design procedure

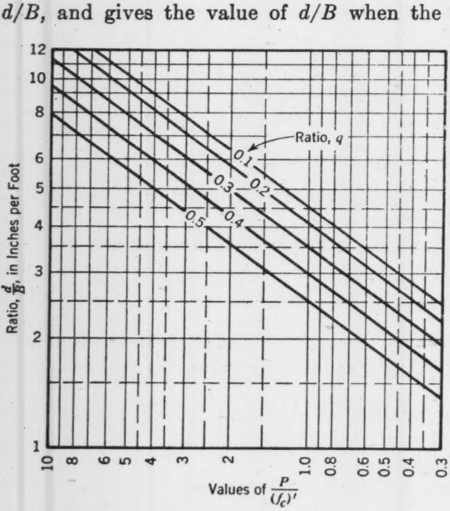


FIG. 5.—PLAIN CONCRETE FOOTINGS

for all types of footings on soil is essentially the same, only one example of a square footing and one of a rectangular footing will be given, together with a brief statement of the modifications necessary to adapt the curves to the design of rectangular, one-way, and continuous footings.

DESIGN OF SQUARE FOOTINGS; NUMERICAL EXAMPLES

Example 1, Square Footing.—Assume a 24-in. by 24-in. square column (area = 4 sq ft), carrying a load of 360 kips. The allowable soil bearing pressure is 6,000 lb per sq ft and the specified compressive strength of concrete $(f_c)'$ at twenty-eight days is 2,000 lb per sq in. The estimated weight of the footing being 300 lb per sq ft, $P = 6{,}000 - 300 = 5{,}700$ lb per sq ft. The area required to support 360 kips, therefore, is $360/5.7 = 62.1$ sq ft—say, an 8-ft square footing with an area of 64 sq ft. A corrected net load is $360/64 = 5.6$ kips per sq ft, from which $P/(f_c)' = 5{,}600/2{,}000 = 2.8$. The width ratio q is $2/8 = 0.25$ and (assuming $y = 0.50$), $P/(f_c)' = 2.8/0.5 = 5.6$.

Enter Fig. 1 at $\dfrac{P/(f_c)'}{y} = 5.6$; follow vertically to $v_c/(f_c)' = 0.03$; thence down along diagonal to $q = 0.25$; from there horizontally to find $d/B = 2.8$ in. per ft. Enter Fig. 2 with $P/(f_c)' = 2.8$; follow diagonal down to left to $q = 0.25$; then down diagonal to right to an intersection with the horizontal $d/B = 2.8$; thence vertically to find $\Sigma o = 7.8$ in. per ft. The use of $\frac{5}{8}$-in. bars at 3-in. centers gives $\Sigma o = 7.8$ in. and $A_s = 1.24$ sq in. per foot width of footing. In Fig. 3, with $d/B = 2.8$ and $q = 0.25$, $\dfrac{f_s A_s}{B P} = 0.36$, from which $f_s = \dfrac{0.36 \times 8 \times 5{,}600}{1.24} = 13{,}000$ lb per sq in.

Example 2, Rectangular Footing.—With the same basic data as in Example 1, assume a maximum footing width W of 7 ft. The area required being 64 sq ft, the length L of the footing is $64/7 = 9.14$ ft. Then $q_L = 2/9.14 = 0.22$; and $q_W = 2/7 = 0.29$. As in Example 1, $P = 5{,}600$ lb per sq ft and $P/(f_c)' = 2.8$. Assume $y = 0.5$, then $\dfrac{P/(f_c)'}{y} = 5.6$.

With the latter value (5.6) enter Fig. 2 and move vertically to $v_c/(f_c)' = 0.03$; thence diagonally downward and to the right, intersecting $q_L = 0.22$ at $d/L = 2.9$ in. per foot length of footing. If $d/L = 2.9$, $d = 2.9 \times 9.14 = 26.5$ in. As a check, enter Fig. 2 as before but intersecting $q_W = 0.29$ at $d/W = 2.7$ in. per foot width of footing. If $d/W = 2.7$, $d = 2.7 \times 7 = 18.9$ in. The greater of the two depths (26.5 in.) is the one selected.

From Fig. 2, $\Sigma o_L = 7.6$ in. per ft, requiring $\frac{5}{8}$-in. bars on 3-in. centers—that is, $A_{sL} = 1.24$ sq in. per ft and (from Fig. 3), $\dfrac{f_s A_s}{L P} = 0.38$. Consequently (from Fig. 3), $f_{sL} = \dfrac{0.38 \times 9.14 \times 5{,}600}{1.24} = 15{,}700$ lb per sq in.

As a check, with $d = 26.5$ in., $d/W = 3.8$ in. per ft. In Fig. 2, begin with $q_W = 0.29$ at $d/W = 3.8$; then Σo_W (from Fig. 2) $= 5.8$ in. per ft. Using $\frac{1}{2}$-in. bars, 3 in. on centers, $\Sigma o_W = 6.3$ in.; that is, $A_{sW} = 0.80$ sq in. per ft, and (from Fig. 3), $\frac{f_s A_s}{W P} = 0.24$. As before, $f_s w = \frac{0.24 \times 7 \times 5{,}600}{0.80} = 11{,}800$ lb per sq in.

Example 3, One-Way and Continuous Footings.—Since one-way and continuous footings are normally of uniform thickness, they will be controlled by bond. The ACI code increases the bond from 0.056 $(f_c)'$ in two-way footings to 0.075 $(f_c)'$ in one-way footings. Therefore, for one-way footings, decrease $P/(f_c)'$ by 25% (in Fig. 3 only) and proceed as for the two-way footing.

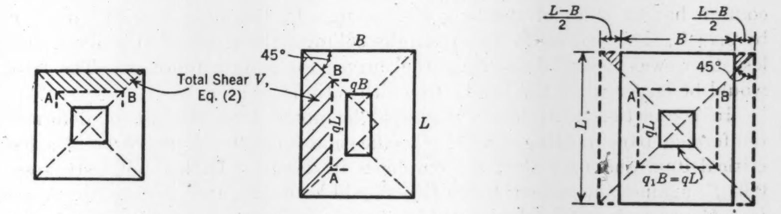
SUMMARY

The curves are more flexible and cover a wider range of design conditions than the most comprehensive tables. Their use entails a minimum of calculation and shows clearly the design procedure and the relation between the design conditions.

The resultant design will be within the practical limits of loading and soil bearing determination.

DISCUSSION

C. Y. Teng,[3] **Jun. ASCE.**—Derivation of Eqs. 2 to 4 is based on geometry which is illustrated in Fig. 6(*a*). These equations and the graphical representation (Fig. 1) are theoretically correct for square footings with square columns. They are also correct for oblong columns and footings in which the dimensions of the column are in the same ratio as that of the footing (Fig. 6(*b*)). In the latter case lines connecting the corners of the column and the corresponding corners of the footing make 45° angles with the sides; therefore, the geometry for square footings and square columns still holds true.



(*a*) SQUARE FOOTING, SUPPORTING SQUARE COLUMN

(*b*) RECTANGULAR FOOTING, SUPPORTING RECTANGULAR COLUMN

(*c*) RECTANGULAR FOOTING, SUPPORTING SQUARE COLUMN

Fig. 6

If the aforementioned geometrical condition is not satisfied, a design based on Eq. 4 or Fig. 1 will be in error but on the conservative side. Fig. 6(*c*) illustrates a footing with dimensions $B \times L$ and column $q_1 B \times q L$. The shaded areas should be deducted from Eq. 2 for the total shear (V) on line AB. Consequently, Eqs. 2 and 4*b* should read

$$V = \frac{P L^2 \left[1 - \left(q + \frac{2d}{L}\right)^2 - \left(1 - \frac{B}{L}\right)^2\right]}{144 \times 4} \qquad \text{(11)}$$

and

$$\frac{v_c}{(f_c)'} = \frac{0.004 \left[36 - \left(6q + \frac{d}{L}\right)^2 - 36\left(1 - \frac{B}{L}\right)^2\right]}{\left(6q + \frac{d}{L}\right)\frac{d}{L}} \left[\frac{\frac{P}{(f_c)'}}{y}\right] \qquad \text{(12)}$$

The 7-ft by 9.14-ft footing in the Example 2 could be 25.2 in. thick, instead of 26.5 in. as determined by Fig. 1.

But the writer does not recommend modifications of the original curves (Fig. 1) developed by the author because in most cases the footings and columns are square or nearly so. These design curves can then be applied directly.

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[3] Structural Engr., Skidmore, Owings & Merrill, Chicago, Ill.

The beauty of the curves is their flexibility and simplicity in use. There is no justification for a set of more complicated curves in order to eliminate an insignificant error. In addition the effort may not necessarily mean more economic design because Σo may control.

Engineers who make or check numerous footing designs will greatly appreciate these curves.

Phil M. Ferguson,[4] Assoc. M. ASCE.—In the case of square footings, the relations in this paper appear to be theoretically correct, and the accuracy of the curves is limited only by the scale to which they are drawn. However, in rectangular footings the theoretical diagonal tension problem is more complex. The long projection is always the critical one, and the critical shear at a given distance from the column is less than it would be for a square column having sides of the long dimension. In Example 2, with sides 7 ft by 9.14 ft, the shear at 26.5 in. from the column (the value of d) is about 11% less than was assumed in using the curves for square footings. The error would be larger when the length-to-width ratio is greater.

In connection with bond, it should be noted that the use of deformed reinforcing bars meeting ASTM Specification 305–49 will make bond a less critical item and may shift the emphasis in design. Higher bond stresses—$0.08\,f'_c$ against the present $0.056\,f'_c$—should be finally approved by the American Concrete Institute during 1951.

Winfield A. McCracken,[5] Assoc. M. ASCE.—The discussers call attention to the fact that for square columns and rectangular footings the curves as presented are in error. The error is on the conservative side and increases with the length-to-width ratio. However, as Mr. Teng indicates, most footings can be designed with a length-to-width ratio close to 1.0, keeping the error within the accuracy of the design assumptions.

Since the bond relation is straight line, the increase in allowable bond stress (from 0.56 to 0.80 f'_c) can be accounted for by multiplying the 0 as obtained from the curves by a factor of 0.7. However, until delivery of bars meeting American Society of Testing Materials Specification 305–49 can be assured, the writer does not recommend revision of the curves for the change in allowable bond.

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[4] Chairman, Civ. Eng. Dept., Univ. of Texas, Austin, Tex.

[5] Cons. Engr., Houston, Tex.

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SHIP RESPONSE TO RANGE ACTION IN HARBOR BASINS

By Basil Wrigley Wilson,[1] Assoc. M. ASCE

Synopsis

The conditions under which a moored ship will respond to sea oscillations in harbor basins are examined on a theoretical basis in this paper. The circumstances in which the ship resonates longitudinally with a periodic surge are deduced and are shown to be dependent primarily on the degree of initial tightness of the moorings, the magnitude of the seiche, and the location of the ship within it. The mass of the ship is of only minor importance in longitudinal motion. Transverse motion is shown to be capable of developing impact forces between ship and quay, which increase approximately as the square of the mass of the ship. Generally, the critical periodicities in both longitudinal and transverse motions are found to lie in the range from 0 min to 2 min. Experimental verification of the theory on models is described, and further evidence supporting the theoretical findings is adduced, indirectly, by induction from an analysis of rope breakages and, directly, by measurement and correlation of the motions of prototype ships with the activating seiches. The importance of tight ropes and of shock-absorbing fenders in helping to circumvent troubles from ship ranging is stressed, but the real solution of undesirable ship motion lies in the subjugation of disturbances that have periodicities of less than 2 min.

1. Introduction

The studies reported in this paper arose as part of the researches into the cause and cure of "range" or surge action in Table Bay Harbor, undertaken at Capetown, Union of South Africa, between 1943 and 1946. The general progress of these researches has been outlined briefly in the contemporary "Annual Reports" of the general manager of the South African Railways and Harbours,

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Note.—Published in November, 1950, as *Proceedings-Separate No. 41.* Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Asst. Research Engr., Chf. Civ. Engr's Office, South African Railways and Harbours, Johannesburg, Union of South Africa.

but a more coherent, although necessarily curtailed, account of the phenomenon and its investigation is contained in a paper describing the new graving dock at Capetown.[2] The complete manuscript, which was the basis of this paper, has been filed in the Engineering Societies Library for reference.

Range action or surge action connotes essentially a motion of water induced by seiches, operating within the confines of a harbor basin. The term "seiche," itself, implies a resonant oscillation of water or "standing wave," sustained by disturbing forces, such as marine or atmospheric wave trains, whose impressed frequency coincides, or closely concurs, with the natural frequency of oscillation of the body of water within the boundaries of the basin. The rhythmic variations of water level at the extremity of Lake Geneva, in Switzerland, were described as seiches more than two hundred years ago.

It usually happens that seiches occur in complex combinations involving both fundamental and higher harmonic modes of oscillation along the principal axes of the basin. The higher frequencies of oscillation are undoubtedly stimulated, for the most part, by visible storm swells, entering the basin, whereas the invisible ground swells of the incoming train of sea waves generally promote the fundamental seiches.

All this is considered to have been adequately established from data collected at Capetown, but it is not within the province of this paper to prove that range action is, in fact, so constituted. Rather, the premise which will be adopted is that the agency responsible for ship movement, when range action occurs, is of the nature of a combination of seiches or standing waves.

The effect of severe range action on shipping berthed alongside solid quays or jetties is always impressive and sometimes alarming. For no apparent reason a ship will describe simultaneous translatory motions in three dimensions within the compass of her mooring ropes. There have been cases, especially with large ships, where this action has been severe and prolonged enough to break all the mooring ropes and to splinter the timber fendering between the ship and the quay. On such occasions the ship's shell plating and bulkheads undoubtedly will also suffer damage of a serious kind.

Early, superficial studies of the range problem at Capetown showed that seiches of about 4-min and 5-min periodicities were very prominent in all the basins, and it was at first assumed that these were the ones responsible for the effects observed. In general, however, the movements of ships were dictated by far more rapid perturbations than these, and, as the researches probed deeper, the significance of the seiches of higher frequency grew more evident. Clear-cut relationships were elusive and were always being occluded by puzzling anomalies of ships not behaving as expected, or otherwise reacting in an unexpected manner. For these reasons, it was very uncertain as to what extent ship disturbances were activated by the strong, transverse seiches with periodicities from 1.7 min to 1.9 min found within the basins, or by their prominent second harmonics. The importance of still higher frequency oscillations was equally uncertain.

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[2] "The Sturrock Graving Dock, Cape Town," by David Eric Paterson, *Journal*, Inst. C. E., October, 1947, pp. 346–348.

Observations of ship motion were made as opportunity offered, but it was soon realized that mere measurements of ship movements could not in themselves provide the answer to this general question unless a sufficient number of ships of all sizes could be observed. Practical difficulties, largely influenced by the unpredictable nature of ship behavior, militated against this statistical approach, and the only satisfactory method of treating the problem appeared to be to examine the relationships theoretically and to check them against such observations as became available.

2. The Spring Analogy of Ship Motion

While observing ship movements in the model of Table Bay Harbor, it became apparent that, under identical conditions, a ship restrained by ropes had a much more violent motion than a ship free to move to the fullest extent in the current of a seiche. The movements in the two cases had an important phase difference, that of the moored ship at times being directly opposed to the movement of the free ship and of the current. It was clear that the effect of the mooring ropes was so important that careful attention would have to be devoted to the nature of their action. The problem, indeed, was at once seen to be analogous to that of a spring-suspended mass, whose point of suspension is subject to a periodic displacement. On this analogy the ship would be represented by the suspended mass, the mooring ropes by the spring, and the seiche current by the periodic disturbing force or displacement.

For every ship, according to her size and the slackness or other characteristics of her mooring ropes, there will exist, on this analogy, a critical period at which she will oscillate dangerously, if there exists also a disturbing impulse of the same periodicity. The disturbing periodicities, of course, are more or less fixed, being the oscillations that develop naturally in the harbor basins according to the size and shape of the basins, and it is only when the critical period of the moored ship is in fairly close agreement with the period of sea oscillation that resonance can take place. When this happens, ropes are likely to be broken; and, if the movement is "off and on," fenders can be crushed. This in essence is the idealized nature of the problem as it was envisaged and as it was subsequently studied.

Notation.—The letter symbols in this paper are defined where they are first introduced, in the text or by the illustrations.

3. The Constraint from Mooring Ropes

In broaching the problem it was necessary to know something of the relationship existing between the retarding pull of a ship's ropes and the longitudinal or lateral displacement of the ship.

Fig. 1 may be assumed to represent a single rope or all the ropes collectively at one end of a moored ship, their lengths being S_0 between A, the rest position of the ship's fair-lead (a rope guide), and B, the bollard on the quay, and their lowest point of sag occurring at C, an intermediate point below the quay level. Assuming for the present that the ropes are of uniform weight per foot run, they will hang in the form of the catenary ACB, the known elements of which (under equilibrium conditions when the ship is at rest) are the vertical and horizontal distances H, h_0 and l_1, l_2, respectively. If the suspension point A suffers a horizontal displacement AA_1 in the plane of the rope, slack is taken up and the rope assumes the configuration of a segment of a new catenary A_1BC_1, whose hypothetical lowest point of sag, C_1, now lies outside of A_1B. The known elements of the catenary in this case are the length of the rope S and the dimensions H and L.

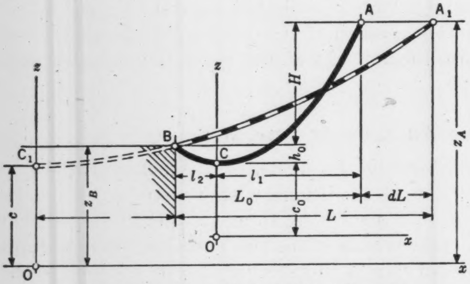

FIG. 1.—GEOMETRY OF MOORING ROPE SUSPENSION

The tensions developing in the rope at any particular stage of displacement of point A are determinable in terms of the density of the rope and the form of the catenary. The latter (in the generalized case when the rope has pulled sufficiently taut to throw C_1 outside of A_1B) may be examined with reference to a coordinate system zOx, the origin of which is taken on the directrix of the catenary A_1BC_1 at a distance c vertically below C_1 (Fig. 1). The general equations of the catenary may then be represented in the forms:

$$z = c \cosh \frac{x}{c} \qquad (1a)$$

and

$$s = c \sinh \frac{x}{c} \qquad (1b)$$

in which s defines the length of the hypothetical rope from point C_1 to any point (x,z) on the catenary. Referring to Fig. 1 and Eq. 1a,

$$z_B + H = c \cosh \frac{x_B + L}{c} \qquad (2a)$$

and

$$z_B = c \cosh \frac{x_B}{c} \qquad (2b)$$

Subtracting Eq. 2b from Eq. 2a,

$$H = c \left(\cosh \frac{x_B + L}{c} - \cosh \frac{x_B}{c} \right) \qquad (3a)$$

Similarly, from Eq. 1b,

$$S = c \left(\sinh \frac{x_B + L}{c} - \sinh \frac{x_B}{c} \right) \qquad (3b)$$

Squaring Eqs. 3, subtracting, and symplifying in terms of standard relationships for hyperbolic functions,

$$S^2 - H^2 = 2\,c^2\left[\cosh\left(\frac{L}{c}\right) - 1\right] \quad \text{(4)}$$

Developing the series expansion for cosh L/c,

$$\begin{aligned} S^2 - H^2 &= 2\,c^2\left[\frac{1}{2!}\left(\frac{L}{c}\right)^2 + \frac{1}{4!}\left(\frac{L}{c}\right)^4 + \frac{1}{6!}\left(\frac{L}{c}\right)^6 + \cdots\right] \\ &= L^2\left[1 + \frac{1}{12}\left(\frac{L}{c}\right)^2 + \frac{1}{360}\left(\frac{L}{c}\right)^4 + \cdots\right] \quad \text{(5)} \end{aligned}$$

and, by transposing,

$$\frac{S^2 - (H^2 + L^2)}{L^2} = \frac{1}{12}\left(\frac{L}{c}\right)^2\left[1 + \frac{1}{30}\left(\frac{L}{c}\right)^2 + \frac{1}{1{,}680}\left(\frac{L}{c}\right)^4 + \cdots\right] \quad \text{(6)}$$

On the right-hand side of Eq. 6, each term in the bracket is negligible except the first, 1. Discarding and solving for c^2,

$$c^2 = \frac{L^4}{12}\left[\frac{1}{S^2 - (H^2 + L^2)}\right] \quad \text{(7)}$$

Since the value of the horizontal tension in a catenary at the lowest point of sag is given by the product $\lambda\, c$, in which λ is the weight per unit length of rope, the horizontal component of rope tension, T_h, at either point A_1 or point B is

$$T_h = \frac{\lambda\, L^2}{\sqrt{12\,[S^2 - (H^2 + L^2)]}} \quad \text{(8)}$$

4. Average Mooring Conditions at Capetown

In South African practice ships of the usual sizes are secured fore and aft to the quays by combinations of strops and ship's wires, springs, and manila ropes. According to current terminology in use at Capetown, a "strop" is a sling of heavy coir rope with a steel thimble at one end, through which a ship's wire is threaded and doubled back to the ship, the strop itself being looped over the quay bollard. A "spring," on the other hand, is a loop of coir rope permanently connected to a single steel wire.

A limited number of "backsprings" or side stays is used amidships to increase the holding power; but, for all practical purposes, the moorings of a ship may be considered to consist of two groups (bow and stern) of more or less equally stressed ropes, whose lines of action are oblique to the longitudinal axis of the ship at an angle of about 20°.

The coir strop fulfils in some measure the functions of a mechanical spring by absorbing, gradually, the shock of any suddenly applied load. At Capetown coir strops are of two sizes, 14 in. and 18 in. in circumference, and the common sizes of steel wire used in conjunction with them are 3 in., 3.5 in., and 4 in. in circumference. Coir strops usually have a sling length of 18 ft, and the 18-in. size would weigh about 18 lb per foot of run as against about 2 lb per foot of run for a single steel wire rope, and 4 lb per ft of run for a double 3.5-in. steel wire rope.

A ship of average size at Capetown (14,200 tons displacement) may be considered to be held with 24 coir ropes of 18-in. size in combination with 18 steel wires of 3.5-in. circumference.

The average parametric dimensions L_0 and H for these moorings are, respectively, 85 ft and 18 ft. The dimensions h_0 and l_2 in Fig. 1 may be assumed, in general, to be, respectively, 2 ft and 10 ft. From knowledge of the dimensions, L_0 and H and l_2 and h_0, the average initial length S_0 of the moorings may be computed to be 88.69 ft.

5. The Elastic Properties of Mooring Ropes

As no reliable information was available in regard to the elastic properties of typical mooring ropes, new and worn samples of coir and steel ropes were subjected to repeated loading tests. To render the results assimilable, mean curves were drawn through the stabilized hysteresis loops of the load-extension diagrams. These gave average relationships between load and extension such as would apply if the ropes were continuously in service. The mean curves for the various ropes tested have been plotted collectively as load-strain diagrams in Fig. 2.

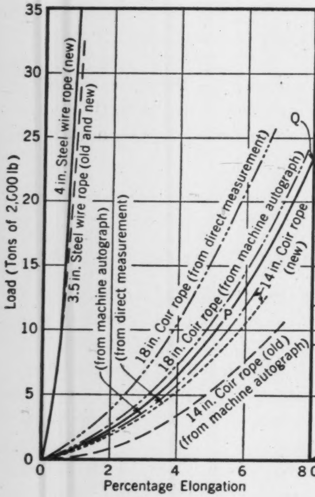

Fig. 2.—Mean Load-Elongation Curves for Coir and Steel Mooring Ropes

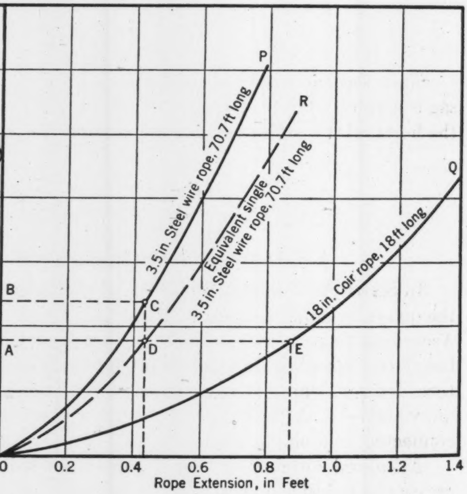

Fig. 3.—Tension-Extension Relationships for Coir-Steel Mooring Combination

Despite divergencies, a band is indicated within which the load-strain relationship for an average coir rope may be expected to lie. Considering the fact that under average mooring conditions at Capetown a certain number of

14-in. strops are likely to be involved, there is justification for taking, as the final criterion of the elastic properties of an 18-in. coir rope, the arbitrary curve OPQ (Fig. 2), which is centrally disposed in the band. Results for the steel ropes are more consistent and show little variation as between old and new, except in ultimate strength. The difference in shock-cushioning capacity between coir and steel is well portrayed in Fig. 2.

Since the equilibrium length of an average mooring is $S_0 = 88.7$ ft, of which 18 ft comprises 18-in. coir rope and the remaining 70.7 ft comprises 3.5-in. steel rope, the extensions that would occur at various rope tensions in the lengths of the coir and steel may readily be deduced. Curves OCP and OEQ in Fig. 3 demonstrate these relationships, in which curve OEQ corresponds with the curve OPQ in Fig. 2.

An average ship of about 14,200 tons displacement, however, is likely to be tied to its berth with 12/9 coir-steel ropes at both bow and stern, and any pull exerted by the ship on the moorings must, in the idealized case here considered, be distributed uniformly among the 12 coir ropes or among the 9 steel ropes, with the result that an individual steel rope is called upon to carry four thirds of the load imposed on an individual coir rope. It will be necessary, therefore, to adjust the extensions on the steel rope to accord with the fact that the load carried will be four thirds of that borne by the coir rope. Curve ODR, Fig. 3, is the result of such an adjustment.

6. The Relation Between Ship Movement and Rope Tension

As the criterion of the actual tension, T, in the rope at any point, it will suffice to consider merely the horizontal component, T_h. (The error in this assumption can be shown to be insignificant.) Accordingly, Eq. 8 may be converted to present needs. Squaring both sides and transposing, the equation takes the form of a quadratic in L^2, the solution of which is

$$L^2 = \frac{6\,T^2_h}{\lambda^2}\left\{\left[+1\,\frac{\lambda^2\,(S^2 - H^2)}{3\,T^2_h}\right]^{\frac{1}{2}} - 1\right\}\ldots\ldots\ldots\ldots(9)$$

If λ is computed as the ordinary arithmetical mean of the weights of the coir and steel ropes, its value is found to be 2.4 lb per ft. Since $(S^2 - H^2)$ is likely to have a fairly constant value in the region of 7,700, the term $\lambda^2\,\frac{(S^2 - H^2)}{3\,T^2_h}$ will therefore be very small compared with unity for all values of T_h greater than, say, 0.5 ton. Whence, with sufficient approximation,

$$L = \sqrt{S^2 - H^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

The implication of this Pythagorean relationship is that the tension on the rope is still comparatively insignificant—less than 0.5 ton—when all slack is taken up and the rope becomes completely taut. Measurable tension then develops only when the rope has been pulled into straight-line suspension. It is now seen that the weight of the rope is of no practical consequence.

By selecting arbitrary values of rope tension, T_h, the corresponding extensions, dS (coir + steel), of an equivalent unit mooring can be read from Fig. 3, and the length of the mooring can be determined as $S = S_0 + dS$. From Eq. 10, L is then simply derived, and the movement of the ship, $dL = L - L_0$, follows at once.

In Fig. 4, a ship is shown, in plan view, moored parallel to a quay at a distance Y_0 from its edge (in the rest position). The moorings AB and A'B' are the projections of the ropes on the horizontal plane and correspond with the length L_0 in Fig. 1, when the ship is at rest. It is obvious than any horizontal movement of the ship under the influence of seiches and mooring restraints may be resolved into longitudinal and transverse components relative to the quay, respectively, u and v.

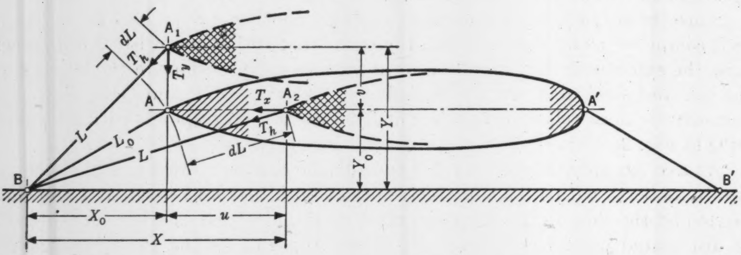

FIG. 4.—GEOMETRY OF LONGITUDINAL AND TRANSVERSE SHIP MOTION

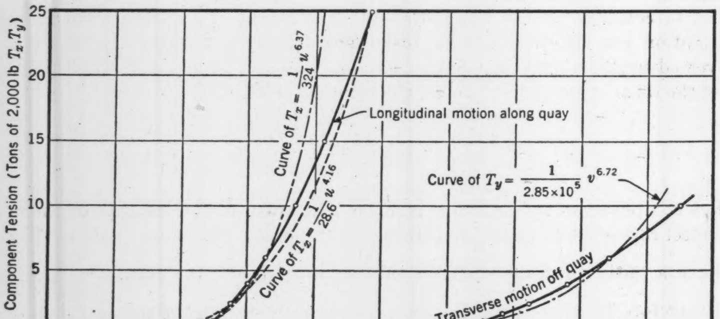

FIG. 5.—RELATIONSHIP BETWEEN ROPE TENSION AND SHIP MOVEMENT

The conversion of dL according to Fig. 4 is represented in Fig. 5. The dimension Y_0 has been assumed equal to 34 ft. Fig. 5 illustrates very clearly that an average ship, moored under the conditions commonly observed at Capetown, is almost completely free to displace longitudinally through a distance of 2.5 ft, and transversely through a distance of 6.5 ft, before any effective rope tension is developed in resisting the movement. The relationships in Fig.

5 conform closely to equations of the following type:

$$T_{x,y} = k\,(u \text{ or } v)^n \ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

in which k is a constant, and n, a numerical exponent, as may be apparent from the superimposed curves in Fig. 5. Of the two functions shown enveloping the curve for longitudinal motion the one with $1/k = 324$ and $n = 6.37$ is considered to be more appropriate as giving greater accuracy over the range of tensions from 0 ton to 10 tons.

7. Longitudinal Ship Motion Under the Stimulus of a Seiche

In the present treatment of this problem the vertical motion of the ship is neglected and that part of the horizontal motion which is parallel to the quay is considered first. The seiche applicable to these conditions is that which has its node at right angles to the side of the basin or berth along which the ship is moored.

Placing the origin O of a system of coordinates ZOX on the still water surface at one end of the dock (assumed rectangular and of uniform depth), and the x-axis horizontal along the side at which the ship is lying, the z-axis being vertical and positive upward, the horizontal displacement (parallel to the quay) of a water particle at any point (x,z) can be written:[3]

$$\bar{x} = A\,\frac{\cosh q(z + d)}{\sinh q\,d}\,\sin q\,x\,\sin\,(p\,t + \epsilon) \ldots\ldots\ldots\ldots (12)$$

in which A is the maximum vertical amplitude (half range) of the seiche; d is the depth of the water in the basin measured from still water level; p is the angular frequency (equal to $2\,\pi/\tau_0$, τ_0 being the period) of the seiche; and q is the nodal frequency for a particular seiche, of value $m\,\pi/D$ (D being the length of the side of the basin and m, an integer defining the nodality of the seiche). In Eq. 12, t has its usual significance of time; and ϵ, of phase angle.

The factor $\dfrac{\cosh q(z + d)}{\sinh q\,d}$ in Eq. 12 obviously varies from $\dfrac{1}{\tanh q\,d}$ at water surface to $\dfrac{1}{\sinh q\,d}$ at sea bottom; and, since the depth d is usually small compared with the dimension D involved in q, the hyperbolic sine and tangent are of the same order, if only the lowest modes of oscillation (such as $m = 1$, 2, and 3) are considered. The horizontal movement of water is thus sensibly the same at all depths and the hyperbolic factor can be replaced by the approximation that $\dfrac{1}{\tanh q\,d}$ tends to become $\dfrac{1}{q\,d}$, when the product $q\,d$ is small.

Differentiating Eq. 12 with respect to t, the horizontal velocity of the water at any point is then given by

$$\frac{d\bar{x}}{dt} = V\cos p\,t \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

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[3] "Hydrodynamics," by Horace Lamb, University Press, Cambridge, England, 1932, p. 356, Article 228.

in which

$$V = \frac{A\,p}{q\,d} \sin q\,x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

The phase angle, ϵ, of the seiche has been discarded for convenience as being unimportant to the argument.

The velocity of the horizontal surge affecting the ship at all depths of its draft at any instant is thus $V \cos p\,t$. The value of V varies, of course, along the length of the ship, but it will be sufficient in most cases to take the value of V corresponding to midposition of the ship and to regard this as representative of water velocity past the ship.

Let the acquired velocity of the ship in the periodic current be v. Then the velocity of the water relative to the ship is $V \cos (p\,t) - v$; and the velocity of the ship relative to the water, $v - V \cos p\,t$.

The force exerted by a stream of water on a submerged cylinder is the product of the mass of the displaced water and the acceleration of the stream in passing the obstruction.[4] It seems justifiable to suppose that the force inducing translation of the ship will be given by the product of the mass of the displaced water (also the mass of the ship M) and the acceleration of the current past the ship.[5] Denoting this force by F,

$$F = M \frac{d}{dt} \left[V \cos (p\,t) - v \right] \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

The forces opposing the motion of the ship comprise the skin-frictional resistance of the water R, and the retarding pull of the ship's ropes, $\Sigma\, T_x$. The resistance R can be expressed as

$$R = Q\,(v - V \cos p\,t)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

in which Q is a constant depending upon the ship being considered.

The velocity $V \cos p\,t$ of the periodic current of a seiche is never likely to exceed about 1 knot at the most in any deep water harbor. That this is true may be gaged (see subsequently in Eq. 27) from the relation $V = A \sqrt{\frac{g}{d}}$. Under what would be regarded as very severe range action, if $A = 1$ ft and $d = 40$ ft, the surge velocity at a node would not exceed $V = 0.9$ ft per sec. Consequently, the curve of Eq. 16 may be accepted as being reasonably straight over the limited range of small velocities and representable as a straight line with the equation:

$$R = K\,(v - V \cos p\,t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

The new constant K depends essentially on the size and shape of the ship.

The longitudinal component of the retarding pull of the ship's ropes, ΣT_x, has been shown to be related to the horizontal movement of the ship along the quay by the equation of the type $\Sigma T_x = C\,u^n$, in which C is a constant depending on the number and size of the ropes and their condition (Section 6).

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[4] "Hydrodynamics," by Horace Lamb, University Press, Cambridge, England, 1932, p. 76, Article 68.

[5] "Theoretical Hydrodynamics," by Louis M. Milne-Thomson, Macmillan Co., London, England, 1938 p. 234, Article 9.22.

It has already been shown that, although there is always tension in the ropes, no matter how slack they may be, such tension is quite negligible compared with the pulls that develop in retarding the movement of the ship. Consequently, when the ship pulls on the stern ropes, the tensions from the bow ropes can be neglected, and vice versa.

By representing ship velocity in terms of the horizontal travel of the ship, Eqs. 15 and 17 may be further developed as follows:

$$F = -M\left(V\,p\sin p\,t + \frac{d^2u}{dt^2}\right) \dots\dots\dots\dots\dots (18a)$$

and

$$R = K\left(\frac{du}{dt} - V\cos p\,t\right) \dots\dots\dots\dots\dots\dots (18b)$$

The equation of motion of the ship along the quay under the stimulus of the periodic surge and the restraints of friction and moorings may now be written as follows:

$$M\frac{d^2u}{dt^2} = F - R - \Sigma T_x \dots\dots\dots\dots\dots\dots\dots (19)$$

which reduces (on substitution of the results of Eqs. 11 and 18) to the differential equation:

$$\frac{d^2u}{dt^2} + \frac{K}{2\,M}\frac{du}{dt} + \frac{C}{2\,M}u^n = \frac{K\,V}{2\,M}\cos p\,t - \frac{V}{2}p\sin p\,t \dots\dots (20)$$

Eq. 20, although representative of a common, natural phenomenon, appears to have no general solution in ordinary transcendental functions. Only in the case when $n = 1$ can the solution be evaluated, such a case representing the damped motion of an ordinary spring-suspended mass under the action of a periodic disturbing force.

8. THE CONDITION OF RESONANCE IN LONGITUDINAL SHIP MOTION

The principal object of the theoretical approach, as outlined in Section 2, was to ascertain the conditions under which ship motion will resonate with the activating periodic surge. The desired information concerns the particular value of the period τ of the seiche which will make the horizontal displacement u, of the ship, a maximum. Any relationship expressing this condition will clearly contain neither of the variables u and t, but only the quantities p (or τ), $\frac{K}{2\,M}$, $\frac{C}{2\,M}$, and V, which are the "foundation stones" of Eq. 20. The elements of the function linking these quantities have been derived by dimensional analysis and the critical periodicity would appear to take the form:

$$\tau = \left(\frac{C}{2\,M}\right)^{-i} V^i \psi\left[\frac{K}{2\,M}\left(\frac{C}{2\,M}\right)^{-i} V^i\right] \dots\dots\dots\dots (21)$$

in which (to avoid complex typography)

$$i = \frac{1}{1 + n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22a)$$

and

$$j = \frac{1 - n}{1 + n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22b)$$

In the particular instance when $n = 1$, Eqs. 22 reduce to $i = \frac{1}{2}$ and $j = 0$; and Eq. 21 becomes

$$\tau = \left(\frac{C}{2\,M}\right)^{-\frac{1}{2}} \psi \left[\frac{K}{2\,M}\left(\frac{C}{2\,M}\right)^{-\frac{1}{2}}\right] \quad \ldots\ldots\ldots\ldots\ldots (23)$$

but, since the known solution for this case, derived from the particular integral of Eq. 20 for $n = 1$, is $\tau = 2\,\pi \left(\frac{C}{2\,M}\right)^{-\frac{1}{2}}$, it follows that the function ψ in Eq. 23 is a constant, of value $2\,\pi$. It is reasonable then to conclude that the function ψ of Eq. 21 is also a constant, of value $2\,\pi$, and the critical resonance condition in the general case thus reduces to

$$\tau = 2\,\pi \left(\frac{C}{2\,M}\right)^{-i} V^{j} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

This result is capable of further development since V is defined in Eq. 14, in which $p = \frac{2\,\pi}{\tau_0}$ and $q = \frac{m\,\pi}{D}$. The periodicity of the seiche τ_0, in the case of a rectangular basin of uniform depth and with vertical walls, such as the Victoria and Duncan docks at Table Bay Harbor, is expressible as[6]

$$\tau_0 = \frac{2\,D}{m(g\,d)^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

The maximum velocity V of the horizontal surge of the seiche, at any point x along the side of the basin where the ship is moored, reduces, in consequence (from Eq. 14) to

$$V = A \left(\frac{g}{d}\right)^{\frac{1}{2}} \sin \frac{m\,\pi\,x}{D} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (26)$$

The worst effects from the periodic current obviously occur when $\sin \frac{m\,\pi\,x}{D}$ in Eq. 26 is equal to 1, or when the center of the ship happens to be coincident with the nodal point of the seiche. When the ship is so situated,

$$V = A \left(\frac{g}{d}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27)$$

If k is the spring constant for an individual mooring rope, as defined in Eq.

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[6] "Hydrodynamics," by Horace Lamb, University Press, Cambridge, England, 1932, p. 284, Article 190.

11, and $N/2$ is half the total number of ropes holding the ship, then,

$$C = \frac{N\,k}{2} \qquad (28)$$

Replacing M in terms of W, the displacement tonnage of the ship, and introducing the results of Eqs. 27 and 28, Eq. 24 can be resolved into its final form:

$$\tau = \frac{2\,\pi}{A}\left(\frac{d}{g}\right)^{\frac{1}{2}}\left(\frac{4\,W\,A^2}{N\,k\,d}\right)^{\frac{1}{2}} \qquad (29)$$

The interpretation to be placed on this result is that, if any of the periodicities τ_0 (Eq. 25) of the seiches to which the harbor basin is addicted (in virtue of its dimensions and depth) agrees with or approximates τ in value (Eq. 29)(the resonant periodicity of the ship according to the manner of its mooring), a very dangerous set of circumstances prevails which might easily result in the ship breaking adrift.

As a check on Eq. 29 there remains another possible line of approach to the problem of resonance. The condition of resonance implies equality between the free period of oscillation of the ship-spring system and the impressed periodicity of the seiche, and this, in turn, requires that the maximum travel of the ship, $u(\max)$ should equal the maximum amplitude of horizontal movement of the water, $\bar{x}(\max)$.

The maximum amplitude of horizontal water movement at the node of the seiche is given by Eq. 12 when $\sin\,(p\,t + \epsilon)$ and $\sin\,q\,x$ both equal unity. By the use of Eq. 25, Eq. 12 is further reducible to the form:

$$\bar{x}(\max) = \frac{A\,\tau_0}{2\,\pi}\left(\frac{g}{d}\right)^{\frac{1}{2}} \qquad (30)$$

Replacing $\bar{x}(\max)$ by $u(\max)$ and solving for τ (τ_0 with suffix discarded), the resonance condition for the ship-spring system is expressed as follows:

$$\tau = \frac{2\,\pi}{A}\left(\frac{d}{g}\right)^{\frac{1}{2}} u(\max) \qquad (31)$$

Comparing Eq. 31 with Eq. 29 it would seem that the latter can be valid only if

$$u(\max) = \left(\frac{4\,W\,A^2}{N\,k\,d}\right)^{\frac{1}{2}} \qquad (32)$$

That this identity is actually true is established by the following derivation. The condition of resonance envisaged in formulating Eq. 31 requires that the acceleration of the ship at the extremity of its travel $u''(\max)$ shall equal the maximum acceleration of the water mass, $\bar{x}''(\max)$. Consequently, differentiating Eq. 13 with respect to t, and putting both $\sin\,q\,x$ and $\sin\,(p\,t + \epsilon)$ equal to unity,

$$u''(\max) = \bar{x}''(\max) = \frac{A\,p^2}{q\,d} \qquad (33)$$

The longitudinal component of the pull which this acceleration imposes on the mooring ropes comprises both the inertia force of the ship, $M\,u''(\max)$, and the pressure of the water, $M\,x''(\max)$, and is therefore

$$\Sigma(\max)\,T_x = \frac{2\,M\,A\,p^2}{q\,d} \quad \text{(34a)}$$

In terms of Eqs. 11 and 28,

$$\Sigma(\max)\,T_x = \frac{N\,k}{2}\,u^n(\max) \quad \text{(34b)}$$

whence, from Eqs. 34,

$$u^n(\max) = \frac{4\,M\,A\,p^2}{N\,k\,q\,d} \quad \text{(35)}$$

By multiplying both sides of Eq. 35 by $u(\max)$ and eliminating the product $p\,\acute{u}(\max)$ on the right-hand side in terms of Eq. 31,

$$u^{1+n}(\max) = \frac{4\,M\,A\,p}{N\,k\,q\,d}\left(\frac{g}{d}\right)^{\frac{1}{2}} A \quad \text{(36)}$$

Since $q = \frac{m\,\pi}{D}$ and $p = \frac{2\,\pi}{\tau}$, the ratio $\frac{p}{q}$ in Eq. 36 can be expressed in terms of m, D, and τ, all of which can be further expressed in terms of g and d by the use of Eq. 25. The result of these changes in Eq. 36 establishes the identity of Eq. 32.

Eq. 29 is confirmed, therefore, and exemplifies the penetrating analytical power of dimensional analysis. Eqs. 29 and 31 both represent the resonant condition of ship motion in the longitudinal direction, within the limits of the assumptions made.

Eq. 31 is extremely useful from the point of view of generalization and lends itself to further manipulation. For this purpose it is noted, as in Fig. 5, that $u(\max)$ depends basically on the amount of movement afforded the ship through slackness of its ropes. Thus, the amount, u_0, by which the ship can displace horizontally to draw the catenary of the moorings into a straight line without rope extension (assuming this were possible), is approximately

$$u_0 = (S^2_0 - H^2)^{\frac{1}{2}} - L_0 \quad \text{(37)}$$

With sufficient approximation for most moored ships, the initial length of mooring rope can be expressed as

$$S_0 = \frac{L_0}{2} + \left(\frac{L^2_0}{4} + H^2\right)^{\frac{1}{2}} \quad \text{(38)}$$

This result derives from assuming that, when the ship is at rest, ropes tend to hang in a parabolic catenary, the lowest point of which, at the quay bollard, has a horizontal slope. Substituting Eq. 38 in Eq. 37 and resolving the roots in terms of the binomial theorem—

$$u_0 = \frac{H^2}{2\,L_0} \quad \text{(39)}$$

—a remarkably simple expression. (In the expansions, terms involving higher powers than the first of H^2/L^2_0 have been discarded since H is usually less than $0.2\ L_0$ for most conditions of mooring.) If the maximum travel of the ship, $u(\max)$, is greater than u_0 by a factor r, then Eq. 31 becomes

$$\tau = \pi r \left(\frac{d}{g}\right)^{\frac{1}{2}} \frac{H^2}{A\ L_0} \qquad (40)$$

9. NATURAL PERIODS OF LONGITUDINAL OSCILLATION FOR SHIPS

It has already been shown in Section 6 that the value of the exponent n for moorings, in the avarage case at Table Bay Harbor, is between 6 and 7. If ropes are slacker than average, n will be greater and, if tighter, n will be less than 6 and 7; but even in the case of initially taut ropes n is not likely to be less than about 1.5. It is clear, then, that even comparatively large variations of the quantities inside the second set of parentheses in Eq. 29 will not appreciably affect the $(n + 1)$th root of the total bracketed quantity. This suggests that, within limits, the mass of the ship and the number of ropes used in mooring it are of minor importance as regards the ship's reaction to the periodic current. By far the most important single factor is the value of n itself. The factor of next importance is the amplitude, A, of the seiche, the periodicity varying roughly in inverse proportion to it.

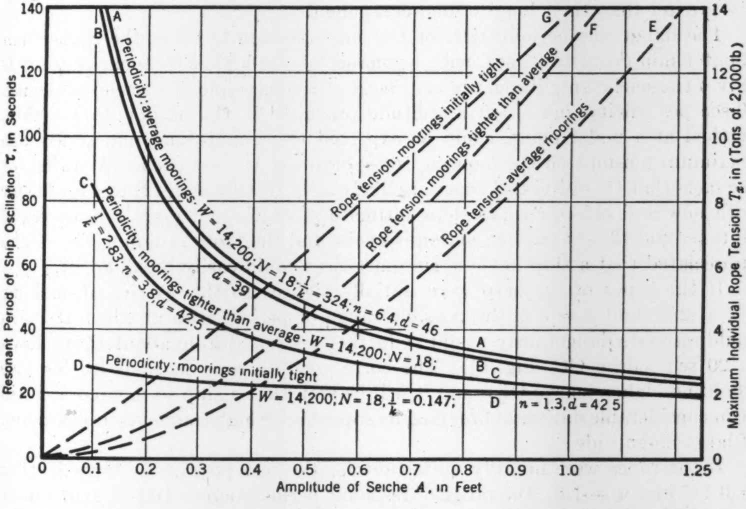

FIG. 6.—INFLUENCE OF ROPE TIGHTNESS ON THE RESONANCE IN LONGITUDINAL SHIP MOTION

It is now necessary to determine the free period of oscillation for the average ship ($W = 14{,}200$ tons), whose manner of mooring ($N = 18$; $1/k = 324$; and $n = 6.4$) has been discussed in Sections 5 and 6. The depth of water in Table

Bay Harbor can be assumed to vary from $d = 39$ ft to $d = 46$ ft, according to the state of the tide. Thus, the variation of free period with amplitude of seiche for these two extremes of tide, as computed from Eq. 29 for the values of the foregoing symbols, is illustrated in Fig. 6 by the curves AA and BB.

These curves are of great interest in so far as they show that seiche periodicities greater than about 2 min are only capable of producing resonance effects in a ship, as normally moored, at very small seiche amplitudes, for which movements would be so feeble as to produce only insignificant rope tensions. This is demonstrated by curve OE, which gives the maximum tension developing in an individual rope at resonance. Data for this curve were computed by Eq. 34*a*, modified by the use of Eq. 25 to the form:

$$\Sigma(\text{max})\ T_x = \frac{2\,\pi\,W}{(g\,d)^{\frac{1}{2}}}\,\frac{A}{\tau} \qquad \text{(41)}$$

The tensions as calculated by Eq. 41 have been multiplied by a factor of 1.07/9 in conformity with the fact that, in the case treated, the total force is divided among 9 ropes (bow or stern), and that actual rope tensions are approximately 7% greater than their longitudinal components.

The dangerous periodicities for the ship are seen to be in the range from about 1 min down to 25 sec, with resonance at the higher frequencies possible only if the seiche amplitudes are very large. For example, if a seiche of about a 30-sec periodicity and a 1-ft amplitude prevailed in the harbor, then a ship, berthed at a nodal point, could be expected to resonate and impose average maximum tensions on the mooring ropes of about 10 tons each. Allowing for the facts that (1) individual ropes are likely to be overloaded; (2) test results for both new and old coir ropes show ultimate strengths not greatly in excess of 10 tons; and (3) the loading action is cyclic and therefore fatiguing—it will be appreciated that a ship in these circumstances could easily break adrift.

If the ropes of the ship were initially tighter to the extent of making $1/k = 2.83$ and $n = 3.8$, the possible range of periodicities at which the ship could resonate dangerously would be further restricted from about 45 sec down to 20 sec (curve CC, Fig. 6). The corresponding rope tensions in this case would be higher (curve OF), and it will be seen that a ship so moored will still be in considerable danger of breaking its ropes under high frequency range action of large magnitude.

If the ropes were initially quite taut for the rest position of the ship ($1/k = 0.147$ and $n = 1.3$), the range of resonant periods (curve DD, Fig. 6) would be limited from about 30 sec to 20 sec. A ship so moored would be virtually immune to all seiche disturbances of greater periodicity; but it would still have dangerous response to these high frequency oscillations, as may be judged from the curve OG, which shows the corresponding rope tensions. The mitigating feature here is that most harbors, although usually incapable of preventing ingress of long-period sea disturbances, show greatest efficiency in blocking the penetration of high frequency wave trains. In a well designed harbor, therefore, the chances of 20-sec to 30-sec seiches reaching any exceptional amplitude are relatively small, and tight moorings, although no cure-all, are thus a fairly effective antidote to longitudinal ship ranging.

On the other hand, if the ship moorings are slacker than avearge ($1/k$ and n considerably greater, say, than 324 and 6.8, respectively), long-period seiches become increasingly more important in the resonant motion. The relationships for such cases would lie above curve AA, Fig. 6, and the dangerous periodicities would cover a range from 30 sec to 80 sec (or more)—frequencies that have far greater penetrative powers in harbors than ordinary storm swells of 10-sec to 20-sec periodicities, and which are capable therefore of attaining fairly large amplitudes. A poorly tied ship with very slack ropes is thus a ready victim for the onset of severe range action.

The foregoing text treats the case of an average ship of 14,200 tons displacement. It might apply equally well to ships both larger and smaller, that have about the same deck height, since the mass of the ship has relatively little effect upon the final result. However, it will be profitable to examine the effects of the size of the ship in so far as it influences the height, H, of the moorings, through the medium of Eq. 40, which expresses the resonance condition in another form.

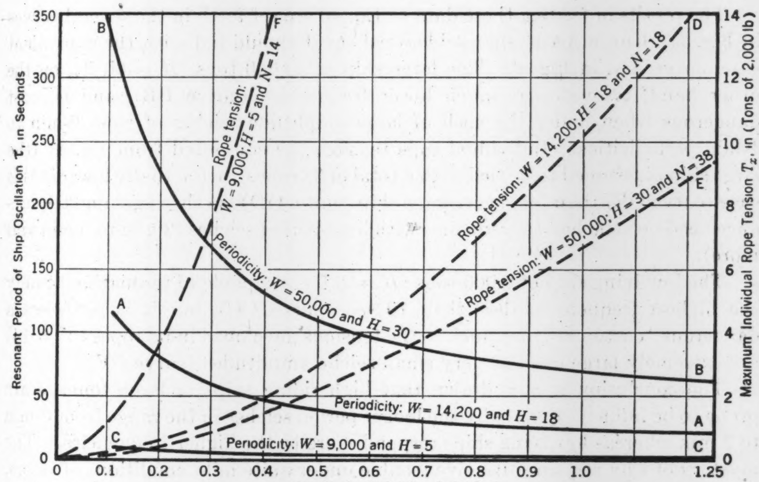

FIG. 7.—INFLUENCE OF SHIP SIZE ON THE RESONANCE IN LONGITUDINAL SHIP MOTION

The criteria are the static parametric dimensions, H and L_0, of the mooring ropes. The distance H is capable of considerable variation according to the size of the ship and the state of its draft; in point of fact, for stern ropes on low lying ships at low tide it can even be negative. The dimension L_0 is less easily affected by ship size and, for convenience, may be considered to remain approximately constant at 85 ft.

The factor r in Eq. 40, being the ratio of u(max) to u_0, will depend on the maximum tension developing in the ropes. An approximate idea of how it varies with seiche amplitude may be gained from Figs. 5 and 6. As an example, note from Fig. 6 that a seiche amplitude of 0.5 ft will induce a maximum rope tension in average moorings of 3.1 tons. This tension corresponds in Fig. 5 with $u(\max) = 2.85$ ft, whereas u_0 is seen to be about 2 ft. The ratio r, accordingly, for $A = 0.5$ ft, is 1.42. In this way r is found to vary from about 1 to 2 over a range of seiche amplitudes from 0 ft to 1.25 ft. The assumption will be made that the same values of r apply for all values of H—that is, for all sizes of ships, as normally moored.

Taking the mean depth of water as $d = 42.5$ ft, it is then possible to investigate the relationship of Eq. 40 in respect to vessels of three sizes—namely, the average ship of $W = 14{,}200$ tons for which $H = 18$ ft, a leviathan of $W = 50{,}000$ tons for which H might be, say, 30 ft, and a low lying tanker of $W = 9{,}000$ tons for which H might be only 5 ft.

The results of feeding these data to Eq. 40 are set forth in the several curves in Fig. 7. Curve AA is almost identical (as it should be) with the equivalent case, curve AA, in Fig. 6. The large ship of 50,000 tons ($H = 30$ ft), on the other hand, resonates at much lower frequencies (curve BB), and is most dangerous when under the spell of large amplitude seiches of from 2-min to 1-min periodicities. Individual rope tensions, as computed from Eq. 41 (the large ship is assumed to be tied with a total of 38 ropes), are actually lower in this case (curve OE) than for the average ship (curve OD), so that again no importance need be attached to resonance with long-period seiches (2 min to 5 min and more).

The low lying ship of 9,000 tons ($H = 5$ ft) is capable of resonating at only the highest frequencies (less than 10 sec—curve CC), but is nevertheless a dangerous "customer" inasmuch as the tensions developed in its ropes ($N = 14$) are extremely large even at very small seiche amplitudes (curve OF).

The conclusion is now drawn that high riding ships of large tonnage are prone to be influenced more by the longer period seiches in the range from 0 min to 2 min whereas low lying ships respond more to the higher frequencies. The mystery of why certain ships give trouble under quite mild conditions of range, while others successfully ride out much heavier incidences, would now also seem to be explained. The location of the ship, whether at a node or at a loop of a seiche; the manner of its mooring, whether with tight or slack ropes—adequate or inadequate in number; the condition of its draft or the design of its decks, whether high or low in the water; and, above all, the magnitude and periodicity of the seiche disturbance, whether large or small—are the essential factors that determine the reaction of the ship.

10. Transverse Ship Motion Under the Stimulus of a Seiche

Consider next the component of horizontal motion of the ship normal to the quay. Enlarging the system of coordinates previously used to a three-dimensional one, with the x-axis horizontal along the side of the dock at which the ship is lying and the y-axis horizontal at right angles thereto, the z-axis being vertical and positive upward, the horizontal displacement (normal to the quay) of a water particle at any distance y from the x-axis can be written as follows:

$$\bar{y} = \frac{A}{q'\,d} \sin q'\,y \sin (p\,t + \epsilon) \ldots\ldots\ldots\ldots\ldots\ldots (42)$$

Eq. 42 is similar to Eq. 12, but refers now to a transverse seiche, whose nodes are at right angles to the y-axis. The symbols have the same designations as before with the difference that q' in this case has the value $m\,\pi/B$, B being the length or width of the side of the dock parallel to the y-axis.

If the center line of the ship be located at a distance Y_0 from the quay wall, as in Fig. 4, the ship will be subject to a periodic lateral displacement, through the off-and-on movement of the seiche, of

$$\bar{y} = \frac{A\,B}{m\,\pi\,d} \sin \frac{m\,\pi\,Y_0}{B} \sin p\,t \ldots\ldots\ldots\ldots\ldots\ldots (43)$$

Now, Y_0 will usually be small compared with B, so that, provided the nodality of the seiche is of a low order ($m = 1$, 2, or 3), the sine involving Y_0 may be replaced by $m\,\pi\,Y_0/B$. Hence,

$$\bar{y} = \frac{A\,Y_0}{d} \sin p\,t \ldots\ldots\ldots\ldots\ldots\ldots (44)$$

For the average ship featured in this paper, Y_0 is about 34 ft; and, if the mean water depth be taken as $d \doteq 42.5$ ft, it is clear that even if the amplitude of the seiche, A, is very large, with a value, say, of 1.25 ft, the maximum amplitude of lateral ship movement cannot exceed about 1 ft. It follows then from Fig. 5 that, since the ship must move laterally from the rest position a distance of about 6 ft before any effective rope tension can come into play, no form of resonant transverse motion of the ship is possible. The circumstances, too, are different from those of longitudinal motion because the only restraints on the movement toward the quay are the virtually inelastic forces in compression of the timber fenders and the ship's shell plating and bulkheads. As often happens, however, the ship may be under the influence of an oblique seiche, especially if located near a corner of the dock, and the longitudinal and transverse components of motion will then synchronize, with the result that the lateral movement of the ship will largely follow the time pattern of the longitudinal movement along the quay.

Most of the severe damage that occurs during range action results from the unsatisfactory cushioning medium to absorb the shock when the ship is impelled toward the quay during the on-movement. It is of interest to estimate the maximum force of impact that a ship may be called upon to absorb when this happens. The worst condition will arise when the ship has a clearance between itself and the fenders exactly equal to the amplitude of the on-movement, $\bar{y}$, for then the impact will occur just as the accelerations of the ship and the water mass reach their peak. These accelerations, given by the second differential coefficient of Eq. 44 with respect to time, are

$$\bar{y}'' = \frac{A\,Y_0\,p^2}{d} \ldots\ldots\ldots\ldots\ldots\ldots (45)$$

The resisting force $P_y(\max)$ called into play must sustain both the inertia of the ship, $M\,\bar{y}''$, and the pressure of the water on the ship, also $M\,\bar{y}''$. Therefore,

$$P_y(\max) = \frac{2\,M\,A\,Y_0\,p^2}{d} \qquad (46)$$

This result may be rendered in different form by the use of Eq. 25, in which B now replaces D, as follows:

$$P_y(\max) = 2\,\pi^2\,m^2\,W\,\frac{Y_0\,A}{B^2} \qquad (47)$$

If the ship also completes a lunge fore or aft at the instant of impact, there will be the additional force of the inward pull of the ship's bow or stern ropes. According to Fig. 4 this force will be Y_0/X of the maximum longitudinal pull, given by Eq. 41. The latter may be somewhat differently stated by substitution from Eq. 25 and, accordingly,

$$\Sigma T_y(\max) = \pi\,m\,W\,\frac{Y_0\,A}{X\,D} \qquad (48)$$

Using now the suffixes x and y to distinguish the amplitudes and nodalities of the longitudinal and transverse seiches, the combination of Eqs. 47 and 48 yields the total force of impact:

$$P(\max) = \pi\,Y_0\,W\left(\frac{2\,\pi\,m^2{}_y\,A_y}{B^2}\right) + \frac{m_x\,A_x}{X\,D} \qquad (49)$$

Eq. 49 reveals at once that it is not the fundamental seiches which have to be feared during range action, but rather their higher harmonics, with nodalities of 2, 3, and higher, because the force increases as the square of the nodality in the principal term. Owing to the approximation made in formulating Eq. 44, Eq. 49 will not be valid for very large values of m; but, in any case, there are limits to the values of m which are applicable, since amplitudes decay with increasing nodality. In any event, the seiche periodicities that will produce the greatest value of $P(\max)$ are between, say, 1 min and 20 sec, for these usually constitute the most active multinodal seiches within a harbor. This deduction is in keeping with the conclusions of Section 9 that the seiches of higher frequency are generally the most dangerous for ships.

If the ship were lying along the shorter side, B, or the y-axis of the dock, the force of impact would be

$$P(\max) = \pi\,Y_0\,W\left(\frac{2\,\pi\,m^2{}_x\,A_x}{D^2} + \frac{m_y\,A_y}{X\,B}\right) \qquad (50)$$

Since B is less than D, it is always easier for a multinodal transverse seiche to maintain itself with larger amplitude than a multitudinal longitudinal seiche of the same periodicity. This fact leads to the general conclusion that Eq. 49 will always give a higher value than Eq. 50, and that damage to ship plating and harbor installations is more likely to occur at berths along the longer side of the dock.

The importance of ship size in lateral impact is a further point of interest, for the force $P(\max)$ increases virtually as the square of the displacement tonnage. This increase occurs because the dimension Y_0 (approximately half the beam of the ship) is itself a function of W. The berthing of very large ships at points in a harbor where the on-movement from transferse seiches is severe should therefore never be allowed unless the berths are equipped with special shock-absorbing fenders.

11. Experiments on Ship Models

An experimental study of the movements of model ships with the object of confirming the theoretical principles evolved in this paper was undertaken at the Range Laboratory at Capetown. Simple apparatus for demonstrating seiches on a model scale, similar to that used by E. Maclagen-Wedderburn,[7] was adapted to good purpose for these experiments. This apparatus took the form of a shallow, glass-sided tank 6 ft long, one end of which was partitioned off as a compartment for a displacer, moving vertically in simple harmonic motion. The partition had a narrow opening at the bottom permitting the flux of water necessary to stimulate a seiche in the remaining part of the trough. A second partition (without an opening) could be moved to any position so as to make the effective length of the tank correspond with the period of the displacer, for the production of a resonant seiche.

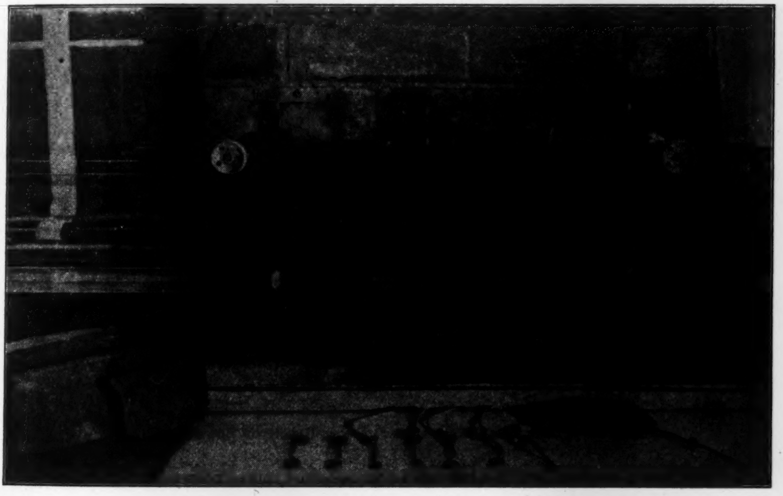

FIG. 8.—MOORING CARRIAGE FOR TANK EXPERIMENTS ON MODEL SHIPS

A mooring carriage (Fig. 8), constructed of toy building parts, was placed across the tops of the glass sides of the tank. The model ships, floating in the water of the trough, were held one at a time with rubber strops to the suspension points below the body of the carriage. These suspension points could be raised or lowered at will, or brought together or farther apart, by the operation of handwheels. By this control the tension in the model ship's ropes could be adjusted to any selected value. The elastic properties of the rubber strops were determined by simple load-extension tests.

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[7] "On the Hydrodynamical Theory of Seiches," by G. Chrystal, *Transactions*, Royal Soc. of Edinburgh, Vol. 41, 1904–1905, Pt. III, pp. 599–649.

The ships, visible in Fig. 8, were modeled in wood with lead inserts to give them the correct buoyancy. The scales used were those of the main harbor model—1/1,200 horizontally and 1/144 vertically. The obvious distortion of the models was of no consequence to the investigation since no attempt was made to translate the model findings to the prototype scale, but, rather, to corroborate theory by a direct application of principles to the model conditions.

Before experiments were made, the critical resonant periodicities at which the various sizes of model ships would oscillate longitudinally, when placed centrally in the nodes of different uninodal seiches, were computed according to the theory evolved in this paper.

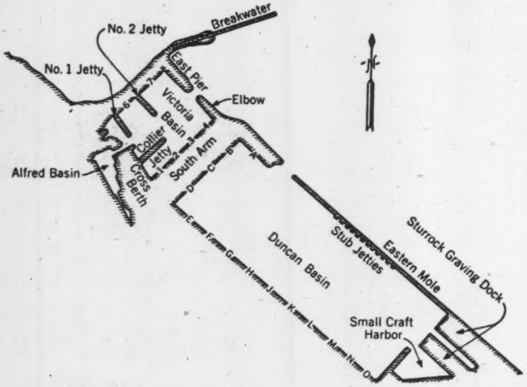


FIG. 9.—TABLE BAY HARBOR, CAPETOWN, UNION OF SOUTH AFRICA

In attempting to check the results in the experimental tank, the procedure followed was to take different lengths of the oscillating basin and adjust the speed and amplitude of the displacer so that a uninodal seiche of amplitude 0.7 cm in a depth of water of 9 cm was set up. The model ship, tied tightly by an appropriate pair of ropes to the mooring carriage, was then placed at the node of the seiche and the suspension points moved inward toward each other so as to slack off the ropes to the degree necessary to cause the ship to oscillate most violently in the periodic current of the seiche. The horizontal distance between suspension points could then be measured accurately across the top of the carriage and the value of L_0, producing critical motion, determined. The measured values of L_0 were then checked against those computed.

The voluminous nature of the calculations for the model experiments prevents their incorporation in this paper, but the results of the tests were more than gratifying and proved that, for the idealized conditions which it treats, the theory is absolutely reliable. One of the most pleasing and impressive demonstrations of the model system was the manner in which it confirmed the importance of tight ropes as a palliative to range action in the mooring of ships.

12. The Statistical Analysis of Rope Breakages at Capetown

The experimental verification of the theoretical approach to the problem of ship ranging permitted the use of the theory in a statistical analysis of the port captain's records of rope breakages at Table Bay Harbor from 1940 to 1946. The length of the paper again precludes the presentation of any details of this application beyond certain findings which have an important bearing on the present theme.

It was found, for instance, that certain berths in Table Bay Harbor are very susceptible to effects from range action, whereas other berths are practically immune. The percentage chances that ships moored at the various berths in the Victoria and Duncan basins (Fig. 9) will part ropes during range action were found on a comparative basis to be as shown in Table 1.

TABLE 1.—Susceptibilities of Berths to Rope Breakage, Table Bay Harbor, Capetown, Union of South Africa

Victoria Basin		Duncan Basin	
Berth	Chance (%)	Berth	Chance (%)
East Pier	11.3	A	21.6
No. 7 Quay	11.1	B	32.7
Outside No. 2 Jetty	20.7	C	18.9
Inside No. 2 Jetty	5.8	D	23.5
North side Collier Jetty	2.0	E	5.3
South side Collier Jetty	9.6	F	6.4
No. 1 South Arm	8.8	G	8.3
No. 2 South Arm	7.2	H	11.0
Nos. 3 and 4 South Arm	17.9	J	9.1
Elbow	20.4	K	6.5
....	....	L	26.3
....	....	M	23.3

In the Duncan Basin, B berth is seen to afford a 1 in 3 chance of rope trouble during range action, and from that point of view quite eclipses any other berth. On the other hand, this berth lies on the node of the second harmonic ($\tau = 0.8 - 1.0$ min) of the transverse oscillation ($\tau = 1.7 - 1.9$ min) for the basin, of which the latter has its node at C berth. It must be inferred, therefore, that the higher frequency seiche must be relatively worse for ships of the size normally accommodated there than for one with a periodicity of about 1.8 min. This inference receives further support from the fact that D berth, coinciding with the other node of the second harmonic, carries also a higher percentage chance of rope breakage than C berth, at the node of the fundamental.

Now the range action model of Table Bay Harbor, besides reproducing the range phenomenon with remarkable accuracy, demonstrated quite conclusively that, although the Victoria Basin could respond to a number of modes of oscillation in the range of periods greater than 4 min, all of these took the form of a general rise and fall of water over the entire area of the dock. In the range of periods of less than 4 min the two most important critical periodicities to which the basin responded covered the bands 2.2 min to 2.5 min and 1.5 min to 1.9 min, of which the latter was much the stronger. Seiches of a periodicity greater than 4 min are obviously incapable of causing rope breakages in the Victoria

Basin and yet, on the whole, the Victoria Basin has fully 70% of the susceptibility of the Duncan Basin to range damage in the form of broken ropes. The only conclusion is that periodicities less than 4 min (and probably less than 2.5 min) are alone responsible for the troubles inherent in range action.

Table 1 records another point of interest in that the inside of No. 2 Jetty and the north side of the Collier Jetty have the lowest susceptibilities of all the berths recorded in the Victoria Basin. Why this should be so may be gleaned from Fig. 9, which shows that No. 2 Jetty and the Collier Jetty form the protecting arms for an inner basin within which the berths in question lie. Since such a system is effective only in damping the penetration of the higher frequency disturbances, the latter are apparently the ones to which ships are most responsive.

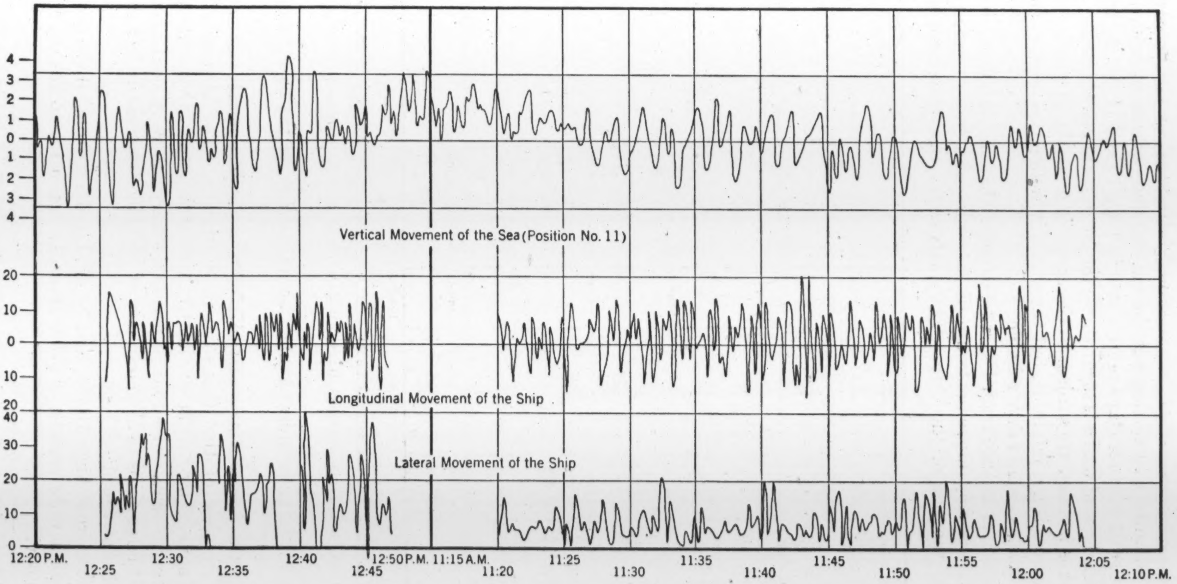
13. Ship Motion by Observation and Measurement

During the course of the researches at Capetown a number of ships of all sizes came under observation. It was the practice, when making measurments of ship motion, to record concurrently the characteristics of the prevailing seiches on "seichometers" or portable tide gages, at points adjoining the berth occupied by the ship. The measurements on the ship comprised simultaneous recordings of the horizontal components of displacement, u and v. Two observers were engaged in this task, one of whom, as the ship reached the extremity of its longitudinal or lateral travel, called out the readings, x or y, on two tapes, one lying on, and parallel to, the edge of the quay, and the other stretched between a point on the ship and the observer's hands. The second observer recorded readings and times to the nearest second. This method, although admittedly subject to error, gave good results in the absence of any complicated self-recording mechanism. No attempt was ever made to record the vertical motions of a ship, since these were considered to be of no particular importance to the investigation.

TABLE 2.—Ships Observed at Capetown, Union of South Africa, June and July, 1945

Ship	Tonnage		Dimensions, in Feet			
	Gross registered	Displacement (when observed)	Length (between perpendiculars)	Beam	Normal loaded draft	Draft (when observed)
Dahlia	5,000	6,400	421	55	25	13
Anna Howard Shaw	8,000	13,400	423	57	27.5	27
H. M. S. Le Tigre	900	2,000	220	35	....	....
Nieuw Amsterdam	36,000	47,700	759	88.3	33.3	33

Typical records for four ships of widely different displacement tonnages (see Table 2), as plotted and correlated with the sea oscillations, are shown in Figs. 10 and 11. The traces in general are complex combinations of harmonic components some of which are easily recognizable, but others of which are

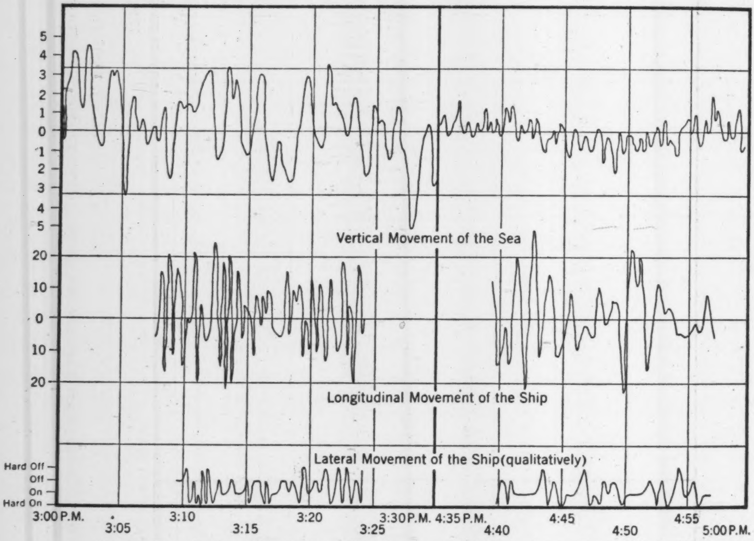


(*a*) ***Dahlia*** in Berth C, July 7 (*b*) ***Anna Howard Shaw*** in Berth D, July 31

FIG. 10.—JULY, 1945, OBSERVATIONS BETWEEN SEA OSCILLATIONS AND SHIP MOVEMENTS

inconspicuous. The method[8] of "residuation," introduced by G. Chrystal was used to good advantage in sorting out the various components of motion, and these, in the case of the four ships represented in Figs. 10 and 11, are depicted in Fig. 12, the time base of which is logarithmic for purposes of condensation. The components thus isolated are subject to the errors of analysis, of course, for the residuation process is not infallible, but the writer is inclined to regard them on a statistical basis as being fairly reliable.

One fact emerges at once from Fig. 12—namely, that the dominant periodic components in the ship movements are of the order of 1 min or less, almost irrespective of the size of the ship. The analyses of the water movements reveal, as expected, the prominent fundamental seiches for the basins and their higher harmonics.



(*a*) *Le Tigre* in Cross Berth, June 20 (*b*) *Nieuw Amsterdam* in Berth A, June 21

FIG. 11.—JUNE, 1945, OBSERVATIONS BETWEEN SEA OSCILLATIONS AND SHIP MOVEMENTS

The case of the *Nieuw Amsterdam* (47,700 tons displacement) is of considerable interest in view of the discussion in Section 9. The strongest seiche affecting this ship seemed to be one of 40 sec; yet it chose to resonate with a binodal transverse seiche of about 1-min periodicity, thereby confirming the deduction that large ships tend to resonate at periodicities greater than those of the average size of vessel.

The evidence from Figs. 10, 11, and 12, and, indeed, from all the measurements made on a wide variety of other ships, is conducive to the conclusions drawn previously.

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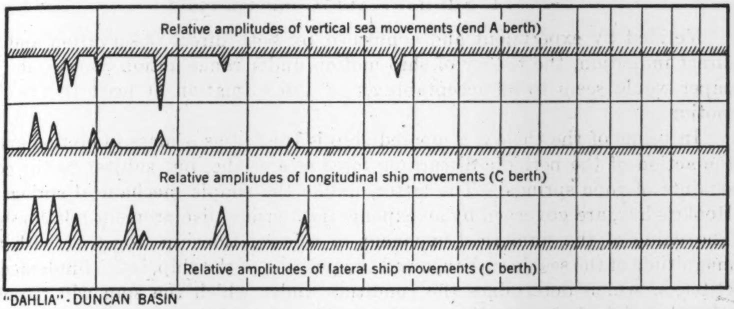
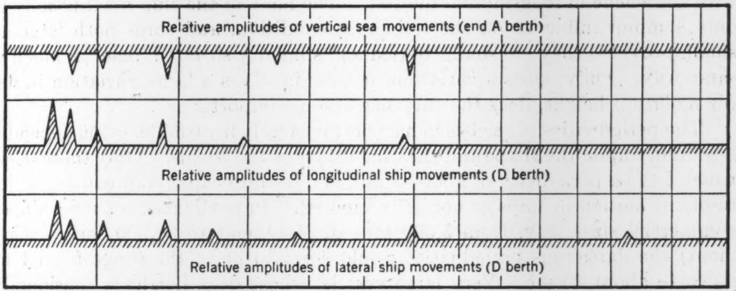
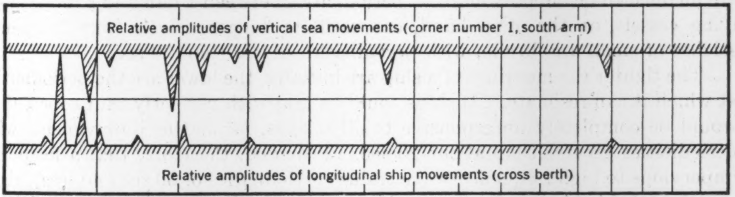
[8] "Investigation of Seiches of Loch Earn by the Scottish Loch Survey," by G. Chrystal, *Transactions*, Royal Soc. of Edinburgh, Vol. 45, 1906, Pt. II, pp. 382–387.



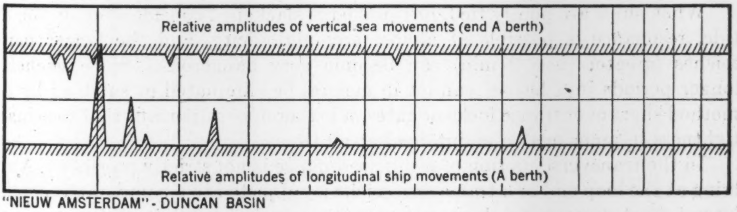
(a) "DAHLIA" - DUNCAN BASIN



(b) "ANNA HOWARD SHAW" - DUNCAN BASIN



(c) H.M.S. "LE TIGRE" - VICTORIA BASIN



(d) "NIEUW AMSTERDAM" - DUNCAN BASIN

FIG. 12.—CORRELATION BETWEEN SHIP MOVEMENTS AND SEA OSCILLATIONS (PERIODICITIES AND AMPLITUDES OF APPARENT COMPONENTS)

14. Summary and Conclusions

Verified by experiment and confirmed by both direct observation and indirect induction, the theory of ship motion under range action evolved in this paper would seem to be acceptable as an approximation at least to the true motion.

In terms of the theory, a moored ship is treated as a mass in motion under the action of the periodic disturbing force of a seiche, but subject to the constraints of rope springs. The latter, unlike the simple mechanical springs of Hooke's law, are governed by an exponential tension-displacement relationship. The value of the numerical exponent in this relationship, together with the magnitude of the sea disturbance and the position of the ship, is the fundamental criterion which determines the condition under which the ship will resonate with the seiche in longitudinal motion. The mass of the ship itself usually has only a minor influence on the resonance condition, and ships both large and small, provided they are similarly tied and similarly situated, react in much the same way. Only when a variation in mass involves a large variation in deck (or mooring) height, does the ship size assume importance.

The periodicities of seiches in harbors are largely invariable, being dependent essentially upon the shape and the dimensions of the basins. Only those seiches which have a periodicity of about 2 min and less are capable of causing serious resonant motion in ships as normally moored. In particular, for most ships of commercial sizes (say, from 5,000 tons displacement to 30,000 tons displacement) the dangerous periodicities would seem to be in the range from 1 min down to about 20 sec. Very large vessels, whose deck height is considerable, are inclined to be affected more by periodicities between 1 min and 2 min. Low lying vessels, on the other hand, can oscillate dangerously in high frequency seiches of small amplitude, with periodicities of less than 20 sec.

The tighter the moorings of a ship are initially, the lower are the periodicities at which it can resonate. If ships could be tied with perfectly taut ropes, they would be completely unresponsive to all seiches, no matter how violent, with periodicities greater than about 20 sec. Provided, therefore, that a harbor is impervious to high frequency wave trains (periodicities of 30 sec and less), tight mooring of ships is an effective antidote to longitudinal range action.

When ships are poorly tied, or ropes have slackened considerably on the ebb tide, resonance is possible at almost any periodicity, and the longer period seiches (greater than 2 min) can become very dangerous. Since seiches of longer periods in a harbor cannot in general be eliminated or subdued by any method short of entrance locks or gates, it is therefore important that reasonable tightness in moorings be maintained at all times.

In the transverse motion of a ship, resonance is not strictly possible. A ship lying at the loop end of a transverse seiche is subjected to a comparatively small lateral displacement which cannot usually exceed about 1 ft, either way from rest position, under the worst conditions, and cannot call into effective play the spring action of the ropes. Nevertheless, this amount of movement is sufficient to occasion very large impact forces between the ship and the quay. The most dangerous periodicities in this action are again in the range from about 1 min down to 20 sec. Of greatest importance is ship size, since the force of impact increases virtually as the square of its mass. Berths located along the longer side of a dock will probably always be more susceptible to the troubles inherent in transverse ranging than berths along the shorter side.

Transverse impact is undoubtedly the most serious feature of range action. Initially tight ropes will not be of much assistance in this case, since the rope tension that can be called into play to prevent lateral movement is extremely limited. If the evil is to be endured, the possible protective measures are the use of shock-absorbing fenders or of bow and stern anchors to hold ships off the quays.

Although it is possible to adopt palliative measures to enable ships to outride range action, the troubles and discomfort inherent in ship surging remain. It is eminently more satisfactory to inhibit the development of the sea disturbances themselves, and this, as this paper seeks to prove, only requires that a harbor should be made reasonably immune to penetration from short-period disturbances of 2 min and less. Oscillations of these frequencies are amenable to treatment by throttling through a chain system of basins, and herein lies the most effective solution of the range problem.

15. Acknowledgment

Grateful acknowledgment is due to W. C. Q. Joosting who assisted the writer at Capetown in securing and plotting measurements on ship motion and in subsequently searching the records for harmonic components. The writer desires also to express his appreciation to the Administration of the South African Railways and Harbours for permission to publish this paper.

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Paper No. 2461

CONSTRUCTION TECHNIQUE OF PASSING FLOODS OVER EARTH DAMS

By Andrew Weiss,[1] Hon. M. ASCE

With Discussion by Messrs. Cleves H. Howell, Gerard H. Matthes, and Andrew Weiss.

Synopsis

The increasing necessity of using earth dams because of the diminishing number of sites suitable for the construction of masonry or concrete dams has created new problems in dam construction, more especially in locations where climatic, geologic, and topographic conditions tend to produce floods which, during construction, are difficult to control by conventional devices, such as tunnels or covered by-passes. The problem is further complicated where scarcities of equipment, repair parts, and trained personnel accentuate the problem. These conditions occur in Mexico, particularly the extraordinary floods, the relative scarcity of equipment, the limited financial resources, and the relative unpreparedness of the construction agency charged with the undertaking. This paper concerns problems of this character, and the solutions adopted.

Irrigation in Mexico

Irrigation is a national necessity in Mexico, a condition quite distinct from irrigation in the United States where the 100th meridian is generally considered to be the limiting line for irrigation practice, and where a 30-in. annual rainfall is ample for successful farming. In Mexico an annual rainfall as great as 6 ft, erratically distributed in a subtropical climate, makes irrigation wholly desirable and even necessary for completely successful agriculture. This has been the experience in such states as Veracruz, Tamaulipas, and Chiapas, as well as in other localities.

Another condition that is quite prevalent in Mexico is the general absence of good reservoir sites in the mountainous regions. Acceptable sites are generally in the lower foothills; but in Mexico such ideal dam sites are most often difficult to find. Generally, the topography of Mexico is characterized

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[1] Cons. Engr., National Comm. of Irrig., Mexico, D. F., Mexico.

by an abrupt transition from the mountain region to the valley lands below, the latter being exposed to violent floods and innundations. Such conditions require multiple-purpose reservoirs which serve irrigation, flood control, and (as soon as the growth of the region justifies the investment) power development. Storage for irrigation is a national need because even the largest rivers have but little water in the rainless period. Ice and snow deposits occur only in a few isolated spots, such as the peaks of Ixtaccihuatl, in the State of Mexico, and Orizaba and Popocatepetl, in the State of Veracruz, which are very limited in extent. In other words, the mountains of Mexico are too far south to serve as collectors of any considerable part of the moisture that might be stored in the form of snow or ice deposits. All the foregoing conditions point to the necessity of artificial storage to provide for complete and successful use of the available water supply.

TABLE 1.—COMPARISON OF DESIGN CHARACTERISTICS OF FOUR EARTH-FILL AND ROCK-FILL DAMS IN MEXICO

Dam	ELEVATIONS, IN FEET ABOVE SEA LEVEL			Catchment area, in square miles	Average annual runoff, in acre-feet	Spillway capacity, in cubic feet per second
	River	Crown	Spillway crest			
(1)	(2)	(3)	(4)	(5)	(6)	(7)
San Ildefonso	7,150	7,353	7,336.6	149.4	76,960	21,180
Azúcar	148.0	287	250.0	11,418.0	987,000	800,000
Palmito	5,068.0	5,346.4	5,314.0	7,000.0	1,068,000	211,800
Valsequillo	6,500.0	6,770.0	6,753.5	1,560.0	316,000	39,000

It was in the creation of storage reservoirs on this considerable scale that the dam construction technique described in this paper was developed. Examples selected for illustration (see Table 1) are: San Ildefonso Dam in the State of Querétaro; Azúcar Dam at Camargo, Tamaulipas; Palmito Dam 100 miles upstream from Torreón, in the State of Durango; and Valsequillo Dam near Puebla, in the State of Puebla.

SAN ILDEFONSO DAM

A rock-fill dam on the Prieto River about 15 miles upstream from the town of San Juan del Río in the State of Querétaro is called "San Ildefonso." It is situated in a narrow canyon that crosses a series of basalt flows intercalated with layers of basaltic agglomerates. This is one of the conventional types of rock-fill dams, provided with a concrete "blanket" on the water side. The maximum height above the river bed is 203 ft, the width of the crown is 16.4 ft, and the side slopes are 1 on 1.4. The upstream concrete face is reinforced with $\frac{3}{4}$-in.-round deformed bars, spaced 1 ft, center to center, in both directions and its thickness varies from 20.5 in. at the base to 8 in. at the crown, a variation of 1.08 in. for every 33 ft upward along its slope (approximately). This concrete blanket rests on a hand-placed rock fill which, in turn, rests on the dumped rock; and the slope of the latter is limited to 1 on 1.27 upstream and 1 on 1.4 downstream. The limiting range of slopes for the hand-placed rock fill is between 1 on 1.27 and 1 on 1.4, with a maximum thickness of 25.6 ft at the foundation and 3 ft at the crown. The spillway is a fixed overflow crest, 16.4 ft below the crown of the dam, designed to discharge 21,180 cu ft per sec when the water depth on the crest is 10 ft. A horseshoe diversion tunnel 12.5 ft in diameter is used for control purposes.

The contractor, provided with only moderate means, soon found himself with insufficient time to complete the dam to a height that would pass the anticipated flood flows. In the circumstances, it was agreed that the contractor should be permitted to continue the rock fill provided that he protected the downstream slope with an exposed mesh of ¾-in. round reinforcing bars. As shown in Fig. 1, the center-to-center spacing was, for horizontal bars, 1 ft on centers and for sloping bars, 4 ft on centers, wired together at the intersections. This steel mesh was thus anchored to the rock fill by 13-ft steel bars, with hooks at both ends, spaced 3 ft vertically and 4 ft horizontally.

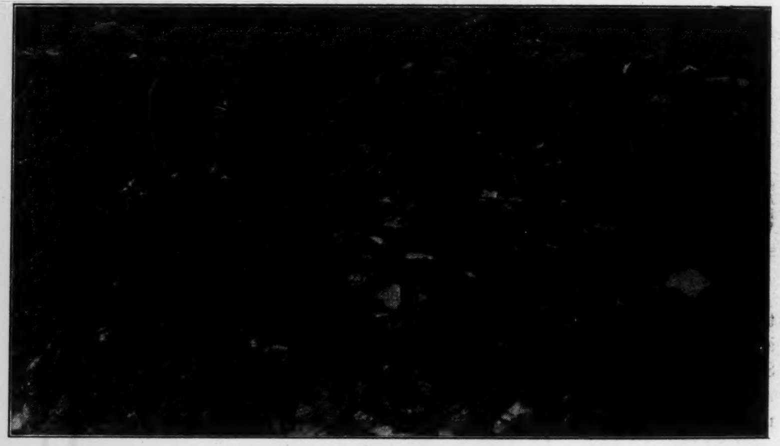


FIG. 1.—EMBANKMENT PROTECTION DURING THE CONSTRUCTION OF SAN ILDEFONSO DAM IN QUERÉTARO

Results.—The rock fill had reached a height of 36 ft on August 2, 1939, when the first flood began to pass over it. From the high water marks on the canyon walls the discharge was estimated at 4,070 cu ft per sec. No additional height was given to this rock fill following this flood because the contractor decided to employ his forces on other work during August and September.

On September 11 a second flood, of greater magnitude, passed over the rock fill, leaving marks on the canyon walls that indicated an overflow depth of 6.56 ft near the downstream crest. Applying the usual formula for broad-crested weirs, this would indicate a total discharge of 8,000 cu ft per sec, of which (possibly) 3,000 cu ft per sec passed through the diversion tunnel. Figs. 1 and 2 showing conditions as of September 22, 1939, demonstrate that neither of these floods left the smallest indication of erosion. The only defect noticeable was a slight bulging along the foot of the downstream slope, where the mesh was slightly displaced because of insufficient anchorage; but no loss was noticeable in the rock fill.

After this second flood the contractor estimated that the flood period was safely past and removed the mesh. This was premature, as demonstrated on October 3, 1939, when a third flood tore away 7,000 cu m (9,000 cu yd) of rock fill; but the reduced height tended to regulate by storage rather than increase the depth of water over the temporary crest. When construction began after this third flood period the protection was considered no longer necessary. As the height of the dam increased, it was estimated that any known or recorded floods, that might be repeated in that season, could thereafter be controlled by the tunnel and the regulating capacity of the reservoir. All the bars were later recovered and used as originally intended, except the anchor bars which remained embedded in the rock fill.

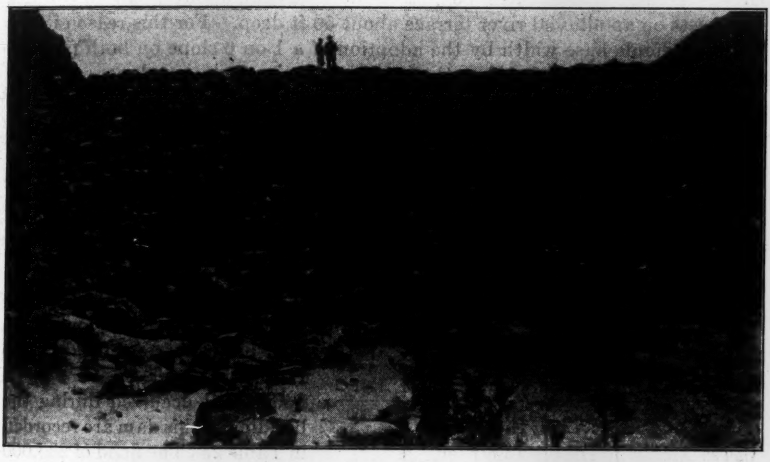


Fig. 2.—Downstream Slope of San Ildefonso Dam in Querétaro, September 22, 1939

Summary.—Weaknesses in the technique, revealed by this experience, suggested the possibility that the anchorage at the bottom of the slope was relatively inadequate, that conditions for cleaning the river bed of loose rock could have been improved, and that the spacing of vertical bars could well have been reduced from 4 ft to 3 ft, center to center. At the bottom of the slope, where insufficient stripping made anchorage difficult, a concrete cover or footing would have protected this part of the mesh from the effects of severe dynamic impact that always threaten to undercut and destroy the protecting device.

Marte Gómez (El Azúcar) Dam

Azúcar Dam is an earth structure about 8 miles upstream from the mouth of the San Juan River, where it joins the Río Grande. The Town of Rio Grande is on the American side and the Town of Camargo, on the Mexican side. The population in the valley below this dam is so dense that no flood risks were even remotely permissible greater than those ordinarily imposed by nature. The embankment slopes were as follows (vertical on horizontal):

Description	Slope
Impervious center, upstream side	1 on 3
Impervious center, downstream side	1 on 0.5
Semi-impervious fill, downstream side	1 on 2
Pervious gravel fill, both sides (to El. 240)	1 on 6

Materials Used in Embankment.—The impervious core is composed of a sandy loam containing from 20% to 30% clay taken from the river terrace along the left side (facing downstream). The central zone is supported on both sides by a gravelly earth fill, taken from the locally known Reynosa formation. On the left side of the stream bed, facing downstream, about a half mile of this dam rests on an alluvial river terrace about 50 ft deep. For this reason it was given an ample base width by the adoption of a 1 on 6 slope on both faces up to an elevation of 42.5 ft below the crown. On the upstream slope, wave protection is provided by a layer of loose rock, 3.28 ft thick. The Reynosa gravels provide the necessary filter between the rock fill and the center core.

This dam presented a special flood control problem. The records of the International Boundary Commission (IBC) revealed one flood of 350,000 cu ft per sec within quite recent times. Little thought was given to flood diversion by tunnels or covered conduits because of the size of the floods to be expected and the fact that there is no well-defined dry season. Almost the only two months free from floods were February and March. The floods that occurred during the building of this dam are recorded in Table 2. The flood of 233,000 cu ft per sec that occurred on August 30, 1938, filled the channel to an elevation of about 65 ft above the river bed and inundated the borrow area on the left side of the stream. When the superintendent undertook to clean the river channel before beginning to excavate, he found it necessary to remove from 15 ft to 20 ft of river debris in order to reach the impervious mudstone suitable for the foundation of the impervious section. A large volume of talus material was also discovered along the right side. These deposits had served as a necessary support to hold the mudstone composing the right river bank from sliding;

TABLE 2.—RECORD OF FLOOD DISCHARGES (CUBIC FEET PER SECOND) DURING THE CONSTRUCTION OF AZÚCAR DAM IN TAMAULIPAS

Month and day	1938	1939	1940	1941
January 27	....	....	....	16,000
March 25	....	....	11,700	....
April 27	....	....	....	16,000
April 30	....	4,520	....	....
May 14	....	23,800	....	....
May 24	6,350	....	....	....
May 25	....	....	38,500	....
June 25	....	....	....	50,000
July 13	....	....	....	12,300
July 25	5,500	....	....	....
August 20	233,000	....	16,200	....
Sept. 1	29,500	....	....	....
Sept. 9	....	....	39,900	....
Sept. 10	....	....	....	72,700
Sept. 30	....	9,850	....	....
Oct. 11	....	65,000	....	....
Oct. 30	....	....	22,600	....
Dec. 22	....	....	21,400	....

and when they were removed the mudstone formed cracks on the right side, parallel to the river bed, tending to permit the formation of open fissures and slides.

To maintain a stable equilibrium, the superintendent began to refill the river bottom with rolled material in the shortest possible time. The center fill was built to a height of 16 ft above the original river bed at the downstream end, or toe, of the dam. Here it was protected against erosion by a rock fill, covered with a network of broken dragline cable and reinforcing rods. The internal water pressure was relieved by drainage through the coarse rock fill. In the hope that it would afford added safety, this rock surface was lined with a layer of lean-mix concrete; but, although it was fairly strong in compression, it proved of little use as protection against erosion.

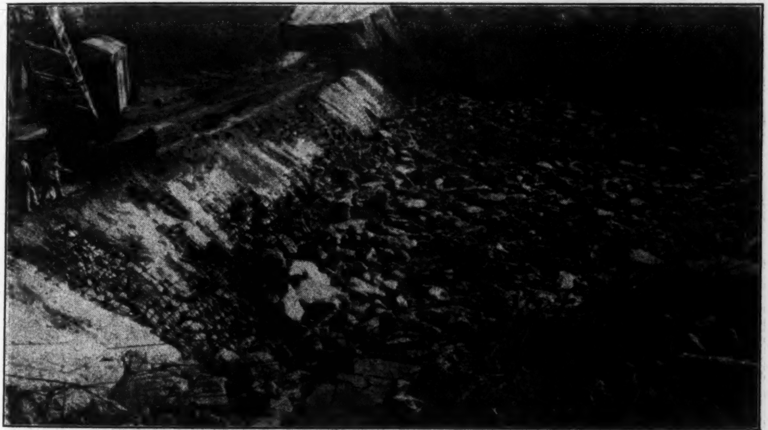


FIG. 3.—LOOSE ROCK FILL ALONG THE TOE OF AZÚCAR DAM IN TAMAULIPAS, JUNE 21, 1940

The building of the central part (that is, the river section) of the dam was suspended during 1940 and was not continued until August, 1941. The center section had reached a height that provided full safety against any further sloughing of the mudstone on the right bank. A construction view as of July, 1940, is shown in Fig. 3. Subsequently, the 1940 autumn floods, and those of the first seven months of 1941 (see Table 2), passed over the dam without the slightest erosion of the main body of the section. The fill in the river had been made on a small upgrade for the purpose of reducing velocities.

As shown in Fig. 4, however, there was some damage to the downstream toe of the rock dike and to the temporary concrete lining that had been poured the previous April 20; but the surviving steel mesh and concrete blanket were still sufficient to afford considerable protection against critical erosion. The lean-mix concrete blanket was a mix of 1.8 sacks (169 lb) of cement per cu yd of concrete in place. Because of its low resistance to erosion, a 4-in. layer of normal concrete was later poured over the weaker layer.

A three-day flood of variable magnitude (an undulating type) began on September 8, 1941, and reached its peak about September 10, when a discharge of 72,700 cu ft per sec was observed about 6 miles upstream by the IBC. Observing the elevation of flow near the crest, and estimating that the control tunnel could have diverted 10,200 cu ft per sec, about 62,500 cu ft per sec must have passed over the dam in this period. The result was that the protective lining (possibly because it was not high enough) was badly eroded. Nevertheless, it is worthy of note that, although the flanking failure of the blanket was in progress for three days, a relatively small volume of the rolled fill was washed downstream. The most serious erosion occurred on a strip of the river bank on the downstream left end of the rolled embankment where the natural uncompacted river terrace afforded the minimum resistance against erosion.

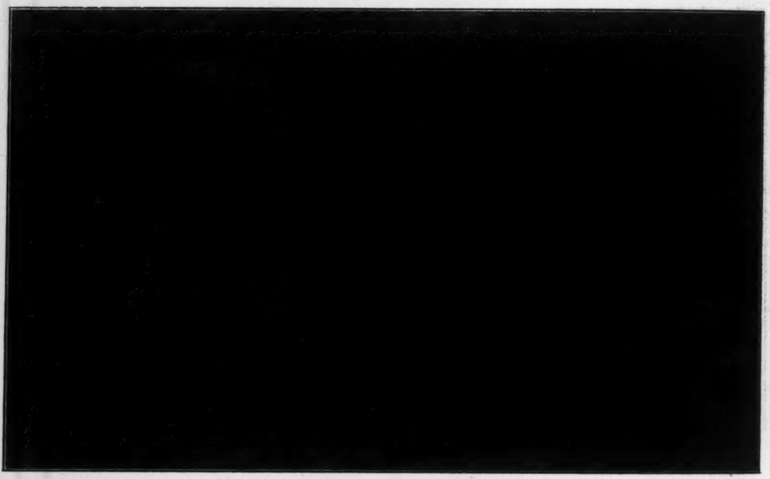


Fig. 4.—Condition of Azúcar Dam in Tamaulipas, October 12, 1940, After the Floods of 1939 and 1940

Summary.—The original plans specified a protective spillway covering not to exceed the height of the by-pass channel leading to the spillway. The construction of a concrete overflow dam 50 ft high, closing the entrance to the spillway crest, was postponed until the embankment and the protective spillway had reached a sufficient height to pass even the record floods to be expected by means of the diversion tunnel and the spillway structure in the process of building without the smallest risk to the densely populated valley downstream from the dams.

At least six lessons were learned or corroborated during the construction of Azúcar Dam—

(1) The contractor must be acutely alert to guard against a flanking washout in the region of erodible material.

(2) When a concrete blanket is used for a protective lining the mix should be of normal quality or be covered with a layer of concrete to insure the required resistance to erosion.

(3) The crest should be raised in relatively small steps unless ample rock protection is available for larger steps, as in the case of Palmito Dam and Valsequillo Dam (described subsequently).

(4) Where the scarcity of rock dictates the use of a concrete cover an ample number of weep holes must be provided for the escape of infiltrating water to prevent uplift of the concrete blanket. This precaution is especially important where the fill is composed of fine or easily eroded material. No indication of uplift was detected at Azúcar Dam.

(5) As construction proceeds, the rolled surface should be sloped so that the downstream end of the embankment is maintained at a level not less than 3 ft higher than the upstream end so as to reduce the velocity of approach to the downstream crest. At Azúcar Dam the downstream level of each layer was always about 3 ft higher than the upstream end.

(6) Where rock or other resistant material is not available at the toe, a concrete apron must be provided to take the impact of the drop and to provide anchorage for the concrete blanket. This apron serves to prevent undercutting, the extent of which will depend on the cohesiveness of the material in the river channel.

Lazaro Cardenas (Palmito) Dam

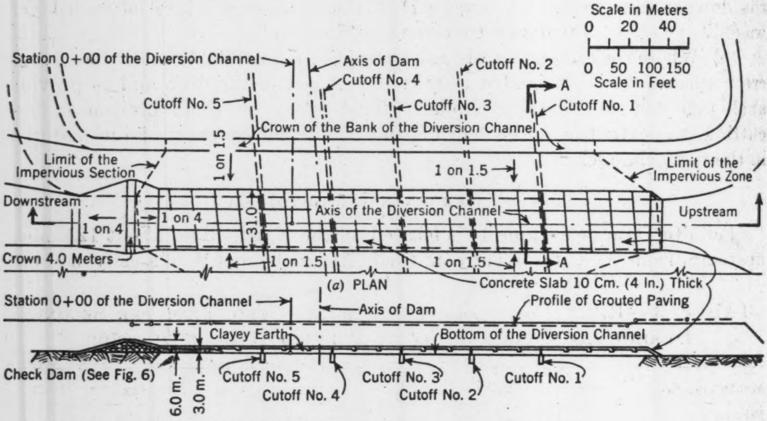
Palmito Dam (see Table 1) is located on the Nazas River about 125 miles upstream from the City of Torreón, Coahuila, at the mouth of a canyon about 1.5 miles downstream from the confluence of its principal tributaries, the Oro and Ramos rivers in the State of Durango. The rocks exposed on both sides of this canyon, and in the river, consist of series of flows of rhyolites intercalated with rhyolitic tuffs of such highly variable structures and textures that

TABLE 3.—Record of Flood Discharges (Cubic Feet per Second) During the Construction of Palmito Dam in Durango

Month and day	1937	1938	1939	1940	1941	1942	1943
Feb. 16	....	....	....	....	....	10,650	....
June 23	....	....	....	....	....	....	7,600
June 30	....	9,750	....	8,790	....	....	....
July 2	....	....	....	....	....	....	10,900
July 27	....	52,500	....	....	9,980	....	....
August	....	....	....	....	16,200	....	....
Aug. 20	....	....	....	....	....	21,600	....
Aug. 21	....	....	21,800	....	....	....	....
Aug. 25	....	....	....	8,700	....	....	....
September	....	....	....	....	....	25,400	....
Sept. 12	....	....	....	....	51,600	....	....
Sept. 14	....	....	5,690	15,200	....	....	....
Sept. 19	49,300	57,000	....	....	....	....	....
Sept. 26	....	....	....	....	....	....	32,500
Oct. 9	....	....	....	....	....	6,150	....
Oct. 11	....	....	....	....	....	....	24,400
Oct. 12	....	....	....	....	6,140	....	....
Dec. 1	....	....	....	5,800	....	....	....
Dec. 12	....	....	....	....	19,200	....	....
Dec. 26	....	....	32,400	....	....	....	....

it was not possible to use the site for building a masonry dam of the gravity type. It was decided to use an earth-fill and rock-fill dam composed of a central impervious core of well compacted, gravelly earth and provided with pervious zones on both sides. On the upstream side the structure was made partly of a dumped rock fill composed of quarry-run rock and gravel; and, in the upper part, it comprised metamorphosed shales derived from the spillway excavation, compacted by rolling. On the downstream side the pervious zone was mostly composed of hard metamorphosed shale. The water side of the dam was further protected with a dumped rock fill placed in the form of an inverted filter with the coarsest material on top. The downstream side was protected against erosion by a hand-placed rock cover.

Because of the broken nature of the foundation rock it was blanket-grouted and provided with five cutoff walls, these provided with a grout curtain to a depth of 50 ft to secure a watertight foundation.


Scale in Meters
0 20 40
0 50 100 150
Scale in Feet
Station 0+00 of the Diversion Channel
Axis of Dam
Cutoff No. 4
Cutoff No. 2
A
Cutoff No. 5
Cutoff No. 3
Cutoff No. 1
1 on 1.5
Crown of the Bank of the Diversion Channel
Limit of the Impervious Section
1 on 1.5
Limit of the Impervious Zone
Downstream
Axis of the Diversion Channel
Upstream
1 on 4
1 on 4
31.0
Crown 4.0 Meters
1 on 1.5
1 on 1.5
A
(a) PLAN
Concrete Slab 10 Cm. (4 In.) Thick
Station 0+00 of the Diversion Channel
Axis of Dam
Profile of Grouted Paving
Clayey Earth
Bottom of the Diversion Channel
Check Dam (See Fig. 6)
6.0 m.
3.0 m.
Cutoff No. 5
Cutoff No. 4
Cutoff No. 3
Cutoff No. 2
Cutoff No. 1

(b) SECTION ALONG THE AXIS OF THE DIVERSION CHANNEL

FIG. 5.—PLAN AND PROFILE OF DIVERSION CHANNEL, PALMITO DAM IN DURANGO

The agricultural development of the region being supplied from Palmito Dam (the so-called Laguna) was begun about 1880, and therefore a long rainfall and runoff record was available. During the construction of the dam the maximum discharges were as given in Table 3. The embankment slopes were as follows:

Description	Slope
Impervious central core, upstream side	1 on 2
Impervious central core, downstream side	1 on 1
Rock fill, downstream side	1 on 2.5
Rock fill, upstream side	1 on 3

The slope of the upstream side was broken at El. 5168 by a berm 33 ft wide. The diversion works comprised three tunnels, each 20 ft in diameter, and an open channel with a bottom width of 82 ft (see Fig. 5). The diversion provided by the three tunnels and the open cut proved quite efficient. By far the greater part of the runoff was passed through the open cut, because the use of the tunnels was generally restricted during the years of construction—first by the excavation and the lining operations, and later by the building of the galleries which are used for the installation of the operating machinery.

All this work was much delayed by conditions existing during World War II. In the spring of 1941, when the end of the dam to the right of the diversion channel had been built to 150 ft above the river channel, it was decided to close the open cut. Hampered by the lack of equipment and by wartime conditions, it was decided to make this closing in successive stages, by a temporary spillway of a type similar to those used at Azúcar Dam, and, subsequently, at the Valsequillo Dam.

A plan was adopted to provide an overflow crest (Fig. 6) made of rock fill and covered with a concrete slab 20 cm (8 in.) thick, anchored to the rock at the foot of the downstream slope and to the rock on the left side and the grouted paving on the right. Also, it was decided to continue the concrete slab to cover the 10-ft thickness of fill in the bottom of the diversion channel, which corresponds to the height of the cutoff walls above the foundation of the impervious fill. The earth fill on the right of the diversion channel was protected with a layer of grouted paving as shown in Fig. 5.

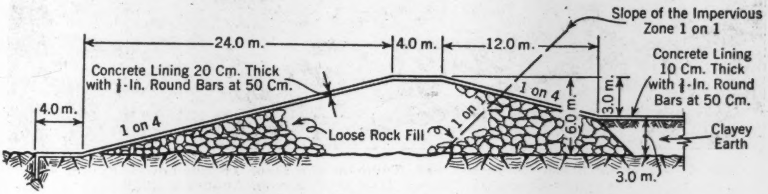

Slope of the Impervious Zone 1 on 1
24.0 m.
4.0 m.
12.0 m.
Concrete Lining 20 Cm. Thick with ⅜-In. Round Bars at 50 Cm.
Concrete Lining 10 Cm. Thick with ⅜-In. Round Bars at 50 Cm.
4.0 m.
1 on 4
1 on 4
1 on 1
6.0 m.
3.0 m.
Loose Rock Fill
Clayey Earth
3.0 m.

FIG. 6.—DETAIL OF OVERFLOW DAM (1 METER = 3.28 FT = 39.37 IN.)

On account of the broken nature of the rock in the bottom of the diversion channel at the lower end of the overflow crest it was decided to put the check dam at the lower end, about 40 ft upstream from the end of the impervious section. A discharge of about 10,500 cu ft per sec flowed through this channel on August 8, 1941, and continued at more or less the same rate for about three or four weeks (see Fig. 7). At this stage only one tunnel was available to supplement the diversion provided by the open channel. In this first stage in the construction of the spillway a maximum flood, on September 11, 1941, was 31,000 cu ft per sec. At the same time, 6,400 cu ft per sec passed through the tunnel, indicating a total of 37,400 cu ft per sec. The functioning of the temporary spillway was completely satisfactory.

The maximum flood for the entire year occurred on September 12. Of the total flow (51,600 cu ft per sec) about 42,000 cu ft per sec passed through the open cut and the remainder, through tunnel No. 3. After this flood, the spillway crest at the lower end of the diversion channel was raised 20 ft. The following flood did not reach the height of this crest, but passed through tunnel No. 3 with the help of the storage regulation provided by the reservoir; however, the flood of December 13, 1941, came close to reaching the height of the new crest.

Summary.—Although the results obtained with this open cut were entirely satisfactory, perhaps too great a risk was taken in making the second lift 20 ft high. A succeeding flood almost overtopped the overflow crest before it was ready. It is essential that the overflow crest be well protected against the danger of concentrated erosion, except where the deck is provided with a masonry or a concrete cover.

FIG. 7.—FLOW OF 10,600 CU FT PER SEC, THROUGH THE DROP AT THE LOWER END OF THE DIVERSION CHANNEL, PALMITO DAM IN DURANGO, AUGUST 8, 1941

After the flood of December 12–13, 1941, and those of January and February following, there was no further need for the temporary open waterway. It was then necessary to remove the protective lining of the spillway channel that passed through the impervious section of the dam and, finally, to complete the dam section as provided in the specifications. In this region the dry period extends from about the end of February until the following June. During this construction period it was estimated that any recorded flood of the preceeding 60 years could have been controlled by the three diversion tunnels, in combination with the impounding capacity of the reservoir.

AVILA CAMACHO (VALSEQUILLO) DAM

Valsequillo Dam is located on the Atoyac River, some 16 miles east of the City of Puebla, which is the capital city of the State of Puebla. It closes a narrow gorge about 50 ft wide with vertical and overhanging rock cliffs. The narrowness of the gorge suggested an ideal site for an arch dam, but this type was rejected after a thorough and careful exploration of the site, and a mixed type of rock-fill and earth-fill dam was selected.

Geologic conditions made this project a difficult one. The river bed had cut a narrow gorge about 100 ft deep into a limestone conglomerate, over which had been deposited beds of tuff and tepetate mixed with sandstone layers. The principal difficulty was the occurrence of many cracks and caverns, generally refilled with clays mixed with crushed rock. In many places these cracks had remained completely open, perhaps by reason of the erosion of such refill during the Quaternary period. The doubt which arose concerning the use of this site caused the National Commission of Irrigation to solicit the help of two consulting geologists (Charles P. Berkey, Hon. M. ASCE, and Kirk Bryan) and a consulting engineer (John L. Savage, Hon. M. ASCE) to make two visits to this site at different stages of the explorations, and the results of their advice proved most valuable.

In addition to the data in Table 1, the slopes of this dam were as follows:

Description	Slope
Mixed earth core, both faces	1 on 1.25
Finished embankment, upstream face	1 on 2.25
Finished embankment, downstream face	1 on 2

The dam as designed had a spillway side channel with a fixed crest to serve a maximum flood height estimated at El. 6763.4.

The impervious section of the dam was built of an altered conglomerate mixed with tuff. The latter was generally found in the borrow area close to the dam in a superficial deposit 10 ft or 20 ft thick underlain by the altered conglomerate. This material made an embankment more dense and impervious than the earth materials of fine texture, such as sandy loams and clay loams, which are usually found in the alluvial river terraces. Nevertheless, the building of the earth fill within the gorge was difficult because of the existence of the aforementioned innumerable fractures and cracks which it was necessary to clean thoroughly, refill, and grout so as to secure an effective closure. In the places where there were overhanging walls cutoff walls were built with the object of guarding against an imperfect closure, such as might occur as the result of a settlement of the compacted fill in succeeding years.

The treatment of the cracks in the canyon walls was so time consuming that the summer floods of 1943 became a menace. It was therefore resolved to apply the protective measures that had been used successfully at San Ildefonso and Azúcar, modified to suit the materials and local conditions. In this case it was expected that floods might overtop the earth fill to a height of 100 ft or more above the river channel. For this reason and because it was feared that trees and stumps might tear away a mesh of reinforcing bars, the engineers had some misgivings in using nothing more than the mesh of reinforcing bars such as were used at San Ildefonso. Therefore, it was resolved to use a concrete slab (see Figs. 8 and 9) reinforced with ⅝-in.-round bars spaced 18 in. on centers in both directions. This deck was 16 in. thick in the lower section and was anchored to a cutoff wall at the bottom of the canyon, using ¾-in.-round bars. The thickness of the slab was maintained at 16 in. for a height of 33 ft measured from the anchor wall, and then it was reduced to 12 in. for the remainder of its height. The entire slab was anchored to the rock fill by ¾-in.-round bars, which extended into the loose rock about 6.5 ft and were also anchored into the canyon walls by bars grouted into the rock, along both sides of the canyon. To prevent the slab from heaving as a result of possible uplift 6-in.-round weep holes were placed 6.5 ft on centers, along the entire surface.

The width of the gorge where the protective slab was placed varied from 40 ft to 50 ft. To provide the necessary access for the trucks which transported the materials and to cause a minimum interference with such transport, the slab was built in two abutting sections, each of which occupied one half of the gorge, and each half was raised alternately 5 ft vertically at a time (see Fig. 10). One half of the overflow crest at the lower end of the dam was thus always 5.0 ft higher than the other half.

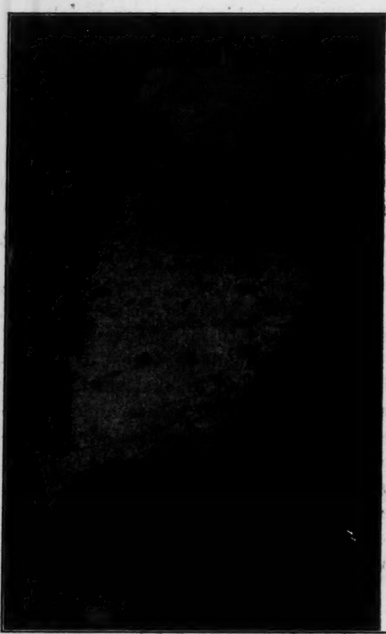


Fig. 8.—Protective Deck, Valsequillo Dam in Puebla, After the Flood of May, 1943

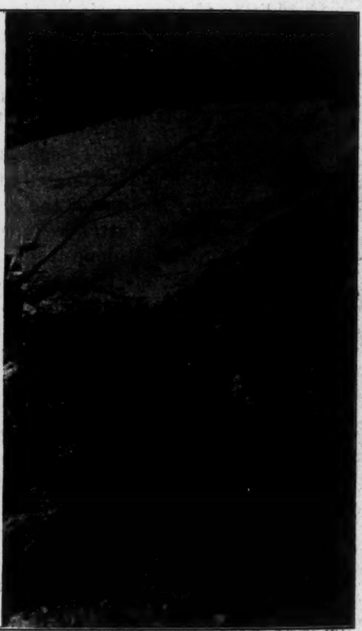


Fig. 9.—Uplift Damage, Valsequillo Dam in Puebla, After the Flood of September, 1943

Three floods overtopped the dam during the construction period in the summer of 1943, when the gorge was being filled. In May a flood overtopped the downstream crest 5.5 ft on the upper half and 10.5 ft on the lower half. The dam had been constructed to more than 52 ft above the stream bed. The width at each temporary crest level was 29.6 ft. The total of the discharge was estimated at 3,500 cu ft per sec.

A second flood, in August, raised the water surface to El. 6622.3, or 122 ft above the river bed. The higher half of the crest was at El. 6614.2 and the lower half, at El. 6609.2. The width of the crest at this stage was 46 ft. The maximum discharge was estimated at 1,306 cu ft per sec.

In September, when the maximum elevation was 6,632 ft (132 ft above the river bed), a flood of 2,605 cu ft per sec occurred. The elevation of the higher half of the downstream crest was then 6,629.0 ft, that of the lower half was 6,624.0 ft, and the width of the gorge was 59 ft. Describing the effects of this flood, Guillermo Lugo, resident engineer, reported:

> "It is necessary to state as an important detail that at more or less half the height of the dam the concrete deck failed to cover the surface of the rockfill, lacking some 200 square ft. This was at a place in the gorge where the rock cliffs were badly fractured. A large portion of the water which passed freely through the downstream rockfill passed through these open spaces."

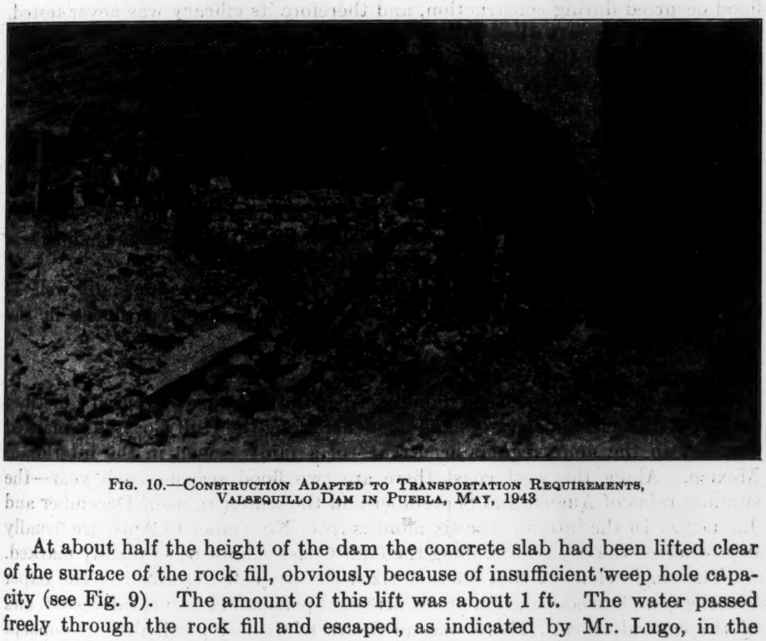


FIG. 10.—CONSTRUCTION ADAPTED TO TRANSPORTATION REQUIREMENTS, VALSEQUILLO DAM IN PUEBLA, MAY, 1943

At about half the height of the dam the concrete slab had been lifted clear of the surface of the rock fill, obviously because of insufficient weep hole capacity (see Fig. 9). The amount of this lift was about 1 ft. The water passed freely through the rock fill and escaped, as indicated by Mr. Lugo, in the openings left in the concrete slab not only at a place approximately half the height of the dam, but also near the bottom of the concrete slab.

In spite of these imperfections (caused by the uplift of the concrete slab and the escape of water through the openings near the rock cliffs) the dam showed no signs of settlement or deterioration, which leads to the belief that a simple mesh of reinforcing bars might have been sufficient to protect the dam, in the same manner as it did at San Ildefonso. The gradation of the rockfill on the downstream section of the dam proved effective protection against any erosion of the impervious core.

Summary.—The region downstream from this dam is sparsely populated and relatively greater risks could be taken without serious danger to human life or property. Furthermore, the rock walls on both sides of the canyon, the relatively small catchment area, and the ample and well-graded rock fill on the downstream slope of the dam, protected by a securely anchored concrete slab, made a failure during construction a virtual impossibility.

SIMILAR PROTECTIVE WORKS AT OTHER DAMS

The Ministry of Hydraulic Resources has found it necessary to adapt similar types of protective measures in the construction of other dams in Mexico. In one of these (the Solis Dam in the State of Guanajuato) a mesh of reinforcing bars was placed similar to that used at San Ildefonso Dam. No flood occurred during construction, and therefore its efficacy was never tested. In another (Cuarenta Dam in the State of Jalisco), the largest flood reached the (then) level of the overflow crest at the downstream face, with insufficient volume to spill over. No doubt the reinforcing bar mesh would have held the rock in place.

A third case of flood protection which was well handled by the contractor was the Sanaloma Dam on the Tamazula River some 16 miles upstream from the City of Culiacan in the State of Sinaloa. In this case a channel was left open along the right end of the dam. The mountain at that end was composed of a weathered granite and the embankment on the left of the open channel was a well-compacted rolled fill made of an altered conglomerate.

In a compacted condition, this material resisted erosion unusually well, as had been previously determined by observing the effects of floods in the preceeding season.

CONCLUSION

As was to be expected, each dam described in this paper presented a distinct problem. The irregularity of flood incidence along the Gulf Coast is in marked contrast to the more regular occurrence of a rainy season in the interior of Mexico. Along the west coast there are two flood seasons each year—the summer rains of August and September and the winter rains of December and January. In the interior, the six months from November to April are usually dry, whereas the dry season along the Gulf Coast cannot be so readily marked.

The single element that characterized the projects described in the paper was the complete cooperation of the principal personal elements interested, and of the central office. There was no desire to display originality of concept involving undue risk, or to curb any individual effort exerted to the desired ends. The only partial failure, at Azúcar Dam, was promptly and most satisfactorily met by the immediate and most effective cooperation of the field forces, in a period of sweltering and enervating heat, at a time just preceding the national holidays of September 15, and 16, when it is customary for all employees to enjoy a well-earned vacation and rest.

There is no doubt that the principles employed can be adapted successfully in many other cases, with notable success and with significant savings in construction cost. It must be clearly understood, however, that these are construction expedients and are not to be accepted as standards for a general application to other cases. Even during construction, it is recommended that no risk be taken that would be even approximately comparable to the breaching of a completed structure. Particularly important is the selection of engineering personnel with innate initiative, experience, and common sense, who are able to interpret their own past experiences and apply them conservatively to new and unexpected field problems.

ACKNOWLEDGMENT.

The dams mentioned in this paper form a part of the irrigation system constructed by the Ministry of Hydraulic Resources, formerly the National Irrigation Commission of Mexico. The National Irrigation Commission was headed by Marte R. Gómez, president, and Adolfo Orive Alba, executive commissioner and minister in charge, with Eugenio Riquelme as assistant minister.

DISCUSSION

Cleves H. Howell,[2] M. ASCE.—This paper by Mr. Weiss describes unique and successful solutions of difficult diversion problems encountered in the construction of four embankment-type dams in Mexico. Flood waters were discharged successfully over the loose-rock, downstream slopes of three partly constructed dams by protecting those slopes by grids of reinforcing bars—and nothing else.

The downstream slope of the fourth dam (Valsequillo Dam near Puebla, in the State of Puebla) was covered with a reinforced concrete slab provided with weep holes. This design seems to have functioned in passing flood water almost as well but certainly no better than the previous ones which provided for reinforcing bar grids only. As suggested by Mr. Weiss, it seems quite reasonable that a grid of reinforcing bars only, as used on the other jobs, would have been quite as efficient and less expensive.

The engineers of Los Angeles (Calif.) County Flood Control District have had wide and instructive experiences with many types of bank and levee protection under high and low velocities in widely different kinds of channels. Several years prior to the great flood of March, 1938, portions of the banks of the San Gabriel and San Antonio Rivers at the mouths of the canyons, and some of the levees below, were protected by rock and wire mattresses. The mattresses were made by placing woven wire on the bank or levee slope, covering the wire with a layer of screened gravel (which, in turn, was covered by woven wire tied to the lower wire by previously placed ties).

During the great flood of March, 1938, this type of protection received a terrific beating, not only from the high velocity of the stream flow but also from the pounding of heavy boulders rolled along the river beds by the fast water. With very minor exceptions all of the installations stood firm. In connection with the facts given, however, it should be noted that, in general, the levees and banks were nearly parallel with the direction of the current. There were no direct head-on attacks.

Mr. Weiss and the other engineers of the Mexican Ministry of Hydraulic Resources, with characteristic clarity of thought and freedom from antique inhibitions, went a long step further. They deliberately discharged large volumes of water over, down, and at right angles to the steep downstream slopes of loose-rock embankments protected only by grids of reinforcing bars. They did this with success. Most engineers would have said it could not be done without disaster.

One is reminded of the numerous things that "could not be done," which were done by Captain James B. Eads, Fellow, ASCE, seven or eight decades ago. "Time marches on."

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[2] Longmont, Colo.

Gerard H. Matthes,[3] **Hon. M. ASCE.**—Necessity may be the mother of invention, but it takes courage to proceed contrary to long established principles. It has been standard practice in the United States never to permit water to flow over an earth embankment either during construction or after completion. This dogma dates back largely to the Johnstown (Pa.) flood (1889) that was caused by high water overtopping an unprotected earth dam. It is refreshing to learn that our neighbors in Mexico have found a way to scrap this time-honored inhibition. Erratic rainfall and runoff conditions coupled with enigmatic subsurface conditions (one never knows in Mexico what troubles may lurk in the space between any two borings) aptly describe the Mexican engineer's routine problems.

The writer was fortunate in visiting several dams with the author, among them the 269-ft earth and rock-fill Valsequillo Dam that had been overtopped by floods three successive times during construction. At the time of this visit the structure had stood completed for more than a year with water close to spillway level. Inspection revealed no leakage at the toe. The limestone conglomerate foundation rock that also composes the walls of the 100-ft deep chasm showed major cracks running in several directions and caused by volcanic disturbance. The sealing of these cracks by grouting, more especially in the vertical chasm walls, had been a chief concern, and to the writer it seemed an almost incredible accomplishment. Aside from this feature of the job the problem of protecting the earth fill against erosion by overtopping had been no new problem to the author as his previous experience with passing floods over earth fills at other dam sites had already pointed the way. In this instance, a concrete slab as described was used to prevent the mesh of reinforcing bars from being demolished by trees and stumps. Inspection of the concrete slab and the manner in which it had been lifted by the force of water for lack of weep hole capacity impressed the writer with its limitations. He felt that no slab but a more secure anchorage for the reinforcing bars would have been a more satisfactory solution in this particular case as there was no scarcity of large rock which could have been held in place by the mesh of bars alone.

Reviewing the experience gained at the various dams described, the important points to be observed in protecting earth dams with rock fills against damage by overflow during construction may be set down as follows:

(*a*) The top of the earth fill should be maintained as flat as practicable, with a light upward grade in a downstream direction. This calls for methodical placing, spreading, and rolling of the materials without interfering with common procedure in these respects except for maintaining the top surface at all times on a slope.

(*b*) Grading of the rock fills (filter fashion) from finest material next to the impervious fill to coarse material at the face requires more than ordinary care in order that no loss of impervious fill material shall occur during periods of overflow.

(*c*) Of prime importance is the necessity of keeping the downstream rock fill at a height well above the level of the earth fill at all times in readiness for a sudden flood. This provision applies likewise to the top of the concrete slab in case one is used.

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[3] Cons. Hydr. Engr., New York, N. Y.

(*d*) Where rocks of sufficiently large sizes are procurable in quantity capable of resisting entrainment by flood flow, no concrete slab is needed.

(*e*) To hold the larger rock-fill material in place, a grillage or mesh of steel bars is required, with meshes to be adequate for preventing any of the larger rock from passing through. The mesh must be held at suitable intervals by means of steel rods fastened to steel disks buried to some depth in the river bed.

(*f*) The crest of the downstream rock fill must be held in place so securely that by no chance can any part of it be eroded during overtopping.

(*g*) Where rocks of large enough sizes are not procurable for the outer face of the upstream rock fill, a concrete slab, provided with a plurality of weep holes, can be substituted. The slab should be reinforced with steel bars tied back by steel rods running to anchors.

(*h*) Customary methods of inspection on earth fill dam construction must be discarded in favor of constant close inspection by engineers qualified to deal promptly with high-water emergencies and capable of making on-the-spot decisions regarding suitability of materials, steel mesh anchorages, proper grading of rock fills, and the like.

(*i*) Ordinary and low river discharges are best handled through a small by-pass tunnel; only flow in excess of by-pass capacity is then permitted to pass over the uncompleted structure.

(*j*) After completion the dam is no longer subject to overtopping by floods, ample spillway capacity being provided.

The writer takes this opportunity to call attention to the use of concrete tetrahedral blocks as admirably suitable instead of large rock at the downstream face. The use that was made of such blocks in building the cofferdam for the Genissiat hydro development on the Rhone River [4,5,6] is recalled. There a row of very large tetrahedrons was placed in the flowing river, each block being held in place by a cable fastened to an anchor at an upstream point. Smaller tetrahedrons were then placed by dumping behind this first row together with rock large enough to lodge firmly. A loose fill of assorted materials followed, and the downstream face of the cofferdam then was carried to higher elevations by means of additional rows of tetrahedrons similarly held in place by anchorages. Earth fill finally made this cofferdam water tight. This method with suitable changes would be well adapted to the Mexican system described.

There appears to be no reason why the construction technique employed in Mexico should not find application in the United States. Superior hydrologic information concerning maximum rates of runoff has naturally led American engineers to by-pass flood flow through costly tunnels or open channels during construction. The experience gained in Mexico with passing flood waters over uncompleted earth structures indicates that flood flow can be made to pass over compacted earth fills without appreciable loss of materials provided the protection at upstream and downstream faces is made adequate. Material savings

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[4] "Barrages en Pierres Lancées," by Charles Blanchet, *La Houille Blanche*, No. 2, 1946, p. 141.

[5] *Ibid.*, No. 4, 1946, p. 393.

[6] *Ibid.*, No. 1, 1947, p. 41.

in cost and time of construction can thus be realized by dispensing with large by-passes.

The author is to be commended for his very frank and unvarnished account of the successes and partial failures experienced in developing this relatively new method of building earth dams. He has blazed a new trail that will be worth following.

Andrew Weiss,[7] Hon. M. ASCE.—The protective works built by the Los Angeles (Calif.) County Flood Control District, described by Mr. Howell, were distinctly successful. The principle that made them such an outstanding success was essentially the same as that applied in the use of a protective grid of reinforcing bars where larger rock is available. The District used a heavy wire mesh to retain the mass of screened gravel, the latter being of sufficient coarseness to be retained by the wire mesh, thus supplying the required weight anchorage and resistance against the transporting forces of the river current. It is a method of protecting riverbanks which is applicable to a much greater extent than in present practice, considering the large number of places where rock of the proper quality and size is too costly for the purpose.

A similar system of protection is in use on the San Lorenzo River in Sinaloa, Mex., where rock "sausages" are made of screened gravel wrapped in a strong wire mesh. However, the "sausages" available for inspection were past their usefulness; the wire meshes were broken, showing the rough treatment to which they had been exposed in a river subject to heavy floods and high velocities.

A similar method of river control involving the use of "gaviones," has been developed in a modified form as river training works in Bolivia. At locations where conditions make the use of these devices economical and practical, this is a logical procedure. It seems that the same system has been adopted, with much greater refinement, in Italy where some construction companies use wire mesh baskets filled with screened gravel which are placed in imitation of dry stone masonry. It seems doubtful that such uses could be applied economically in Mexico, especially when the method is to be used as a construction technique and in no case as a part of the finished design. It is never suggested that wire mesh units (which are often as expensive to build as the storage works themselves) shall be incorporated as an integral part of the permanent spillway structure.

On his visit to several of the dams and dam sites in Mexico, Mr. Matthes had a good suggestion to offer, and he described a precaution to be applied in using this technique on the standard type, rolled earth fills such as the Valsequillo Dam. It is really suprising how simple is the problem of sorting and grading the transition zone, thus defeating any tendency of parts of the dam to become eroded during construction. No special screening of materials is needed. Anyone employed in building dams in climates and conditions prevalent in certain parts of Mexico and in building similar types of dams elsewhere is very likely to encounter such conditions. It is not so common to find such climatic conditions in the United States. It is certain that no one should go out of his way to seek such opportunities of doing extra work and spending funds unless the conditions warrant such precautions; but the engineer is quite likely to find them sooner than he expects, and he should be prepared to meet them as well as is possible.

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[7] Cons. Engr., National Comm. of Irrig., Mexico, D. F., Mexico.

The use of tetrahedrons for anchoring such a rock fill appears attractive where prevailing conditions make this method preferable. It might be less attractive in places like Mexico on account of the greater scarcity of the materials needed and the lack of suitable roads. Doubtless the contractor or his client would be obliged to take these conditions into account and use their own good judgment before selecting tetrahedrons.

The partial failure that the administration encountered on the El Azúcar (Marte Gómez) Dam was doubtless due to the insufficient height of the slope protection which permitted the flood waters to invade an unprotected bordering terrace deposit; but even after an overflow period of about 3 days, the damage was unimportant in comparison with the resulting economies secured. This experience emphasizes the need of confining such measures so that resulting damages may be kept within the extent of destruction that would result from natural floods in the same valley. Reliable agencies have recorded a number of such natural floods; and similar records are generally available in regions where irrigation is a known practice or where some reliable gaging station has been established, such as that of the International Boundary Commission, 6 miles upstream from the El Azúcar Dam.

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TRANSACTIONS

Paper No. 2462

LATERAL BUCKLING OF ECCENTRICALLY LOADED I-SECTION COLUMNS

BY H. N. HILL,[1] M. ASCE, AND J. W. CLARK,[2] JUN. ASCE

WITH DISCUSSION BY MESSRS. JACK R. BENJAMIN, JACOB KAROL, AND H. N. HILL AND J. W. CLARK.

SYNOPSIS

This paper describes an experimental investigation of the behavior of I-section columns loaded eccentrically in the plane of the web. All the specimens tested failed by combined bending and twisting at stresses within the elastic range of the material.

The test results confirm theoretical work that has been done on the subject and indicate that a previously proposed design formula to cover this type of failure is unconservative.

INTRODUCTION

The ultimate strength of a member under an axial compressive force is reduced if the member is at the same time required to resist transverse bending moments. Conversely the ability to resist transverse bending is impaired by the presence of an axial compressive force. Consequently, in designing a member to be subjected simultaneously to axial compression and transverse bending, it is necessary to reduce the allowable end load and the allowable bending moment below the values which would be permitted if either occurred alone.

A member under end load and bending may fail in either of two general ways: (1) By excessive bending in the plane of the applied bending moment; or (2) by lateral buckling, involving twisting of the member and bending in a direction other than the plane of the applied bending moment. This second type of failure is the subject of this paper.

A design formula to cover the case of lateral buckling of a member under simultaneous end load and bending was published in 1938.[3] The formula was

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NOTE.—Published in September, 1950, as *Proceedings-Separate No. 34*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Asst. Chf., Eng. Design Div., Aluminum Research Laboratories, Aluminum Co. of America, New Kensington, Pa.

[2] Research Engr., Eng. Design Div., Aluminum Research Laboratories, Aluminum Co. of America, New Kensington, Pa.

[3] "Structural Aluminum Handbook," Aluminum Co. of America, Pittsburgh, Pa., 1938.

based on a theoretical solution of the problem. At the time a test program was planned to supply experimental evidence of the adequacy of the formula. The test program was considerably delayed and in the meantime a more rigorous theoretical solution of the problem was published.[4,5] On completion of the test program it seemed advisable to reexamine the design formula in the light of the test results and the more recent theoretical treatment.

Object of the Investigation

The object of the investigation described in this paper was to study the behavior of I-section members under simultaneous end load and bending in the plane of the web. For convenience in testing, the specimens were bent by applying end loads with eccentricity in the plane of the web. The specimens tested were therefore eccentrically loaded columns.

Description of Specimens

The average measured dimensions for each of the extruded I-section specimens employed in this investigation and section elements based on these dimensions are given in Table 1. The areas corresponding to the measured dimensions agreed with areas computed on the basis of the weight and specific gravity of the specimens within 1% in all cases.

The specimens were of aluminum alloy 27S-T6. At the time the tests were started, this alloy was commonly used in structural applications.[6] It is no longer a standard product, however, having been largely replaced in the structural field by alloy 14S-T6. Tensile tests (on standard specimens[7] $\frac{1}{2}$ in. wide) and pack compressive tests[8] were made, the specimens being cut in a longitudinal direction from the flanges of the I-sections. The results of these tests are given in Table 1. The properties are typical of 27S-T6 alloy. The compressive modulus of elasticity of this alloy is 10,700,000 lb per sq in. and Poisson's ratio is $\frac{1}{3}$.

The measured crookedness, listed in Table 1, was obtained in most cases by placing the specimens on a flat surface and measuring the maximum clearance between the surface and the flanges of the specimen by a feeler gage. The crookedness of specimen 13 and of specimen 10, after straightening, was obtained by supporting the specimen as a beam at the ends and measuring the distance to a flat surface at points along the length with a dial gage. The specimens were then turned over and the measurements repeated to eliminate the effect of deflection caused by the weight of the specimens.

Description of Tests

The columns were tested in a 40,000-lb Amsler machine, using trunnion type ball bearing heads of 10,000-lb capacity. These heads provided a hinged

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[4] "The Buckling of Compressed Bars by Torsion and Flexure," by J. N. Goodier, *Bulletin No. 27*, Cornell Univ. Eng. Experiment Station, Ithaca, N. Y., 1941.

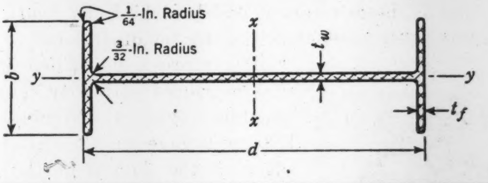
[5] "Flexural-Torsional Buckling of Bars of Open Section," by J. N. Goodier, *Bulletin No. 28*, Cornell Univ. Eng. Experiment Station, Ithaca, N. Y., 1942.

[6] "Light-Weight Structural Alloys," by Zay Jeffries, C. F. Nagel, and R. T. Wood, in "Structural Application of Steel and Light-Weight Alloys," *Transactions*, ASCE, Vol. 102, 1937, p. 1279, Table 7(*b*).

[7] "Standard Methods of Tension Testing of Metallic Materials," *1946 Book of A.S.T.M. Standards Including Tentatives*, Pt. I-B, 1946, A.S.T.M. designation: E8-46, p. 297, Fig. 2.

[8] "The Pack Method for Compressive Tests of Thin Specimens of Materials Used in Thin-Wall Structures," by C. S. Aitchison and L. B. Tuckerman, *Technical Report No. 649*, National Advisory Committee for Aeronautics, Washington, D. C., 1939.

TABLE 1.—ELEMENTS OF SECTIONS AND MECHANICAL PROPERTIES OF 27S-T6 EXTRUDED I-SECTION SPECIMENS



Specimen	Depth, d, in inches	Flange width, b, in inches	Thickness, in Inches		Area, in square inches	Moment of Inertia, in Inches^4	
			Flange, t_f	Web, t_w		I_x	I_y
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	3.990	1.383	0.0960	0.0775	0.5676	1.388	0.04248
2	3.991	1.383	0.0958	0.0787	0.5716	1.392	0.04239
3	3.993	1.382	0.0963	0.0763	0.5638	1.387	0.04253
4	3.998	1.381	0.0963	0.0780	0.5696	1.390	0.04246
5	4.001	1.387	0.0972	0.0817	0.5882	1.430	0.04340
7	3.994	1.383	0.0963	0.0768	0.5660	1.391	0.04261
9	3.996	1.384	0.0972	0.0817	0.5873	1.424	0.04315
10	3.993	1.382	0.0963	0.0780	0.5702	1.395	0.04255
11	3.992	1.382	0.0962	0.0788	0.5730	1.397	0.04251
12	3.993	1.383	0.0959	0.0768	0.5647	1.386	0.04240
13	3.993	1.382	0.0962	0.0778	0.5692	1.393	0.04246

TABLE 1.—(*Continued*)

Specimen	Torsion factor, J, in inches^4	Tensile strength, in pounds per square inch	Yield Strength, in Pounds per Square Inch[a]		Crookedness,[b] in Inches	
			Tension	Compression	West flange	East flange
(1)	(9)	(10)	(11)	(12)	(13)	(14)
1	0.001435	63,700	55,000	55,000	0.012 S	0.013 N
2	0.001459	64,680	56,000	55,500	....	....
3	0.001414	63,270	54,900	55,300	0.006 S	0.006 S
4	0.001453	63,950	56,000	56,000	....	....
5	0.001576	63,270	54,900	55,500	0.006 S	0.023 N
7	0.001427	63,300	54,800	55,500	0.045 N	0.016 N
9	0.001574	63,270	54,900	55,500	0.018 N	0.014 N
10	0.001455	63,950	56,000	56,000	0.022 S[c]	0.016 S[c]
11	0.001472	64,680	56,000	55,500	0.017 S	0.016 S
12	0.001416	61,820	53,000	54,200	0.026 N	0.018 S
13	0.001447	63,510	52,900	52,300	0.095 S	0.032 N

[a] Set = 0.2%. [b] N and S indicate maximum bow to the north and south, respectively. [c] These are the values before straightening. After straightening the "crookedness" was reduced to 0.008 in. and 0.006 in., for the west flange and east flange, respectively.

condition in the direction of the axis in the web (with the center of rotation in the planes of the ends of the specimen) and a flat-end condition in the direction of the axis normal to the web. The test setup is illustrated in Fig. 1, with specimen 10, Table 1, in position for testing at an eccentricity of 1 in.

Columns having lengths of 40 in., 60 in., 70 in., and 100 in. were tested, with eccentricities of load in the plane of the web of 0 in., $\frac{1}{4}$ in., $\frac{1}{2}$ in., and 1 in. Eccentricity was measured with reference to concentric circles inscribed on the faces of the heads. The maximum eccentricity was within the core of the section—that is, within a region such that there was no tendency to develop tensile stresses between the specimen and the platens. In this region of small eccentricities (that is, large ratios of end load to Euler load), the maximum safe stress is appreciably lower than that for bending alone.

The stresses developed in the columns were measured with four Huggenberger tensometers (type A: Nominal magnification 1,200 and gage length 1 in.) clamped to the edges of the flanges at the centers of the specimens as shown in Fig. 1.

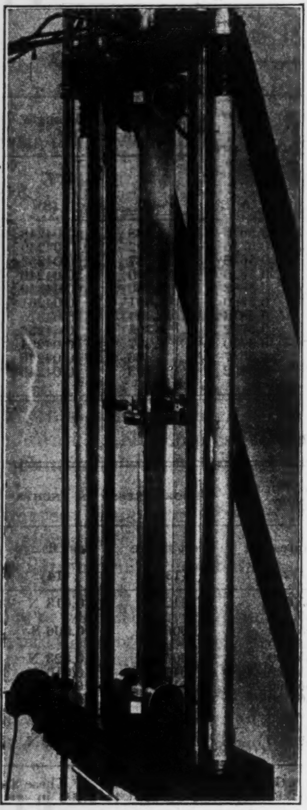


Fig. 1.—Eccentrically Loaded Column Specimen 10 in Position for Testing

Deflections were measured with the aid of mirrored scales and taut wires. In the tests in which the load was applied with no eccentricity, deflections perpendicular to the web were measured on both flanges, 2 in. below the center. When an eccentric loading was used, the deflection perpendicular to the web was measured on the flange with the greater compression, and the deflection at the center in the direction of the web was also measured.

In all cases, loading was continued in increments until a definite maximum had been reached, as indicated by a falling off of the load despite continued movement of the head of the testing machine.

The 40-in. specimens were each tested only once since failure was accompanied by some permanent set. Failure was elastic in the remaining columns so that the same specimen could be tested at different eccentricities. Specimen 10 was retested after having been straightened by loading the specimen as a beam perpendicular to the plane of the web.

Results and Discussion

Table 2 lists the loads at failure. Typical plots of the measured deflections and the stresses corresponding to the measured strains are shown in Figs. 2 and 3 for specimen 11. Deflection curves, both in the plane of the web and normal to this plane, are shown in Fig. 4 for all the eccentricities of load under which this specimen was tested. It can be seen from the curves of deflection perpendicular to the web (Fig. 4(*b*)) that failure in each case was by lateral buckling. This was true of all specimens tested.

TABLE 2.—LATERAL BUCKLING OF ECCENTRICALLY LOADED I-SECTION COLUMNS

Specimen	Length, L, in inches	Eccentricity, e, in inches	Loads, P, at failure, in pounds	COMPUTED CRITICAL COLUMN LOAD, IN POUNDS		CRITICAL BENDING MOMENT	COMPUTED LOADS AT FAILURE, IN POUNDS			PERCENTAGE VARIATION[b]		
							Handbook formula[a] (Eq. 3)	Goodier Solution		Handbook formula[a] (Eq. 3)	Goodier Solution	
				$K=0.5$	$K=$test value	$K=$test value		Before adjustment	Adjusted for deflection		Before adjustment	Adjusted[c]
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
Coefficient $K=0.5175$—												
1	39 $\frac{15}{16}$	0	10,500	11,250	10,500	....	....	....	....	....	....	....
2	39 $\frac{31}{32}$	$\frac{1}{4}$	10,250	11,205	10,458	21,840	10,312	10,140	10,076	+ 0.6	−1.1	−1.7
3	40	$\frac{1}{2}$	9,230	11,229	10,480	21,842	9,938	9,446	9,295	+ 7.7	+2.3	+0.7
4	40	1	7,750	11,210	10,463	21,822	8,772	7,970	7,758	+13.2	+2.8	+0.1
5	39 $\frac{31}{32}$	1	8,140	11,472	10,707	22,480	8,993	8,192	7,997	+10.5	+0.6	−2.0
Coefficient $K=0.5046$—												
7	60	0	4,910	5,000	4,910	....	....	....	....	....	....	....
		$\frac{1}{4}$	4,780		4,910	10,949	4,850	4,793	4,768	+ 1.5	+0.3	−0.3
		$\frac{1}{2}$	4,380		4,910	10,949	4,685	4,517	4,450	+ 7.0	+3.1	+1.6
		1	3,645		4,910	10,949	4,191	3,879	3,773	+15.0	+6.4	+3.5
Coefficient $K=0.5079$—												
9	60 $\frac{1}{32}$	0	4,900	5,057	4,900	....	....	....	....	....	....	....
		$\frac{1}{4}$	4,835		4,900	11,064	4,841	4,790	4,766	+ 0.1	−0.9	−1.4
		$\frac{1}{2}$	4,540		4,900	11,064	4,680	4,526	4,462	+ 3.1	−0.3	−1.7
		1	3,840		4,900	11,064	4,195	3,901	3,798	+ 9.2	+1.6	−1.1
Before Straightening, Coefficient $K=0.5341$—												
10	69 $\frac{15}{16}$	0	3,220	3,674	3,220	....	....	....	....	....	....	....
		$\frac{1}{4}$	3,170		3,220	7,632	3,185	3,158	3,146	+ 0.5	−0.4	−0.8
		$\frac{1}{2}$	3,005		3,220	7,632	3,088	3,005	2,971	+ 2.8	0	−1.1
		1	2,610		3,220	7,632	2,790	2,621	2,561	+ 6.9	+0.4	−1.9
After Straightening, Coefficient $K=0.5248$—												
10	69 $\frac{15}{16}$	0	3,335	3,674	3,335	....	....	....	....	....	....	....
		$\frac{1}{4}$	3,280		3,335	7,861	3,298	3,270	3,257	+ 0.5	−0.3	−0.7
		$\frac{1}{2}$	3,100		3,335	7,861	3,197	3,109	3,072	+ 3.1	+0.3	−0.9
		1	2,680		3,335	7,861	2,886	2,708	2,643	+ 7.7	+1.0	−1.4
Coefficient $K=0.5308$—												
11	69 $\frac{31}{32}$	0	3,255	3,668	3,255	....	....	....	....	....	....	....
		$\frac{1}{4}$	3,225		3,255	7,714	3,220	3,193	3,180	− 0.2	−1.0	−1.4
		$\frac{1}{2}$	3,055		3,255	7,714	3,122	3,038	3,003	+ 2.2	−0.6	−1.7
		1	2,655		3,255	7,714	2,820	2,651	2,589	+ 6.2	−0.2	−2.5
Coefficient $K=0.5147$—												
12	100	0	1,690	1,791	1,690	....	....	....	....	....	....	....
		$\frac{1}{4}$	1,645		1,690	4,525	1,675	1,668	1,663	+ 1.8	+1.4	+1.1
		$\frac{1}{2}$	1,570		1,690	4,525	1,635	1,610	1,594	+ 4.1	+2.5	+1.5
		1	1,375		1,690	4,525	1,503	1,444	1,411	+ 9.3	+5.0	+2.6
Coefficient $K=0.5201$—												
13	100 $\frac{1}{16}$	0	1,655	1,791	1,655	....	....	....	....	....	....	....
		$\frac{1}{4}$	1,650		1,655	4,479	1,641	1,634	1,630	− 0.5	−1.0	−1.2
		$\frac{1}{2}$	1,610		1,655	4,479	1,602	1,578	1,564	− 0.5	−2.0	−2.9
		1	1,435		1,655	4,479	1,475	1,419	1,389	+ 2.8	−1.1	−3.2

[a] "Alcoa Structural Handbook," Aluminum Co. of America, Pittsburgh, Pa., 1945, p. 58. [b] Percentage variation of computed values from the actual loads at failure. [c] The Goodier solution adjusted for deflection.

Curves of calculated bending deflection in the plane of the web are shown in Fig. 4(a). Deflections were computed from the formula[9]:

$$\delta = e\left(\sec \frac{\pi}{2}\sqrt{\alpha} - 1\right) \qquad (1)$$

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[9] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, p. 13.

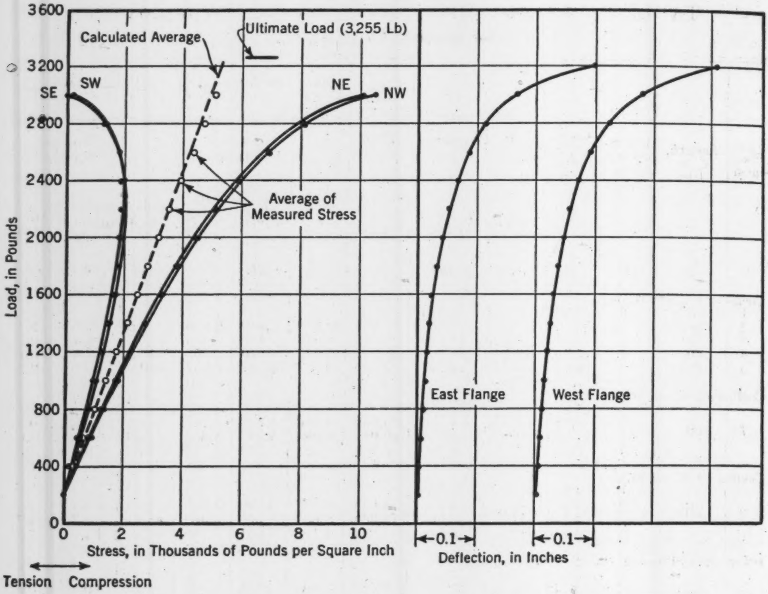


FIG. 2.—MEASURED STRESSES AND DEFLECTIONS IN SPECIMEN 11 (SEE TABLE 2), WITH NO ECCENTRICITY

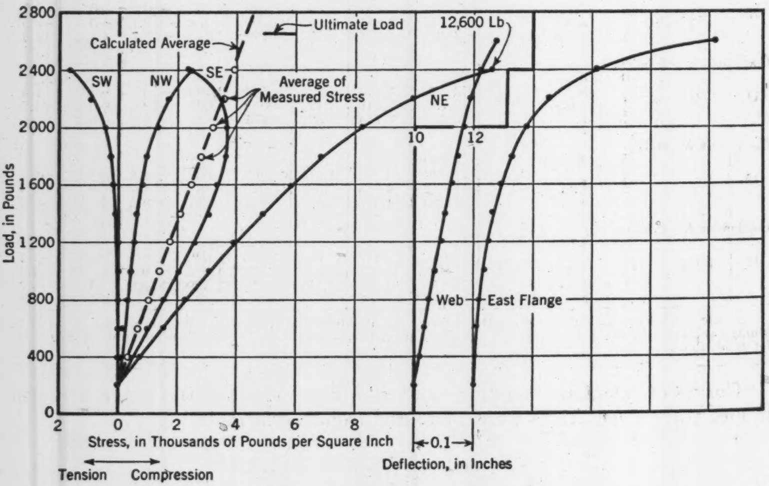


FIG. 3.—MEASURED STRESSES AND DEFLECTIONS IN SPECIMEN 11 (SEE TABLE 2), WITH 1-IN. ECCENTRICITY

in which δ is the deflection at the center in the plane of the web, in inches; e is the eccentricity of load, in inches; and α is the ratio of load on the column to the Euler critical load for a pinned-end column. Figs. 5 and 6 show representative comparisons between measured and calculated bending stresses in the plane of the web. Bending stresses were computed by the formula:

$$f_b = \frac{P\,y}{I}\,(e + \delta)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

in which f_b is the maximum bending stress, in pounds per square inch; P is the load on the column, in pounds; and y is the distance from the neutral axis to the point of attachment of the tensometers, in inches. There was good agreement between the test results and the calculated values of bending stress and deflection in the plane of the web. This agreement can be considered as indicating that the eccentricities of loading were very close to the desired values.

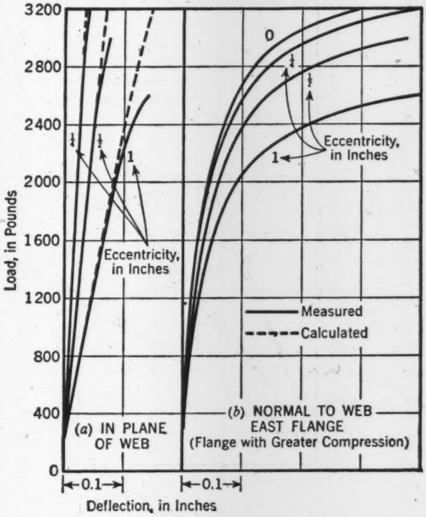


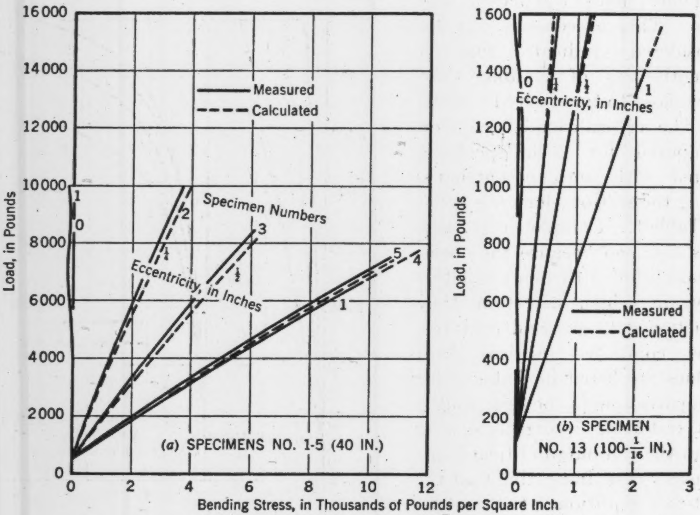
Fig. 4.—Deflections of Web and East Flange; Specimen 11

The critical load has been computed for each specimen tested with zero eccentricity, using the section elements given in Table 1. Complete end fixity was assumed because the specimens all failed by bending laterally, in which direction they functioned as flat-end columns. The computed critical load values are listed in Table 2 for comparison with the test results. The test loads are slightly smaller than the computed critical loads in every case, indicating that the flat-end condition of the tests resulted in less than complete fixity. The fixity coefficients (K-values) corresponding to the test results were computed and are shown in Table 2. These values range from 0.505 to 0.531 (the value for complete fixity being 0.5; and that for pinned ends, 1.0). Variations in the value of the fixity coefficient for the different tests are probably associated with initial crookedness, and with deviations of the ends of the specimens from the desired condition of being parallel and normal to the axis. The ends of these specimens were finished in a milling machine and checked for parallelism by measuring the length at the four corners with an outside caliper incorporating a 0.001-in. dial indicator. The range of fixity factors encountered is typical of previous experience on columns so machined. In more recent investigations more nearly perfect end conditions have been obtained by turning the ends of the specimens on centers or on an arbor in a lathe.

A working specification[10] for allowable bending stress in the presence of an axial load, for the case involving lateral buckling, permits a maximum bending stress (compression) on the extreme fiber, at or near the center of the unsupported length, in addition to direct compression, expressed by

$$\frac{Mc}{I} = f_a \sqrt{1 - \frac{\frac{P}{A}}{f'_c}} \quad \text{.......................} (3)$$

In Eq. 3 (in pounds per square inch), $\frac{P}{A}$ is the average compressive stress on the cross section of the member produced by column load; f_a is the allowable compressive working stress for a member considered as a beam; and f'_c is the allowable working stress for a member considered as a column tending to fail in the direction normal to the plane of the bending forces. Eq. 3 was based on the theoretical solution given by S. Timoshenko[11] for lateral buckling of deep rectangular beams, under simultaneous end load and pure bending. This solution can be expressed as



(a) SPECIMENS NO. 1-5 (40 IN.)

(b) SPECIMEN NO. 13 (100-$\frac{1}{16}$ IN.)

FIG. 5.—BENDING STRESSES IN THE PLANE OF THE WEB

$$\frac{P}{P'} + \left(\frac{M}{M'}\right)^2 = 1 \quad \text{.......................} (4)$$

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[10] "Alcoa Structural Handbook," Aluminum Co. of America, Pittsburgh, Pa., 1945, p. 58.

[11] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, p. 243.

in which P is the axially applied end load, in pounds; P' is the critical end load for buckling in the direction of least stiffness, in pounds; M is the bending moment applied in the direction of greatest stiffness, simultaneously with the end load P, in inch-pounds; and M' is the critical bending moment for the beam under pure bending alone, in inch-pounds. Bruce Johnston,[12] M. ASCE, has derived the same relationship for I-section members. Eq. 3 is derived from Eq. 4 by applying safety factors and expressing the end load and the bending in terms of unit stresses.

For the purpose of comparing the results of these tests with the foregoing treatment, the maximum load values have been expressed as ratios $\frac{P}{P'}$ and $\frac{M}{M'}$ and are plotted in Fig. 7. The applied bending moment is, of course, $M = P\,e$. The critical values of axial load (P') and bending moment[13] (M') were calculated from the following equations:

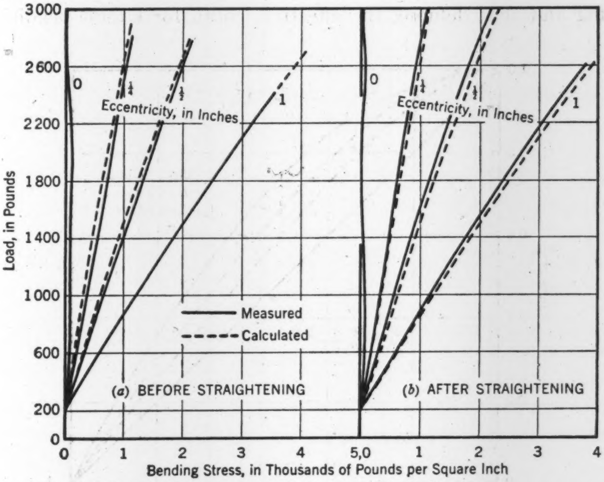

FIG. 6.—SPECIMEN 10 (69$\frac{13}{16}$ IN.); BENDING STRESSES IN THE PLANE OF THE WEB

$$P' = \frac{\pi^2 E\,I}{(K\,L)^2} \qquad (5)$$

and

$$M' = \frac{\pi \sqrt{E\,I\,G\,J}}{K\,L} \sqrt{1 + \left(\frac{\pi\,d'}{2\,K\,L}\right)^2 \frac{E\,I}{G\,J}} \qquad (6)$$

in which E is the modulus of elasticity, in pounds per square inch; I is the moment of inertia in a lateral direction, in inches^4; d' is the depth of the beam

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[12] "Lateral Buckling of I-Section Column with Eccentric End Loads in Plane of the Web," by Bruce Johnston, *Journal of Applied Mechanics*, December, 1941, p. A-176.

[13] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, p. 261.

(distance between the centroids of the flanges), in inches; L is the laterally unsupported length of the beam, in inches; J is the torsion factor, in inches[4]; and G is the modulus of rigidity, in pounds per square inch:

$$G = \frac{E}{2\,(1 + \mu)} \quad \text{.......................} (7)$$

in which μ is Poisson's ratio. The K-values used in calculating P' and M' for the various specimens were those determined from the tests of the specimens under zero eccentricity. Since the exact factors causing K to be greater than 0.5 were not known, and since the values of K were all close to 0.5, no attempt was made to differentiate between the K-value defining the effective length for twisting and that defining the effective length for lateral bending. It is recognized, however, that for certain conditions the effective length may be different for twisting than for lateral bending. In the case of the 40-in. lengths for which a different specimen was used for each eccentricity, the K-value determined for the axially loaded case was used throughout. Section elements determined from measurements on each specimen (Table 1) were used in calculating the P'-values and the M'-values.

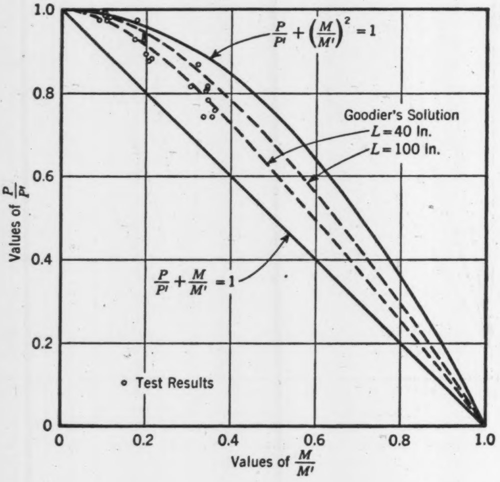

FIG. 7.—INTERACTION CURVES OF $\frac{P}{P'}$ VERSUS $\frac{M}{M'}$ FOR ECCENTRICALLY LOADED COLUMNS

It will be noted in Fig. 7 that the points representing the test results generally fall below the parabola representing Eq. 4, indicating that the theoretical solution on which Eq. 3 was based gives nonconservative results.

Perhaps a more direct comparison between the test results and the theory can be obtained by comparing values of maximum load directly. For an eccentrically loaded column Eq. 4 can be expressed as

$$P = P'\left[\frac{2}{1+\sqrt{1+4\left(\frac{e\,P'}{M'}\right)^2}}\right]\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Table 2 lists the critical load values calculated in accordance with Eq. 8 and the percentage variation of these values from the test results. The test results are generally lower than the calculated values, the discrepancy increasing with increasing eccentricity. The maximum difference occurred for specimen 7, a 60-in. column, when tested with 1-in. eccentricity. The calculated critical load was about 15% higher than the load at which the specimen failed.

J. N. Goodier[14] has furnished a more general solution to the problem of the buckling of a member under end load and bending, including in his derivation the hypothesis by H. Wagner describing the torsional effect of the thrust. Mr. Goodier's solution for an eccentrically loaded column of the type employed in this investigation may be expressed as

$$P = P'\left\{\frac{2}{1+\left(\rho\frac{P'}{M'}\right)^2+\sqrt{\left[1-\left(\rho\frac{P'}{M'}\right)^2\right]^2+4\left(\frac{e\,P'}{M'}\right)^2}}\right\}\ldots(9)$$

in which ρ is the polar radius of gyration, in inches; Eq. 9 differs from the previous expression for critical load (Eq. 8) by the introduction of the term $\left(\rho\frac{P'}{M'}\right)^2$ which is related to Mr. Wagner's critical thrust for torsional buckling. Expressed in the form of Eq. 4, Eq. 9 may be written

$$\frac{P}{P'}+\left(\frac{M}{M'}\right)^2 = 1-\frac{P}{P'}\left(\rho\frac{P'}{M'}\right)^2\left(1-\frac{P}{P'}\right)\ldots\ldots\ldots\ldots(10)$$

For a particular column specimen Eq. 10 can be plotted as a curve of $\frac{P}{P'}$ versus $\frac{M}{M'}$ similar to the parabola of Eq. 4. Fig. 7 shows typical curves obtained from Eq. 10 for the 40-in. and 100-in. specimens of this investigation. Curves for the remaining specimens would fall between the two shown. It can be seen that these curves lie below the one representing Eq. 4 and agree more closely with the test results.

Table 2 lists the critical loads for the columns tested, computed by Eq. 9, and the percentage differences between the test results and these calculated values. The values predicted by Eq. 9 vary from 2.0% less than the test value, for specimen 13 with ½-in. eccentricity, to 6.4% greater than the test value, for specimen 7 tested with a 1-in. eccentricity of load.

The difference between the test results and the theoretical values of critical load may be further reduced by consideration of the effect of deflections in the plane of the web. Eqs. 8 and 9 are based on the assumption that the moment of the load is constant along the length of the member. Actually the moment arm of the load in the plane of the web is equal to the eccentricity plus the deflection in this direction. The values of critical load in Table 2, Cols. 10 and 13, were computed by assuming the effective eccentricity of the load to be equal to the original eccentricity plus two thirds of the calculated center deflection in the plane of the web at failure. The value of two thirds times the maximum deflection was chosen to give an approximation of the average deflection over the length of the specimen. The variation between test results and the critical loads computed by this method ranges from − 3.2% for specimen 13 tested with a 1-in. eccentricity of load to + 3.5% for specimen 7 with a 1-in. eccentricity.

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[14] "Flexural-Torsional Buckling of Bars of Open Section," by J. N. Goodier, *Bulletin No. 28*, Cornell Univ. Eng. Experiment Station, Ithaca, N. Y., 1942, p. 12.

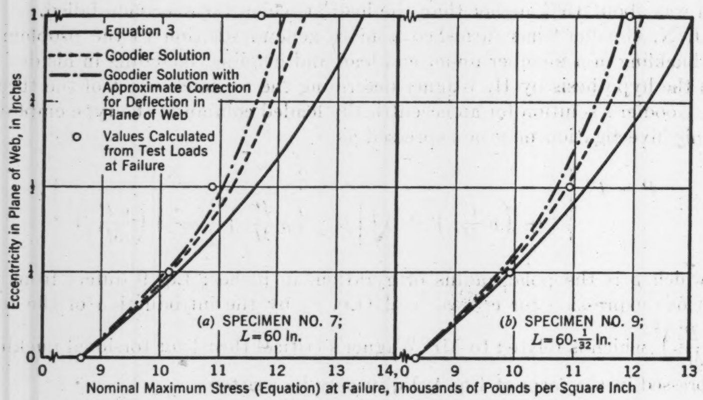

FIG. 8.—ECCENTRICITY OF LOAD VERSUS MAXIMUM STRESS AT FAILURE; 60-IN. SPECIMENS

Since design formulas for eccentrically loaded columns are expressed in terms of the maximum allowable nominal stress—

$$f_n = \frac{P}{A} + \frac{P\,e\,c}{I} \qquad (11)$$

—the critical loads as determined by a representative few of these tests and the various theoretical solutions have been converted into maximum nominal stresses and plotted against eccentricity in Fig. 8. The agreement between test results and the various theories can be clearly seen in this illustration and in others not published.

The reason that the test results do not agree with the parabolic curve in Fig. 7 is found in the incompleteness of the theory on which the expression for the parabola is based. The test results are in satisfactory agreement with the theoretical treatment of Mr. Goodier. The writers have been unable, however, to express this solution in a form simple enough for design purposes.

The straight line in Fig. 7 represents the relationship:

$$\frac{P}{P'} + \frac{M}{M'} = 1 \qquad (12)$$

which corresponds to the equation for combined axial and bending stresses given in some specifications.[15,16] As used in Fig. 7, the P'-value is the critical load for lateral buckling under axial end load alone. The plotted points representing the test results fall above this straight line and below the parabola representing Eq. 4 which was intended to cover this type of failure.

CONCLUSIONS

The following conclusions seem justified on the basis of the foregoing results and discussion:

1. When loaded eccentrically in the plane of the web, the extruded I-section columns of this investigation failed by lateral buckling at loads that were generally lower than those predicted by the theoretical solution which was the basis for the design formula, Eq. 3. In the extreme case, the theoretical value was about 15% higher than the test load at failure.
2. The test results were in good agreement with the theoretical solution of Mr. Goodier.[14] The critical loads as predicted by the Goodier solution ranged from 2.0% less to 6.4% greater than those determined by the tests.
3. The range of variation between test results and theoretical values of critical load was reduced to − 3.2% and to + 3.5% by modifying the Goodier solution to take approximate account of the effect on moments of the deflections in the plane of the web at the maximum load.
4. The critical loads determined in these tests were all higher than values predicted by the straight-line formula for interaction between axial load and bending moment.

Preliminary results of additional tests undertaken to extend the range of the investigation described in this paper, confirm the solution, as modified to account for deflection in the plane of the web. They show that, for certain proportions of eccentrically loaded I-section columns, failure may occur by lateral buckling at loads considerably less than those predicted by the straight-line interaction formula, Eq. 12. This equation, however, will yield conservative results if the deflection of the member in the plane of bending is included in computing the bending moment.

ACKNOWLEDGMENTS

The investigation described in this paper was conducted at the Aluminum Research Laboratories in New Kensington, Pa., under the direction of E. C. Hartmann, M. ASCE, chief of the Engineering Design Division, and R. L. Templin, M. ASCE, assistant director of research and chief engineer of tests of the Aluminum Company of America.

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[15] "Specification for the Design, Fabrication, and Erection of Structural Steel for Buildings," AISC, New York, N. Y., February, 1946, Section 12.

[16] "Design Specifications for Bridges and Structures of Aluminum Alloy 27S-T," prepared by Leon S. Moisseiff for the Aluminum Co. of America, 1940, article 205.

DISCUSSION

Jack R. Benjamin,[17] Assoc. M. ASCE.—The study of lateral buckling (torsional failure) of I-sections is greatly assisted by the paper. However, the writer notes some serious shortcomings. Generally, if a theoretical check of experimental results is to be made, the theory must agree with the experimental conditions. The theoretical results quoted do not agree with the experiment performed. It is not sufficient to apply arbitrary factors of effective length and eccentricity without examining the implications of these factors.

The column under test will rotate torsionally under all loads. This rotation modifies the usual concepts. The rotation of column sections modifies the effective length. The point of zero moment about the minor axis is above the quarter point because of the rotation producing a moment which equals the major axis moment times the angular rotation.

The theoretical formulas quoted are not based on constant moment along the length of the member. They contain the influence of deflection on moment. The theory requires the use of the end moments at the ends of the effective length. These moments are experimentally applied and include the influence of a deflection. Thus the eccentricities in Table 2 should include this item if a check of the theory is desired.

The check between theory and experiment is remarkable considering the assumptions made. The theoretical solution of the experimental column can be made numerically without excessive work by determining the required equilibrium conditions at sections along the column length and then examining the compatible distortions.

The profession of structural engineering is greatly in need of some simple, rational, approximate theory of torsional failure. The presentation of empirical interaction formulas is not sufficient. Design requires an understanding of structural action much more than the ability to turn the crank on a formula. The writer hopes that the authors' studies may evolve some simple, rational, approximate theory.

Jacob Karol,[18] M. ASCE.—A valuable addition to the literature on lateral buckling is made by this paper. It is doubly interesting because the particular test data cited agree very well with the modified general theory of J. N. Goodier, and the theory explains why the previously suggested solutions (Eqs. 4 and 12) do not fit the test data. The authors state that they have been unable to express the solution in a form simple enough for design purposes. In the ensuing discussion a chart will be presented that the writer believes gives a solution conveniently and accurately.

A minor change in the modified general theory is proposed to permit the use of this theory for I-beams whose proportions may vary considerably from those tested by the authors. A more accurate expression for the critical value of the bending moment (M_{cr}) is

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[17] Asst. Prof., Civ. Eng., Stanford Univ., Stanford, Calif.

[18] Design Engr., Howard, Needles, Tammen & Bergendoff, Kansas City, Mo.

$$M_{cr} = \frac{M'}{\sqrt{1 - \frac{I_y}{I_x}}} \quad \text{(13)}$$

in which M' is the value given by Eq. 6.

The effect of the deflections in the plane of the web on the moments at the critical section can be determined by the following formula. The relation of the maximum moment ($M_{\max}$) to the applied moment (M) is given by

$$\frac{1 - \alpha}{1 + 0.272\,\alpha} = \frac{M}{M_{\max}} \quad \text{(14)}$$

The average moment over the length of the specimen is given by

$$M_{av} = M + 0.636\,(M_{\max} - M) = \frac{M\,(1 - 0.191\,\alpha)}{1 - \alpha} \quad \text{(15)}$$

Mr. Goodier's general theory as modified by the authors and the writer may then be written

$$\frac{P}{P'} + \left(\frac{M_{av}}{M_{cr}}\right)^2 = 1 - \beta\,\frac{P}{P'}\left(1 - \frac{P}{P'}\right) \quad \text{(16)}$$

in which

$$\beta = \left(\frac{\rho\,P'}{M_{cr}}\right)^2 \quad \text{(17)}$$

An important fact concerning Eq. 17 is that β is nondimensional. Its value is determined from Eqs. 5 and 13:

$$\beta = \frac{\left(\frac{2\,\rho_x}{d'}\right)^2\left[1 - \left(\frac{I_y}{I_x}\right)^2\right]}{1 + \frac{2\,J}{\pi^2\,(1 + \mu)\,I}\left(\frac{K\,L}{d'}\right)^2} \quad \text{(18)}$$

The numerator of the right hand side cannot exceed one. The denominator will vary from a minimum of one for a short deep beam with wide flanges to a large number for a long shallow beam with narrow flanges. The value of the parameter β approaches one in the first case, and Eq. 16 reduces to a modification of Eq. 12. In the second case, the value of the parameter approaches zero, and Eq. 16 reduces to a modification Eq. 4. Since the values of the parameter for the test specimens lie between 0.33 and 0.56, it is obvious that Eqs. 4 and 12 are not applicable. It should be noted that, since P' and M_{cr} must be calculated in the solution of a particular problem, Eq. 17 is more convenient for the calculation of β than Eq. 18.

Eqs. 15 and 16 have been plotted in Fig. 9. Two examples of the use of the chart as applied to the test data in Table 2 follow:

1. For specimen 13. Given $P' = 1{,}655$ lb, $M_{cr} = 4{,}479 \times 1.015 = 4{,}546$ in.-lb, $\rho = 1.588$ in., and $KL = 52.04$ in., find the maximum theoretical load for 1-in. eccentricity. Compute the Euler load in the plane of the web, P_e =

14,700 lb, and compute $\beta = 0.334$. The solution must be made indirectly. Estimate that $P = 1{,}396$ lb, for which $\alpha = P/P_e = 0.095$, and $P/P' = 0.845$. For these values, the chart shows $M/M_{av} = 0.92$ and $\frac{M_{av}}{M_{cr}} = 0.333$. Hence $M = 0.92 \times 0.333 \times 4{,}546 = 1{,}396$ in.-lb, and $e = 1$ in. The actual test load was 1,435 lb so the error equals -2.7%.

2. For specimen 7. Given $P' = 4{,}910$ lb, $M_{cr} = 10{,}949 \times 1.015 = 11{,}110$ in.-lb, $\rho = 1.592$ in., and $KL = 30.28$ in., find the maximum theoretical load

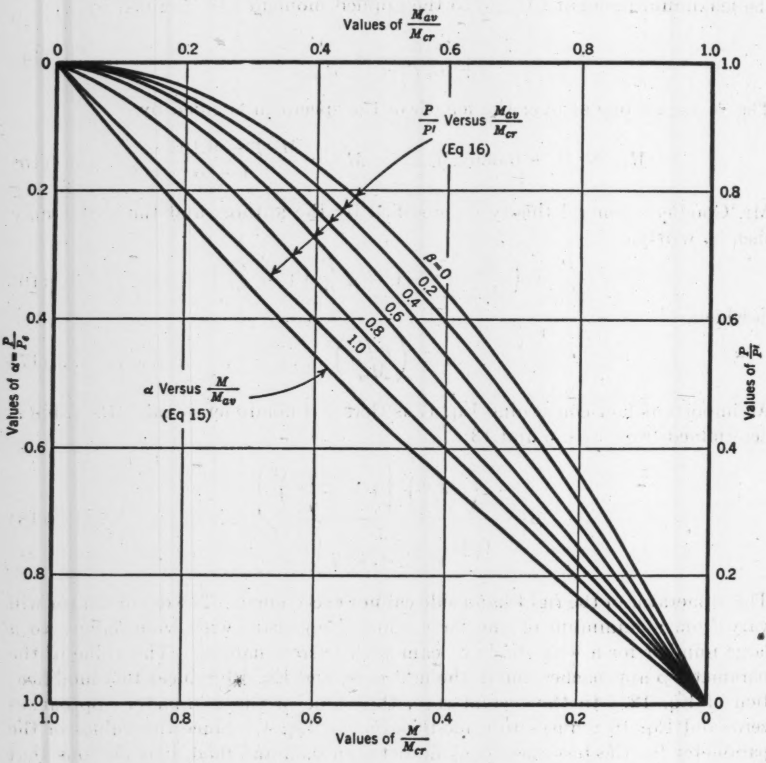

Fig. 9

for 1-in. eccentricity. Compute $P_e = 40{,}700$ lb and $\beta = 0.495$. Assume that $P = 3{,}810$ lb, hence $\alpha = 0.094$, and $P/P' = 0.776$. For these values the chart indicates that $M/M_{av} = 0.923$ and $M_{av}/M_{cr} = 0.372$. Hence $M = 0.923 \times 0.372 \times 11{,}110 = 3{,}810$ in.-lb and $e = 1$ in. The test load was 3,645 lb, giving an error of $+4.5\%$.

It is thus evident that use of the chart gives results substantially in agreement with those obtained by the authors. Because of the nondimensional parameters, the chart is generally valid for the analysis of any I-beam eccentrically loaded in the plane of the web. For short beams where the critical stresses will be in the inelastic range, the writer suggests the use of the tangent modulus in solving Eqs. 5 and 13. It should be remembered, however, that in calculating the term α the Euler load is computed using the initial modulus.

Fig. 9 is an interaction (or stress-ratio) curve similar to those curves generally used in aircraft stress analysis. An equation of the working stress type can also be written as

$$\frac{M_c}{I} = \frac{1 - \alpha}{1 - 0.191\,\alpha} f_a \sqrt{\left(1 - \frac{P/A}{f'_c}\right)\left(1 - \beta\,\frac{P/A}{f'_c}\right)} \ldots\ldots\ldots (19)$$

The danger in using this expression is that α must be computed for the actual working load (P) multiplied by the factor of safety used in the specifications, a point that may be overlooked by some designers. Furthermore, the factor of safety must be the same for bending and compressive stresses for Eq. 19 to be applicable. Assuming that the proper values are used, solution of a particular problem by Fig. 9 or Eq. 19 becomes a matter of individual preference.

The use of Fig. 9 in solving a particular problem is quite simple. Once a value of P has been assumed, α and P/P' are calculated directly. For the value of α, read the value M/M_{av} directly from the chart. For the values of P/P' and β, read the value of M_{av}/M_{cr} directly from the chart. Then

$$M = \frac{M_{av}}{M_{cr}} \times \frac{M}{M_{av}} \times M_{cr} = M \ldots\ldots\ldots\ldots\ldots\ldots (20)$$

and $e = M/P$. If this checks the required value of e, the solution is correct. Otherwise, a new value of P must be assumed and the calculations repeated.

It should be noted that for a given value of P in a particular problem, the value of e is given directly by the procedure outlined above.

If the additional tests mentioned by the authors have been completed, it is hoped that they will be presented in the closing discussion.

H. N. HILL,[19] M. ASCE, AND J. W. CLARK,[20] JUN. ASCE.—Mr. Benjamin's statements concerning the differences between the experimental conditions and the assumptions on which the quoted theory is based are open to some question. The primary difference between theoretical and test conditions is that in the theory the bending moment about the major axis is assumed to be constant along the length of the member, whereas the bending moment in an eccentrically loaded column varies because of the deflection in the plane of the applied bending moment. Mr. Benjamin's suggestion advocating the use of the bending moment at the points of inflection rather than at the ends would be one way of taking approximate account of this effect for cases in which the ends of the member are fixed in the lateral direction.

Eqs. 9 and 10 may be derived on the basis of Mr. Goodier's equations of equilibrium, assuming the ends of the member to be either hinged, or fixed against lateral bending and rotation. The effect of the torsional rotation on the bending moment about the minor-axis is included in the derivation.

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[19] Asst. Chf., Eng. Design Div., Aluminum Research Laboratories, Aluminum Co. of America, New Kensington, Pa.

[20] Research Engr., Eng. Design Div., Aluminum Research Laboratories, Aluminum Co. of America, New Kensington, Pa.

As Mr. Karol points out, an average bending moment, such as that defined by Eq. 15, may be substituted for M in Eqs. 9 and 10. In further analysis of the tests reported in this paper and subsequent test data[21] the writers have used an average moment defined by

$$M_{av} = \frac{M}{1 - \alpha} \qquad \text{(21)}$$

This expression gives satisfactory agreement between measured and calculated critical loads and is somewhat simpler than Eq. 15. In view of this agreement the writers suggest that a design formula for columns that are subject to eccentric loading and that fail by lateral buckling be based on the equation

$$\frac{M}{M'} = (1 - \alpha)\left(1 - \frac{P}{P'}\right) \qquad \text{(22)}$$

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[21] "Lateral Buckling of Eccentrically Loaded I- and H-Section Columns," by H. N. Hill and J. W. Clark, *Proceedings*, U. S. National Congress of Applied Mechanics, June, 1951 (publication pending).

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HYDROLOGY OF MEXICO

BY ANDRÉS GARCÍA-QUINTERO,[1] ASSOC. M. ASCE

WITH DISCUSSION BY MESSRS. ANDREW WEISS, MILTON O. SCHMIDT, AND GERARD H. MATTHES

SYNOPSIS

Rainfall is the principal source of water in Mexico and the principal factor in its hydrology; consequently the main purpose of this paper is to review the hydrometeorological and geophysical aspects, in order to explain why water, the most precious of all natural resources, is so scarce.

GENERAL DESCRIPTION

Geographically, Mexico is between the latitudes 14° north and 33° north within which parallels (both north and south of the equator) the desert belts of the world are known, roughly, to occur (the Sahara Desert in Africa, the Arabia, Iran, and Gobi deserts in Asia, and the American deserts in the northern hemisphere; and the Atacama and Patagonia deserts in South America, the Kalahari Desert in Africa, and the Australian deserts in the southern hemisphere).

In a classification of the areas of the world into areas of relative rainfall, zone 1 being the wettest,[2] the geographic position of Mexico may be considered as zone 4—dry the entire year, between latitudes 19° north and 31° north—that is, more than 75% of the total area is in zone 4; and so, from the geographical point of view, Mexico may be classified as not rainy even when, in some southeastern regions, there are tropical rains and tropical climates.

Orography.—The mountain system of Mexico has been divided into six groups or ranges, extending more or less along the coast lines: (*a*) Sierra Madre Oriental, (*b*) Sierra Madre Occidental, (*c*) Sierra Madre del Sur, (*d*) Sierra Volcanica Transversal, (*e*) Sierra de Chiapas, and (*f*) Sierra de Baja California.

(*a*) Sierra Madre Oriental.—This range of mountains running along the coast of the Gulf of Mexico faces the maritime air masses coming inland from

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[1] Director, Hydrology Section, Water Resources Ministry of Mexico, Mexico D. F., Mexico.

[2] "Introduction to Meteorology," by Sverre Petterssen, 1st Ed., McGraw-Hill Book Co., Inc., New York, N. Y., 1941.

the Gulf of Mexico, the Caribbean Sea, or the Atlantic Ocean (see Fig. 1). Its rapid increase of altitude from sea level to more than 7,250 ft results in a total decrease in temperature of nearly 20° F (a 1° F drop for every 366-ft increase in altitude). This topographic environment produces abundant condensation and rainfall on the windward side of the mountains; and, as the stream of humid, warm air masses is more or less permanent all the year, this coastal region receives the greatest rainfall in Mexico. On the other side of the mountains, in the central plateau, the rainfall is reduced to the minimum and increases again on the east flanks of the other mountain system known as the Sierra Madre Occidental.

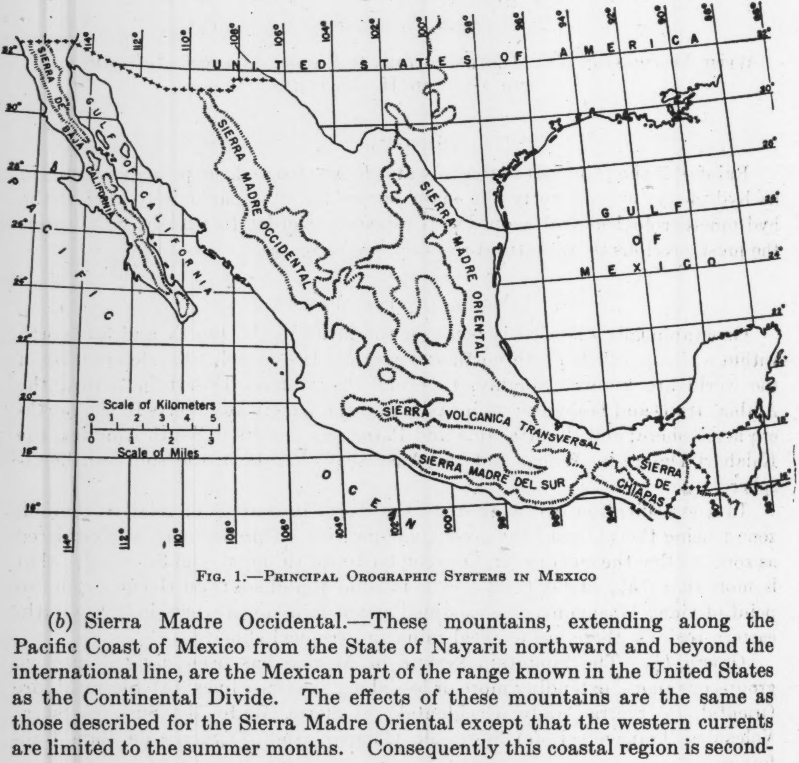
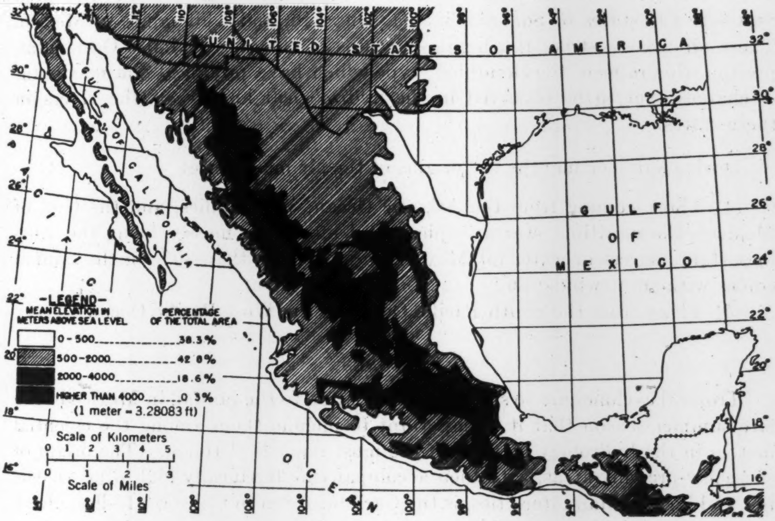

UNITED STATES OF AMERICA
GULF OF MEXICO
PACIFIC OCEAN
SIERRA DE BAJA CALIFORNIA
GULF OF CALIFORNIA
SIERRA MADRE OCCIDENTAL
SIERRA MADRE ORIENTAL
SIERRA VOLCANICA TRANSVERSAL
SIERRA MADRE DEL SUR
SIERRA DE CHIAPAS
Scale of Kilometers
0 1 2 3 4 5
Scale of Miles
0 1 2 3
114° 112° 110° 108° 106° 104° 102° 100° 98° 96° 94° 92° 90° 88° 86°
32° 30° 28° 26° 24° 22° 20° 18° 16°

FIG. 1.—PRINCIPAL OROGRAPHIC SYSTEMS IN MEXICO

(*b*) Sierra Madre Occidental.—These mountains, extending along the Pacific Coast of Mexico from the State of Nayarit northward and beyond the international line, are the Mexican part of the range known in the United States as the Continental Divide. The effects of these mountains are the same as those described for the Sierra Madre Oriental except that the western currents are limited to the summer months. Consequently this coastal region is secondary to region (*a*) in terms of mean annual rainfall.

(*c*) Sierra Madre del Sur.—This mountain range runs from the Tehuantepec Isthmus through Oaxaca and Guerrero states in a general direction which parallels the air currents of the Pacific and which is not favorable for rainfall production. Only the minor branches of this system have topographic effects on the humid air masses coming from the west, as a result of which the southern coasts of Mexico have the lowest precipitation. Also, this condition explains why in the Balsas River Basin, surrounded by mountains with peaks more than 8,200 ft high, the rainfall is least in the lower valleys where arid and desert climates are found and highest in the cool climates in the higher parts.

(*d*) Sierra Volcanica Transversal.—These mountains embrace a belt that starts at Zempoáltepec, Oaxaca, and that, in a general direction to the west, goes on to the Pacific Coast. This orographic system is also known as the Volcanic Tranverse Mountains because all the active volcanoes, or those of relative recent age, are found along latitude 19° north. The principal volcanoes from west to east are: Jorullo, in the State of Colima; Paricutin, in the State of Michoacan; Toluca Nevado, Popocatepetl, Iztaccihuatl, and Ajusco, in the State of Mexico; and Citlaltepetl, in the State of Veracruz. Because the southern side of these mountains receives the humid air currents from the Pacific, the major rainfall occurs there, even when a part of the precipitation is discharged on the coastal side of the Sierra Madre del Sur.


UNITED STATES OF AMERICA
GULF OF CALIFORNIA
GULF OF MEXICO
PACIFIC OCEAN
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-LEGEND-
MEAN ELEVATION IN METERS ABOVE SEA LEVEL
PERCENTAGE OF THE TOTAL AREA
0 - 500 38.3%
500-2000 42.8%
2000-4000 18.6%
HIGHER THAN 4000 0.3%
(1 meter = 3.28083 ft)
Scale of Kilometers
0 1 2 3 4 5
0 1 2 3
Scale of Miles

FIG. 2.—MEAN ALTITUDES OF MEXICO

(*e*) Sierra de Chiapas (Chiapas Mountain System).—This range is part of the Continental Divide extending northward from Central America. Its geographical position between the air flows of the Caribbean Sea on the north and those of the equatorial areas of the Pacific Ocean on the south assures abundant condensation and precipitation. In this mountain range (or in its vicinity) the maximum point rainfall has been observed in Mexico, reaching an average of more than 177 in. per year. It is notable in Fig. 2 that the coastal plains on the Pacific Coast (El. 0 to 500 meters or 1,645 ft) are narrower than those on the coast of the Gulf of Mexico and that the narrowest coastal plains are on the South Pacific Coast and along both sides of the lower California Peninsula.

Air Masses and Fronts

An important percentage of rainfall in the northern part of Mexico has its source in the continous "battle fronts" of air masses from polar continents and seas coming in contact with air masses from equatorial areas. Sverre Petterssen has published graphic illustrations of the air-mass sources for the northern hemisphere in summer[3] and in winter.[4]

In winter months the air masses that enter Mexico are:

(1) Those from the great American plains, permeated with snow or frost;

(2) Those from the Atlantic Ocean, formed by tropical maritime air, which is humid and warmer than air that has passed over land; and

(3) Air masses from the Pacific Ocean, in the months of December, January, and February, passing through the states of California and Arizona in the United States to the states of Sonora, Sinaloa, Durango, and Chihuahua in Mexico. These air masses strike the high mountains of the Sierra Madre Occidental, precipitating rains of the cloudburst type which have produced the maximum discharges ever to be observed in the hydrological history of the rivers in these states.

During summer months the origins of the air masses are:

(4) Those coming from the Atlantic Ocean, West Indies, and the Gulf of Mexico—the maritime, warm tropical, and humid air masses being the most important sources of water for Mexico, because more than 80% of the rainfall comes with these winds; and

(5) Those from the South Pacific Ocean or the West Pacific Ocean.

Cyclones

Tropical cyclones are so significant in relation to the rainfall in Mexico in the late summer season that it is important to include them among the essential factors in the hydrology of some of the most important rivers. The origin of these low-pressure areas, with their accompanying terrifically high wind velocities and high rainfall intensities, is the Caribbean Sea in the West Indies, along the equatorial doldrums. Their general path flows to the west and passes over the Yucatan Peninsula. Then, in the Gulf of Mexico, it deviates to the northwest, going inland at the coast of the State of Tamaulipas or at the northern coast of the State of Veracruz in Mexico or through the states of Texas, Louisiana, Mississippi, Alabama, and Florida in the United States. The cyclonic season generally covers the months of August and September, but storms of maximum intensity have occurred in the last days of August or in the first days of September.

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[3] "Weather Analysis and Forecasting," by Sverre Petterssen, McGraw-Hill Book Co., Inc., New York, N. Y., 1940, p. 160, Fig. 89.

[4] *Ibid.*, p. 165, Fig. 90.

Since a cyclone is an enormous low-pressure area, several hundred miles in extent, all the air masses in the areas contiguous to it tend to be drawn in. In terms of rainfall this tendency means:

1. There is an abnormal increase in rains all along the parts of the cyclone;
2. When the cyclone is intense, the beneficial effects of its rain load are practically reduced to nothing and the damages from flooding residential, industrial, and farm lands are tremendous. (Examples of the catastrophic nature of these revolving monsters are the storms of August 29, 1909, and August 29, 1938, coming from the Caribbean Sea. After the cyclones crossed the Gulf of Mexico, their natural violence was increased by the steep slopes of the Sierra Madre Oriental, which precipitated cloudbursts and flood conditions).
3. The major cyclones occur in the summer months, when there is maximum evaporation from the equatorial land area and sea surface; also, at this time of the year, Mexico receives the greatest proportion of the humid air masses from the eastern bodies of maritime water.

When a cyclone is being generated in the Caribbean Sea or in the Gulf of Mexico, many of the air currents swirl into the cyclone center, leaving the Mexican sky clear—without clouds and without rainfall—for many days and even weeks. When the cyclone is "on the march," however, it is a most calamitous threat, rather than a beneficial factor, in the production of rainfall. In planning hydraulic works for conservation and for the use of surface waters, cyclones impose very unfavorable conditions, requiring very large spillway and flood-control capacities.

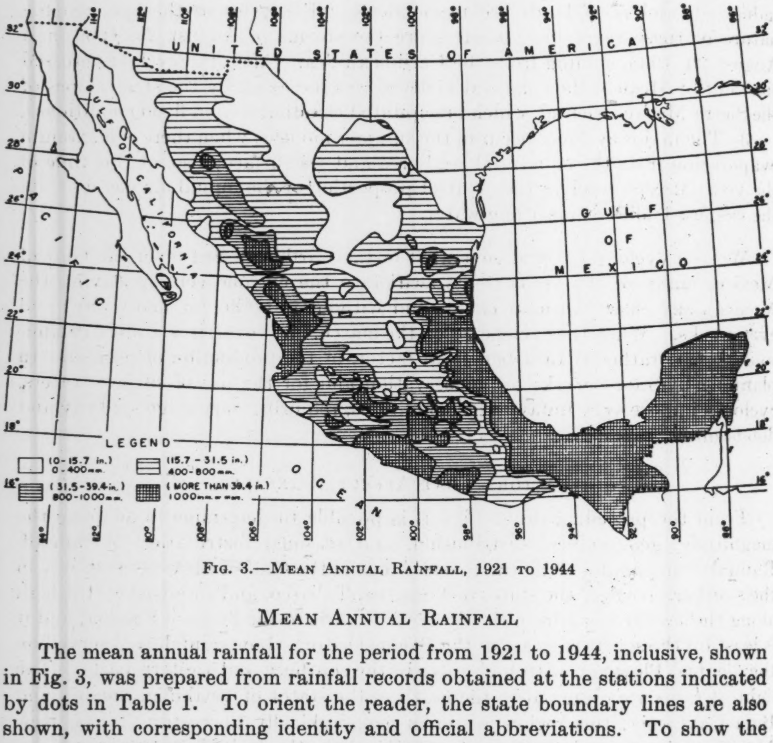
FACTORS THAT AFFECT RAINFALL

From the preceding description it is possible to determine in advance the magnitude, geographical distribution, and seasonal distribution of rainfall. Rainfall is maximum along the coastal lands of the Gulf of Mexico, especially in the southern parts of the states of Veracruz, Tabasco, and Campeche; it is high along the Pacific coast from the State of Nayarit to the State of Sinaloa; and it is least on the southern coast of the Pacific Ocean. Low rainfall is observed on the Central Pleateau, with high rates in the southern and western parts. The lightest rains are observed to the north in the states of Coahuila, Sonora, and Baja California; this region is similar geographically to western Texas, New Mexico, Arizona, and southeastern California in the United States.

Most of the rains in Mexico are "topographic" in character—that is, they are precipitated by mountain systems. There are rains caused by fronts in the constant clashes of air masses such as when the tropical maritime air masses impinge upon masses coming from the cold polar regions of the north; these rains are observed in winter in the northern and northwestern parts of Mexico.

The occurrence of precipitations of convective origin is observed in humid regions such as the states of Veracruz, Tabasco, and Campeche; in the Yucatan Peninsula, where there are no mountains, the rainfall is semiconvective because the high temperature of the sun-heated lands produced vertical air flows which are replaced by humid air masses coming from the east. When these ascending currents reach a critical altitude and temperature, relatively high rainfall results.

Cyclonic rains are observed along the paths of hurricanes; the areas more seriously affected are the northeastern part of Mexico, the Yucatan Peninsula, the central coastal region of the Gulf of Mexico, the Isthmus of Tehuantepec, the Western Pacific Coast, and the southern extreme of Lower California.


UNITED STATES OF AMERICA
GULF OF CALIFORNIA
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LEGEND
(0-15.7 in.) 0-400 mm.
(15.7 - 31.5 in.) 400-800 mm.
(31.5-39.4 in.) 800-1000 mm.
(MORE THAN 39.4 in.) 1000 mm. or more.

FIG. 3.—MEAN ANNUAL RAINFALL, 1921 TO 1944

MEAN ANNUAL RAINFALL

The mean annual rainfall for the period from 1921 to 1944, inclusive, shown in Fig. 3, was prepared from rainfall records obtained at the stations indicated by dots in Table 1. To orient the reader, the state boundary lines are also shown, with corresponding identity and official abbreviations. To show the seasonal distribution, some rainfall stations were selected as typical of the various areas in Mexico, and the monthly distribution for these stations is shown in Fig. 4. More than 75% of the annual rainfall occurs in the summer months of June, July, August, and September; only in the northern part of lower California does most of the precipitation occur in the winter months of December, January, February, and March. In Fig. 5, the shaded areas indicate a monthly mean rainfall of 25 mm or 1 in. or more.

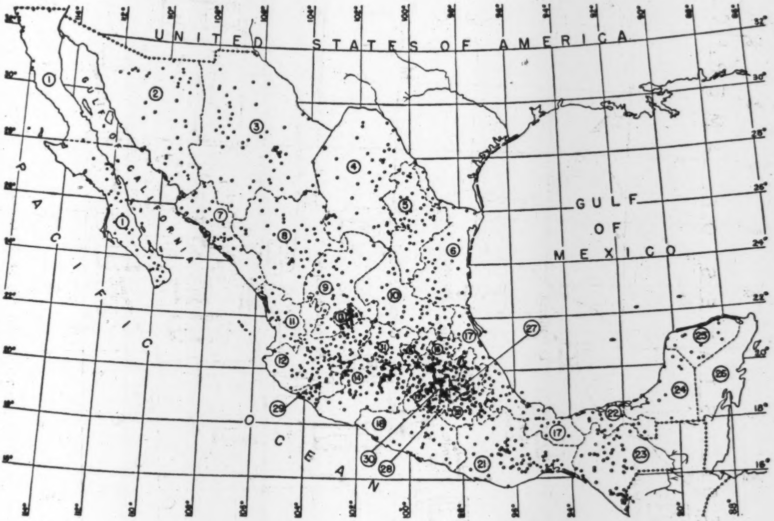
RAINFALL BY HYDROGRAPHIC REGIONS

With the foregoing descriptions in mind, Mexico can be divided into a relatively large number of regions in terms of the difference in climate, as affected by altitude, mountains, and air currents. Mean annual rains for every region are summarized in Table 2, Cols. 4 and 5.

SURFACE RUNOFF

The amount of rainfall, its variations and distribution, both geographical and seasonal, combined with the geophysical factors of geographical position, topography, air masses, and fronts, as well as cyclonic disturbances, result in surface runoff properties more or less similar to those known by American civil engineers who have worked for hydraulic developments in the southwestern United States. In general, the characteristics of Mexican rivers are: (*a*)

TABLE 1.—THE UNITED STATES OF MEXICO: THEIR NAMES, OFFICIAL ABBREVIATIONS, AND RELATIVE DISTRIBUTION OF RAINFALL STATIONS


UNITED STATES OF AMERICA
GULF OF CALIFORNIA
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GULF OF MEXICO
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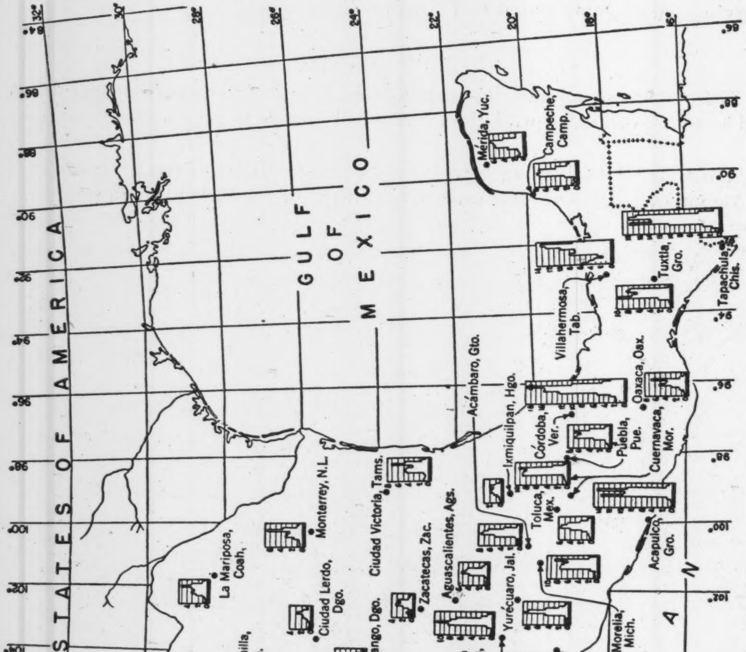
31

No. (1)	Name (2)	Abbr.[a] (3)	No. (1)	Name (2)	Abbr.[a] (3)	No. (1)	Name (2)	Abbr.[a] (3)
1	Baja California	B.C.	12	Jalisco	Jal.	22	Tabasco	Tab.
2	Sonora	Son.	13	Aguascalientes	Ags.	23	Chiapas	Chis.
3	Chihuahua	Chih.	14	Michoacan	Mich.	24	Campeche	Camp.
4	Coahuila	Coah.	15	Querétaro	Qro.	25	Yucatan	Yuc.
5	Nuevo Léon	N.L.	16	Hidalgo	Hgo.	26	Quintana Roo	Q.R.
6	Tamaulipas	Tams.	17	Veracruz	Ver.	27	Tlaxcala	Tlax.
7	Sinaloa	Sin.	18	Guerrero	Gro.	28	Morelos	Mor.
8	Durango	Dgo.	19	Mexico	Mex.	29	Colima	Col.
9	Zacatecas	Zac.	20	Puebla	Pue.	30	Distrito Federal	D.F.
10	San Luis Potosi	S.L.P.	21	Oaxaca	Oax.	31	Guanajuato	Gto.
11	Nayarit	Nay.						

[a] Official abbreviation


UNITED STATES OF AMERICA
GULF OF MEXICO
PACIFIC OCEAN
GULF OF CALIFORNIA
Mexicali, B.C.
Ensenada, B.C.
Altar, Son.
Carbo, Son.
El Aguila, Son.
El Oregano, Son.
Casas Grandes, Chih.
Navojoa, Son.
San Blas, Sin.
La Paz, B.C.
Culiacán, Sin.
El Palmito, Dgo.
La Boquilla, Chih.
La Mariposa, Coah.
Ciudad Lerdo, Dgo.
Durango, Dgo.
Monterrey, N.L.
Ciudad Victoria, Tams.
Zacatecas, Zac.
Aguascalientes, Ags.
Santiago Ixcuintla, Nay.
Guadalajara, Jal.
Yurécuaro, Jal.
Colima, Col.
Morelia, Mich.
Acámbaro, Gto.
Ixmiquilpan, Hgo.
Toluca, Mex.
Acapulco, Gro.
Córdoba, Ver.
Puebla, Pue.
Cuernavaca, Mor.
Oaxaca, Oax.
Villahermosa, Tab.
Tuxtla, Gro.
Tapachula, Chis.
Mérida, Yuc.
Campeche, Camp.
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98°
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106°
108°
110°
112°
114°

FIG. 4.—MONTHLY DISTRIBUTION OF RAINFALL, IN INCHES

Intermittent currents with the highest percentage of runoff occurring during summer months; (*b*) very small or no discharge in the low-water season; (*c*) very rapid floods with great discharges; and (*d*) long periods of drought during several years, in many basins in the Northern Central Plateau and in the northwestern region. There are only three or four navigable rivers emptying into the Gulf of Mexico; and, in the case of the remainder of the rivers, it is not possible to hope for perennial discharges.

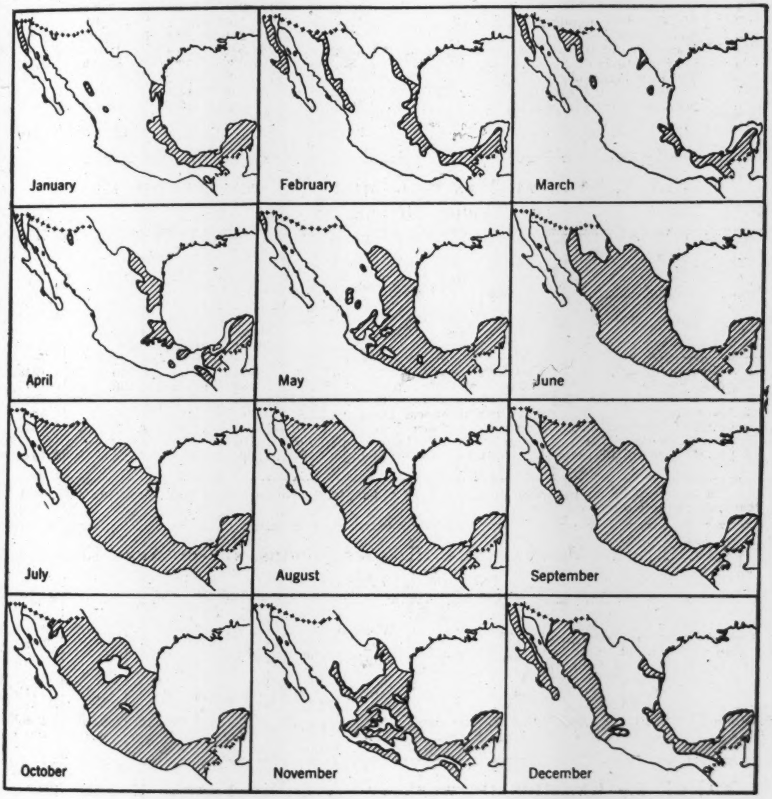

FIG. 5.—GEOGRAPHICAL DISTRIBUTION OF MONTHLY RAINFALL

In the states of Sonora and Sinaloa there are years in which the winter rainfall produces great flood conditions in all the rivers, and maximum discharges have been observed in these winter months.

To measure the surface runoff of its rivers, Mexico has established (through its Water Resources Ministry) an hydrometric service with nearly 450 mea-

TABLE 2.—Rainfall and Runoff in Mexico

No.	Region	Area, in square miles	Mean Annual Rainfall		Mean Annual Runoff	
			Volume, in thousands of acre-feet	Depth, in inches	Volume, in thousands of acre-feet	Percentage
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	Northwestern	183,480	174,300	17.8	17,820	12.1
2	Mid-Pacific (including Balsas basin)	126,300	226,190	33.6	21,170	14.4
3	South Pacific	36,300	97,280	50.3	6,680	4.5
4	Northeast (including Rio Grande in Mexico)	149,870	184,840	23.1	12,840	8.7
5	Gulf of Mexico	83,820	297,530	66.6	85,680	58.3
6	Yucatan Peninsula	47,130	136,200	54.2	...	...
7	North Central Plateau	21,250	16,210	14.3	1,720	1.2
8	South Central Plateau	107,770	99,960	17.4	1,170	0.8
	Total	755,920	1,232,510	30.6	147,080	100.0

TABLE 3.—Annual Maximum, Minimum, and Average Runoff of Typical Rivers in Mexico

No.[a]	Region	River and Site	Drainage area, in square miles	Annual Runoff in Thousands of Acre-Feet			Period
				Maximum	Minimum	Average	
1	Northwestern	Yaqui at Tecori	28,420	3,800	1,180	2,240	1929–1945
2	Mid-Pacific	Culiacan at Culiacan	6,290	5,090	1,300	3,100	1924–1945
3	South Pacific	Tehuantepec at Las Cuevas	3,480	2,690	290	1,070	1936–1945
4	Northeast	San Juan at Azucar Dam	11,420	3,390	330	990	1924–1943
5	Gulf of Mexico	Tonto at Temascal Dam[b]	1,270	8,700	2,340	4,490	1927–1948
7	North Central Plateau	Nazas at Palmito Dam	6,990	1,970	270	1,070	1929–1945
8	South Central Plateau	Lerma at Tepuxtepec Dam	2,400	1,189	235	544	1932–1945

[a] See corresponding numbers in Table 2. [b] Estimated with annual rainfall, drainage area, and 2 years of complete observed data.

TABLE 4.—Mean Annual Runoff (Thousands of Acre-Feet) in the United States

Description	North Atlantic	South Atlantic and Gulf	Saint Lawrence	Mississippi	Western Gulf	Colorado	Great Basin	Columbia	California	Total
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Runoff	94,110	112,940	16,510	416,280	25,920	13,030	2,360	169,400	38,370	888,920
%	10.6	12.7	1.9	46.8	2.9	1.5	0.2	19.1	4.3	100.0

TABLE 5.—Relation Between the Average Annual Runoff for the United States and Mexico

Table	Country	Area, in square miles	Runoff, in thousands of acre-feet	Unit runoff, in acre-feet per square mile
(1)	(2)	(3)	(4)	(5)
2	Mexico	755,930	147,060	194.541
4	United States	3,026,790	888,920	293.684
$\frac{\text{United States}}{\text{Mexico}}$		4.00	6.04	1.51

surement stations, located in basins with an area equivalent to 40% of the total land area of Mexico (see Table 2). An idea of the annual variation in some of the typical rivers is given in Table 3.

Data on the hydraulic surface resources of the United States are presented in Table 4.[5] Table 5 indicates that the United States, with an area

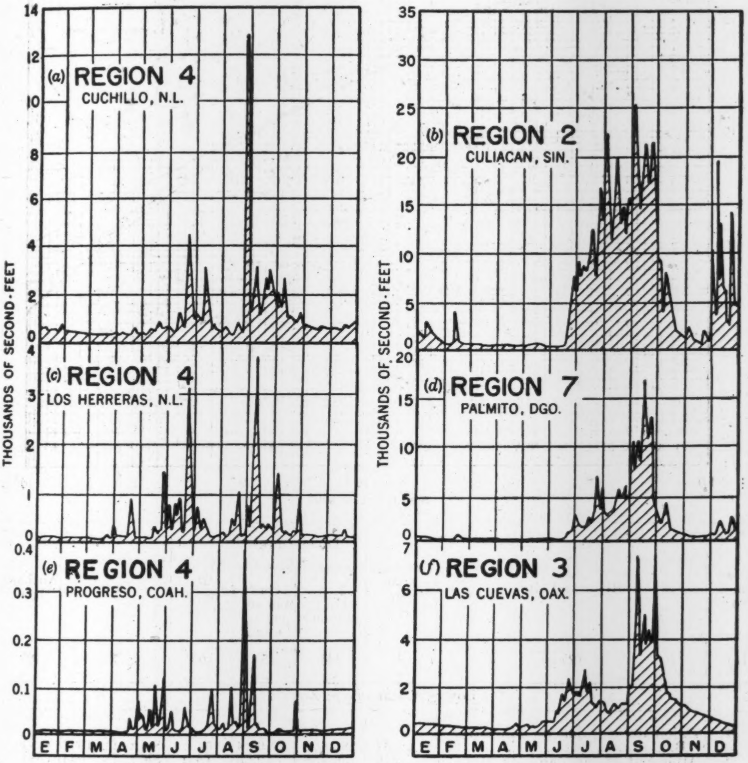

Figure	River	Gaging station[a]	Period of record	Area[b]
a	San Juan.	Cuchillo, N.L.	1934–1943	3,570
b	Culiacan.	Culiacan, Sin.	1934–1943	6,290
c	San Juan.	Los Herreras, N.L.	1940–1943	6,900
d	Nazas.	Palmito, Dgo.	1934–1943	6,990
e	Nadadores.	Progreso, Coah.	1937–1943	6,060
f	Tehuantepec.	Las Cuevas, Oax.	1936–1943	3,480

[a] See Table 2. [b] Drainage area, in square miles.

FIG. 6.—HYDROGRAPHS OF SOME TYPICAL MEXICAN RIVERS

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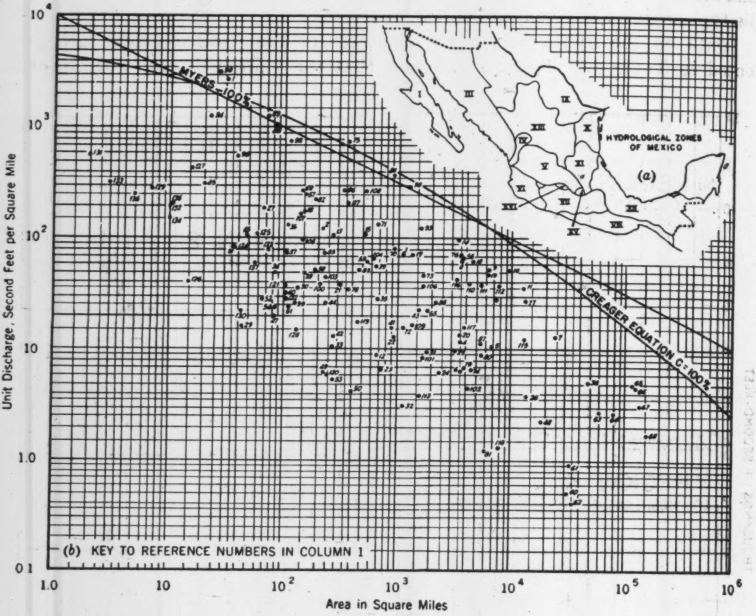
[5] "Floods; Their Hydrology and Control," by H. K. Barrows, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1948, p. 52.

TABLE 6.—MAXIMUM DISCHARGES OBSERVED IN MEXICO



Reference No.[a]	River and site	Drainage area (sq miles)	Discharge[b] (Cu ft per sec)	Discharge[b] (Cu ft per sec per sq mile)	Myers rating, *p* (%)	Date	Authority[c]
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
	(*c*) LOWER CALIFORNIA, ZONE I[d]						
1	Cajoncito at Cajoncito dam site, B. C.	31	85,602	27,61.4	154.0	Sept. 11, 1941	CNI
2	Mulejé at Mulejé dam site, B. C.	216	26,486	122.6	18.0	Sept. 13, 1941	CNI
	(*d*) NORTH PACIFIC, ZONE III						
3	Altar, Altar, Son.	1,120	81,223	72.5	24.3	Aug. 2, 1939	CNI
4	Sonora, El Orégano, Son.	3,508	40,859	11.6	6.9	Sept. 27, 1943	CNI
5	Bavispe, La Angostura Dam, Son.	6,988	73,525	10.5	8.8	Sept. 8, 1932	CNI
6	Papigochic, La Junta, Chih.	3,552	22,743	6.4	3.8	July 25, 1938	CNI
7	Yaqui, El Aguila, Son.	24,749	324,187	13.1	20.6	Dec. 24, 1914	CNI
8	Mayo, Los Mezcales, Son.	3,745	240,139	64.1	39.3	Dec. 24, 1914	CNI
9	Fuerte, Sn. Francisco, Sin.	6,332	324,893	51.3	41.9	1919	CNI
10	Fuerte, Huites, Sin.	9,861	507,681	51.5	51.2	Dec. 9, 1943	CNI
11	Fuerte near San Blas, Sin.	12,841	447,611	34.9	39.5	Dec. 9, 1943	CNI
12	Ocoroni Creek near Naranjo, Sin.	695	61,306	8.8	23.3	Oct. 11, 1943	CNI
13	Cabrera Creek near Zopilote, Sin.	274	28,923	105.6	17.5	Oct. 11, 1943	CNI
14	Sinaloa at Jaina dam site, Sin.	3,421	328,425	96.0	56.2	Dec. 9, 1943	CNI
15	Mocorito near Guamuchil, Sin.	517	54,737	105.9	24.1	Sept. 26, 1943	CNI
16	Humaya near Palos Blancos, Sin.	4,734	294,523	62.2	42.9	Dec. 9, 1943	CNI
17	Tamazula at Sanalona dam site, Sin.	1,390	98,881	71.1	19.1	Oct. 8, 1945	CNI
18	Bledal Creek at Bledal, Sin.	150	27,404	182.7	22.4	Dec. 9, 1943	CNI
19	Culiacán, at Culiacán, Sin.	6,283	388,460	61.8	49.1	Dec. 9, 1943	CNI
20	San Lorenzo near Santa Cruz, Sin.	3,549	79,281	22.3	13.3	....	CNI

TABLE 6.—(*Continued*)

Reference No.[a]	River and site	Drainage area (sq miles)	Discharge[b]		Myers rating, p (%)	Date	Authority[c]
			(Cu ft per sec)	(Cu ft per sec per sq mile)			
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
	(*e*) San Pedro River Basin, Zone IV						
21	Santiago near Refugio Salcido, Dgo.	320	12,007	37.5	6.7	Oct., 1943	CNI
22	El Tunal near San Felipe, Dgo.....	937	11,936	12.7	3.9	Oct., 1943	CNI
23	La Sauceda at Peña Blanca, Dgo....	727	4,873	6.7	1.8	Sept., 1944	CNI
	(*f*) Lerma–Santiago River Basin, Zone V						
24	Lerma at Tepuxtepec Dam, Mich..	2,400	14,691	6.1	3.0	Aug., 1935	CNI
25	Lerma at Solis Dam, Gto..........	3,239	31,077	9.6	5.5	Oct. 12, 1941	CNI
26	Lerma at Yurecuaro, Mich.........	13,827	51,277	3.7	4.3	Sept., 1926	CNI
27	Tarandacuao near Tarandacuao, Gto..........................	69	13,102	189.9	15.8	July, 1931	CNI
28	Tigre Creek near Munguía, Gto.....	154	7,663	49.8	6.2	Sept., 1941	CNI
29	Casa Blanca near Casa Blanca, Mich........................	43	706	16.4	1.1	Sept., 1934	CNI
30	Tlalpujahua near Pateo, Mich.....	135	4,732	35.1	4.1	July, 1935	CNI
31	Laja at Begonia dam site, Gto......	1,815	16,880	9.3	4.0	Sept., 1944	CNI
32	Turbio at Las Adjuntas, Gto......	1,168	3,567	3.1	1.0	Sept., 1944	CNI
33	Duero at Zamora, Mich...........	272	2,860	10.5	1.7	Sept., 1939	CNI
34	Tlazazalca at Urepetiro, Mich.....	85	4,202	49.4	4.6	Sept., 1943	CNI
35	Lagos at Cuarenta Dam, Jal.......	695	19,635	28.3	7.5	Sept., 1944	CNI
36	Teocaltiche near Calera, Jal.......	113	14,867	131.6	14.0	July 13, 1937	CNI
37	Del Valle, near Valle de Guadalupe, Jal.........................	112	8,299	74.1	7.8	June 23, 1941	CNI
38	Santiago at Yago, Nay...........	47,876	239,114	5.0	10.9	Oct., 1943	CNI
	(*g*) Midpacific, Zone VI						
39	Ayuquila near Tacotán, Jal........	658	37,645	57.2	14.7	Sept., 1944	CNI
40	Tapalpa at El Nogal, Jal..........	113	3,531	31.2	3.3	Aug., 1944	CNI
41	Tuxpan near Quito, Jal...........	900	13,702	15.2	4.6	Sept., 1944	CNI
	(*h*) Balsas River Basin, Zone VII						
42	Atoyac near El Carmen, Pue......	291	3,602	12.4	2.1	July, 1940	CNI
43	Atoyac at Tejaluca, Pue..........	1,559	34,926	22.4	8.9	1919	CNI
44	Zahuapan at Panotla, Tlax........	242	6,321	26.1	4.1	Sept. 23, 1941	CNI
45	Tembembe at Tembembe, Mor....	49	5,368	109.6	7.7	June, 1935	CNI
46	Cotija, at Cotija, Mich...........	20	6,109	305.5	13.7	July, 1936	CNI
47	Quitupan at Quitupan, Jal........	85	1,660	19.5	1.8	Sept., 1944	CNI
48	Balsas at Mezcala, Gro...........	18,218	41,777	2.3	3.1	Oct. 5, 1941	CFE
49	Valle de Bravo at Valle de Bravo, Mich.........................	232	1,413	6.1	0.9	Aug. 21, 1942	CFE
50	Malacatepec at Malacatepec, Mich.	405	1,660	4.1	0.8	Sept. 27, 1943	CFE
51	Verde at Station No. 4...........	116	2,790	24.1	2.6	June 28, 1943	CFE
52	Ixtapan del Oro at Station No. 7....	66	1,872	28.4	2.3	Sept. 5, 1943	CFE
53	El Salitre at Station No. 5, Mich....	280	1,519	5.4	0.9	Sept. 25, 1943	CFE
54	Cupatitzio at Station No. 9, Mich...	85	1,942	22.8	2.1	July 3, 1941	CFE
	(*i*) South Pacific, Zone VIII						
55	Tehuantepec near Nejapa, Oax.....	1,795	38,846	21.6	9.2	Sept., 1944	CNI
56	Tehuantepec at Tehuantepec, Oax.........................	3,660	247,202	67.5	40.9	Sept., 1944	CNI
57	Los Perros Creek at Ixtepec, Oax. ..	154	38,493	250.0	31.0	Sept., 1944	CNI
58	Chicapa near Union Hidalgo, Oax...	193	10,065	52.2	7.2	Sept., 1944	CNI
59	Suchiate at Paso de Guarumaro, Chis.........................	564	34,290	60.8	14.4	Sept. 22, 1944	CNI

TABLE 6.—(*Continued*)

Reference No.[a]	River and site	Drainage area (sq miles)	DISCHARGE[b]		Myers rating, p (%)	Date	Authority[c]
			(Cu ft per sec)	(Cu ft per sec per sq mile)			
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
	(*j*) RIO GRANDE BASIN IN MEXICO AND INTERNATIONAL LINE, ZONE IX						
60	Grande at Fort Quitman, Tex.	31,990	16,986	0.5	0.9	June 20, 1905	IBC
61	Grande at La Nutria, Tex.	33,924	30,017	0.9	1.6	....	IBC
62	Grande at Presidio Arriba, Tex.	35,000	15,185	0.4	0.8	June 12, 1912	IBC
63	Grande at Presidio Abajo, Tex.	59,767	162,094	2.7	6.6	Sept. 11, 1904	IBC
64	Grande at Langtry, Tex.	79,298	204,118	2.6	7.3	June 18, 1922	IBC
65	Rio Grande at Del Rio, Tex.	121,513	600,347	4.9	11.3	Sept. 1, 1932	IBC
66	Rio Grande at Eagle Pass, Tex.	125,261	568,563	45.	16.1	Sept. 2, 1932	IBC
67	Rio Grande at Laredo, Tex.	130,616	402,585	3.1	10.9	Sept. 3, 1932	IBC
68	Rio Grande at Zapata, Tex.	154,184	261,327	1.7	6.7	Sept. 4, 1932	IBC
69	Las Vacas Creek at Villa Acuña, Coah.	146	39,199	268.5	32.5	April 5, 1940	IBC
70	San Diego near Jimenez, Coah.	931	75,220	80.8	24.7	Sept. 18, 1941	IBC
71	San Rodrigo, near El Mural, Coah.	586	81,223	138.6	33.6	Sept. 7, 1932	IBC
72	Escondido at Villa de Fuerte, Coah.	1,130	17,693	15.7	5.3	May 14, 1935	IBC
73	Alamo near Ciudad Mier, Tams.	1,675	76,632	45.8	18.8	Sept. 7, 1933	IBC
74	Salinas at Cienega de Flores, N. L.	454	30,194	6.6	4.5	May, 1941	CNI
75	Station Catarina at Monterrey, N. L.	366	278,031	759.6	145.6	Aug. 26, 1909	ENR
76	San Juan at El Cuchillo, N. L.	3,578	252,922	70.7	42.3	Aug. 29, 1938	CNI
77	San Juan at Santa Rosalia, Tams.	13,000	353,145	27.2	31.0	Aug. 30, 1909	IBC
78	Pilon at Montemorelos, N. L.	366	127,132	34.7	66.6	Aug. 30, 1938	CNI
79	San Pedro at Villalba, Chih.	3,838	26,521	6.9	4.3	Sept., 1942	CNI
80	Sabinas at Sabinas, Coah.	5,471	48,663	8.9	6.6	Aug., 1944	CNI
81	Nadadores near Progreso, Coah.	6,027	7,098	1.2	0.9	Sept., 1944	CNI
82	Parral, at Parral, Chih.	187	42,377	226.6	31.0	Sept. 8, 1944	CNI
83	Florido near San Gabriel, Dgo.	480	24,720	51.5	11.3	Sept. 7, 1944	CNI
84	Florido near San Antonio, Dgo.	913	33,549	367.5	11.1	Sept. 7, 1944	CNI
85	Primero near Mouth Rio Florido, Dgo.	230	17,657	76.8	10.5	Sept. 7, 1944	CNI
86	Florido near Corrales, Chih.	2,174	56,503	26.0	12.1	Sept. 8, 1944	CNI
	(*k*) NORTH OF GULF OF MEXICO, ZONE X						
87	San Fernando near San Fernando, Tams.	5,452	61,730	11.3	8.4	Aug., 1938	CNI
88	Camacho at Bridge in Road No. 1, N. L.	86	91,217	1,060.7	98.5	Aug., 1938	DNC
89	Ramos at Bridge in Road No. 1, N. L.	71	89,875	1,265.8	106.7	Aug., 1938	DNC
90	Blanquillo at Bridge in Road No. 1, Tams.	27	86,379	3,199.2	166.4	Aug., 1938	DNC
91	Lajas at Bridge in Road No. 1, Tams.	36	3,037	84.4	5.1	Aug., 1938	DNC
92	Linares near Linares	119	90,052	756.7	82.6	Aug., 1938	DNC
93	Purificación near Barretal, Tams.	1,559	193,170	123.9	49.0	Aug., 1938	CNI
94	La Presa at Bridge in Road No. 1, Tams.	22	27,369	1,244.0	58.5	Aug, 1938.	DNC
95	San Marcos at Bridge in Road No. 1 Tams.	39	22,601	579.5	36.2	Aug., 1938	DNC
96	Santa Engracia at Road No. 1., Tams.	339	92,171	271.9	50.1	Aug., 1938	DNC
97	Cabezones at Bridge in Road No. 1 Tams.	354	73,666	208.1	39.2	Aug., 1938	DNC
98	Guayalejo at Bridge in Road No. 1 Tams.	1,274	368,330	289.1	103.2	Aug., 1938	DNC
	(*l*) PANUCO RIVER BASIN, ZONE XI						
99	Tepeji at Taxhimay Dam, Hgo.	125	3,249	26.0	2.9	Oct., 1941	CNI
100	Tepeji at Tepeji, Hgo.	212	8,087	38.1	5.6	Sept., 1936	CNI
101	Tula near Binola Ranch, Hgo.	1,674	13,985	8.4	3.4	July, 1937	CNI
102	Tula near Ixmiquilpan, Hgo.	4,161	18,893	4.5	2.9	Oct., 1941	CNI
103	Tlautla near Jasso, Hgo.	239	10,736	44.9	6.9	July, 1938	CNI
104	Metztitlán near Venados Ranch, Hgo.	620	42,377	68.4	17.0	Oct., 1930	CNI
105	San Ildefonso at San Ildefonso Dam, Mex.	149	14,867	99.8	12.1	Aug., 1937	CNI
106	Guayalejo at San Gabriel dam site, Tams.	1,660	59,752	36.0	15.1	June, 1944	CNI

TABLE 6.—(*Continued*)

Reference No.[a]	River and site	Drainage area (sq miles)	Discharge[b]		Myers rating, *p* (%)	Date	Authority[c]
			(Cu ft per sec)	(Cu ft per sec per sq mile)			
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
	(*m*) GULF OF MEXICO, ZONE XII						
107	Tomatan at Station No. 8½, Ver.	145	24,438	168.5	20.3	Sept. 22, 1944	CFE
108	Los Bobos at Martinez de la Torre, Ver.	532	141,258	265.5	61.3	Sept. 22, 1944	CNI
109	La Antigua at Villa Cardel, Ver.	1,342	22,495	16.8	6.1	July, 1936	CNI
110	Papaloapan at Quiotepec, Oax.	4,226	165,625	39.2	25.5	Sept. 22, 1944	CP
111	Papaloapan at Tuxtepec, Oax.	5,647	206,590	36.6	27.5	Sept. 24, 1944	CP
112	Papaloapan at Papaloapan, Oax.	7,834	284,282	36.3	32.2	Sept. 24, 1944	CP
	(*n*) NORTH CENTRAL PLATEAU, ZONE XIII						
113	Casas Grandes near Casas Grandes, Chich.	1,630	6,286	3.9	1.6	Sept. 30, 1941	CNI
114	Nazas at Palmito dam site, Dgo.	7,431	216,478	29.1	25.1	Sept. 8, 1944	CNI
115	Nazas at Fernández Canyon, Dgo.	12,560	151,358	12.1	13.5	Sept, 1917	CNI
116	El Oro, near Jesús Maria Canyon, Dgo.	3,293	134,548	40.9	23.4	Sept. 8, 1944	CNI
117	Ramos at Potrerillo Canyon, Dgo.	3,911	61,271	15.7	9.8	Sept. 8, 1944	CNI
118	Aguanaval near La Flor, Dgo.	7,942	10,206	1.3	1.1	Sept., 1941	CNI
119	Los Lazos at El Sauz, Zac.	472	8,264	17.5	3.8	Sept., 1943	CNI
	(*o*) CUITZEO LAKE BASIN, ZONE XIV						
120	Grande de Morelia at Cointzio Dam, Mich.	243	1,448	6.0	0.9	Aug., 1931	CNI
121	Querendaro near Querendaro, Mich.	106	3,920	37.0	3.8	Aug., 1941	CNI
	(*p*) VALLEY OF MEXICO, ZONE XV						
122	Cuautitlán at Guadalupe Dam, Mex.	110	3,108	28.3	3.0	July, 1937	SCOP
123	Remedios at Molino Blanco, Mex.	75	5,933	79.1	6.9	May, 1939	SCOP
124	Tlalnepantla near Madin, Mex.	37	3,108	84.0	5.1	Oct., 1936	SCOP
125	Hondo at Rio Hondo, Mex.	59	6,710	113.7	8.7	June, 1941	SCOP
126	Chico de los Remedios at Cristo Ranch, Mex.	15	600	40.0	1.6	June, 1935	SCOP
127	Texcoco near Texcoco, Mex.	16	6,957	434.8	17.4	July, 1938	CNI
128	Ameca near Temamatla, Mex.	135	2,013	14.9	1.7	Sept., 1941	SCOP
129	Guadalupe Creek near Atlamaya, D. F.	7	1,942	277.4	7.4	Sept., 1942	SCOP
130	Magdalena at Anzaldo Dam, D. F.	42	918	21.9	1.4	Sept., 1933	SCOP
131	Texcalatlaco at Texcalatlaco Dam, D. F.	2	1,165	582.5	8.2	Sept., 1933	SCOP
132	Piedad at Colonia del Valle, D. F.	10	2,013	201.3	6.4	Sept., 1938	SCOP
133	Becerra near Viaducto, D. F.	3	953	317.7	5.5	Sept., 1935	SCOP
134	Totolica near Totolica, D. F.	10	1,519	151.9	4.8	July, 1941	SCOP
135	Tacubaya at Belem, D. F.	5	1,236	247.2	5.5	Sept., 1933	SCOP
136	Mixcoac at Mixcoac, D. F.	11	2,295	208.6	6.9	July, 1937	SCOP
137	Churubusco at Xoco, D. F.	58	3,496	60.3	4.6	July, 1931	SCOP

[a] See numbers in the key, inset Table 6(*b*). [b] Maximum instantaneous discharge. [c] Authorities: CNI—Comisión Nacional de Irrigación (National Commission of Irrigation); CFE—Comisión Federal de Electricidad (Federal Commission of Electricity in Mexico); IBC—International Boundary Commission, United States and Mexico; ENR—*Engineering News-Record;* DNC—Dirección Nacional de Caminos (National Direction of Roads); CP—Comisión del Río Papaloapan (Papaloacpan River Commission); and SCOP—Secretaria de Comunicaciones y Obras Públicas (Communications and Public Works Ministry).
[d] Inset Table 6(*a*) is the key for zone numbers.

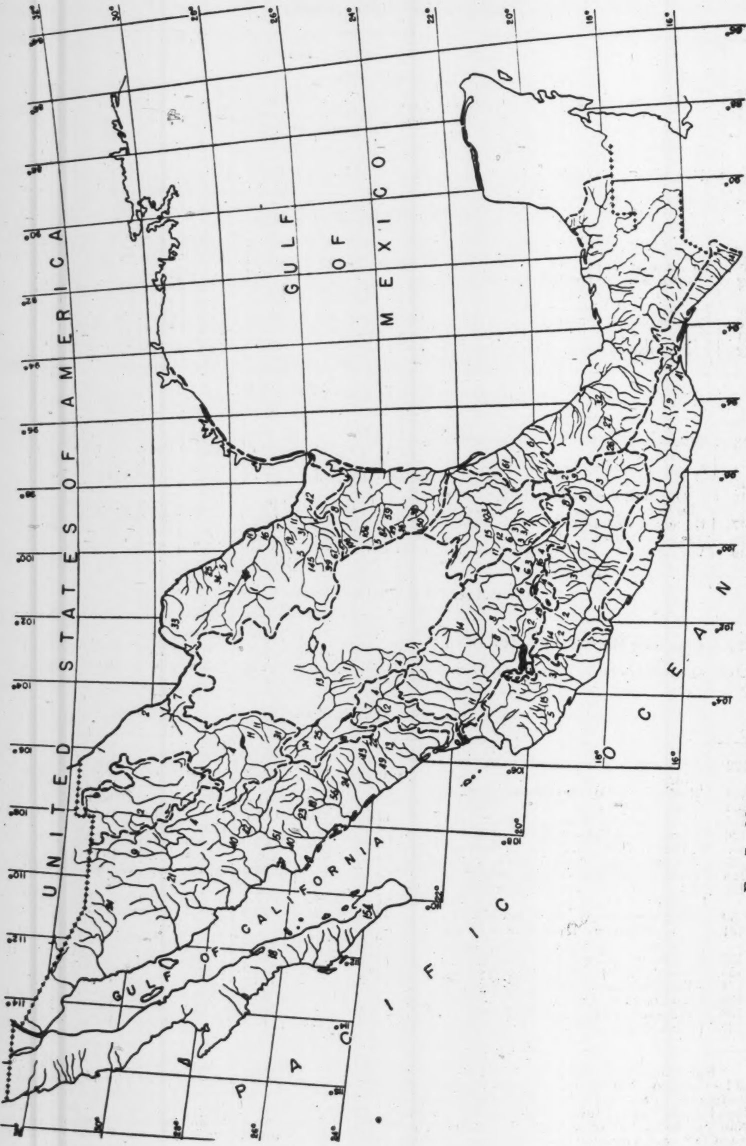

UNITED STATES OF AMERICA
GULF OF MEXICO
GULF OF CALIFORNIA
PACIFIC OCEAN
32°
30°
28°
26°
24°
22°
20°
18°
16°

Fig. 7.—Maximum Flood-Flow Percentage Ratings, p, in the Myers Formula

four times larger than that of Mexico, has more than six times its average annual surface runoff, and that the runoff unit area is one and one-half times larger.

RIVER DISCHARGES

An estimate of the variation of discharge in Mexican rivers is possible by reference to Fig. 6, which includes mean hydrographs of some of the principal rivers. The great concentration of runoff in summer months occurs from May or June to September or October, which period accounts for more than 75% of the annual yield.

MAXIMUM DISCHARGES

Although, in the northwestern part of Mexico, maximum observed discharges have occurred in winter months, in the remainder of the nation the major discharges and floods are observed in summer months, with the appearance of humid winds and air masses. Even tropical cyclones are recorded in August and September. The data in Table 6 were prepared to show the unit discharge for the various drainage areas and for each of the regions into which Mexico has been divided. Fig. 7 shows the rating percentages—

$$p = \frac{Q}{100\sqrt{M}} \qquad \text{(1)}$$

—in the Myers formula[6] of flood flows in Mexico, for comparison with other world areas. In Eq. 1, in addition to the rating percentage p, the flow Q is in cubic feet per second; and M is the area in square miles.

It is important to emphasize that the maximum discharges ever recorded in Mexico came from the rivers in the northeast region which originated in the storms produced by cyclonic disturbances. The next discharges, in order of maximum intensity came from the northwestern rivers where the humid air masses coming from the Pacific Ocean strike the sierra and deliver rainfalls of the cloudburst type. The intermountain plateau is more or less protected against violent storms such as those observed in the coastal regions and the maximum discharges are relatively minor.

ACKNOWLEDGMENTS

The revised text and material presented in this paper were prepared by the writer after presentation to the ASCE Mexico City Meeting in July, 1949, before the Irrigation Division. The subject of the paper is part of the writer's work as chief of the Hydrology Section of the Water Resources Ministry of Mexico, of which Adolfo Orive Alba is the secretary. Aurelio Benassini, Assoc. M. ASCE, is chief engineer of the River Control and Irrigation Department. In preparing this paper, and in collecting data, the writer has been assisted by numerous persons without whose help it could not have been completed.

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[6] "Hydrology," Oscar E. Meinzer, editor, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1942, p. 535.

DISCUSSION

ANDREW WEISS,[7] Hon. M. ASCE.—A concise and complete description of the hydrologic resources of the Republic of Mexico is presented in this paper, thus affording a comparison of these resources with those of the United States. A careful reading of the paper will reveal that Mexico can never hope to rank very high as a producer of agricultural wealth, and that it is in need of developing its agriculture to as high a degree as possible. It is a country blessed with a magnificent climate that will always attract people seeking such comforts as the State of California provides in the United States. Similar to California, the areas of most attractive climate in Mexico have resulted in a certain shift in the density of population. For these reasons the unit cost of developing irrigation and colonization resources in suitable climates may reach costs considerably higher than the average, a fact which has already been demonstrated in Mexico. It is to be expected that many parts of Mexico will attract a select type of colonist, even from outside its border, who may wish to enjoy the healthful environment. Of the several interesting topics suggested by this paper, seven may be emphasized as follows:

(1) Mexico occupies a relatively dry zone on the face of the globe, dry the entire year between latitudes 19°N to 31°N. More than 75% of the nation is in this belt; and the wettest region is to be found in zone XII (Item No. 5, Table 2). Mexico may be classified as not rainy, despite the fact that tropical rains occur in some southeastern regions.

(2) The orography and geologic structure of Mexico is the cause of the reduced runoff, a condition that is especially noticeable in the preponderant limestone structure so prevalent in the central plateau and the "carstic" regions distributed throughout the republic.

(3) Attention is directed to the relatively short records of river measurements—generally a period of about twenty years, which covers the period during which runoff records have so far been obtained. Doubtless, in the future, there will be some surprises in the occurrence of both maximum and minimum observations.

(4) In general, the Republic of Mexico is not blessed with any large percentage of good agricultural land, such as that which occurs in abundance in large regions of the southern, eastern, and midwestern parts of the United States. Therefore, it will always be necessary to husband the resources represented by its waters and soils so as to develop an agriculture sufficient to support the population which its climatic resources will always invite.

(5) It is clear from Table 3 that storage of the flood waters is an unavoidable necessity in Mexico if its hydraulic resources are to be utilized in conformity with its climatic conditions.

(6) A certain increase of agricultural production may be obtained by pumping the shallow water table over relatively large areas in the so-called Bajio region, an extensive agricultural area in the central part of the republic.

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[7] Cons. Engr., Dept. of Water Resources, Mexico, D. F., Mex.

Large tracts of the present irrigable area in the region of Lower California are now covered by a network of canals for irrigation and drainage in the section watered by the international diversion dam south of Yuma, Arizona. Increased agricultural production is possible in this region. Similarly, most of the agricultural lands on the Yucatan Peninsula must be irrigated by pumping.

(7) A large part of the irrigated and irrigable areas of the republic is located on the flat areas that link the Sierra Madre Occidental with the Gulf of California. Development in this region will require considerable improvement in transportation facilities before products can be moved to the central regions where the principal markets are located.

Milton O. Schmidt,[8] Assoc. M. ASCE.—An admirable attempt to treat the broad subject of the hydrology of Mexico within the compass of a few pages has not been fully successful in this paper. It still is incomplete, and it lacks the quality of conclusiveness. Its cardinal purpose, as stated in the Synopsis, is "* * * to explain why water * * * is so scarce" in Mexico. In support of this statement the author needs to name the factors that control and affect the disposition of precipitation. In short, he has presented rainfall and runoff data but has cast no light on the effects exercised by such factors as geology, evaporation in its relation to temperature, soil, vegetation, and ground slopes. Hence, one can make only vague conjectures as to why many Mexican streams are not perennial and why the average yield from the nation's rivers is rather small. Explanations of the rainfall-runoff relation are the core of applied hydrology, and this essential point has not been covered by the author.

Tables 2 to 5 need comment, for the conclusions reached by the author in connection with their use are subject to question. Table 2 contains no explanation of the absence of runoff data for the Yucatan Peninsula. Is it to be inferred that no gaging stations are operative here or that the topography consists of "sinks" that make customary surface drainage impossible? Nevertheless, runoff data for some 47,130 sq miles are missing. The author does not reveal how the discharges observed at "* * * nearly 450 measurement stations, located in basins with an area equivalent to 40% of the total land area of Mexico * * *" have been translated into mean annual runoff for the entire country containing 755,920 sq miles. The total mean annual runoff of 147,080,000 acre-ft is readily converted to inches of depth over the entire area and equals 3.6 in. The use of inches of depth to express runoff is much more indicative of relative magnitude than is the use of acre-ft and much more readily compared with inches of depth of rainfall.

The author has used data from "Floods; Their Hydrology and Control"[5] and has failed to note that its author, H. K. Barrows, M. ASCE, attributes the total mean annual runoff for the United States to only 2,428,123 sq miles or to about 80% of the total area (3,000,000 sq miles). More important, Mr. Garcia-Quintero has used a source of information which contains numerous errors.[9] It is to be hoped that, in the closing discussion, the author will reconcile his paper by reference to the original water supply papers or to other studies prepared by the United States Geological Survey.

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[8] Associate Prof. of Civ. Eng., Univ. of Illinois, Urbana, Ill.

[9] "Floods; Their Hydrology and Control," by H. K. Barrows, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1948 (Col. 5, Table 4-6, change the mean daily flow from the St. Lawrence Basin from "22,800 cu ft per sec" to "228,000 cu ft per sec").

The work of Walter B. Langbein, Assoc. M. ASCE, and others reported in 1949,[10] reliably indicates that the country-wide average annual runoff is 8.5 in. The ratio of average annual runoff in the United States to that in Mexico is, therefore, $\frac{8.5}{3.6}$ or 2.35, not 1.51 as given by the author in Table 5.

The writer would feel grateful to Mr. Garcia for including in his closure some information about ground-water hydrology and its relation to dry-weather flow of streams in Mexico. The writer would also like some explanation as to why substantially the same mean annual rainfall in the United States and Mexico (30 in.) produces 8.5 in. and 3.6 in., respectively, of runoff in these two countries.

GERARD H. MATTHES,[11] HON. M. ASCE.—The engineering profession is indebted to the author for his informative review of hydrologic conditions in Mexico. Little has appeared in print on this subject in American literature, aside from a few records published in the Water Bulletins of the International Boundary Commission of the United States and Mexico. Therefore, his review is doubly welcome to American engineers. The author makes clear that the primary objective of Mexico's engineers in establishing rainfall and stream-flow measuring stations has been to obtain data for use in effecting the fullest practicable utilization of Mexico's scant water resources for agricultural purposes. Only 6.8% of Mexico receives rainfall that is sufficient (or adequately distributed seasonally) for maturing crops. The few Mexican rivers that have perennial flows are located in this 6.8% area, which is shown in Figs. 3, 4, and 5 to lie chiefly along the southern Gulf coast.

Since the author's tabulations did not afford space to show the periods covered by the records, the writer wishes to point out that, in the matter of stream-flow records, stations were established in Mexico as early as 1900 and 1901. However, it was not until 1926, when Mexico's National Irrigation Commission was created, that the study of its water resources received its main impetus. Many of the meteorologic and hydrometric stations listed by the author were established by this commission. At the end of 1940, 205 rainfall-measurement stations and 199 stream-gaging stations had been placed in operation, and by 1946 their number had grown to 235 and 239, respectively.[12] The Water Resources Ministry, which took over the work of the commission in 1946, continued the establishment of stations until by 1950, as stated by the author, 450 gaging stations were in operation in Mexico. They supply data for use in planning irrigation projects. In so far as the writer is aware, gaging stations for use in flood control projects, like the one on the Papaloapan River in the State of Veracruz, were not established until much later. In short, there are still large areas for which no hydrologic records are available. As Mexico is a country subject to cataclysmic floods which annually take a large toll of human lives, flood control projects in time will call for the establishment of additional gaging stations.

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[10] "Annual Runoff in the United States," by Walter B. Langbein and Others, *U. S. Geological Survey Circular No. 52*, Washington, D. C., June, 1949.

[11] Retired Cons. Hydr. Engr., New York, N.Y.

[12] "Labor de la Comisión Nacional de Irrigación en sus Veintiun Años de Vida," by Adolfo Orive Alba, "Irrigación en México," Vol. 27, No. 4, 1946, p. 13.

The author has wisely refrained from deducing relationships between rainfall and runoff for specific watersheds, as affected by evaporation and different soil types. Owing to the peculiar climate, topography, and disturbed geological formations characteristic of Mexico, academic studies of this kind are likely to be of small help in practical problems. One factor, which American engineers have to contend with in rare cases but one which affects Mexican hydrology in many sections, is the occurrence of large closed basins that contribute no visible surface runoff to any river system. To illustrate, the drainage area of the Rio Bravo ("Rio Grande" in the United States) within the rim of the Rio Grande basin measures 335,000 sq miles, of which 163,100 sq miles yield no runoff to the river. Some of these closed basins are situated in parts of Chihuahua, Coahuila, and Durango on the Mexican side; other closed basins exist in Texas and New Mexico on the United States side of the river. Whatever ground-water movement finds its way from these basins into stream channels is as yet largely conjectural.

The relatively short periods covered by most records and, more especially, the unpredictable character of flood-flow intensities were responsible for the adoption of the system of building earth dams with rock fills in such a manner as to be capable of withstanding overtopping by floods during construction.[13]

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[13] "Construction Technique of Passing Floods over Earth Dams," by Andrew Weiss, *Proceedings-Separate No. 40*, ASCE, Vol. 76, October, 1950.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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TRANSACTIONS

Paper No. 2464

UPLIFT PRESSURES IN CONCRETE DAMS

By Kenneth B. Keener,[1] M. ASCE

With Discussion by Messrs. Ross M. Riegel, William P. Creager, A. Warren Simonds, Fairfax D. Kirn, Serge Leliavsky Bey, A. H. Davison, Duff A. Abrams, and Kenneth B. Keener.

Synopsis

Measurements of uplift pressures in concrete dams constructed by the Bureau of Reclamation (USBR) of the United States Department of the Interior have been made continuously since 1926. The justification for providing methods for, and making, such measurements and the keeping of complete records are discussed. A description of the installation and location of equipment for observing foundation uplift is given. The frequency of the readings, the over-all results obtained, and the limitations of the procedure are portrayed. A case history is included for Hoover Dam in Arizona-Nevada. A brief recounting is given of the small amount of experimental work done to determine the uplift within the concrete. The methods, together with examples, of relieving excessive foundation uplift pressures are cited. Finally, a short recital is made of the uplift assumptions used in the design of concrete dams.

General

In the design of transverse cross sections of concrete gravity dams, in addition to resistance to overturning, at least two safety factors—f_s, a sliding factor, and f_τ, a shear-friction factor[2]—are considered of importance:

$$f_s = \frac{H}{W - U} \qquad (1)$$

and

$$f_\tau = \frac{(W - U)\,\phi + A_h\,\tau}{H} \qquad (2)$$

in which H is a horizontal force; W is the weight; U is the uplift; ϕ is the coefficient of internal friction; A_h is the horizontal area; and τ is the unit shear resistance.

Note.—Published in June, 1950, as *Proceedings-Separate No. 25*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Chf., Dams Div., Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo.

[2] "Stability of Straight Concrete Gravity Dams," by D. C. Henny, *Transactions*, ASCE, Vol. 99, 1934, p. 1041.

It is considered good and safe practice to keep the sliding factor f_s well below unity; and the shear-friction factor f_r should be five or greater. Noting the position of U, the "uplift," in Eqs. 1 and 2, one can readily determine the effect of that item in influencing the final cross section of the dam. If the assumed uplift is higher than necessary, it will be reflected in an unduly large cross section and a more costly dam. Conversely, if the assumed uplift is too low, undesirable encroachment will be made on the normally used values of the factors of safety.

Investigations of uplift pressures in dams completed by the USBR have consisted of procedures for making measurements along the contact between the concrete and the foundation rock, along construction joints in the mass concrete, and in the concrete between construction joints. Recently, some testing has been done in the USBR laboratory to determine the effect of uplift pressure on the shearing strength of concrete.

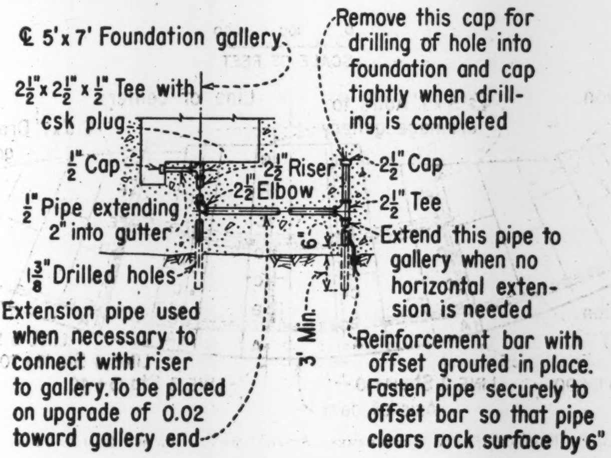

℄ 5'x 7' Foundation gallery
2½" x 2½" x ½" Tee with csk plug
½" Cap
½" Pipe extending 2" into gutter
1⅜" Drilled holes
Extension pipe used when necessary to connect with riser to gallery. To be placed on upgrade of 0.02 toward gallery end
2½" Riser
2½" Elbow
6"
3' Min.
Remove this cap for drilling of hole into foundation and cap tightly when drilling is completed
2½" Cap
2½" Tee
Extend this pipe to gallery when no horizontal extension is needed
Reinforcement bar with offset grouted in place. Fasten pipe securely to offset bar so that pipe clears rock surface by 6"

FIG. 1.—TYPICAL HOLE AND PIPE DETAILS, HUNGRY HORSE DAM IN MONTANA

The purpose of the investigations was twofold: (1) To determine what uplift assumptions should be made in the future in designing concrete dams; and (2) to observe regularly the uplift conditions at completed dams so that if uplift forces dangerously exceed the design assumptions, drainage or grouting, or both, may be accomplished to relieve such a condition.

INSTALLATIONS FOR OBSERVATION OF FOUNDATION UPLIFT

Beginning in about 1925 at American Falls Dam in Idaho, systems of vertical wells have been placed across the contacts of the bases of concrete dams with the foundation rock. The number of holes placed to date (1950) is 360 at fifteen dams, an average of 24 holes for each dam. The maximum number at any one dam was 60 at Marshall Ford in Texas, and the minimum was 6 at Olympus in Colorado. Grand Coulee in Washington has 38 holes and Shasta in California, 53 holes. At Hungry Horse in Montana 35 holes are being drilled and there will be 24 holes at Canyon Ferry in Montana.

The layout of foundation uplift pressure pipes at Hungry Horse Dam is used for purposes of illustration of all installations. Typical hole and pipe details are shown in Fig. 1. The holes, drilled approximately 3 ft into foundation rock and having a minimum diameter of $1\frac{5}{8}$ in., are not drilled until after the $2\frac{1}{2}$-in. standard black iron riser pipe has been embedded in concrete and until foundation grouting in the surrounding area has been accomplished. Before placing concrete the lower end of the riser pipe is anchored about 6 in. above the foundation rock, by tack welding it to an offset reinforcement bar grouted in place in the foundation. Uplift pressure pipes not located directly under foundation galleries are extended to them by horizontal and vertical piping, thus permitting all uplift pressure measurements to be made within the galleries. A $2\frac{1}{2}$-in. by $2\frac{1}{2}$-in. by $\frac{1}{2}$-in. tee with a $2\frac{1}{2}$-in. countersunk (csk) plug is installed at the top of each riser pipe immediately below the gallery floor. A $\frac{1}{2}$-in. pipe extends horizontally from the tee to the gallery gutter where it is capped.

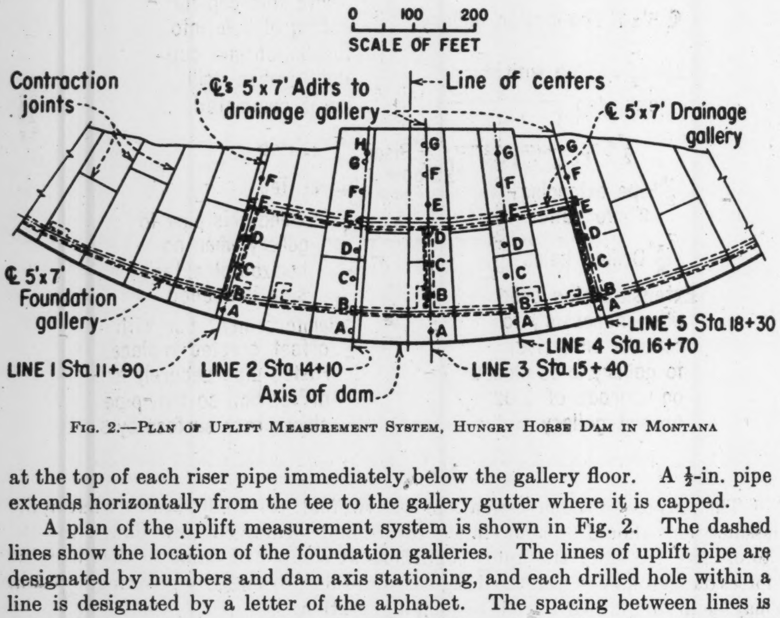
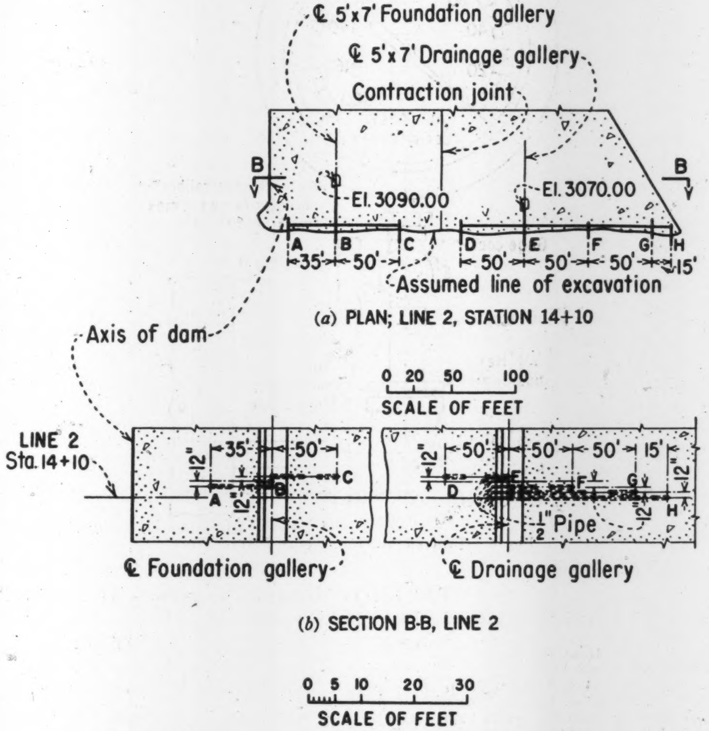


FIG. 2.—PLAN OF UPLIFT MEASUREMENT SYSTEM, HUNGRY HORSE DAM IN MONTANA

A plan of the uplift measurement system is shown in Fig. 2. The dashed lines show the location of the foundation galleries. The lines of uplift pipe are designated by numbers and dam axis stationing, and each drilled hole within a line is designated by a letter of the alphabet. The spacing between lines is determined partly by a study of the shape of the axis profile and partly by the geological characteristics of the foundation. Should all these be relatively uniform, nearly equal spacing is adopted. The spacings in this actual illustration may be noted as 220 ft, 130 ft, 130 ft, and 160 ft. The spacings between holes within a line are made nearly equal. The upstream and downstream holes are drilled within the range of from 10 ft to 50 ft from the finished faces of the dam at the foundation-concrete contacts. Naturally, the general purpose in locating uplift pressure holes is to cover the lowest areas of the foundation as well as those of possible geological weakness where the uplift pressures would be expected to be greatest. For instance, Hole H, Fig. 2, was located within a large fault area. The profile and plan of line 2 at Station 14+10, in Fig. 3, show the locations of the holes and piping in relation to other features of the lower part of the dam.

When the reservoir water surface back of a dam reaches an appreciable height, observers begin recording uplift pressures. Initially, the plug is removed and any water that may have seeped into the hole or connecting pipe during construction is removed by pumping, whereupon the plug is replaced. Altitude gages, calibrated in feet of water, are connected through a stopcock to the ½-in. pipes that terminate in the gallery gutter. However, the gages are omitted in the case of pipes which do not carry water or indicate pressure several days after the pumping out operation. Where the water surface in such


℄ 5'x7' Foundation gallery
℄ 5'x7' Drainage gallery
Contraction joint
B
B
El. 3090.00
El. 3070.00
A
B
C
D
E
F
G
H
35'
50'
50'
50'
50'
15'
Assumed line of excavation
Axis of dam
0 20 50 100
SCALE OF FEET
LINE 2
Sta. 14+10
12"
35'
50'
C
B
A
12"
12"
50'
50'
50'
15'
D
F
G
12"
12"
H
½" Pipe
℄ Foundation gallery
℄ Drainage gallery
0 5 10 20 30
SCALE OF FEET

(a) PLAN; LINE 2, STATION 14+10

(b) SECTION B-B, LINE 2

FIG. 3.—LOCATION OF HOLES AND PIPES IN THE LOWER PART OF HUNGRY HORSE DAM (LINE 2, STATION 14+10) IN MONTANA

pipes is accessible, its elevation is determined by suspending a sounding bell on a graduated tape.

The first set of readings is made after the gages (Fig. 4) have been connected and the stopcocks have been open for about two weeks. The readings are

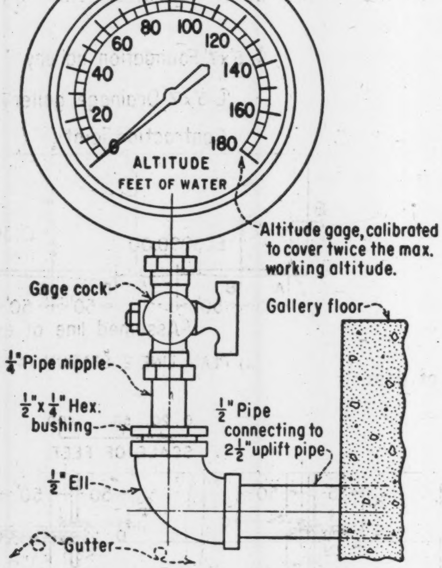
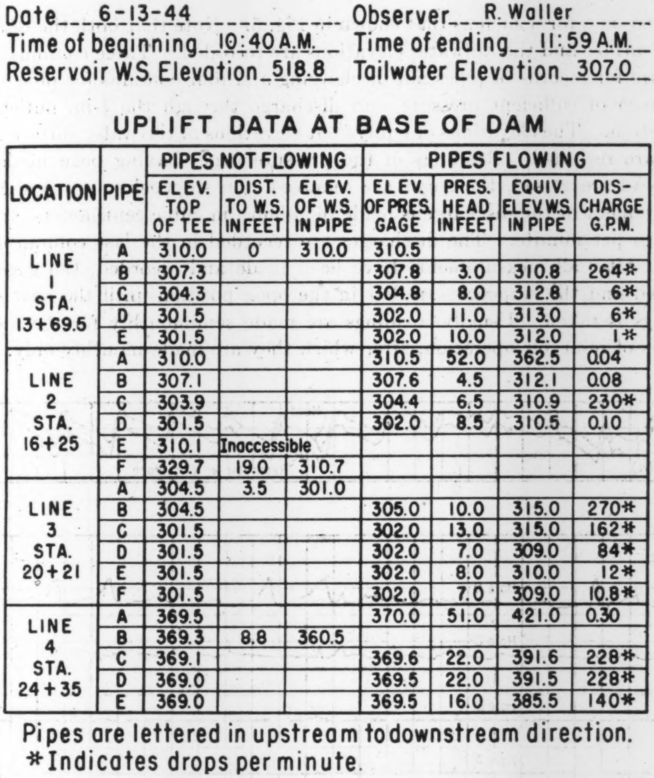

0
20
40
60
80
100
120
140
160
180
ALTITUDE
FEET OF WATER
Altitude gage, calibrated to cover twice the max. working altitude.
Gage cock
Gallery floor
$\frac{1}{4}$" Pipe nipple
$\frac{1}{2}$" x $\frac{1}{4}$" Hex. bushing
$\frac{1}{2}$" Pipe connecting to $2\frac{1}{2}$" uplift pipe
$\frac{1}{2}$" Ell
Gutter

FIG. 4.—TYPICAL ALTITUDE GAGE INSTALLATION

TABLE 1.—UPLIFT PRESSURES AT FRIANT DAM IN

Date of record	Head, in feet, lines 1, 2, 3	LINE 1					LINE 2					
		A	B	C	D	E[a]	A	B	C	D	E	F
March 11	120.30	3.74	3.99	4.82	4.99	0.42	26.18	....	4.07	3.33	....	....
March 15	126.90	2.68	2.92	3.70	3.86	−2.02	25.14	0.79	3.39	1.89	....[b]	2.44
April 14	132.30	1.89	2.12	3.25	2.65	1.89	23.43	1.97	2.57	1.51	....[b]	1.89
May 10	158.10	1.77	1.64	2.91	3.04	1.77	22.33	1.52	1.71	1.45	....[b]	2.09
June 13	211.80	1.42	1.79	2.74	2.83	2.36	26.20	2.41	1.84	1.65	....[b]	1.75
July 29	209.20	1.24	1.63	2.58	2.68	3.15	23.47	2.96	1.67	1.72	1.29	1.43
August 25	185.80	1.61	2.05	2.85	2.96	2.69	23.41	2.74	1.56	1.61	....[b]	2.31
September 23	163.40	1.77	1.65	2.88	3.00	3.00	23.50	2.75	2.02	1.77	....[b]	1.90
October 28	147.70	2.64	2.84	3.86	3.66	5.69	27.01	2.71	1.90	1.96	....[b]	2.44
November 28	170.30	2.86	2.99	3.88	3.99	5.75	24.54	3.76	2.76	1.94	....[b]	2.64
December 30	195.60	2.51	2.66	3.68	3.53	5.57	25.77	3.32	2.20	1.74	....[b]	2.51
Maximum	211.80	3.74	3.99	4.82	4.99	5.75	27.01	3.76	4.07	3.33	....	2.64
Minimum	120.30	1.24	1.63	2.58	2.65	0.42	22.33	1.52	1.56	1.45	....	1.89
Average	165.58	2.19	2.39	3.38	3.38	3.54	24.63	2.68	2.34	1.87	....	2.14

[a] Negative values of uplift pressure are shown only to indicate that pressures are below tailwater elevation



Date 6-13-44 Observer R. Waller
Time of beginning 10:40 A.M. Time of ending 11:59 A.M.
Reservoir W.S. Elevation 518.8 Tailwater Elevation 307.0

UPLIFT DATA AT BASE OF DAM

LOCATION	PIPE	PIPES NOT FLOWING			PIPES FLOWING			
		ELEV. TOP OF TEE	DIST. TO W.S. IN FEET	ELEV. OF W.S. IN PIPE	ELEV. OF PRES. GAGE	PRES. HEAD IN FEET	EQUIV. ELEV. W.S. IN PIPE	DIS-CHARGE G.P.M.
LINE 1 STA. 13+69.5	A	310.0	0	310.0	310.5			
	B	307.3			307.8	3.0	310.8	264*
	C	304.3			304.8	8.0	312.8	6*
	D	301.5			302.0	11.0	313.0	6*
	E	301.5			302.0	10.0	312.0	1*
LINE 2 STA. 16+25	A	310.0			310.5	52.0	362.5	0.04
	B	307.1			307.6	4.5	312.1	0.08
	C	303.9			304.4	6.5	310.9	230*
	D	301.5			302.0	8.5	310.5	0.10
	E	310.1	Inaccessible					
	F	329.7	19.0	310.7				
LINE 3 STA. 20+21	A	304.5	3.5	301.0				
	B	304.5			305.0	10.0	315.0	270*
	C	301.5			302.0	13.0	315.0	162*
	D	301.5			302.0	7.0	309.0	84*
	E	301.5			302.0	8.0	310.0	12*
	F	301.5			302.0	7.0	309.0	10.8*
LINE 4 STA. 24+35	A	369.5			370.0	51.0	421.0	0.30
	B	369.3	8.8	360.5				
	C	369.1			369.6	22.0	391.6	228*
	D	369.0			369.5	22.0	391.5	228*
	E	369.0			369.5	16.0	385.5	140*

Pipes are lettered in upstream to downstream direction.
*Indicates drops per minute.

FIG. 5.—UPLIFT PRESSURE READINGS AT FRIANT DAM IN CALIFORNIA (TABLE 1)

CALIFORNIA EXPRESSED AS PERCENTAGES OF HEAD (1944)

LINE 3						LINE 4					Head, in feet, line 4	Date of record
A[a]	B	C	D[a]	E	F[a]	A	B[a]	C	D	E		
....	....	....	....	....	....	....	....	....	....	....	....	March 11
−6.38	3.47	3.47	3.07	0.71	−0.08	30.57	−1.32	11.47	20.26	19.58	72.74	March 15
−6.35	2.65	2.65	1.89	0.38	0	39.52	−1.21	10.55	21.18	19.28	79.04	April 14
−4.74	3.04	3.04	2.09	0.82	0.19	37.54	−0.82	19.93	19.84	17.93	104.54	May 10
−2.83	3.78	3.78	0.94	1.42	0.94	38.12	−0.16	19.51	19.45	15.65	158.04	June 13
−1.39	3.15	3.39	0.29	1.24	0.76	39.30	0.47	19.79	19.08	15.23	155.84	July 29
4.04	3.23	3.23	0	1.08	0.54	40.32	0.64	19.95	18.74	15.33	132.04	August 25
2.39	3.00	2.69	−0.67	0.55	0.24	42.14	0.95	19.90	17.99	16.17	109.74	September 23
3.32	3.32	3.16	0.27	0.95	0.61	42.71	1.33	19.60	17.99	16.92	93.04	October 28
3.70	3.70	3.70	1.64	1.35	1.06	39.43	0.99	17.29	15.90	14.15	114.74	November 28
1.48	3.78	3.78	0.97	1.74	1.23	36.62	1.10	17.75	13.39	13.39	139.94	December 30
4.04	3.78	3.78	3.07	1.74	1.23	42.71	1.33	19.95	21.18	19.58	158.04	Maximum
1.48	2.65	2.65	0.94	0.71	0.54	30.57	0.47	10.55	11.33	13.39	72.74	Minimum
2.99	3.31	3.29	1.77	1.16	0.86	38.63	0.91	17.57	17.72	16.36	115.97	Average

and are omitted in computations for average values for year. [b] Pipe inaccessible for measurement.

recorded on a form such as that shown in Fig. 5. Note that both the reservoir water surface and the tailwater elevations are recorded. Three columns of the form are for records of pipes not discharging and four columns for pipe which, by reason of sufficient pressure, can discharge through the ½-in. outlet pipe connection. The elevations or equivalent elevations of the water surface in the pipes are recorded. Readings of the pressure gages having been made, the stopcocks are closed, the gages are removed, the stopcocks are opened, and measurement of the discharge is made in gallons, in cubic centimeters, or even in drops per minute. The discharges are recorded in the last column of the form. After all measurements have been made and recorded, the gages are replaced and the stopcocks are left in the open position until the next set of readings is taken. Usually, readings are made semimonthly for the first six months of reservoir operation, after which they are made monthly only.

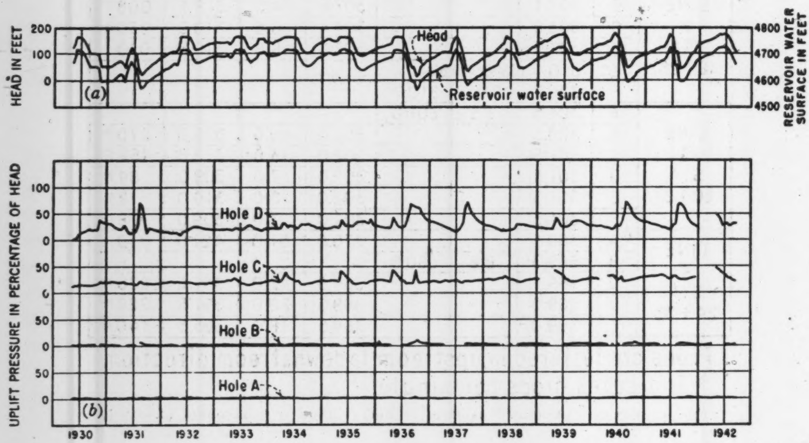


Fig. 6.—Uplift Pressures at the Base; Line 2, Gibson Dam in Montana

For illustrative purposes, although it is the actual record at Friant Dam in California during 1944, Table 1 shows uplift pressures expressed as a percentage of head. The uplift pressure at each hole in the foundation is reduced to a percentage of total head acting on the dam at the line of pipes within which the hole is located. Total head is taken as the difference between the reservoir water surface and the tailwater surface elevations; or, if there is no tailwater at the line of holes, the total head is the difference between the reservoir water surface elevation and the elevation of the concrete-rock contact. The latter elevation is taken as the average of all concrete-rock contact elevations as determined from the uplift holes within a line of holes. At the bottom of Table 1, the maximum, minimum, and average values of the uplift pressures, in percentage of head for each hole for the year, are noted. Graphs are prepared and kept up to date showing a running record of the uplift pressure in percentage of head for each hole. Fig. 6 shows such graphs for the holes in line 2 at Gibson Dam in Montana. Records of the hydraulic heads and of the elevations of the reservoir water surface are shown in Fig. 6(*a*). The entire history of recorded uplift pressures for a dam may be reviewed on such graphs.

Fig. 7 shows two charts, typical of what may be done in analyzing uplift pressure values. These were prepared from uplift pressure readings at American Falls Dam from 1926 to 1939. Fig. 7(*a*) shows the point-to-point variation across the foundation of the dam for the average uplift pressures at each hole within a line of holes for the period of record. Each line of holes at the dam is included and the average of all lines is shown by the heavier full line. Fig. 7(*b*) is the same, except that the plotting is the result of averaging the yearly maximum uplift pressure readings at each hole regardless of the date in which the maximums were recorded. For comparative purposes, full, two-thirds, and one-third uplift pressures varying linearly from 100% reservoir water pressure at the upstream face of the dam to zero pressure at the downstream face are shown on both charts.

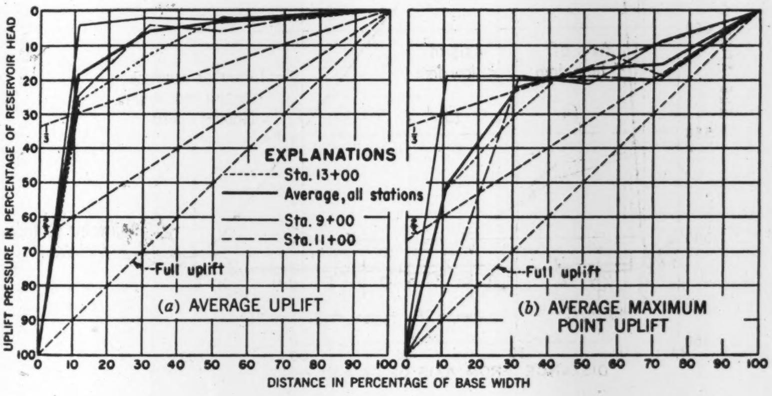
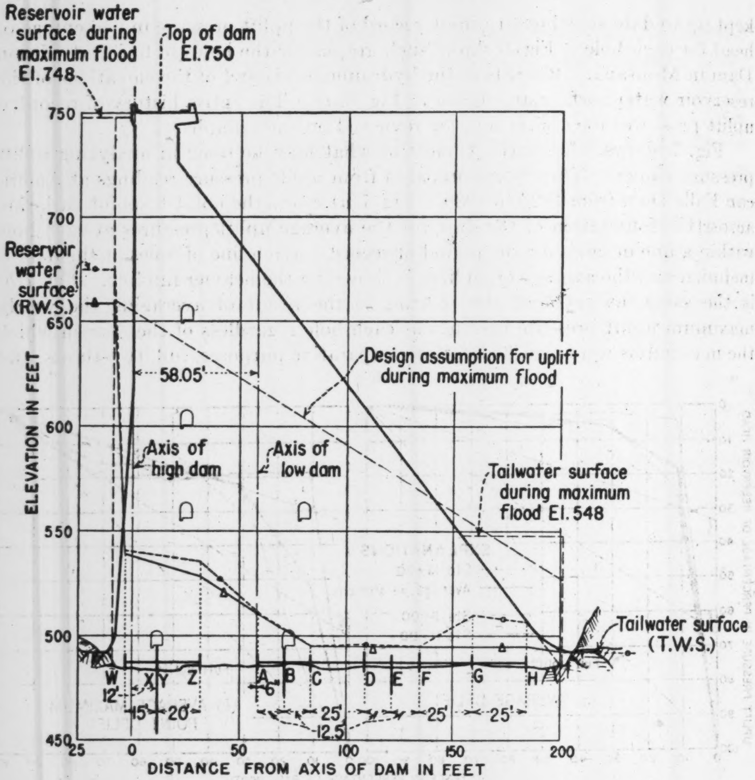


FIG. 7.—UPLIFT PRESSURE AT BASE OF DAM, 1927 TO 1939, AMERICAN FALLS DAM IN IDAHO

Fig. 8 is an example of a sectional view along a line of uplift holes. This particular illustration is for line 3 at Marshall Ford Dam. Plottings of the uplift pressures for two different reservoir water surface elevations have been made on the section, as well as a line showing the design assumption for uplift during a period of maximum flood. In plotting the uplift pressures the results from holes Y, B, and E have not been used as they were drilled considerably beyond the conventional depth of 3 ft below the foundation in order to determine the uplift experimentally in such locations. The upper strata were cased off so that actual foundation pressures are not reflected in the observations. Pressures at holes X and Y in the same location differed in that the observations at the latter (which is about 22 ft deeper than hole X) were from 15 ft to 19 ft

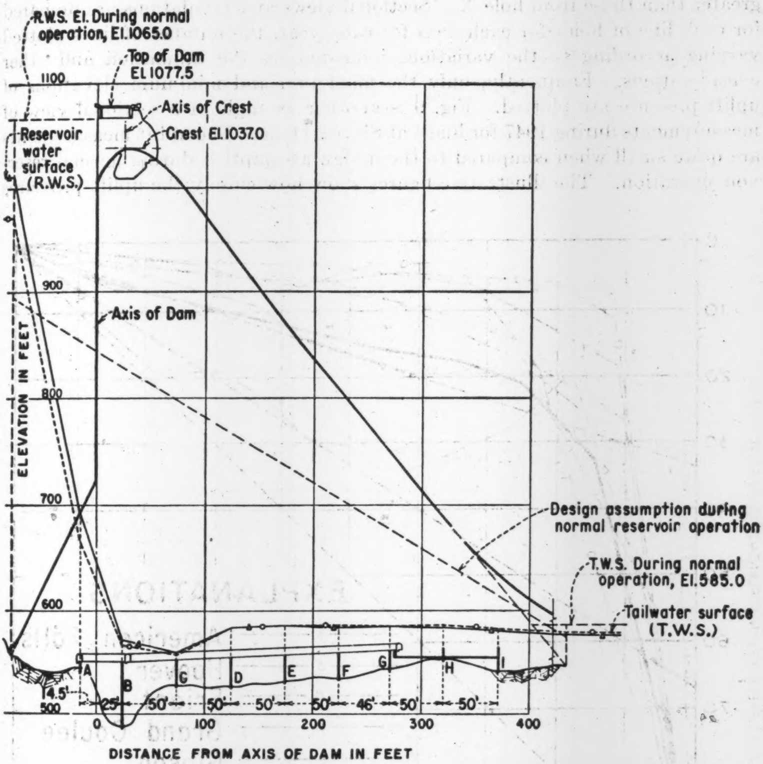


DATE	R.W.S. EL.	EL. OR EQUIVALENT EL. OF W.S. IN PIPE												T.W.S. EL.	LEGEND
		W	X	Y	Z	A	B	C	D	E	F	G	H		
JAN. 18, 1946	664.04	539	504	523	524	510	495	495	496	N.W.	495	508	505	492.20	
FEB. 20,	667.36	539	505	523	529	510	495	495	495	488	495	508	504	492.40	
MAR. 21,	669.80	541	505	524	534	510	495	495	495	N.W.	495	510	507	492.60	
APR. 23, 1946	674.89	541	510	524	536	511	495	495	496	N.W.	495	510	507	492.20	--○--
MAY 20,	677.53	543	509	524	537	510	495	495	495	N.W.	495	511	508	492.80	
JUNE 20,	674.42	542	509	524	534	511	495	495	495	N.W.	495	510	507	492.32	
JULY 22,	673.26	542	510	524	534	511	495	495	495	N.W.	495	495	495	491.92	
AUG. 20,	668.97	541	509	524	533	511	495	495	495	N.W.	495	495	495	493.16	
SEPT. 19,	662.35	539	509	524	531	—	495	495	495	N.W.	495	503	495	492.62	
OCT. 21, 1946	658.67	539	509	524	529	510	495	495	495	N.W.	495	495	495	492.60	—△—
NOV. 21,	661.24	539	509	524	529	510	495	495	495	N.W.	495	503	495	492.00	
DEC. 19, 1946	664.38	537	509	524	524	510	495	495	495	N.W.	495	503	495	492.54	

N.W. Indicates no water at measurable depth.

LINE 3 - STA. 40 + 23.5

FIG. 8.—UPLIFT PRESSURES AT FOUNDATION; LINE 3 (STATION 40+23.5), MARSHALL FORD DAM IN TEXAS (MEASUREMENTS DURING 1946)



DATE	R.W.S. EL.	EL. OR EQUIVALENT EL. OF W.S. IN PIPE									T.W.S. EL.	LEGEND
		A	B	C	D	E	F	G	H	I		
JAN. 22, 1947	964.0	668	569	559	583	586	585	586	586	583	580.9	
FEB. 19,	974.1	673	569	559	585	587	586	587	586	583	577.8	
MAR. 18,	995.2	680	571	560	587	587	586	586	586	582	577.8	
APR. 16, 1947	1010.6	689	572	560	584	586	585	586	585	582	577.8	—●—
MAY 28,	1006.1	690	572	560	584	587	586	586	586	583	581.4	
JUNE 25,	1006.4	689	572	560	584	587	587	587	586	584	581.8	
JULY 23,	996.4	680	571	560	584	586	585	586	586	583	582.1	
AUG. 20,	984.9	673	570	559	583	586	586	586	586	583	581.7	
SEPT. 17,	976.1	668	569	559	582	585	585	585	585	583	579.8	
OCT. 15, 1947	972.3	665	568	559	584	586	586	586	586	583	579.7	----○----
NOV. 26,	973.1	664	569	559	583	586	583	585	585	582	579.3	
DEC. 23, 1947	972.8	664	569	559	583	585	585	586	585	582	578.2	

LINE 4 - STA. 26+06.75 ON ℄ OF TRANSVERSE GALLERY

FIG. 9.—UPLIFT PRESSURES AT FOUNDATION; LINE 4 (STATION 26+06.75) ON CENTER LINE OF TRANSVERSE GALLERY, SHASTA DAM IN CALIFORNIA (MEASUREMENTS DURING 1947)

greater than those from hole X. Sectional views and tabulations are prepared for each line of holes for each dam for each year, the number of lines plotted varying according to the variations indicated by the tabulation and other considerations. Frequently, only the maximum and minimum elevations of uplift pressure are plotted. Fig. 9 is another example of a sectional view of measurements during 1947 for line 4 at Shasta Dam. The uplift measurements are quite small when compared to the design assumption during normal reservoir operation. The illustrative figures show how closely the uplift pressures can be watched in order to detect any unusual conditions which may develop and which deserve remedial attention.

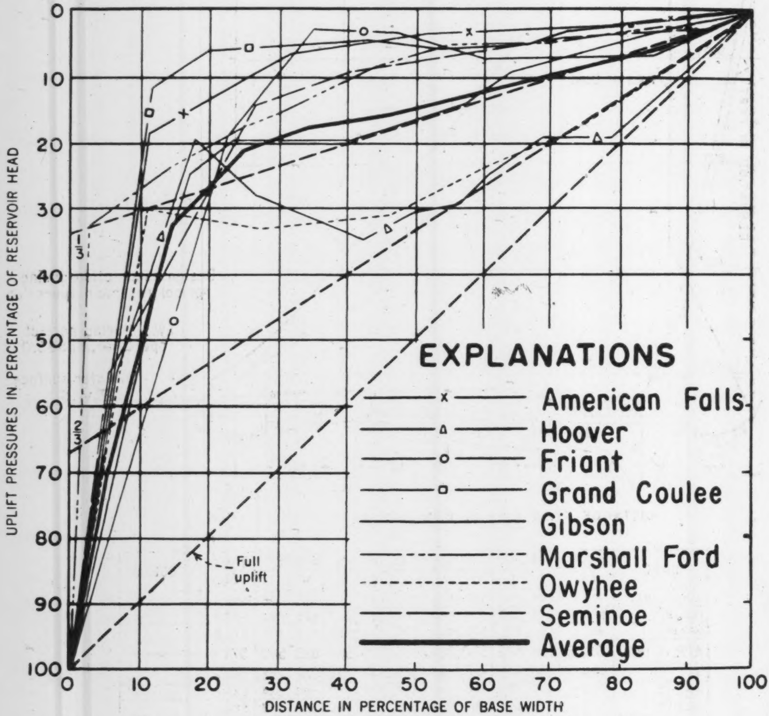


Fig. 10.—Average Uplift Pressure at the Base of Several Dams

Fig. 10 presents a chart prepared similarly to the charts in Fig. 7 but which gives the average foundation uplift pressure, to the time of preparation of the chart, for eight dams. The average of all lines is represented by the heavy full line. That line is indicative of the general trend of foundation uplift pressures transversely across concrete dams, the foundations of which are provided with drains and grout curtains. It might be used in considering design assumptions for future concrete dams. It is interesting to note that the measured pressures are generally less than a two-thirds uplift pressure assumption, but that, as might be expected, the higher pressures are near the upstream face of a dam.

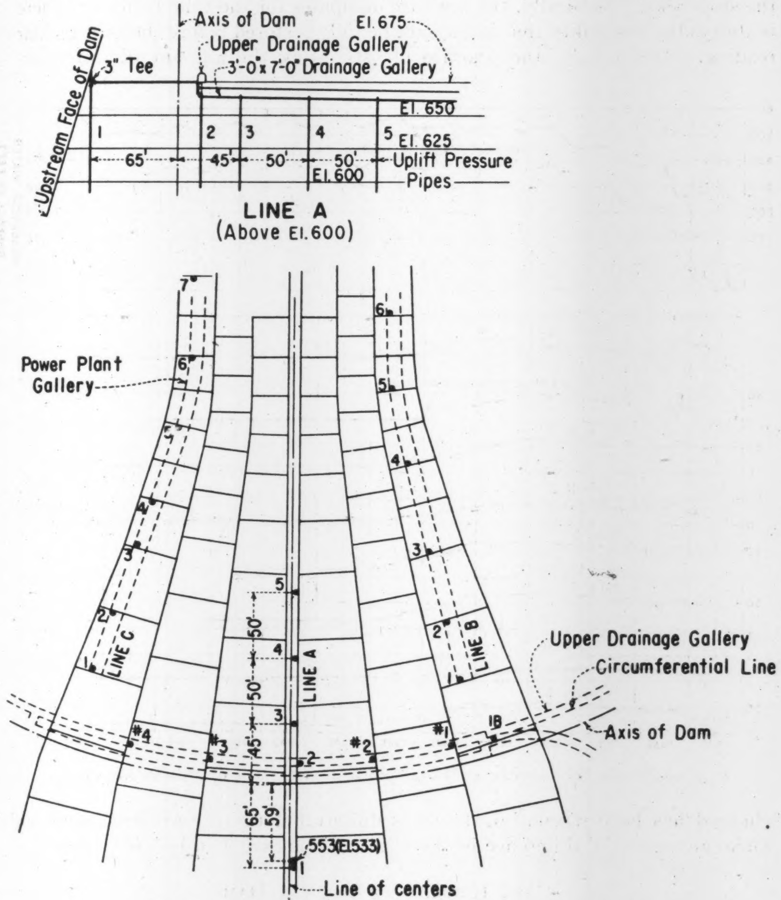

Axis of Dam
El. 675
Upper Drainage Gallery
Upstream Face of Dam
3" Tee
3'-0" x 7'-0" Drainage Gallery
El. 650
1
2
3
4
5
El. 625
65'
45'
50'
50'
Uplift Pressure Pipes
El. 600
LINE A
(Above El. 600)
Power Plant Gallery
7
6
5
4
3
2
1
LINE C
6
5
4
3
2
1
LINE B
LINE A
50'
50'
45'
65'
59'
553(El. 533)
#4
#3
#2
#1
1B
Upper Drainage Gallery
Circumferential Line
Axis of Dam
Line of centers

FIG. 11.—LOCATION OF UPLIFT PRESSURE HOLES, HOOVER DAM IN ARIZONA-NEVADA

The method described for observing foundation uplift pressure has some limitations. Although widely dispersed, the holes cover only a small percentage of the foundation area. Other than those holes which have direct vertical access to galleries, it is difficult, if not impossible, to clean out holes that have become clogged. There has been some difficulty in keeping the gages in perfect operating condition, because the moving parts tend to corrode in the damp atmosphere of the galleries. They should be recalibrated every few months. As mentioned previously, after the pressure observations have been made, it has been the practice to remove the altitude gages and measure the discharge from the stopcocks. Naturally, the pressure dissipates for the time being and there is always the possibility that it may not be fully restored before the next regular reading. Accordingly, the practice of estimating the amount of water discharged has been discontinued. It is doubtful if the revised procedure will make any material difference in the observations except in isolated cases.

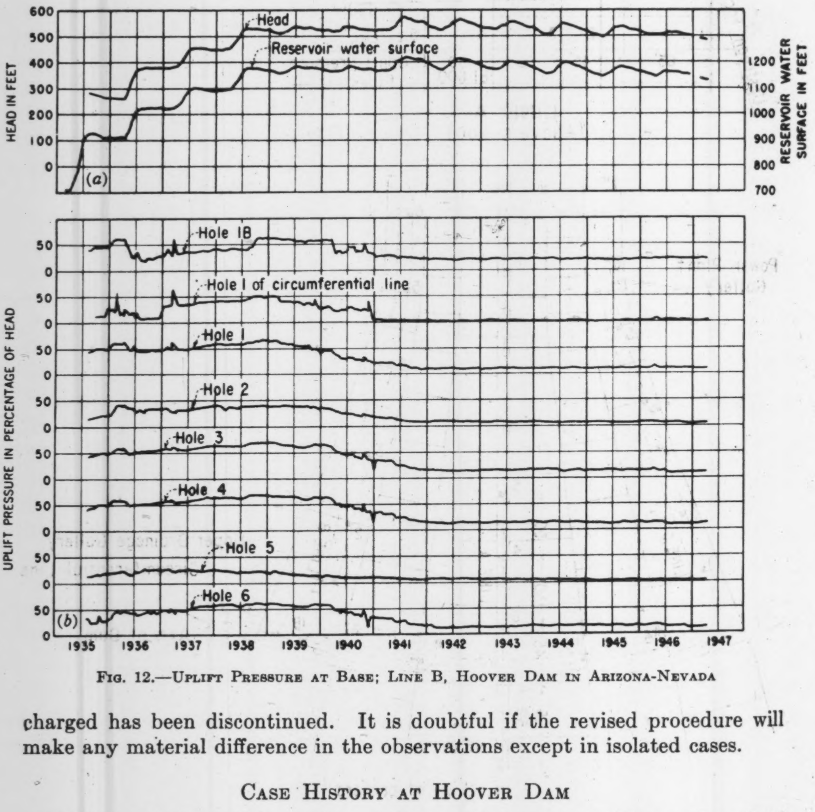
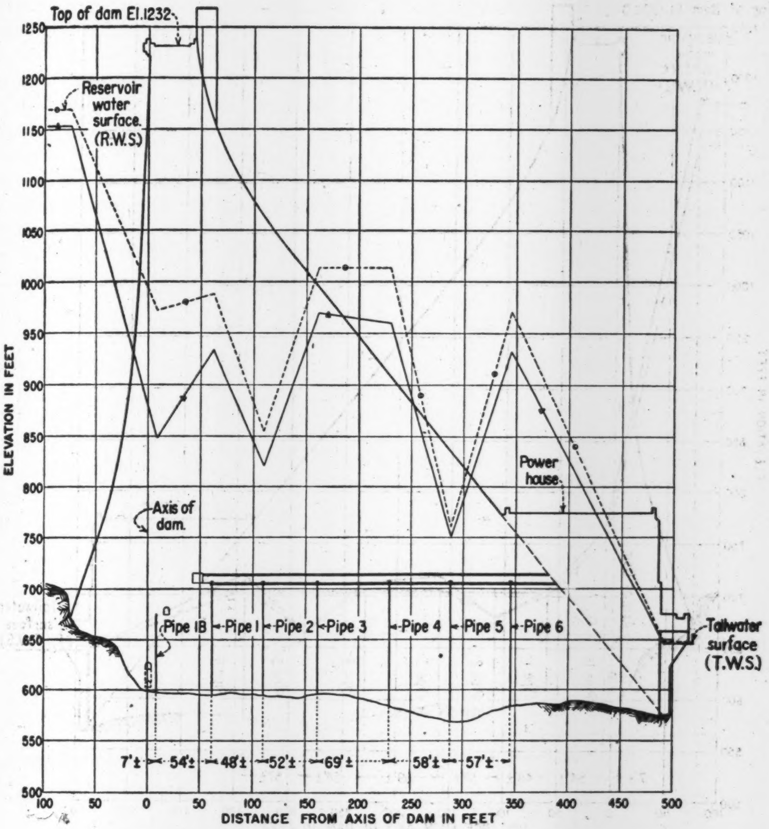


Fig. 12.—Uplift Pressure at Base; Line B, Hoover Dam in Arizona-Nevada

Case History at Hoover Dam

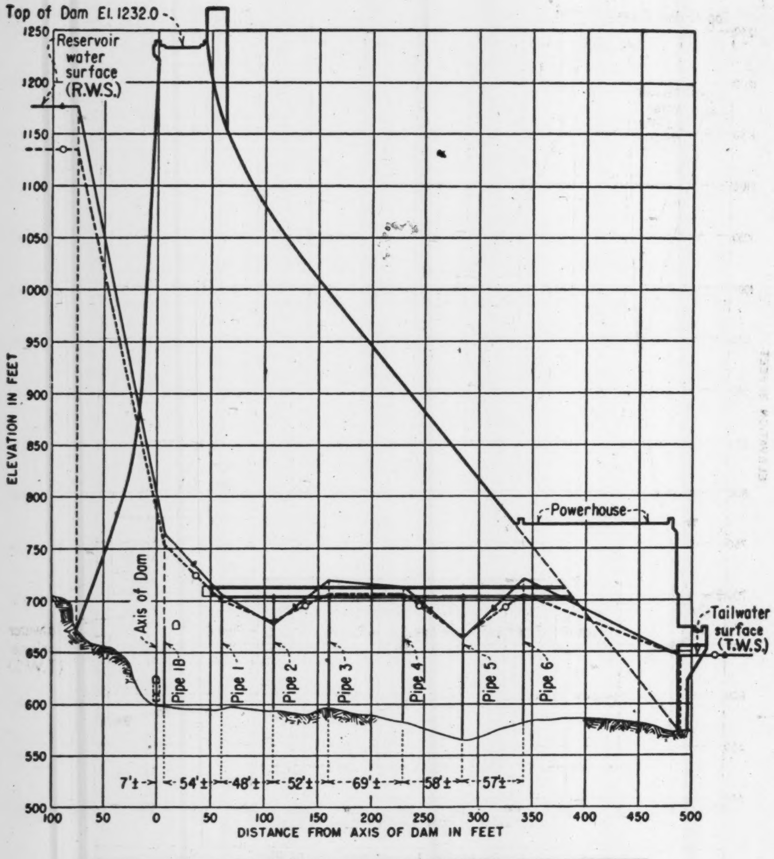
The essential case history of foundation uplift pressures at Hoover Dam is of interest. Fig. 11 shows the location of the 24 foundation uplift pressure holes—three radial lines and one circumferential line. Uplift pressures were first recorded in August, 1935, and have been continued to date. The higher pressures occurred during the first five years. Fig. 12, a running record for the holes in line B, reveals an increase in pressures until 1939 when remedial measures



DATE	R.W.S. EL.	EL. OR EQUIVALENT EL. OF W.S. IN PIPE							T.W.S. EL.	LEGEND
		1B	1	2	3	4	5	6		
JAN. 14, 1938	1095.33	822.05	909.78	825.67	933.92	938.00	757.06	906.18	645.47	
FEB. 15,	1094.65	821.05	904.78	821.67	930.92	935.00	750.06	909.18	645.60	
MAR. 15,	1099.15	814.05	909.78	818.67	934.92	937.00	750.06	911.18	645.05	
APR. 15,	1100.75	836.05	909.78	819.67	938.92	940.00	749.06	915.18	646.75	
MAY 13,	1119.90	838.05	918.78	819.67	944.92	948.00	750.06	922.18	645.55	
JUNE 15, 1938	1152.93	848.05	933.78	819.67	970.92	960.00	750.06	932.18	645.70	—•—
JULY 15,	1172.59	853.05	957.78	843.67	983.92	987.00	760.06	953.18	646.06	
AUG. 15,	1172.85	859.05	974.78	856.67	1000.92	999.00	761.06	966.18	645.95	
SEPT. 15,	1173.03	890.05	978.78	857.67	1006.92	1007.00	763.06	970.18	646.40	
OCT. 14,	1172.81	970.05	987.78	855.67	1008.92	1013.00	766.06	977.18	647.00	
NOV. 14, 1938	1169.19	972.05	988.78	854.67	1013.92	1014.00	761.06	971.18	647.12	- - -○- - -
DEC. 16, 1938	1168.37	972.05	987.78	854.67	1007.92	1010.00	755.06	966.18	646.84	

LINE B ALONG 10-FOOT GALLERY FROM ELEVATOR TO POWER HOUSE-NEVADA SIDE

FIG. 13.—UPLIFT PRESSURES AT FOUNDATION; LINE B, HOOVER DAM IN ARIZONA-NEVADA (MEASUREMENTS DURING 1938)

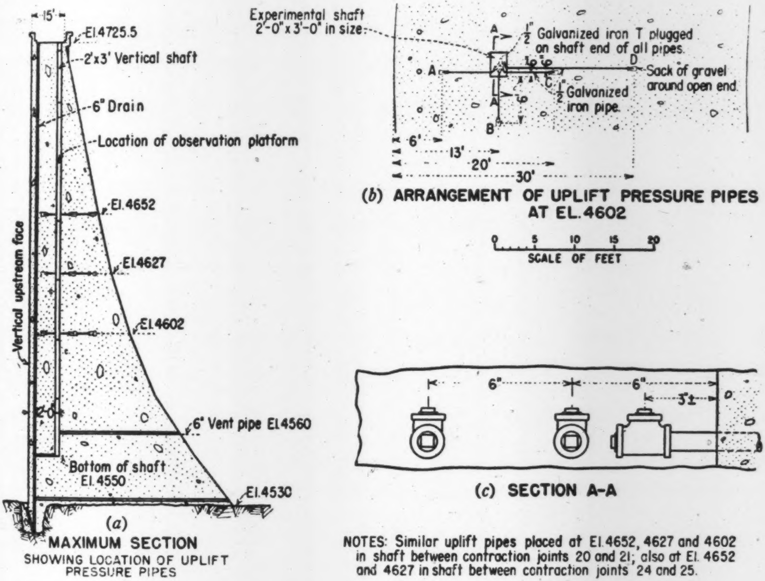


DATE	R.W.S. EL.	EL. OR EQUIVALENT EL. OF W.S. IN PIPE							T.W.S. EL.	LEGEND
		1B	1	2	3	4	5	6		
JAN. 15, 1947	1146.22	754	700	682	707	706	666	707	647.00	
FEB. 14,	1140.40	754	701	682	707	707	665	707	647.00	
MAR. 14,	1136.08	755	700	681	707	707	665	707	647.00	
APR. 15, 1947	1135.12	755	700	681	707	707	665	707	647.00	--○--
MAY 15,	1141.03	755	700	681	707	707	665	707	647.00	
JUNE 16,	1160.52	758	702	680	707	707	664	707	647.00	
JULY 15,	1175.65	762	704	680	707	707	664	707	647.00	
AUG. 15,	1177.23	766	706	680	719	714	665	721	647.00	
SEPT. 15,	1179.40	766	707	680	720	714	665	722	647.00	
OCT. 15,	1176.78	767	707	680	720	714	664	722	647.00	
NOV. 14, 1947	1177.01	766	707	679	720	714	664	722	647.00	—△—
DEC. 15, 1947	1173.55	766	707	679	720	714	664	722	647.00	

LINE B ALONG 10-FOOT GALLERY FROM
ELEVATOR TO POWERHOUSE-NEVADA SIDE

Fig. 14.—Uplift Pressures at Foundation; Line B, Hoover Dam in Arizona-Nevada (Measurements During 1947)

were undertaken by drilling more drain holes and by further foundation grouting. Measurements and experimental observations made during 1936 and 1937 indicated that, when the drains were closed intentionally, the uplift pressures increased and that, when they were opened, there was some decrease in pressure. After the drilling of additional drain holes, the pressures gradually decreased and the decrease was more rapid after supplemental grouting of the foundation. By 1942 the pressures had been reduced to nominal amounts that have remained generally constant since about 1942. Fig. 13 shows the uplift pressure measurements on line B during 1938; and, in contrast, Fig. 14 shows the same during 1947.


15'
El.4725.5
2'x3' Vertical shaft
6" Drain
Location of observation platform
El.4652
El.4627
El.4602
Vertical upstream face
20'
6" Vent pipe El.4560
Bottom of shaft El.4550
El.4530
Experimental shaft 2'-0"x3'-0" in size
A
A
B
C
D
1" Galvanized iron T plugged on shaft end of all pipes
Sack of gravel around open end
½" Galvanized iron pipe
6'
13'
20'
30'
0 5 10 15 20
SCALE OF FEET
6"
6"
3½"

(*a*) MAXIMUM SECTION SHOWING LOCATION OF UPLIFT PRESSURE PIPES

(*b*) ARRANGEMENT OF UPLIFT PRESSURE PIPES AT EL.4602

(*c*) SECTION A-A

NOTES: Similar uplift pipes placed at El.4652, 4627 and 4602 in shaft between contraction joints 20 and 21; also at El. 4652 and 4627 in shaft between contraction joints 24 and 25.

FIG. 15.—PIPES FOR MEASURING UPLIFTS IN CONCRETE, GIBSON DAM IN MONTANA

UPLIFT WITHIN THE CONCRETE

Equipment to determine the uplift within the mass concrete at horizontal construction joints has been installed in Gibson, Owyhee (in Oregon), and Hoover dams. Fig. 15 shows the installation at Gibson Dam. Sacks of gravel were embedded in the concrete around the open ends of ½-in.-diameter pipe, which led to a small vertical shaft where measurements of seepage or uplift could be made. There were five systems or sets of uplift pipes, identical with the arrangement shown in Fig. 15(*b*), installed at various locations throughout the dam. Frequent observations were made on these uplift units beginning in May, 1930. There was no indication of uplift pressure in any of the units. During June and August, 1930, a slight amount of water appeared in the pipes

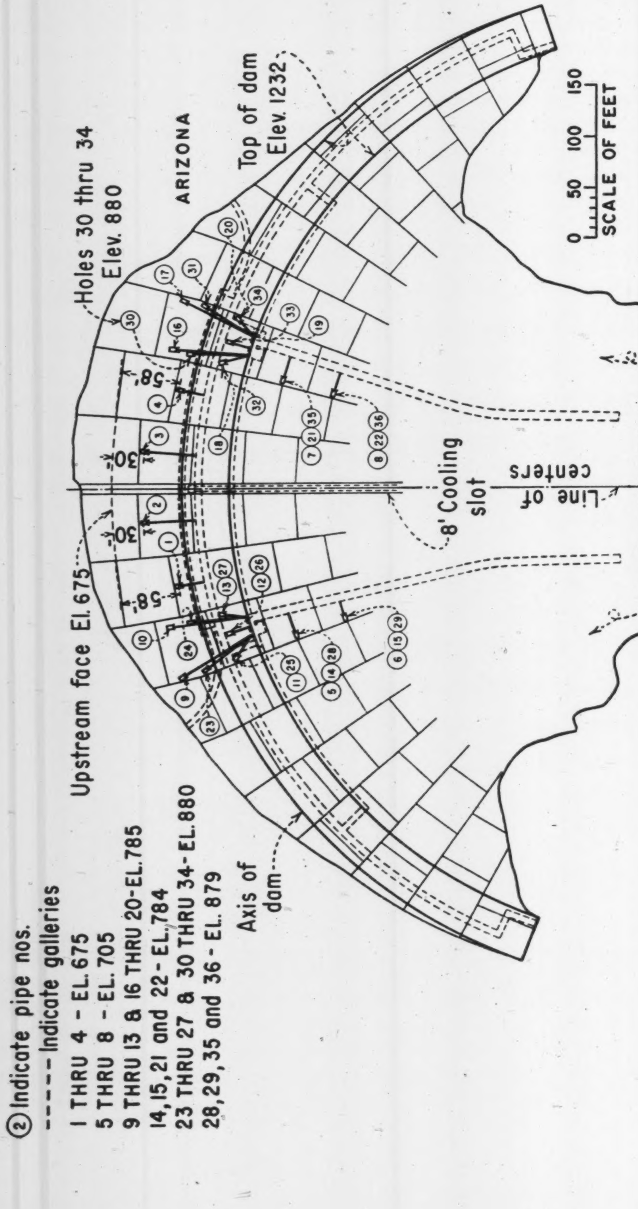

② Indicate pipe nos.
----- Indicate galleries
1 THRU 4 - EL. 675
5 THRU 8 - EL. 705
9 THRU 13 & 16 THRU 20-EL.785
14,15,21 and 22-EL.784
23 THRU 27 & 30 THRU 34-EL.880
28,29,35 and 36-EL.879
Upstream face El. 675
Holes 30 thru 34 Elev. 880
ARIZONA
Top of dam Elev. 1232
Axis of dam
30'
30'
58'
58'
8' Cooling slot
Line of centers
Construction joints not shown
PLAN
0 50 100 150
SCALE OF FEET

FIG. 16.—LOCATION OF UPLIFT PRESSURE PIPES AT HORIZONTAL CONSTRUCTION JOINTS, HOOVER DAM IN ARIZONA-NEVADA

from units A and B at El. 4602, and those pipes were noted as damp in March and April, 1931. At two other units at El. 4627 and El. 4652, a few drops of water were recorded in June, 1930. Otherwise, all units were dry and, after monthly observations had been made for eleven years with no signs of moisture in any of the units, observations were discontinued in September, 1942, except for quarterly check readings.

Similar installations were made at Owyhee Dam to the extent that it was possible to measure the uplift at twenty-four different locations throughout the mass concrete. Observations since May, 1943, have shown no uplift pressure or even any indication of moisture.

Fig. 16 shows the locations of thirty-six uplift units installed at horizontal construction joints within the mass concrete of Hoover Dam. With almost negligible exceptions, since observations began in August, 1935, no uplift or moisture has been indicated.

Some comparatively recent tests were made in the USBR laboratories of 337 specimens to determine the effective area over which uplift pressure acts. Methods and results of those tests have been described by Douglas McHenry,[3] M. ASCE. The method of determining the effective area was indirect, consisting of measuring the effect of induced pore pressure on shearing strength. Specimens were tested with and without pore pressure, and from the observed reduction in shearing strength the uplift force was found. By the Coulomb equation and by various combinations of testing specimens of the same material with different combinations of direct axial and circumferential fluid loadings, the effective area of pore pressure at failure was determined. As a result of those indirect tests (of which Mr. McHenry noted that there were many possibilities of inaccuracy) it was indicated that the pore pressure was effective over almost the entire surface of the area of failure. It is well known that many capable engineers have studied extensively the vexing problem of determining the effective area over which uplift acts. However, this paper is confined to activities of the USBR and does not include the many endeavors of others on this subject. It is possible that a more direct solution will be evolved.

RELIEF FROM EXCESSIVE UPLIFT

Everyone is familiar with the procedure of preparing the foundation for a dam so that uplift pressure will be alleviated as much as practicable. This involves deep curtain grouting near the axis and a row of drainage holes immediately downstream. Where the geological formations are porous, badly jointed, or otherwise comparatively permeable, there will be blanket grouting over the entire area before concrete is placed. Immediately before placing concrete, the foundation rock surface is thoroughly cleaned of all foreign matter and covered with a thin layer of mortar, from ½ in. to 1 in. thick, which is thoroughly brushed into all crevices and over the entire surface of the foundation. The mortar is the same as that of the regular concrete mixture. The mortar layer is intended to provide a tight contact or bond between the mass concrete and the foundation rock.

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[3] "The Effect of Uplift Pressure on the Shearing Strength of Concrete," by Douglas McHenry, *Report No. 48*, Vol. I, International Comm. on Large Dams, Paris, France, 1948.

After a dam has been completed, observations of foundation uplift pressure sometimes indicate that the design assumptions have been exceeded to such an extent that the uplift should be reduced. In such event, access through various galleries in the dam make it fairly easy to tighten or extend the grout curtain, to clean out the drain holes by reboring, or to drill additional drain holes.

Reduction of uplift pressure at Hoover Dam by supplemental grouting and drainage has been illustrated by reference to conditions at line B: Fig. 12 shows the record before and after the supplemental work, which was completed about 1941; Fig. 13, the uplift during 1938; and Fig. 14, the same during 1947.

A program of supplemental grouting and drainage was conducted at Owyhee Dam during 1936 and 1937, about four years after completion of that structure. Although the primary purpose was to reduce seepage of water through the left abutment rock, undesirable uplift more pronounced on that part of the dam near the left abutment was reduced.

Observations on one of the smaller concrete gravity dams showed that the uplift pressure in several locations was greater than that used in the design assumption. As a first step in reducing the pressure, additional drain holes were drilled between the existing drain holes which are on 10-ft centers. That process resulted in sufficient reduction. Had it not, the grout curtain could have been tightened or extended. At another dam where uplift pressures have increased since construction beyond those which are considered to be desirable, it is hoped to accomplish satisfactory reduction merely by reaming out the existing drainage holes which give indications of having become plugged with natural deposits.

Uplift Design Assumptions

The maximum and most common uplift design assumption used by the USBR in the past for concrete gravity dams is that uplift pressure on the base varies uniformly from full-reservoir pressure at the upstream toe to tailwater elevation or zero, as the case may be, at the downstream toe and that the pressure acts over two thirds of the base area. Some of the exceptions to that common assumption were noted by Ivan E. Houk, M. ASCE, 1932.[4] For example, it was assumed for Owyhee Dam that the pressure would vary uniformly from full-reservoir head at the upstream toe to one half of that head at the line of foundation drains (located 6.5 ft downstream from the axis) and from there uniformly to tailwater head at the downstream toe, and that such pressures would act over the entire base of the dam. The common assumption of full-reservoir head varying uniformly to tailwater elevation and applied to two thirds of the base area was used in the designs of Hoover, Marshall Ford, Friant, Grand Coulee, Shasta, Keswick (in California), Kortes (in Wyoming), and Canyon Ferry dams.

Observations of uplift on existing dams not only are of value in detecting unusually high pressures, but also are of value in considering uplift design assumptions for proposed dams. Such assumptions may best be determined by the designer after judging the past performance of uplift on existing dams with similar foundations in conjunction with the definite plans for grouting and draining the foundation at the proposed dam. It is readily recognized that the uplift design assumption for a dam to be constructed on a porous or badly jointed rock member should be higher than for one on a tight geological formation. The possibility that uplift pressures may be unusually slow in acting and may not be effective until after a long period must be given some weight.

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[4] "Measurements at Existing Structures," by Ivan E. Houk, *Civil Engineering*, September, 1932, p. 578.

Considering the extraordinarily low uplift pressures so far indicated over a short period for Shasta Dam, as shown in Fig. 9, and the average uplift pressures indicated by Fig. 10 for several dams over comparatively long periods, one is faced with the question of whether the uplift assumptions of the past have, in general, been too conservative. A higher-than-necessary uplift assumption results in actual factors of safety in the completed dam which are beyond the intention of the designer. With such an assumption, the dam is made heavier than necessary to resist overturning (should overturning be a governing factor), the shear-friction factor is actually greater than five (which is considered ample), and the sliding factor is much less than the value intended for safety. In brief, an uplift assumption more in keeping with the observations that have been made would result in smaller cross sections and appreciable reduction in cost. Should there be some skepticism regarding the adequacy of an uplift design assumption, if one has confidence in the reliability of the equipment and observations, then for dams with foundation galleries there is the insurance that uplift higher than that assumed may be relieved by supplemental grouting and drainage. Thus, a first-cost saving in reducing the section of a dam might or might not be fully realized in any particular case. However, it would appear that for a number of dams there would be a resultant saving in cost.

Conclusion

An attempt has been made to give factual data on the experience of the USBR with uplift pressures on concrete dams. The greater part of such experience has consisted of the installation of equipment for, and the observation of, foundation uplift pressures. The value of those observations may be separated into two categories: First, the periodic observation of uplift pressures is an operational guide for determining the necessity of any remedial measures that should be undertaken to reduce undue pressures in existing dams; and, second, the observations may be utilized in considering uplift design assumptions for proposed dams.

Acknowledgment

This paper, in considerably more detail, was presented to the Power Division at the 1949 Fall Meeting of ASCE in Washington, D. C., at the request of the chairman of the Joint Committee on Masonry Dams.

DISCUSSION

Ross M. Riegel,[5] M. ASCE.—A composite diagram of foundation uplift observations in four dams of the Tennessee Valley Authority (TVA) is shown in Fig. 17. The dams with their respective maximum heights are as follows: Fontana, 480 ft; Hiwassee, 307 ft; Cherokee, 202 ft; and Douglas, 175 ft. This composite is made from measurements upon fourteen sections in the four structures. The measurements were made by methods similar to those described by Mr. Keener.

In each dam a thorough job of cutoff grouting has been done. Drains downstream from the cutoff have been provided at intervals ranging from 7.2 ft to 10 ft on centers, the average being about 8.25 ft. The sections entering into the composite were taken at the highest reservoir stages available, but tailwater elevations were moderate. Nothing is available at high stages of tailwater.

The solid line in Fig. 17 represents composite readings of intensity in the area of contact or within the top 2 ft to 3 ft of foundation. Heavy dotted lines are shown extending to a point representing the composite position of drainage outlets in a gallery. The drain outlets do not coincide as a rule with the measuring sections. There is some justification for the view that the pressure gradient should be no higher than the drain outlets, and the heavy dotted lines represent a gradient corresponding to this view.

The composite is plotted with the same scale system as that used by Mr. Keener and, therefore, it is easy to transfer the average curve in Fig. 10 to that of the eight dams in Fig. 17. Study of Fig. 10 and reference to the author's description of the Hoover Dam (Arizona-Nevada) measurements indicate the following: If the most recent measurements at Hoover Dam (after the later drilling and grouting) were used, and if Owyhee Dam, in Oregon (which seems to be something of a "freak"), were omitted, the two average diagrams would be much closer together. The two curves show the marked effect of combined cutoff grouting and drainage.

Fig. 17 also shows an intensity assumed for design in the four TVA dams. Actually the gradient used terminates at maximum assumed tailwater elevation, but, of course, no observations have been made for such conditions. The gradient shown merely indicates the comparison between the observed data and a design gradient referred to the same tailwater.

Data on Norris Dam in Tennessee are not included in the TVA composite. This dam was designed by the Bureau of Reclamation (USBR) of the Department of the Interior but was built by the TVA. The original design called for a grouted cutoff and for so-called observation holes downstream from the cutoff into a drainage gallery at intervals of 50 ft. Other drain holes were added during construction and numerous drain holes were drilled in the spillway apron. Observations on uplift cells showed reduction of uplift pressure as the additional holes were drilled. At present the uplift pressure diagram closely resembles the average diagram plotted by Mr. Keener. Probably an important factor is the extensive program of "consolidation" grouting which was applied to the general foundation of the dam in addition to the cutoff grouting.

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[5] Head Civ. Engr., Design Dept., TVA, Knoxville, Tenn.

The author has referred to tests conducted by closing drain holes temporarily. In June, 1936, a test was made at Norris Dam by closing all the drains for 24 hr. During this period increases of 30% were observed at cells 18 ft and 35 ft from 1 drain hole and lesser effects were observed at other cells. At Hiwassee Dam in North Carolina tests were made on different days on four separate blocks in which uplift cells had been established by closing all the drain holes in each block. The plugs were removed as soon as noticeable increases in pressure were observed. This took place at intervals of from 2 hours to 22 hours in the respective blocks.

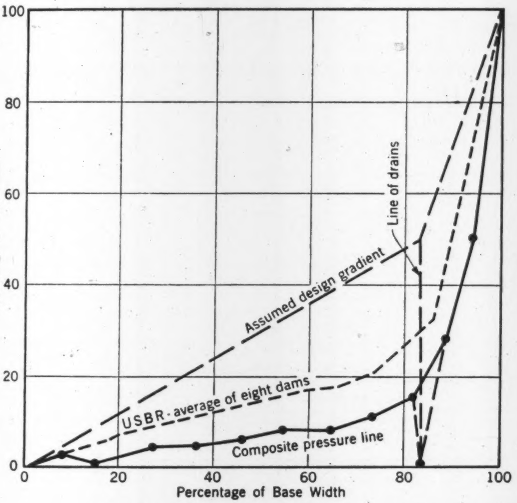

FIG. 17.—COMPOSITE FOUNDATION UPLIFT INTENSITY FOR FOUR DAMS OF THE TENNESSEE VALLEY AUTHORITY

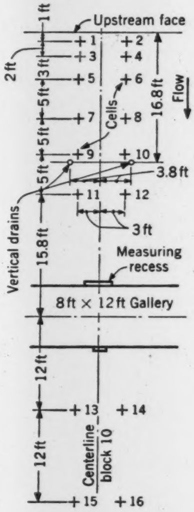

FIG. 18.—PLAN OF LOCATION OF PRESSURE CELLS, HIWASSEE DAM IN NORTH CAROLINA

Measurements of Pore Pressure in Concrete.—The author has described certain attempts to measure pore pressure at Hoover Dam. Similar attempts were made at Norris Dam. In each of three blocks of this dam six cells, for the observation of pore pressure, were installed on lift joints at El. 875 to El. 880, which is at a depth of 145 ft below the spillway crest, and where the external static pressure would have a maximum value of 63 lb per sq in. with lesser values at lower reservoir elevation. These cells consisted of porous concrete blocks 12 in. square, each of which was connected with an observation point in a gallery by two small pipes. During a period of four years no pressures were ever observed on sixteen of these eighteen cells. In one cell, in block 37, at a distance of 2.5 ft from the water surface, a maximum pressure of 13 lb per sq in. was observed in June, 1936, which was the month in which the reservoir was filled for the first time. Subsequently, lower pressures were noticed ranging from 0 lb per sq in. to 8 lb per sq in. during the four-year observation period. The observer concluded that a small crack connected this cell with the water surface and that this crack subsequently closed up—probably from a swelling effect at the surface of the concrete.

In another cell in block 43, at a distance of 48 ft from the surface, a maximum pressure of 13.5 lb per sq in. was observed in May, 1939. In this cell the water flowed from the connecting pipes at times, whereas at other times there was a flow toward the cell and water had to be supplied to the pipes to keep them full. It was concluded that a crack connected this cell with a contraction joint which was known to contain water under variable pressure conditions.

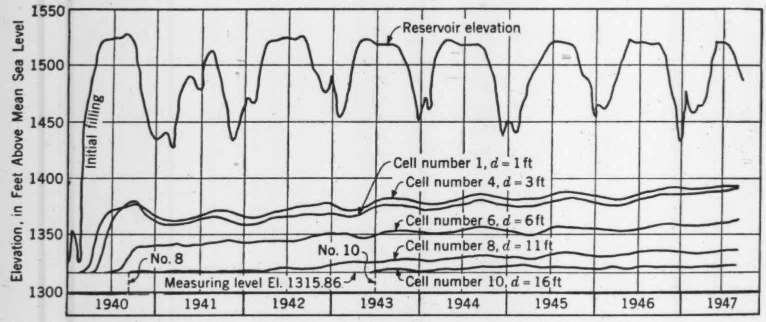

FIG. 19.—RECORD OF PORE PRESSURE MEASUREMENTS, HIWASSEE DAM IN NORTH CAROLINA

In another block at Norris Dam a set of six sacks of gravel was placed at locations between the lift joints in a manner similar to that used at USBR's Gibson Dam (in Montana), and no pressure was ever recorded from these sacks. In short, no definite values of pore pressure were obtained in this dam during the four-year period of observation.

On Hiwassee Dam, however, some very definite results were obtained. Fig. 18 shows the location of a number of cells placed at one level in the middle of a lift. These cells were developed by the USBR primarily for measuring pore pressures in earth. They were connected by an insulated wire inside a metallic tube to a measuring station, and their principle of operation was similar to that of the Goldbeck cell. The application of a carefully controlled air pressure will cause an electric circuit to be broken when internal and external pressures on the diaphragm in the cell balance.

The record of these cells is shown,[6] for a seven-year period in Fig. 19. The depths *d* indicate the relative levels at which the cells were placed. Only the higher of the two pressures measured at each distance is recorded. They show a consistent pattern of slowly rising pore pressures. Fig. 18 shows the relation of vertical drainage wells in the concrete to the measuring cells. No appreciable pressures have yet been detected downstream from these drainage wells.

At the Fontana project in North Carolina another installation was made as shown in Fig. 20. The small square symbols indicate the location of the same kind of pore pressure cells used on the preceding projects, whereas the cells marked PT were the so-called piezometer tips, also obtained from the USBR. This tip is merely a small chamber protected by a porous stone and connected by two tubes with an observing point. The use of the two tubes permits both of them to be filled with water, after which one is closed by a valve and the other is permanently connected to a Bourdon pressure gage. It was expected that the piezometer tips would record higher pressures and that the other cells would record lower pressures in a manner similar to those obtained at Hiwassee Dam. However, the concrete at Fontana Dam, at least in the block observed, appeared to be more permeable than that at Hiwassee Dam. The pore pressure cells showed a rather rapid rise and then began to fall—apparently from overstress. The USBR stated that these cells were to be limited to 200 ft of water pressure and this limit seems to be somewhat on the high side.

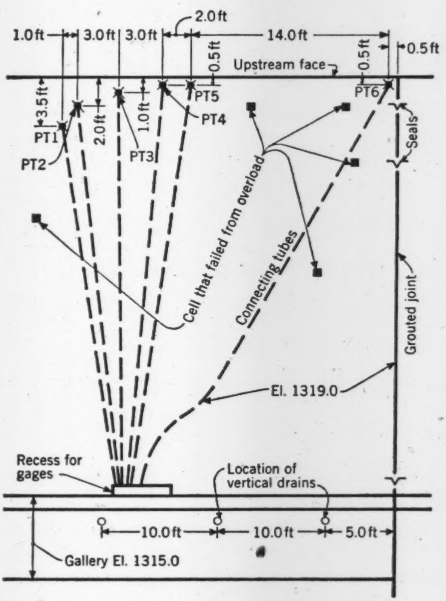
2.0 ft
1.0 ft
3.0 ft
3.0 ft
14.0 ft
0.5 ft
0.5 ft
0.5 ft
Upstream face
3.5 ft
2.0 ft
1.0 ft
PT1
PT2
PT3
PT4
PT5
PT6
Seals
Cell that failed from overload
Connecting tubes
Grouted joint
El. 1319.0
Recess for gages
Location of vertical drains
10.0 ft
10.0 ft
5.0 ft
Gallery El. 1315.0

FIG. 20.—LOCATION OF PIEZOMETER TIPS, FONTANA DAM IN NORTH CAROLINA

The piezometer tips, however, afforded the record shown in Fig. 21, which is reasonably consistent. The record marked CT is from a Carlson-Terzaghi cell similar to that referred to by the author, which was installed in another block of the dam but at the same elevation as the piezometer tips. The measuring disk of this meter was at a depth of 18 in. from the surface. The PT installation, however, encountered trouble. The tips should have been connected with the gages by copper tubing, but, since the installation was made during wartime, the copper tubing could not be obtained and a plastic tubing was substituted. This tubing had to be bent outside the concrete to make satisfactory connection with the gages. After about eighteen months, the period shown on the record, the plastic tubes began to fail from the combined effects of flexure and pressure. Repairs were finally made on five of the six cells by substituting copper tubing. However, about one year of record was lost while these repairs were made. Thereafter, the pattern of pressures was about the same as indicated in Fig. 21, except that the Carlson-Terzaghi cell showed about 95% of the external hydraulic pressure whereas the highest piezometer tip reading showed only about 90%. Pressures at greater depth are continuing to rise but very slowly.

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[6] "Methods and Instruments for the Measurement of Performance of Concrete Dams of the TVA," by C. E. Blee and R. M. Riegel, *Proceedings*, Conference on Large Dams, Stockholm, Sweden, June, 1948, p. 5 (R45), Fig. 1.

The history of these measurements indicates some of the difficulties that must be anticipated in making measurements of this kind; and such investigations are much needed. On the three dams, TVA engineers have one quite satisfactory record, one failure, and one partly successful record—the over-all efficiency being about 50%. It can be stated, however, that these measurements have demonstrated that pore pressures of material intensity do exist in the region adjacent to the reservoir water and that a gradient is apparent between the reservoir and drainage openings or galleries. Whether there is any observable pressure downstream from the drainage wells is not known.

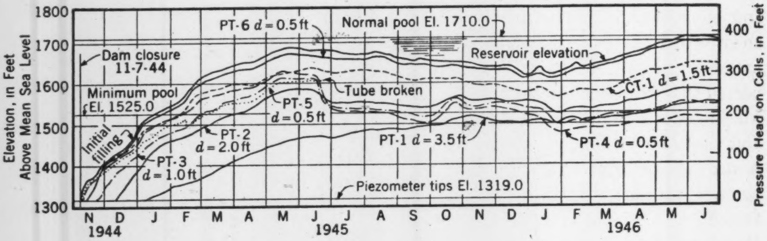

FIG. 21.—RECORD OF PORE PRESSURE MEASUREMENTS, FONTANA DAM IN NORTH CAROLINA

The foregoing data are intended to amplify and confirm the data submitted by the author and it is believed that they have considerable value from this standpoint.

WILLIAM P. CREAGER,[7] HON. M. ASCE.—A valuable addition to the existing data regarding uplift pressures under masonry dams founded on rock has been presented by Mr. Keener. This paper, together with data offered by Ivan E. Houk,[4] M. ASCE, and others, assists materially in estimating the pressures to be expected under such structures.

The author has stated (under the heading "Uplift Design Assumptions") that the

> "* * * most common uplift design assumption used by the USBR in the past * * * is that uplift pressure on the base varies uniformly from

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[7] Cons. Engr., Buffalo, N. Y.

[4] "Measurements at Existing Structures," by Ivan E. Houk, *Civil Engineering*, September, 1932 p. 578.

full-reservoir pressure at the upstream toe to tailwater elevation or zero, as the case may be, at the downstream toe and that the pressure acts over two thirds of the base area."

The writer believes that there is now sufficient evidence that uplift pressure is exerted over 100% of the base or at least close enough to that value to use for all practical purposes. Under this assumption, the uplift pressure may be divided into two parts:[8,9]

1. Tailwater pressure over the entire area of the base, plus
2. A pressure equal to the net head on the dam at the upstream toe, varying uniformly to a percentage, p, of the net head at the grouted cutoff and drains and thence varying uniformly to zero at the downstream toe.

For practical purposes, the uplift pressure may be considered to be equal to the tailwater pressure plus a percentage, p, of the net head on the dam at the upstream toe, varying uniformly to tailwater pressure at the downstream toe, all exerted over 100% of the area.

The percentage, p, has a value which depends on the effectiveness of the grouted cutoff and the drains. It is obviously a function of the relative permeability of the grouted cutoff and the drains, and the condition of the foundations downstream from the cutoff. This value is not known beforehand. However, the previously mentioned tests on seven dams reported by Mr. Houk, six additional tests by the author, and about five others known to the writer indicate conclusively that the value of the percentage, p, may be expected not to exceed 50%, if the foundation is well grouted and drained.

Incidentally, since the tailwater uplift is known, Fig. 10 would be more enlightening if it recorded the percentage of the net head on the dam instead of the percentage of reservoir head to show the aforementioned second division of uplift pressure.

A. Warren Simonds,[10] M. ASCE.—An excellent summary of the experiences of the USBR in observing uplift pressures beneath concrete dams is contained in this paper. The installation of measuring equipment, method of making observations, and results of the measurements are presented. For design purposes, the observed uplift pressures at the various dams are expressed as averages plotted in percentage of reservoir head against percentage of base width. This tends to reduce the magnitude and distribution of pressures beneath the dams to a common base for comparison.

In using the graphs shown in Fig. 10 for design purposes, it should be remembered that the plotted results are average values and therefore will not necessarily furnish an exact comparison for an entire foundation area at some other site. Foundations vary widely; geological irregularities or defects may make a large difference over a local area in the resulting uplift pressures due to full reservoir head. Furthermore the uplift pressures vary from time to time. The magnitudes of the pressures fluctuate normally with the variations in

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[8] *Transactions*, ASCE, Vol. 111, 1946, p. 1202.

[9] *Ibid.*, Vol. 114, 1949, p. 215.

[10] Engr., Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo.

reservoir water surface after an appropriate lag in time which is dependent on the degree of permeability of the foundation.

The case history of the uplift pressure at Hoover Dam in Arizona-Nevada (which the author describes briefly) is of unusual interest because of its relation to certain physical and geological conditions encountered in that foundation. The uplift pressure at this dam was caused by water from two different sources—cold water from the normal percolation of reservoir water into the foundation, and warm alkaline water coming from springs which apparently originated at a considerable depth beneath the surface.

During the excavation of the foundation area, several warm springs were encountered along the Nevada side of the canyon. These springs were opened by drilling and pipe connections were made to them. The pipes were led into one of the lower galleries of the dam for grouting purposes after a sufficient cover of concrete in the dam had been placed over the areas from which the springs emerged. Other physical defects, in addition to the usual joints and cracks in the foundation rock, were several shear zones in the right central part of the foundation area. Some of these shear zones were in close proximity to the warm alkaline springs which occurred beneath the uplift pressure pipes in line B, Fig. 11.

After sufficient concrete had been placed above the warm spring area, an attempt was made to grout the springs by pumping neat cement grout through the pipes previously installed for this purpose. When the cement grout came in contact with the warm alakline water, a flash set of cement occurred before the springs could be grouted. This resulted in plugging the pipes that provided the only means of drainage from the warm spring area. The warm water subsequently found its way into the shear zones in the foundation area and caused the high uplift pressure in 1938 which was indicated by the installation along line B, Fig. 11. Subsequent grouting operations using special cements and retarders, and the drilling of foundation drains in strategic places reduced the magnitude of the uplift pressure until it was no longer excessive. It is interesting to note that on the Arizona side of the canyon, where there were no warm springs or shear zones, the uplift pressure along line C, Fig. 11, was much less than that which occurred along line B.

It may be well to emphasize that uplift pressure patterns beneath concrete dams do not vary as a straight line from the upstream face to the downstream face of the structure. All the patterns of observed pressures show a range from full reservoir pressure at the upstream face to a marked reduction at or near the line of foundation drains. From the line of drains to the downstream face, there are some irregularities, but, under normal conditions, the maximum pressures in this area are fairly small.

Uplift pressures beneath concrete dams can be controlled to a large extent by a reasonable program of grouting and drainage. Adequate grouting will minimize the flow of water beneath the structure and drainage will prevent the development of high pressures. During the construction period of a concrete dam, it happens frequently that some areas of the foundation receive inadequate treatment. It is only normal, therefore, to supplement the original foundation treatment by additional grouting and drainage after the reservoir has filled.

The need for such additional treatment is indicated by the uplift pressures observed at the base of the dam. For this reason the observations for uplift pressure serve as a valuable indicator of conditions in the foundation beneath a dam after the reservoir has filled.

FAIRFAX D. KIRN.[11]—Valuable information regarding uplift pressures measured at some of the USBR dams now in service is presented in this paper. These measurements clearly show the effect of adequate drainage systems in reducing the uplift pressure at the line of drains. Foundation grouting was also shown to be a useful medium for relieving unduly large uplift pressures. The measured uplift pressures do not, of course, give any information regarding the extent of the area over which the uplift acts.

Under the heading "Uplift Design Assumptions," Mr. Keener states:

> "The maximum and most common uplift design assumption used by the USBR in the past for concrete gravity dams is that uplift pressure on the base varies uniformly from full-reservoir pressure at the upstream toe to tailwater elevation or zero, as the case may be, at the downstream toe and that the pressure acts over two thirds of the base area."

Diagrams determined from measured uplift pressures under dams in service as given in Mr. Keener's paper show that the average of the observed pressures is generally well under the foregoing design assumption. Based on these observed measurements and also on the knowledge that excessive uplift pressures can be prevented, or controlled if they occur (by an adequate drainage system and foundation grouting), it would seem logical to assume that the uplift pressure varies uniformly from full reservoir head at the upstream face of the dam to some percentage of the difference between reservoir head and tailwater head plus tailwater head at the center line of drains, and then from there uniformly to tailwater head at the downstream face. The area over which the uplift pressure acts is something of a guess. The trend at the present time is to assume that the uplift pressure acts over 100% of the area of the base. However, regardless of published laboratory tests (which are too few in number to be conclusive) and hypothetical theories and until more exact methods are developed for determining the area over which uplift pressures act, the writer can not subscribe to the 100% area assumption. As long as cohesion exists, it would seem impossible that uplift pressure can act over 100% of the area of contact whether this area be considered to be in the concrete, the rock, or the bond between the concrete and rock.

In computing safety factors for transverse cross sections of gravity dams, it should be noted from Eqs. 1 and 2 that high stability is indicated not only by a high shear friction factor but also by a low sliding factor. Eq. 1 is the generally accepted sliding factor equation. It is believed, however, that a more satisfactory equation for computing the stability of the dam against failure by sliding can be obtained directly from Eq. 2. If the cohesion (unit shear resistance) is assumed to be zero in Eq. 2, it reduces to

$$f_r = \frac{(W - U)\,\phi}{H} \quad \text{.......................} (3)$$

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[11] Head, Trial Load Group, Dams Div., Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo.

Eq. 3 takes into account the coefficient of internal friction, ϕ, of the concrete and rock. When f_r, computed from Eq. 3, is greater than 1, safety against failure by sliding is indicated the same way as for Eq. 2. Eqs. 1 and 3, however, should rarely, if ever, be used for determining the stability of a gravity dam against failure by sliding, since the cohesion will always have some value if the rock at the dam site is at all suitable for a gravity structure.

In discussing the safety factor equations, Mr. Keener states (under the heading "General"):

> "Noting the position of U, the "uplift" in Eqs. 1 and 2, one can readily determine the effect of that item in influencing the final cross section of the dam. If the assumed uplift is higher than necessary, it will be reflected in an unduly large cross section and a more costly dam. Conversely, if the assumed uplift is too low, undesirable encroachemnt will be made on the normally used values of the factors of safety."

In addition to the uplift, Eq. 2 contains two other terms which also influence the final cross section of the dam and for which values must be assumed at the present time. These are the coefficients of internal friction, ϕ, and of cohesion, τ. Both terms, of course, affect the magnitude of the shear friction factor. The cohesion, in particular, has a marked influence, since it is multiplied by the area, A_h, of the contact plane. Laboratory and field tests made to determine actual values for these terms would eliminate the necessity of using assumed values in Eq. 2 with the consequent penalty of having to design the cross section of the dam for higher factors of safety. Investigational work of this nature is now (1950) being undertaken by the USBR. When sufficient information is available regarding the distribution of the uplift force and the area over which it acts, the magnitude of the cohesion, and the coefficient of internal friction for any particular concrete and foundation rock, it will then be possible to reduce the requirement that the shear friction factor be five, or greater, as mentioned by Mr. Keener. The results will be a more economically designed dam.

The writer believes that the gravity method of analysis should be used only to determine an approximate cross section for the dam. A gravity dam is a statically indeterminate structure the same as an arch dam, and any analysis that neglects effects of the transfer of load horizontally to the abutments by twist and beam action gives erroneous stresses and safety factors. For substantiation of this statement one need only look at the diagonal cracking that occurs near the abutments of practically all gravity dams. A simple gravity analysis requires that a slice of the dam be analyzed as an isolated structure with no consideration of the effect of load transferred by horizontal elements to the canyon walls. Such an analysis gives stresses and safety factors against failure by sliding which are identical at the same elevation for every similar cross section in the dam regardless of its location in the structure. An analysis of the dam acting as a monolithic structure by the trial load method, however, discloses that this is not the case. The trial load analysis shows that safety factors at the inclined abutment planes of the shorter cantilever sections which base on the canyon walls are the critical ones and not those which occur at the bases of the higher sections in the central part of the dam.

If the term, sec ψ, is inserted in Eq. 2 and the area, A_h, replaced by the area, A, for the inclined base plane of the cantilever, the equation becomes

$$f_\tau = \frac{(W - U)\,\phi \sec \psi + A\,\tau}{H} \qquad (4)$$

in which ψ is the angle between the vertical and the inclined abutment plane.

Eq. 4 gives the shear friction factor at the inclined abutment plane for any cantilever section with its base on the canyon wall. Since sec ψ is never less than 1 and the area, A, is always greater than A_h, Eq. 4 gives a value for the shear friction factor on the inclined base plane of the cantilever which is greater than that computed for a horizontal plane at the same elevation from Eq. 2. This statement, however, is not true if the factor is computed for the load distribution determined from a trial load analysis of the structure. The transfer of load horizontally to the abutments by twist and beam action increases the magnitude of the denominator, H, in Eq. 4 to a greater extent than the numerator is increased, and the resulting shear friction factor is considerably less than the factor determined from the same equation for a gravity analysis.

SERGE LELIAVSKY BEY,[12] M. ASCE.—The author of this interesting paper deserves credit for presenting, in a concise and lucid style, a number of measurements of pore pressure observed at the dams of the USBR. Within its scope this paper has certainly achieved its purpose, and although the observations forming its subject matter are more in the nature of maintenance routine than pioneer work, they constitute a valuable contribution to the subject under discussion.

From another standpoint, also, the paper is important; that is, as a standard showing how such measurements must be observed and recorded. It is to be hoped that it will encourage other bodies in charge of dam maintenance to undertake and publish similar experiments.

The following points are suggested not as an adverse criticism, but as improvements that may possibly enhance its importance. In the first instance, the author seems to be rather reserved in stating his conclusions explicitly. The fact that they are negative—that no new point has been discovered—does not reduce their weight and value, particularly if opposed to the painstaking care with which these experiments were conducted. The diagrams in Fig. 10 are indeed very much the same as, or equivalent to, some published earlier.[4] The fact is rather instructive and deserves particular mention as a significant conclusion.

Another point calling for comment concerns the term "uplift," as used in the title and text of the paper. Modern dam design distinguishes between the uplift force, as it appears, for example, in Eqs. 1 and 2, and the pore pressure (or interstitial pressure) which is a hydraulic concept and is supposed to be investigated quantitatively by means of pipe measurements, such as those described in the paper. Admittedly, the two would be numerically equal if the effective area were assumed to be 100%, but the writer has shown, experimentally, that this assumption was not realized in nature; in fact, according to the results of his tests, the effective uplift area to be included in the calculation of a dam is 85%.[13] Attention may also be called to the fact that, as stated by the author under the heading "Uplift Design Assumptions," in a number of dams built by USBR, the effective area was taken to be only two thirds. The distinction between the two concepts in question (that is, hydraulic pore pressure and uplift force) is, however, much deeper than this arithmetical difference, for one of them is a scalar quantity whereas the other is a vector.

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[12] Director, Designing Service, Projects Dept., Ministry of Public Works, Cairo, Egypt.

It follows, therefore, that as an insurance against possible misunderstandings, the title of the paper could have been altered into "Pore Pressures in Concrete Dams," with corresponding corrections in the text. This would also have the advantage of avoiding the feeling of disappointment on the reader's part when he discovers the subject of a paper is different from its title.

The third point involves Eqs. 1 and 2 used by the author in estimating the safety coefficient, as influenced by the uplift factor (see under the heading "General"). These formulas are not intimately correlated with the main subject of the paper but, nevertheless, call for comment as being somewhat reminiscent of the early period (middle nineteenth century) when structures, such as bridges and dams, were commonly designed from the principle of the breaking load. Total weight, total shearing force, and various other similar concepts were characteristic of the period. Criteria based on such "total" values and their counterparts, the "average" stress and safety factor, have since been replaced by design methods controlled by local stress intensities. From this standpoint the principle of returning back to the total values (as those in Eqs. 1 and 2) does not seem to be fully justified.

A. H. Davison,[14] M. ASCE.—For its presentation of factual data on experience with uplift pressures in concrete dams, as well as for the suggestion that such data are useful in considering uplift assumptions for proposed dams in order to avoid extravagant dam sections in design, this paper is a valuable addition to the literature.

Avoidance of extravagant dam sections in design may be attained by substituting a rational approach to the determination of uplift pressure for an assumed uplift pressure.[15]

A direct approach to the problem of determining uplift pressures involves, first, the computation of the hydraulic gradient through the dam and then the application of the corresponding uplift shown by the computed hydraulic gradient. Taking Fig. 8 as an example of factual data applicable to Marshall Ford Dam (in Texas), and applying the suggested procedure for comparison, the computed results check fairly closely with the factual data and indicate that the design assumption for uplift shown in Fig. 8 was unnecessarily conservative by about 10%.

Design considerations applicable to this example are:

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[13] "Experiments on Effective Uplift Area in Gravity Dams," by Serge Leliavsky Bey, *Transactions*, ASCE, Vol. 112, 1947, p. 444.

[14] Cons. Engr., Glen Falls, N. Y.

[15] *Transactions*, ASCE, Vol. 114, 1949, p. 239.

a. The section of dam considered is of homogeneous mass concrete, standing on rock foundation, and must be adequate to withstand the maximum design flood.

b. A design criterion for the adequacy of structure is that the resultant of overturning forces shall pass through the downstream limit of the middle third of the base of the dam. It is assumed that the concrete structure is set into the rock foundation sufficiently by construction methods to prevent sliding.

c. Natural foundation materials and materials of construction are elastic and compressible to a degree dependent on the material and on the amount and distribution of superimposed weight.

d. The Darcy law assumes that lost head through the structure increases uniformly along a straight line from the headwater to the tailwater in proportion to the distance along the line of creep of percolating water that would affect the hydraulic gradient. The assumption with respect to variation in the value of lost head is accepted as correct for the case of the theoretical rectangular wall, which represents uniform distribution of superimposed weight over any horizontal plane of the structure.

e. For other cross-sectional shapes than the theoretical rectangular wall, the variation of the lost head will be affected by: (1) The distribution of the weight of the structure across the width of the section, and (2) the variation in the position of the center of gravity of accumulated sectional area across the width of the section as denoted by coordinates measured from the source of hydrostatic pressure.

Fig. 22(*a*) has been drawn as a practical duplicate of Fig. 8, showing the typical dam section and adopting El. 486 as the average elevation of the base. Fig. 22(*b*) shows the theoretical rectangular wall constructed to the same critical dimensions as the practical dam section for the maximum design flood condition. Reference to Fig. 22(*b*) shows that the base width is 0.845 h to satisfy the design criterion for resistance to overturning, thus establishing the theoretical straight-line hydraulic gradient.

For a 1-ft length of homogeneous concrete gravity dam, l is the width and A is the area of progressive strips of the section, with x and y the corresponding coordinates of the center of gravity.

Friction head loss through the structure may be expressed by

$$h_f = K\,[w\,A\,(y + x)]^n \qquad (5)$$

in which h_f is the friction head, in feet; w is the weight of structure, in pounds per cubic foot; n is an exponent that varies with circumstances; and K is a dimensional coefficient that varies with circumstances. For a homogeneous structure like the concrete dam under discussion, Eq. 5 may become

$$h_f = K_1\,[A\,(y + x)]^n \qquad (6)$$

or as modified to apply it to this specific problem.

Tabulating the different elements for various values of l in Fig. 22(*b*) and satisfying the criterion that h_f must vary in a straight line from the reservoir

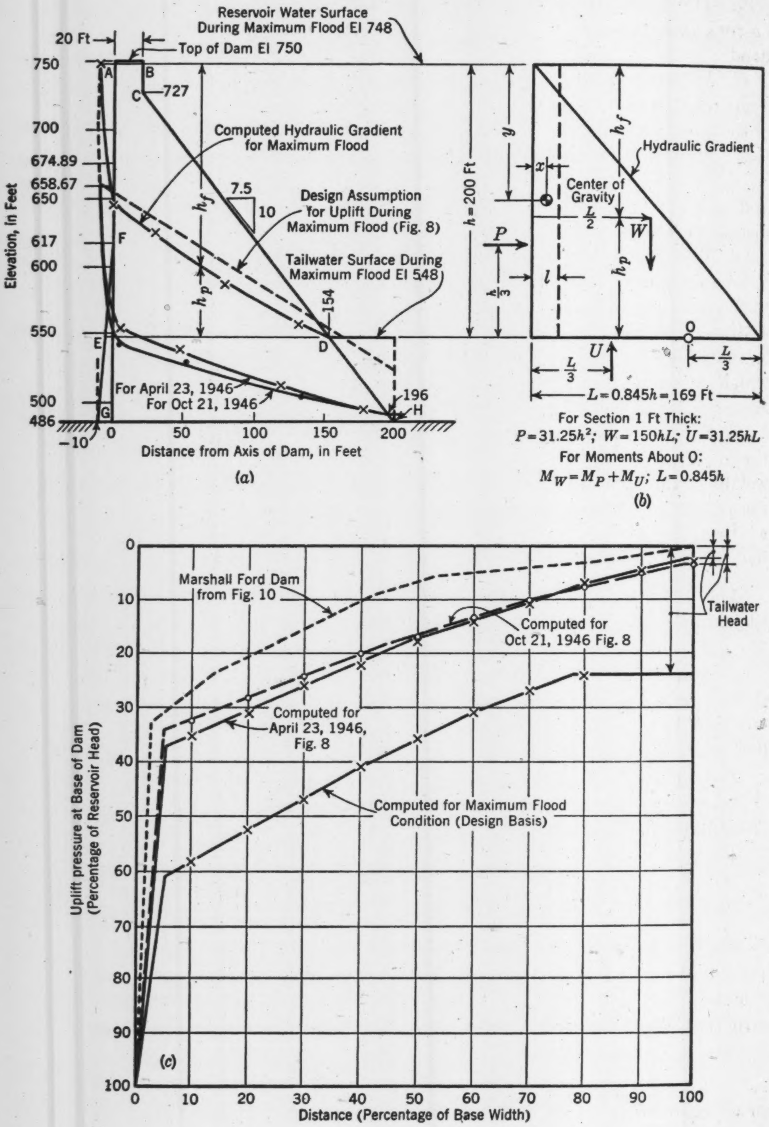
Reservoir Water Surface During Maximum Flood El 748
20 Ft
Top of Dam El 750
750
700
674.89
658.67
650
617
600
550
500
486
-10
A
B
C
727
D
E
F
G
H
154
196
Elevation, in Feet
Computed Hydraulic Gradient for Maximum Flood
7.5
10
Design Assumption for Uplift During Maximum Flood (Fig. 8)
Tailwater Surface During Maximum Flood El 548
h_f
h_p
For April 23, 1946
For Oct 21, 1946
0
50
100
150
200
Distance from Axis of Dam, in Feet
(a)
$h = 200$ Ft
y
x
Center of Gravity
$\frac{L}{2}$
P
W
$\frac{h}{3}$
l
U
O
$\frac{L}{3}$
$\frac{L}{3}$
h_f
h_p
Hydraulic Gradient
$L = 0.845h = 169$ Ft
For Section 1 Ft Thick:
$P = 31.25h^2$; $W = 150hL$; $U = 31.25hL$
For Moments About O:
$M_W = M_P + M_U$; $L = 0.845h$
(b)
0
10
20
30
40
50
60
70
80
90
100
Uplift pressure at Base of Dam (Percentage of Reservoir Head)
Marshall Ford Dam from Fig. 10
Computed for Oct 21, 1946 Fig. 8
Computed for April 23, 1946, Fig. 8
Computed for Maximum Flood Condition (Design Basis)
Tailwater Head
(c)
0
10
20
30
40
50
60
70
80
90
100
Distance (Percentage of Base Width)

FIG. 22

water surface to the tailwater surface, means that computed values of the compound factor $[A\ (y + x)]^n$ or some modification of that factor across the section must fall on a straight line. It is determined that the required factor is $\left[\frac{A\ (y + x)}{100{,}000} - 1.2\right]^{0.73}$ for this example.

The value of K_1 in Eq. 6 is established by considering that part of Fig. 22(*a*) which is bounded by the limiting elevations of the reservoir water surface and the tailwater surface for the maximum design flood. Tabulation of elements is made as before, and the value of K_1 is found by substituting the known value of 200 ft for h_f when $l = 154$ ft from the axis of the dam. The value of K_1 is thus determined as 16.9, and the formula applicable to Fig. 22(*a*) becomes

$$h_f = 16.9 \left[\frac{A\ (y + x)}{100{,}000} - 1.2\right]^{0.73} \qquad \text{(7)}$$

Application of Eq. 7 to the computation of the hydraulic gradient applicable to the total section area of Fig. 22(*a*) develops the three hydraulic gradients which conform to the three sets of factual data cited in Fig. 8. Also plotted for comparison is the "design assumption for uplift during maximum flood," taken from Fig. 8.

Fig. 22(*c*) shows a diagram of uplift pressure on a percentage basis similar to Fig. 10, reproducing the graph for Marshall Ford Dam. For comparison, the graphs corresponding to the computed hydraulic gradients in Fig. 22(*a*) have also been plotted.

Incidentally, the factual-data graph for April 23, 1946, in Fig. 8 appears somewhat erratic, possibly because insufficient time was allowed to establish equilibrium before the series of readings was taken. Possibly tailwater conditions, following or during a spring flood, had not become completely adjusted to headwater conditions.

DUFF A. ABRAMS[16] M. ASCE.—No answers to the vital questions of hydrostatic uplift under large concrete dams have been presented in this paper. The only statement that can be called a general conclusion (under the heading "Uplift Design Assumptions") is:

> "Considering the extraordinarily low uplift pressures so far indicated over a short period for Shasta Dam, as shown in Fig. 9, and the average uplift pressures indicated by Fig. 10 for several dams over comparatively long periods, one is faced with the question of whether the uplift assumptions of the past have, in general, been too conservative."

Admittedly readings at Shasta Dam (in California) were extraordinarily low, but it will be shown that they prove something more than that.

The author is faced with the question of whether uplift assumptions have been too conservative. That is exactly the question that faced an earlier generation of engineers when they inaugurated USBR's extensive program of research on uplift in 1915.

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[16] Cons. Engr., New York, N. Y.

The paper did bring out the disturbing fact that water under pressure was found at every point under all USBR concrete dams. Because of unscientific methods used, it is doubtful whether the full pressure of this water was determined in any instance. Examination of the data on uplift shows that the author misinterpreted the measurements and that the conclusions in the paper are erroneous and misleading.

Uplift Problems Not Solved.—Under the heading "Case History at Hoover Dam," the author states:

> "The essential case history of foundation uplift pressures at Hoover Dam is of interest. * * * Uplift pressures were first recorded in August, 1935, and have been continued to date. The higher pressures occurred during the first five years. Fig. 12, a running record for the holes in line B, reveals an increase in pressures until 1939 when remedial measures were undertaken by drilling more drain holes and by further foundation grouting."

In other words, Hoover Dam was no sooner in service than uplift became so alarming that search was begun for a "foundation treatment." After three or four years of "cut and try," grouting was adopted. This work was then so urgent that, in spite of shortages of labor and equipment, it was continued throughout World War II. Such episodes prove that uplift is real and that its problems have not been solved.

Averages of Averages.—The data supporting the paper consist of Fig. 10 and selected fragmentary uplift readings from five dams. Fig. 10 involves a bewildering array of averaging-out processes:

1. Averaging early semimonthly with later semiannual uplifts;
2. Averaging uplifts at three to seven different points in a given dam;
3. Averaging uplifts at pipes at widely different elevations;
4. Averaging readings over a wide range of reservoir heads;
5. Averaging uplifts measured by different methods;
6. Averaging uplifts before and after "foundation treatment";
7. Averaging all of the foregoing averages for a given dam;
8. Averaging of averages for small, medium, and large concrete dams.

This unscientific averaging is the principal basis for the conclusions in the paper. A low uplift in a small dam in Wyoming (Seminoe) cannot possibly offset the dangerous uplift under a large dam in Arizona-Nevada (Hoover), as implied by the average curve in Fig. 10. The author not only depends on the average curve but also draws it in disregard of the individual curves. At 20% of the base width the average uplift for eight dams is 22% of head; the average curve erroneously shows 27%.

At Hoover Dam on November 14, 1938, pipes B4 and C5 showed:

Elevations, in feet	B4	C5
Water in pipes	1,014	667
Tailwater	647	647
Uplift (feet of water)	367	20

Pipe B4 recorded eighteen and one half times as much uplift as pipe C5, but both pipes are about the same distance from the heel of the dam; hence they appear only as an average in Fig. 10. Pipe B4 revealed an uplift of 71% of the difference between lake and tailwater, or nearly three times the value assumed in design.

Fig. 10 shows that the downstream 60% of Hoover Dam had an average uplift from semimonthly, monthly, quarterly, and semiannual readings for fourteen years, equal to or greater than the "two-thirds full uplift" assumed in design. In October, 1938, measured uplift in pipe B6, 421 ft from the heel (Fig. 13) was 330 ft. The assumed uplift was 55 ft. In other words, with the lake only three-fourths full, the measured uplift was six times that assumed. All evidence suggests that, at full lake, the uplift at this point is about ten times that assumed in design.

For the first five years uplift under Hoover Dam was alarmingly high. It is stated that, for the next seven years, uplift was negligible; but in Fig. 10 the high and low values were averaged. The averaging-out processes conceal these dangerous conditions. The three instances cited may not represent the worst conditions at Hoover Dam.

The most important consideration here is that structures (including large concrete dams) are not designed for average loads.

Even the period covered by Fig. 10 is open to considerable doubt. The paper states that the period covered was "* * * to the time of preparation of the chart * * *." One might guess that meant to the end of 1949. The only instance in which an independent clew can be applied shows that guess is erroneous. The average curve for the American Falls Dam in Idaho (Fig. 7(*a*)) is reproduced in Fig. 10, but it included no data later than 1939.

TABLE 2.—EQUIVALENT ELEVATION OF WATER SURFACE IN PIPES

Pipe (Fig. 5) (1)	Elevation (ft) (2)	Δ (ft) (3)	Δ^2 (4)
1A	310.0	−1.5	2.25
1B	310.8	−0.7	0.49
1C	312.8	1.3	1.69
1D	313.0	1.5	2.25
1E	312.0	0.5	0.25
2B	312.1	0.6	0.36
2C	310.9	−0.6	0.36
2D	310.5	−1.0	1.00
2F	310.7	−0.8	0.64
3B	315.0	3.5	12.25
3C	315.0	3.5	12.25
3D	309.0	−2.5	6.25
3E	310.0	−1.5	2.25
3F	309.0	−2.5	6.25
Mean	311.5	0.0	3.47

Friant Dam (*in California*).—This section is confined largely to lines 1, 2, 3 (Fig. 5). Line 4 is 65 ft higher and, therefore, contributed little to the uplift under the dam. A statistical analysis of the equivalent elevations of the water surfaces in pipes for fourteen of the sixteen pipes in lines 1, 2, and 3 (June 13, 1944), is given in Table 2. The mean elevation of the water in the pipes was 4.5 ft above tailwater. The author computes uplift as $4.5/212 = 2.1\%$ of head. The square-root-mean-squared difference is the standard deviation, $D = (3.47)^{\frac{1}{2}} = 1.9$ ft. The coefficient of variation, $V = 1.9/311.5 = 0.6\%$. This low coefficient is the first thing to arouse one's suspicions. The readings are the more suspect since water was discharged from the pipes. Fig. 5 is the USBR standard uplift form; it was described under the heading "Installations for Observation of Foundation Uplift":

> "Three columns of the form are for records of pipes not discharging and four columns for pipe which, by reason of sufficient pressure, can discharge through the ½-in. outlet pipe connection."

Pipe 1A was entered in Fig. 5 as "not flowing," although water stood exactly at the top of the tee. The observer could not determine where to enter his record without first opening the stopcock. That dropped the pressure to 0; hence, the gages measured only small pressures built up during the few minutes required to prepare for readings.

As applied to concrete dams subject to uplift, two profound hydraulic principles are: (1) With a fountain running, the pressure at discharge is negligible; and, (2) with discharge cut off, the pressure rises instantly to that of the source. Uplift pipes are fountains designed for another purpose. In concrete dams the channels between the lake and the uplift pipes are more devious and constricted. The better the design and workmanship, the fewer and smaller the channels will be.

The entire treatment of the paper is based on two assumptions, first, that a large volume of water is required to produce uplift; and, second, that a large volume of water is always present at the base of a dam. Neither assumption is necessarily correct; in important instances both are erroneous. A film 0.00001 in. thick will transmit uplift. Movement toward an outlet would be extremely slow, since most of the head is lost as friction. Assume an initial flow of 11 in. per min at an orifice 4 in. wide at the bottom of an empty pipe. The discharge being 0.00044 cu in. per min, a period of 231/0.00044 = 525,000 min, or 1 year, will be required to discharge the first gallon. The rate would become slower as the pipe filled. Such a film can transmit dangerous uplift, but many years or centuries would be required to fill some of the long pipes; hence the pressure would never show on USBR gages. That is one reason why uplift was "slow in acting" in many instances.

The inside volume of a 2½-in. pipe is 0.4 gal per ft. Pipe B5 at Hoover Dam is 140 ft long and holds a barrel of water. Perhaps that explains why uplift always lagged behind (see Figs. 12, 13, and 14). Marshall Ford pipe 3Y is 40 ft long; Shasta pipe 4B is 65 ft long; both lagged behind the general trend. Small copper tubing should have been used. Cheaper and more elastic than steel, noncorroding copper tubing comes in rolls, can be strung anywhere without danger of leaks, and holds 1/600 as much water as a 2½-in. pipe.

In the opinion of the writer, Friant Dam readings are worthless as measures of uplift. Instead of uplifts of 1% to 3% of the head, all evidence indicates that, in the absence of bleeding of pipes, uplift would have been nearly 100% of the head over the entire base of the dam. The procedure at Friant Dam illustrates many of the unscientific features of USBR's uplift methods.

American Falls Dam.—The heavy zigzag lines in Fig. 7 show that the "average maximum uplift" was two and one half to four times the "average uplift." A comparison of the fourteen-year average with the "maximum point uplift" at Station 11 is given in Table 3. The maximum point uplift was 3.2 to ten times the fourteen-year average uplift. That is the true measure of the value of average curves in Fig. 10.

Marshall Ford Dam.—The author interprets Fig. 8 as proof of low uplift; but that is questionable in the face of the monotonous repetition of 495 ft and 496 ft, which appear fifty-seven times in 132 readings. Such wholesale coincidences in six pipes scattered over the base of a large dam do not represent uplift. The answer is to be found in the fact that the gallery floor is at El. 493 ft. Bleeding of the pipes is the only possible explanation. Gage readings of 10 ft to 50 ft represent the small build-up of pressures before readings were completed. True uplift (lake full) may be any value up to 748 − 548 = 200 ft of water over the entire base of the dam.

TABLE 3.—COMPARISON OF FOURTEEN–YEAR AVERAGE WITH THE MAXIMUM POINT UPLIFT

Description	Pipe Location[a]			
	12	32	52	73
14-yr average uplift (percentages from Fig. 7*a*)	18	6	3	2
Maximum point uplift (percentages from Fig. 7*b*)	83	19	19	20

[a] Pipe location as a percentage of the base width from the heel of the dam.

Shasta Dam.—"* * * the extraordinarily low uplift pressures * * * for Shasta Dam" were referred to at the beginning of this discussion. In 1947 the seventy-two monthly readings of six pipes 4D to 4I (Fig. 9), representing the downstream 350 ft of width of dam were:

Elevation, in feet	Number of occurrences
582	6
583	12
584	6
585	13
586	27
587	8

The mean elevation for the seventy-two readings is 585 ft; $D = 1.5$ ft; and $V = 1.5/585 = 0.3\%$.

Readings over the period of a year at six points under a large dam that show no response to changes of 47 ft and 39 ft in the lake, but come out with $V = 0.3\%$, cannot be accepted as measures of uplift. It is as likely that two thirds of all trees in a forest will be of uniform height as measured by $V = 0.3\%$. Some other explanation must be found.

As for Table 4, bleeding was practiced two years earlier on the same project; it was standard practice three years later. Analogy shows that Shasta pipes were bled before the gages were read. True uplift (lake full) may be anything up to 1065 − 585 = 480 ft of water over the entire base, or ninety-two times what the author interpreted as "the extraordinarily low uplift pressures."

Hoover Dam.—In the preceding paragraphs attention has been called to the wide discrepancy between uplifts in two similarly located pipes in Hoover Dam, to the excessive pressures in pipes B4 and B6, and to the erroneous practice of averaging high uplifts before, with low uplifts after, foundation treatment.

The author points to Fig. 14 to support the conclusion (under the heading "Case History at Hoover Dam") that "By 1942 the pressures had been reduced to nominal amounts that have remained generally constant since about 1942." In July, 1938, and in July, 1947, lake elevations were essentially the same. Uplift readings at the seven pipes are presented in Table 5.

TABLE 4.—COMPARISON OF UPLIFT READINGS IN TWO DAMS IN THE CENTRAL VALLEY OF CALIFORNIA

Dam (1)	Year of observation (2)	No. of pipes considered (3)	No. of uplift readings (4)	Coefficient of variation V (%) (5)	Water in pipes above tailwater (ft) (6)	Pipes were bled before readings (7)
Friant	1944	14	14	0.6	4.5	yes
Shasta	1947	6	72	0.3	5.2	not stated

The author's inquiry stopped there with the conclusion that the drops of 91 ft to 280 ft represent a decrease from "drilling more drain holes and by further foundation grouting." The writer's conclusions are entirely different.

The five pipes with highest 1938 readings were essentially uniform in 1947, suggesting that they had been "trimmed" artificially. For pipes B1, B3, B4, B6, twenty-four of the forty-eight monthly readings in 1947 were 707 ft. Those "generally constant" values should have attracted the author's attention. From the standpoint of probability there is about 1 chance in 40,000 that such a condition could occur. Readings at Friant Dam and Marshall Ford Dam pointed unerringly to the galleries. The paper identifies the Hoover gallery, where readings were made only as a powerhouse gallery. Official plans show the level floor of this gallery to be at 705.00 ft. The "generally constant" 707 ft is within a few millimeters of the exact elevation of the gage connections.

TABLE 5.—DROP IN HOOVER DAM UPLIFT READINGS IN NINE YEARS

Date	Lake elevation, in feet	WATER ELEVATION IN PIPES, IN FEET						
		1B	B1	B2	B3	B4	B5	B6
July 15, 1938......	1173	853	958	844	984	987	760	953
July 15, 1947......	1176	762	704	680	707	707	664	707
Drop............	....	91	254	164	277	280	96	246

Bleeding of pipes is the only possible explanation. In 1942 Hoover Dam pipes were bled without reading gages. The determining influence of the gallery elevation on uplift readings at Hoover Dam is shown also by the following comparison of heights of water, in the pipes, above tailwater:

Dam	Year	Height, in feet
Friant	1944	4½
Marshall Ford	1946	6
Shasta	1947	5
Hoover	1947	60

The only reason that the Hoover Dam pipes did not show the same small height above tailwater as the three other dams is that the gallery is 55 ft too high. Instead of a "nominal amount," true uplift (lake full) may be anything up to 1229 − 647 = 582 ft of water over the entire base of the dam.

The four pipes in the foregoing list define the uplift pattern for downstream as 80% of the dam. Little weight was given to pipes 1B, B2, and B5. Pipe 1B was read from another gallery and cannot be identified positively with a physical feature of the dam. Pipes B2 and B5 always lagged considerably behind (Fig. 12) and, hence, do not participate in the general pattern, probably because of leaks or other causes that the author should explain.

In the writer's opinion, there is no scientific evidence that the ten years of effort spent on supplemental grouting of the foundation had any effect in reducing uplift at Hoover Dam. Uplift was undoubtedly reduced somewhat by the repeated bleeding of pipes and additional drains, but chief reliance seems to have been placed on a systematic lowering of the lake elevation.

Bleeding of Pipes.—One version of this operation (as stated in the paragraph preceding Table 1) is:

> "Readings of the pressure gages having been made, the stopcocks are closed, the gages are removed, the stopcocks are opened, and measurement of the discharge is made in gallons, in cubic centimeters, or even in drops per minute."

After thus showing that bleeding of pipes was standard practice in 1950, the author states (under the heading "Installation for Observation of Foundation Uplift"):

> "As mentioned previously, after the pressure observations have been made, it has been the practice to remove the altitude gages and measure the discharge from the stopcocks. Naturally, the pressure dissipates for the time being and there is always the possibility that it may be fully restored before the next regular reading. Accordingly, the practice of estimating the amount of water discharged has been discontinued. It is doubtful if the revised procedure will make any material difference in the observations except in isolated cases."

No dams are named, and no dates are given. Bleeding of pipes is the dominating feature of USBR method. Hence the questions raised would have been most appropriate before the investigation had begun, but phrases such as "always the possibility," "may not be fully restored," "doubtful if," "any material difference," and "except in isolated cases," are strange expressions at the end of a scientific report on a thirty-five year research program.

Concerning Hoover Dam in 1936, the author writes (under the heading "Case History at Hoover Dam"):

> "* * * when the drains were closed intentionally, the uplift pressures increased and * * * when they were opened, there was some decrease in pressure."

Why, then, does the author in 1950 speculate as to the effect of opening uplift pipes?

At Hoover Dam April 15 to July 15, 1938, the lake rose for three months and then was stationary for three months. A brief study of the effect of this record on uplift is given in Table 6.

TABLE 6

Lake surface	Increase in Uplift, in Feet		
	Pipe 1B	Pipe B3	Pipe B6
Lake rose 72 ft in three months	17	45	38
Lake stationary for three months	117	25	24

Pipe 1B is nearest the heel, pipe B3 is nearest the middle, and pipe B6 is nearest the toe of the dam. As the lake rose 72 ft, water in pipes B3 and B6 rose two to three times as much as it did in pipe 1B. With the lake stationary, water in pipe 1B rose seven times as much as it did while the lake was rising 72 ft. There is evidence that uplift would have increased further had the lake been stationary longer. Bleeding of pipes would have destroyed all significant features of these tests. With those data in the books for eleven years it is difficult to understand either the author's speculations on delayed uplift action or the philosophy behind the practice of bleeding pipes while endeavoring to measure uplift.

Various phases of bleeding of uplift pipes are summarized as follows:

Year	Comment
1938	Hoover Dam (Fig. 13). No evidence of bleeding of pipes.
1940	Hoover Dam. Fig. 12 indicates that bleeding began coincident with "foundation treatment," and continued through 1947.
1942	Hoover Dam. Horizontal joint pipes bled without any reading of gages.
1944	Friant Dam (Fig. 5). Fourteen pipes gave a coefficient of variation *V* equal to 0.6%; the paper states that there was bleeding after the reading of pressure gages. Other evidence shows bleeding before a reading of the gages.
1946	Marshall Ford Dam (Fig. 8). El. 495 or El. 496 appear fifty-seven times in 132 readings of eleven pipes, and unerringly fix the elevation of the gallery; bleeding before reading the gages is the only possible explanation.

1947 Shasta Dam (Fig. 9). Seventy-two readings of six pipes gave a value of V equal to 0.3%; bleeding before reading the gages is the only possible explanation.

1947 Hoover Dam (Fig. 14). For four pipes, twenty-four of the forty-eight readings were 707 ft. This fact points unerringly to the elevation of the gallery floor; bleeding of pipes before reading is the only possible explanation.

1950 The paper gave four versions of USBR bleeding practices:
(*a*) Bleeding without reading gages (Hoover Dam, 1942).
(*b*) Bleeding before reading gages.
(*c*) Bleeding after reading gages.
(*d*) Bleeding of pipes has been discontinued.

Not only do the various versions in the paper conflict, but also the statement on which practice (*d*) is based is open to different interpretations. The writer interprets it to mean that bleeding has been discontinued; but the statement (paragraph preceding "Case History at Hoover Dam") was: "* * * the practice of estimating the amount of water discharged has been discontinued."

Repeated trials should not be necessary to convince an engineer that a pipe filled with compressed water will show discharge if a stopcock is opened. It is not clear what use has been made of discharge data. Whether the discharge is small or large does not influence uplift pressure. Bleeding converted the pipes into pressure-relieving drains.

Design Assumption.—There is no basis in the selected fragmentary data in the paper for either the uplift curves or the design assumption. The entire shape of the curves is dictated by two points—100% uplift at the heel and zero uplift at the toe—but both of those points are fictional. That is, 100 times 0 is the same as 0 times 100, and neither one contributes in the least to the uplift under the dam. The paper was made unnecessarily confusing by plotting uplift curves in Figs. 7 and 10 in the fourth quadrant and Figs. 8, 9, 13, and 14 in the first quadrant.

At Hoover Dam in 1938 (the only data free from evidence of bleeding), ten of the twelve readings of pipe B6 (nearest the toe) were higher than for pipe 1B (nearest the heel); in August, the difference was 107 ft. For 1936 to 1940 (before bleeding; see Fig. 12) the uplift near the toe was higher than near the heel. In other words, the first quadrant curves slope upward, whereas the author assumes a downward slope. With sufficient time, the uplift will be 100% of the head over the entire base of the dam, or three times what the author mistakenly calls two-thirds full uplift.

Origin of Uplift.—A dam that is nearly right is not good enough. Dams must be built safely or not at all. The designer cannot work intelligently until he knows what he is designing for or against. With concrete that is essentially impermeable, with several grout curtains, numerous drains, and an intimate bond between dam and rock, hydrostatic uplift that threatens to float the dam downstream is a decided anomaly. The paper provides no hint of the origins of uplift under USBR dams. Uplift cannot be treated as an isolated phenomenon. It is closely associated with: (1) Expansion of concrete near the rock contact because of temperature rise from delayed hydration of too much of the wrong kind of cement; (2) horizontal movement of a dam due to the thrust of a lake; (3) vertical movement due to compression of concrete and rock; and (4) tilting due to a combination of items 1, 2, and 3. In the writer's opinion, cause (1) is the most important. All those topics have been under intensive study since the the beginning of Hoover Dam in 1933; none was mentioned in the author's paper under the heading "Case History at Hoover Dam."

Kenneth B. Keener,[17] M. ASCE.—The observations, reported by Mr. Riegel, of the foundation uplift pressures under four high concrete dams constructed by the TVA are valuable in that they confirm the general results obtained by the USBR. By Fig. 17 it is seen that the average or composite pressure line for the TVA dams closely parallels that for the USBR dams although it is about 10% lower. Mr. Riegel's statement is unquestionably true: that the two lines would be closer together if only the measurements made at Hoover Dam, after supplemental drilling and grouting, were used and if the measurements at Owyhee Dam were omitted. One might go further and attempt to analyze the difference by the relative natural tightness of the foundations, by the degree of thoroughness in grouting the various foundations, or by a comparison of the foundation drainage systems. However, such a detailed study will not be undertaken, since it is sufficiently satisfying that Fig. 17 gives added reason for questioning "* * * whether the uplift assumptions of the past have, in general, been too conservative." (quoted from the paper under the heading "Uplift Design Assumptions"). Definitely, it appears that they have.

It seems reasonable to expect that some seepage will pass through the pores in the concrete between drains. With such an assumption the composite pressure line could not drop to zero where it crosses the line of drains. True, the pressure will be zero at any particular drain; but between drains it will be considerably higher, influenced by the size and spacing of the drain holes, by the water surface elevations of the headwater and tailwater, and by the permeability of the concrete and foundation rock. This is illustrated in Fig. 9: two rows of 3-in. drainage holes on 10-ft centers were drilled 50 ft deep from each of the drainage galleries in the region of pipes A and B, and yet the uplift pressures are considerably higher than the elevations of the galleries.

One is impressed with both the rapidity and effect on the uplift pressures obtained by closing drains at Norris and Hiwassee Dams. Those tests support the value of the continued operation of keeping all drains free and open.

The provisions for measuring pore pressure in the concrete of Norris Dam, similar to those employed at Gibson, Owyhee, and Hoover Dams, seem quite simple and should be conducive to direct results. Mr. Riegel reports that "* * * no definite values of pore pressure were obtained in this dam (Norris) during the four-year period of observation." His accounting of the methods and results of obtaining pore pressure seems to warrant a statement to the effect that at twenty-two different locations in the concrete of Norris Dam there was no pore pressure during the four-year period, and that pressures recorded at two other locations could be reasonably accounted for by cracks that led directly to definite water supplies or reservoirs.

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[17] Chf., Dams Div., Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo.

Less simple means were used to determine the pore pressures in the concrete in Hiwassee and Fontana Dams. The seven-year record of results (Fig. 19) at Hiwassee Dam on cells near the upstream face is enlightening, as is the fact that no appreciable pressures were detected downstream from the drainage wells. The latter information is evidence of the usefulness of foundation drains in reducing pore pressures in the concrete in addition to their primary purpose of reducing foundation uplift pressure. Noticeable pore pressures are to be expected in concrete dams near their upstream faces, dependent naturally on the relative permeability of the concrete. The important question is "How far downstream do these pore pressures exist?"

The difficulties encountered with measuring equipment at Fontana Dam accentuate the problems involved in obtaining lengthy records of uplift pressures in concrete when somewhat complicated and delicate means are utilized.

Mr. Creager has raised the subject of area of intensity of uplift pressure. The writer did not intend to open this subject for discussion, since it has been so thoroughly covered by L. F. Harza,[18] M. ASCE. However, in completing the record of work done by the USBR on uplift pressures, a brief accounting was given of tests made in the USBR laboratories to determine the effective area over which uplift pressure acts. In addition, a factual statement (quoted by Mr. Creager) was made of the maximum and most common assumptions used by the USBR in the past. Both references justified Mr. Creager in discussing that much-debated subject.

The USBR has never made any measurements of uplift on its existing dams which would pretend to show the percentage of area over which uplift forces act. It is doubtful that a procedure for such measurements could be devised. The measurements (made continuously since 1926, of the intensity of uplift at various points under its concrete dams) cannot be analyzed to determine the effective area. Undoubtedly, there are many highly qualified engineers who frankly admit that they do not know closely the percentage of effective area and who await with technical interest the results of testing that will solve this vexing problem. If the theory had been proved conclusively, there would not be such advocating of testing to solve this problem.

Because of the data obtained in the measurement of uplift by the USBR and the TVA, the shape of the uplift pressure diagram given by Mr. Creager is a logical choice. Any assumed percentage of the product of the maximum theoretical intensity and the entire area should reflect not only the characteristics of the foundation rock and the concrete but also the locations and depths of foundation grouting and drainage systems. The uplift assumption then becomes a problem for each particular dam, one to be solved by those who are familiar with the local information, with the construction designs, and with past experience under similar conditions.

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[18] "The Significance of Pore Pressure in Hydraulic Structures," by L. F. Harza, *Transactions*, ASCE, Vol. 114, 1949, p. 193.

Mr. Leliavsky states that the writer "* * * seems to be rather reserved in stating his conclusions explicitly." The paper is largely a recital of numerous results obtained by the measurement of uplift under concrete dams. The data obtained augment those published seventeen years earlier. It would appear that the earlier data were not sufficient to convince designers that the uplift pressure diagrams could be modified and the total uplift reduced. Thus, for almost two decades the ultraconservative assumptions of the past were continued, as may be seen in Figs. 7, 8, 9, and 10. It was thought that the designer could quickly draw his own conclusions and the writer might appear presumptuous in even suggesting such apparent ones. Nevertheless, the writer did state (under the heading "Uplift Design Assumptions") that

> "* * * an uplift assumption more in keeping with the observations that have been made would result in smaller cross sections and appreciable reduction in cost."

The title "Pore Pressures in Concrete Dams," suggested by Mr. Leliavsky, would have been unacceptable because few data relative to the pore pressure within the mass of the concrete were presented in the paper. Most of the observations (and all those that were positive) were taken at the foundation contact, or 3 ft into rock.

Mr. Leliavsky's comment that "Criteria based on * * * the 'average' stress and safety factor, have since been replaced by design methods controlled by local stress intensities" arouses one's curiosity as to what degree that trend has become established. Such an application seems impracticable at the present time. Although additional stresses due to the structural action of the concrete mass of the dam may be predicted approximately, the haphazard stresses introduced by the exigencies of construction by variable local temperature conditions and by changed foundation topography encountered only after excavation would seem to preclude any but an extremely rough control of design methods by local stress intensities.

Mr. Kirn subscribes to a valuable use of the recorded uplift observations by modifying the assumed uplift pressure diagram from the former straight-line distribution to one that recognizes the effect of adequate drainage systems and foundation grouting. Mr. Kirn's opinion (that Eq. 1 and Eq. 3 should seldom, if ever, be used for determining the stability of a gravity dam against failure by sliding) is agreed with by the writer for the reasons so ably presented. The probability that, upon obtaining the additional information now being gathered, the shear friction factor may be reduced below five is encouraging; reduction (for example, to four) would reduce the cost of concrete dams appreciably.

Mr. Kirn's discussion on the analysis of concrete gravity dams, although not particularly pertinent to the subject of the paper, is a worthy addition to technical literature on the design of such dams. He has explained admirably and satisfactorily why a gravity dam is a statically indeterminate structure.

Mr. Davison has presented an interesting discussion of his theory that the shape of the pressure distribution curve through a dam is dependent predominantly upon the weight distribution. He ignores the effects of drainage systems and grout curtains in the foundation. It might be called to mind again that the observations reported by the writer were of pressures between the foundation rock and concrete and not of pressures throughout the mass concrete. To the writer's knowledge, the assumption that pressures vary according to local stress intensities has not been proved by any laboratory or theoretical evidence that would indicate permeabilities to be modified greatly by small changes in stresses at the small unit loads used in dam design. It would appear that such factors as the degree of hydration of the cement, the concrete-mix proportions, and the duration of saturation of the concrete would affect permeability more than local stress intensity.

Nevertheless, Mr. Davison's conclusion (by his method of computation) that the design assumption for uplift, shown in Fig. 8, was conservative only by about 10% should be comforting to the designers of Marshall Ford Dam.

The writer (and it is ventured to include all the many engineers connected with the uplift observation program) makes no claim to perfection in the investigation described, as the reader may observe from the factual data presented. There was no attempt to conceal facts. However, to have recorded all data collected in the past would have required a volume of such size as to be prohibitive.

Mr. Abrams is disturbed over the fact that water under pressure was found at every point under all USBR dams. To the writer this discovery is as enlightening as (and no more disturbing than) a much earlier discovery that water under pressure is found against the upstream faces of all dams. The main purpose of the paper was to show the magnitude of the water pressure (truly a normal condition) under concrete dams as well as the distribution of those pressures as reflected by actual measurements. Such information would be of value to designers of dams. The reader has the opportunity of judging whether or not the methods were scientific, for those methods were explained in considerable detail.

Because of Mr. Abrams' deduction that in all cases water was tapped off immediately prior to taking the uplift pressure readings, it appears that he doubts the value of the observations. A rereading of the paragraph regarding the manner in which the readings are made (see the text following Fig. 4) should clarify this question. Discharge from uplift pipe installations was measured after the uplift pressure readings had been made. It is true that in a certain particularly tight foundation, it was observed that pressures did not come up sufficiently at the end of 15 days to match readings that formerly had been made at 30-day intervals. Hence, as stated later in the paper, the practice of obtaining discharge from uplift pressure pipes was discontinued in all installations. This was to some extent an improvement in the methods, although entirely unnecessary at most installations.

Mr. Abrams is quite critical of Fig. 10. As stated in the paper, the drawing was prepared in a manner similar to Fig. 7, which in turn is typical of what can be shown graphically with data from uplift readings. Charts for all of the dams listed in Fig. 10 were made in a manner identical to Fig. 7. The individual plottings in Fig. 10 are the averages from charts similar to Fig. 7. The average, which may be termed the grand average, on Fig. 10 indicates a general trend, and that is all it was intended to show. It may be seen from it that uplift pressure variation does not follow a straight line but is hyperbolic or exponential in form.

Mr. Abrams has questioned the use of iron pipe rather than small copper tubing for uplift equipment installations. It should be remembered, as stated in the paper, that holes are drilled through the uplift pipe into the foundation rock after initial grouting. If no pressure is shown at the gallery elevation and if pipes are conveniently situated, soundings are made by dropping a bell down the pipe. Both of those procedures would be extremely difficult, if not impossible, if copper tubing were utilized.

In closing his discussion on uplift at Hoover Dam, Mr. Abrams makes the bold prognostication that "With sufficient time, the uplift will be 100% of the head over the entire base of the dam * * *" In the face of all tests and theory on the flow of liquids through permeable media, it is impossible to maintain that pressures must not drop in some manner from headwater to tailwater. They cannot at any time be 100% of the total pressure throughout the permeable medium.

Perhaps if Mr. Abrams had had the opportunity to read Mr. Simonds' able and clarifying discussion prior to preparing his own, his sweeping criticisms of the grouting and drainage program at Hoover Dam would have been modified. In any event Mr. Simonds, unknowingly, did answer many of them.

Mr. Simmonds' discussion is a valuable addition to the paper in that it shows the effect of local geological and construction conditions upon the development of uplift pressure at Hoover Dam. It also shows how excessive pressures may be reduced by corrective measures.

Much has been learned from the extensive program of observations of uplift under concrete dams. The discussions have contributed greatly to the practical and theoretical knowledge of this subject, so necessary to those who have the responsibility of designing dams that will be secure from failure and will be economical in construction. Laboratory testing or theoretical studies can never be as satisfying as actual observations on structures in service.

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EFFECT OF SKEW ANGLE ON RIGID-FRAME REACTIONS

BY WALTER C. BOYER,[1] JUN. ASCE

WITH DISCUSSION BY MESSRS. D. A. NETTLETON, MAURICE BARRON, AND WALTER C. BOYER.

SYNOPSIS

Modern high-speed traffic requires grade-separation structures that conform to essential road alinement. The two-hinged, rigid-frame bridge is highly regarded for such use; but, because of past failures, some doubt exists as to the applicability of the skewed type of structure. The investigation on which this paper is based was made to confirm the validity of the analytical method of solution for reactive forces in a skewed structure with models. The variation of these forces was studied for a complete range of skew angles.

INTRODUCTION

The problem of the skew rigid frame has always held a prominent place in engineering study and research. It assumed notoriety in its initial stages when several failures, entailing loss of life, resulted from fallacious design procedure. Following this period of skew span failures, the designing engineer resorted to another design which became perpetuated in structures that exist today as monuments to his inability to cope with the skew bridge problem—that is, the sharp-bend bridge, which reduced the design to a right frame but which created a traffic hazard of the first magnitude.

The profession can view with pride the first complete and logical mathematical solution of the skewed span problem presented by J. Charles Rathbun,[2] M. ASCE, in 1924. In this analysis Mr. Rathbun correctly included the effects of torsion on the redundant reactions and the moments inherent in a skewed arch. The validity of his solution was questioned and consequently tests were conducted by the late George E. Beggs,[3] M. ASCE. The series of tests were

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[1] Asst. Prof., Civ. Eng., The Johns Hopkins Univ., Baltimore, Md.

[2] "Analysis of the Stresses in the Ring of a Concrete Skew Arch," by J. Charles Rathbun, *Transactions*, ASCE, Vol. LXXXVIII, 1924, p. 611.

[3] "Skew Arch Reactions Measured by Reciprocal Method," by George E. Beggs. *Engineering News-Record*, July 21, 1927, pp. 106–107.

made for a 30° angle of skew and for varying roadway widths. Since the publication of this work, doubt has still been raised concerning the validity of the mathematical solution when the skew angle becomes large and the slab wide. Thus, it is evident additional experimental research is required as a comparison to the analytical solution.

STATEMENT OF THE PROBLEM

This paper is concerned primarily with testing the validity of Mr. Rathbun's theory when applied to a two-hinged skewed rigid structure, and with showing the variation of the various reactive moments and forces with the angle of skew. The basic structure used had a slab 100 ft long with a leg height of 22 ft. This structure, when scaled down, was ideal for model study. It was tested for the effects of skew by varying the angle from 0° to 50°, in increments of 10°. The roadway slab width was varied from 20 ft to 40 ft in increments of 10 ft. The solution by these data was compared to the solution of the reactions of the structure by mathematical analyses.

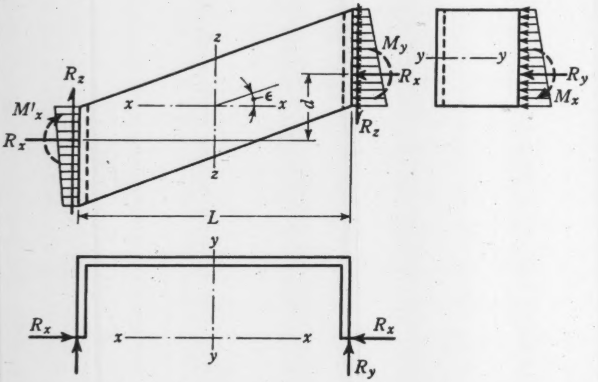

FIG. 1.—REACTIVE FORCES AND MOMENTS IN A TWO-HINGED SKEWED FRAME

The mathematical design procedure used was basically that proposed by Mr. Rathbun and applied by Arthur G. Hayden,[4] M. ASCE, to the two-hinged rigid frame. The nomenclature used for reactive forces is depicted in Fig. 1.

LABORATORY SETUP

The fundamental method used for determining the reactive elements was based on the deformeter method presented by Mr. Beggs.[5] This system has the advantage of yielding influence lines for reactions and moments by introducing controlled deformations at the reactions. In contrast to the procedure, followed by Mr. Beggs, of introducing extremely small deflections, the jig developed for this series of tests employed relatively larger deflections as prescribed in the simplification of William J. Eney,[6] M. ASCE. A diagram of the movable abutment through which controlled deformations and moments were introduced into the structure is shown in Fig. 2. A thrust along axis z to determine R_z may be applied by sliding the base plates along angle slides A, and maintaining this deformation by the gage blocks behind butt angle B. A thrust along axis x to determine R_x may be applied by sliding the channel section along the normal blocks, guided by the moment shaft. The deformation is controlled by placing gage blocks behind butt stop C. Moment M_x is applied by removing the normal blocks and rotating the channel section. The angle of twist is adjusted by the calibrated radial pitchholes in butt stop C. A pin is used to maintain this deformation. Moment M_y is applied by turning one base plate over the other about the moment fulcrum D. The angle is adjusted by the fixed radial pitch in the upper base plate and is held by a pin.

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[4] "The Rigid-Frame Bridge," by Arthur G. Hayden, John Wiley & Sons, Inc., New York, N. Y., 1941, pp. 137–182.

[5] "The Use of Models in the Solution of Indeterminate Structures," by George E. Beggs, *Journal of the Franklin Institute*, March, 1927, p. 375.

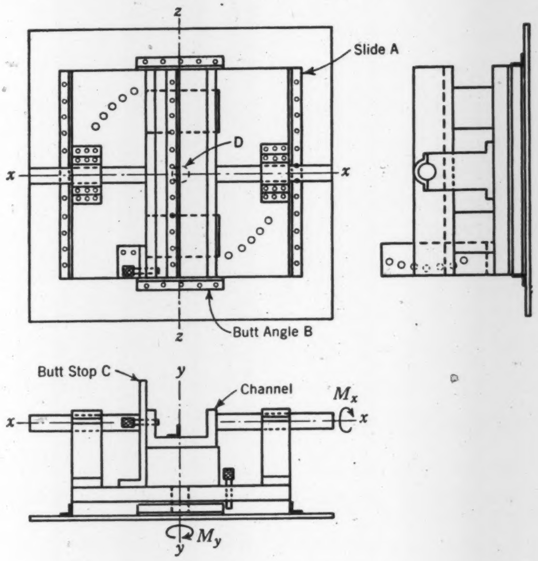

FIG. 2.—DIAGRAM OF TEST STAND

A converted hydraulic point gage, accurate to ±0.003 in. was used to measure vertical deflections along the slab of the model. To bring the point gage into contact with the model slab without deforming the surface, a conducting silver paste was applied to the surface of the model. A "magic eye" tube (6E5) was used to determine the exact contact of the point gage with the silvered slab surface.

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[6] "Stress Analysis with Elastic Models and a New Deformeter Apparatus," by William J. Eney, *Report No. 214.19*, Lehigh Univ., Bethlehem, Pa., 1941.

The models were constructed to a scale of 1 in. = 5 ft, thus giving a span of 20 in. for a prototype of 100 ft. The ratio of the moment of inertia of the slabs to the moment of inertia of the legs was 1:8. The slab was connected to the legs by steel clamps making a rigid joint. Hinged ends were contrived by adopting a bank of cabinet hinges, carefully selected, to reduce friction to a minimum.

Several materials were tested for model construction, and grade XXX phenolic resin was ultimately selected since it exhibited a minimum of cold flow characteristics.

Fig. 3 shows the grid system used to give complete deflection coverage of the slab. After a controlled movement had been introduced by the test frame, vertical deflection readings were taken at each of the grid points by the point gage mounted on a fixed and independent datum plane. The datum plane used was a 3-ft by 3-ft steel surface plate, 1.5 in. thick, accurate to 0.001 in. A photograph of the test apparatus in normal position is shown in Fig. 4.

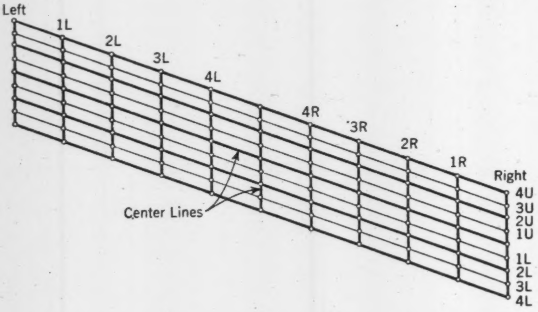

FIG. 3.—PLAN OF A SKEWED FRAME SHOWING THE GRID SYSTEM FOR DEFLECTION READINGS

EXAMPLE OF A 20-FT ROADWAY WITH A SKEW ANGLE OF 30°

To determine the thrust along R_x, the test stand was given a deflection of 0.5 in. along the x-axis from the normal position, this deflection being introduced in both directions. Vertical deflection readings were taken at all grid points along the surface from the normal position in both directions. Thus, the difference in readings represented the vertical deflections of the slab for an abutment deflection of 1 in. The readings taken for the center line of the model were as recorded in Cols. 2 and 3, Table 1.

Since the point gage reads in centimeters the difference values (Col. 4, Table 1) must be divided by 2.54 to convert them to influence ordinates. The influence ordinates along other longitudinal sections have the same values as those indicated for the center line. The influence ordinates for R_z were found in the same manner as those for R_x.

To determine the moment M_x, the channel was turned first in a clockwise direction. Vertical deflection readings were taken at all grid points along the surface of the model slab. The channel was then turned in a counterclockwise direction and readings were taken again. The readings thus taken along line 4U, Fig. 3, were as shown in Cols. 5 and 6, Table 1.

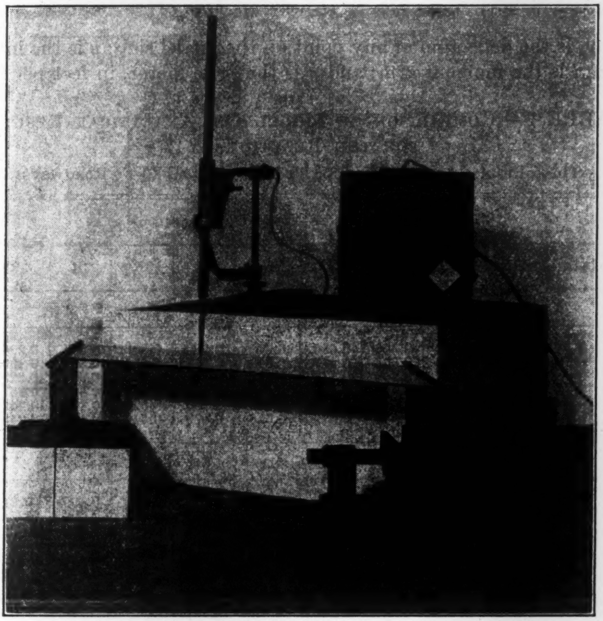

FIG. 4.—TEST APPARATUS IN NORMAL POSITION

TABLE 1.—TESTS IN A MODEL OF A 30° SKEW BRIDGE, 20 FT WIDE

Road section (Fig. 3)	THRUST ALONG R_x			MOMENT, M_x			Influence ordinate (line 2, Fig. 3)
	Movement left	Movement right	Difference	Clockwise	Counterclockwise	Difference	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Left end	19.55	19.55	0	19.56	19.55	0.01	0
1L	19.17	18.54	0.63	19.16	19.09	0.07	− 0.64
2L	19.09	18.07	1.02	19.15	18.90	0.25	− 2.25
3L	19.05	17.73	1.32	19.15	18.75	0.40	− 3.67
4L	19.01	17.52	1.49	19.23	18.61	0.62	− 5.68
Center	19.01	17.46	1.55	19.27	18.51	0.76	− 6.96
4R	19.03	17.54	1.49	19.34	18.46	0.88	− 8.07
3R	19.06	17.75	1.31	19.45	18.45	1.00	− 9.16
2R	19.11	18.09	1.02	19.57	18.46	1.11	−10.20
1R	19.14	18.50	0.68	19.67	18.47	1.20	−11.00
Right end	19.46	19.46	0	20.00	18.74	1.26	−11.52

To convert the difference values to inches, they must be divided by 2.54. The influence ordinates for moment were then determined by the relationship:

$$M_x = 1 \times \frac{y_1}{y} \times l \times n \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which y_1 is the deflection of any point on the model slab; y is the introduced deflection; l is the moment arm; and n is the scale factor, in feet per inch.

TABLE 2.—COMPARISON OF MODEL AND ANALYTICAL RESULTS FOR THE REACTIONS R

(These Results Are the Same for 20-Ft, 30-Ft and 40-Ft Roadways)

Skew angle	INFLUENCE POINTS ALONG CENTERLINE OF SLAB										
	Left	1L	2L	3L	4L	Center	4R	3R	2R	1R	Right
					(*a*) MODEL RESULTS, R_x						
0	0	0.228	0.394	0.506	0.580	0.612	0.580	0.506	0.394	0.228	0
10	0	0.209	0.330	0.505	0.577	0.612	0.577	0.505	0.330	0.209	0
20	0	0.228	0.394	0.506	0.571	0.612	0.577	0.506	0.394	0.221	0
30	0	0.247	0.402	0.520	0.588	0.610	0.588	0.518	0.402	0.252	0
30	0	0.236	0.406	0.510	0.586	0.615	0.590	0.524	0.414	0.242	0
50	0	0.248	0.430	0.545	0.579	0.618	0.579	0.545	0.430	0.248	0
					(*b*) ANALYTICAL RESULTS, R_x						
0	0	0.224	0.399	0.523	0.598	0.623	0.598	0.523	0.399	0.224	0
10	0	0.224	0.399	0.523	0.598	0.621	0.598	0.523	0.399	0.224	0
20	0	0.224	0.400	0.524	0.598	0.622	0.598	0.524	0.400	0.224	0
30	0	0.224	0.399	0.523	0.598	0.622	0.598	0.523	0.399	0.224	0
40	0	0.224	0.400	0.523	0.598	0.623	0.598	0.523	0.400	0.224	0
50	0	0.224	0.398	0.523	0.598	0.623	0.598	0.523	0.398	0.224	0
					(*c*) MODEL RESULTS, R_s						
0	0	0	0	0	0	0	0	0	0	0	0
10	0	0.039	0.067	0.091	0.098	0.101	0.098	0.091	0.067	0.039	0
20	0	0.083	0.138	0.188	0.212	0.224	0.212	0.186	0.138	0.083	0
30	0	0.122	0.220	0.287	0.319	0.331	0.319	0.287	0.220	0.122	0
40	0	0.161	0.315	0.405	0.471	0.480	0.471	0.405	0.315	0.161	0
50	0	0.252	0.470	0.582	0.680	0.705	0.680	0.582	0.470	0.252	0
					(*d*) ANALYTICAL RESULTS, R_s						
0	0	0	0	0	0	0	0	0	0	0	0
10	0	0.039	0.070	0.092	0.105	0.109	0.105	0.092	0.070	0.39	0
20	0	0.081	0.144	0.190	0.217	0.227	0.217	0.190	0.144	0.081	0
30	0	0.129	0.230	0.301	0.345	0.359	0.345	0.301	0.230	0.129	0
40	0	0.188	0.334	0.439	0.502	0.523	0.502	0.439	0.334	0.188	0
50	0	0.266	0.474	0.623	0.713	0.743	0.713	0.623	0.474	0.266	0

The introduced deflection was 2.4 in. with a moment arm of 11.2 in. The scale of the model was 1 in. = 5 ft. An example of the conversion of difference values, for 1R, is: $M_x = \frac{1.20}{2.54} \times \frac{1}{2.4} \times 11.8 \times 3 = 11.0$ ft-lb. The influence ordinates along line 4U are given in Col. 8, Table 1.

Moment M_y was determined in the same manner as M_x. For moment applications that did not strain the model, no appreciable deflections could be

TABLE 3.—Comparison of Model and Analytical Results for Moment, M_x at Right Abutment

(Roadway Width, 20 Ft)

Transverse section	Influence Ordinates at Grid Points of Slab										
	Left	1L	2L	3L	4L	Center	4R	3R	2R	1R	Right
(*a*) Model Results; Skew Angle, 0°											
4U	0	−0.55	−1.30	−2.28	−3.50	−5.04	− 6.50	− 7.50	− 8.70	− 9.40	−10.00
3U	0	−0.46	−0.94	−1.64	−2.55	−3.94	− 4.76	− 5.94	− 6.80	− 7.50	− 7.50
2U	0	−0.28	−0.63	−1.00	−1.64	−2.55	− 3.12	− 4.03	− 4.49	− 5.04	− 5.00
1U	0	0.18	−0.32	−0.52	−0.76	−1.10	− 1.74	− 2.11	− 2.24	− 2.50	− 2.50
Center	0	0	0	0	0	0	0	0	0	0	0
1L	0	+0.18	+0.32	+0.52	+0.76	+1.10	+ 1.74	+ 2.11	+ 2.24	+ 2.50	+ 2.50
2L	0	+0.28	+0.63	+1.00	+1.64	+2.55	+ 3.12	+ 4.03	+ 4.49	+ 5.04	+ 5.00
3L	0	+0.46	+0.94	+1.64	+2.55	+3.94	+ 4.76	+ 5.94	+ 6.80	+ 7.50	+ 7.50
4L	0	+0.55	+1.30	+2.28	+3.50	+5.04	+ 6.50	+ 7.50	+ 8.70	+ 9.40	+10.00
(*b*) Analytical Results; Skew Angle, 0°											
Center	0	0	0	0	0	0	0	0	0	0	0
(*c*) Model Results; Skew Angle, 10°											
4U	0	−0.73	−1.83	−3.12	−4.31	−5.77	− 6.77	− 7.70	− 8.80	− 9.70	−10.13
3U	0	−0.64	−1.56	−2.57	−3.57	−4.49	− 5.32	− 6.14	− 6.87	− 7.60	− 7.60
2U	0	−0.55	−1.19	−1.74	−2.75	−3.21	− 3.84	− 4.49	− 4.85	− 5.32	− 5.04
1U	0	−0.27	−0.73	−1.10	−1.56	−1.92	− 2.20	− 2.47	− 2.66	− 2.74	− 2.56
Center	0	−0.18	−0.36	−0.55	−0.64	−0.64	− 0.54	− 0.55	− 0.27	− 0.27	0
1L	0	0	+0.09	+0.37	+0.37	+0.82	+ 1.10	+ 1.37	+ 1.74	+ 2.20	+ 2.56
2L	0	+0.09	+0.55	+1.01	+1.47	+2.11	+ 2.75	+ 3.30	+ 3.84	+ 4.58	+ 5.04
3L	0	+0.55	+1.10	+1.93	+2.75	+3.57	+ 4.58	+ 5.32	+ 6.23	+ 7.15	+ 7.60
4L	0	+0.73	+1.65	+2.66	+3.85	+4.86	+ 5.86	+ 7.05	+ 8.43	+ 9.45	+10.15
(*d*) Analytical Results; Skew Angle, 10°											
Center	0	−0.011	−0.019	−0.031	−0.030	−0.028	− 0.030	− 0.031	− 0.019	− 0.011	0
(*e*) Model Results; Skew Angle, 20°											
4U	0	−0.46	−1.83	−3.21	−4.58	−6.05	− 7.51	− 8.52	− 9.43	−10.50	−10.65
3U	0	−0.37	−1.56	−2.84	−3.94	−5.13	− 6.14	− 7.05	− 7.60	− 7.98	− 7.98
2U	0	−0.37	−1.28	−2.20	−3.04	−3.58	− 4.58	− 5.22	− 5.68	− 5.58	− 5.32
1U	0	−0.28	−0.83	−1.47	−2.02	−2.38	− 2.93	− 3.20	− 3.44	− 3.30	− 2.66
Center	0	−0.09	−0.28	−0.64	−1.10	−1.20	− 1.10	− 0.64	− 0.28	− 0.28	0
1L	0	+0.09	0	+0.09	+0.18	+0.27	+ 0.55	+ 0.82	+ 1.28	+ 1.83	+ 2.66
2L	0	+0.45	+0.46	+0.82	+1.12	+1.56	+ 2.20	+ 2.84	+ 3.68	+ 4.30	+ 5.32
3L	0	+0.73	+0.91	+1.74	+2.47	+3.20	+ 4.04	+ 5.10	+ 6.14	+ 7.24	+ 7.98
4L	0	....	+1.74	+2.66	+3.67	+3.76	+ 5.95	+ 7.05	+ 8.42	+ 9.90	+10.65
(*f*) Analytical Results; Skew Angle, 20°											
Center	0	−0.025	−0.047	−0.067	−0.063	−0.059	− 0.063	− 0.067	− 0.047	− 0.025	0
(*g*) Model Results; Skew Angle, 30°											
4U	0	−0.64	−2.25	−3.67	−5.68	−5.96	− 8.07	− 9.16	−10.20	−11.00	−11.52
3U	0	−0.64	−1.93	−3.57	−4.76	−5.68	− 6.78	− 7.33	− 8.07	− 8.52	− 8.64
2U	0	−0.46	−1.55	−2.94	−3.85	−4.30	− 4.86	− 5.40	− 5.32	− 5.50	− 5.76
1U	0	−0.27	−1.19	−2.20	−2.84	−2.93	− 3.20	− 3.30	− 3.30	− 3.12	− 2.88
Center	0	−0.18	−0.64	−0.83	−1.55	−1.73	− 1.55	− 0.83	− 0.64	− 0.18	0
1L	0	−0.18	−0.27	−0.27	−0.27	0	+ 0.46	+ 1.19	+ 1.56	+ 2.38	+ 2.88
2L	0	−0.18	+0.09	+0.46	+1.00	+1.74	+ 2.56	+ 3.48	+ 4.40	+ 5.50	+ 5.76
3L	0	0	+0.82	+1.28	+2.28	+3.30	+ 4.40	+ 5.68	+ 6.96	+ 8.33	+ 8.64
4L	0	+0.27	+1.28	+8.20	+3.48	+4.86	+ 6.24	+ 7.88	+ 9.25	+11.10	+11.52

TABLE 3.—(*Continued*)

Transverse section	Influence Ordinates at Grid Points of Slab										
	Left	1L	2L	3L	4L	Center	4R	3R	2R	1R	Right
(*h*) Analytical Results; Skew Angle, 30°											
4U	0	−1.32	−2.66	−3.96	−5.23	−6.47	−7.79	−9.07	−10.32	−11.55	....
Center	0	−0.036	−0.072	−0.113	−0.099	−0.091	−0.099	−0.113	−0.072	−0.036	0
4L	0	−0.05	+1.17	+2.43	+3.71	+5.03	+5.26	+7.54	+8.54	+10.17	....
(*i*) Model Results; Skew Angle, 40°											
4U	0	−1.11	−3.58	−5.13	−6.86	−8.42	−9.52	−10.72	−11.44	−12.30	−13.05
3U	0	−1.00	−3.30	−4.76	−6.14	−7.15	−7.95	−8.70	−8.96	−9.61	−9.89
2U	0	−1.00	−2.94	−4.03	−4.95	−5.50	−6.31	−6.50	−6.50	−6.67	−6.60
1U	0	−1.00	−2.65	−3.20	−3.93	−3.93	−4.30	−4.30	−4.10	−3.48	−3.94
Center	0	−1.00	−2.00	−2.28	−2.65	−2.38	−2.65	−2.19	−2.10	−1.19	0
1L	0	−0.91	−1.28	−1.37	−1.19	−0.64	−0.27	+0.82	+1.74	+2.75	+3.94
2L	0	−0.46	−0.55	−0.18	+0.46	+1.28	+2.29	+3.48	+4.76	+6.14	+6.60
3L	0	−0.09	+0.36	+0.64	+2.20	+3.12	+4.49	+5.77	+7.79	+9.43	+9.89
4L	0	+0.46	+1.00	+2.20	+3.30	+5.04	+6.41	+8.34	+10.35	+12.35	+13.05
(*j*) Analytical Results; Skew Angle, 40°											
Center	0	−0.060	−0.115	−0.139	−0.135	0.134	−0.135	−0.139	−0.115	−0.060	0
(*k*) Model Results; Skew Angle, 50°											
4U	0	−1.30	−3.32	−5.24	−7.21	−8.80	−10.00	−11.48	−12.61	−13.87	−15.59
3U	0	−1.18	−3.20	−5.12	−7.10	−8.10	−9.20	−10.46	−11.00	−11.20	−11.70
2U	0	−1.10	−3.04	−5.00	−7.00	−7.50	−8.00	−8.30	−8.30	−8.30	−7.79
1U	0	−1.00	−2.76	−4.04	−6.20	−6.50	−6.00	−6.00	−5.20	−4.36	−3.90
Center	0	−1.00	−2.24	−3.26	−4.32	−4.64	−4.32	−3.26	−2.24	−1.00	0
1L	0	−0.76	−1.56	−2.55	−3.00	−3.00	−2.25	−1.00	+1.00	+3.21	+3.90
2L	0	−0.64	−0.93	−1.17	−1.00	−0.64	+1.34	+3.21	+5.70	+6.80	+7.79
3L	0	−0.42	−0.09	+0.09	+1.00	+2.42	+3.94	+7.00	+9.84	+11.05	+11.70
4L	0	+0.09	+0.64	+1.35	+2.84	+4.92	+7.40	+10.81	+12.88	+15.15	+15.59
(*l*) Analytical Results; Skew Angle, 50°											
Center	0	−0.079	−0.164	−0.205	−0.196	−0.189	−0.196	−0.205	−0.164	−0.079	0

detected in the slab. For the model slab, the function M_y was thus taken as zero.

Test Results.—The basic frame tested had a span length of 100 ft constructed to a scale of 1 in. 5 ft. Roadway widths of 20 ft, 30 ft, and 40 ft were considered. The angle of skew was varied from 0° to 50° in 10° increments. The results of this test program are reported in Tables 2, 3, and 4, and influence contour lines are drawn for the function M_x since this is the only function that gives variable influence ordinates along transverse sections of the bridge slab. Analytical results are given first in each case so that comparisons may be made with the model results in alternate lists.

The model results for R_x and R_z reported in Table 2 are those determined for the 20-ft roadway. Results for the 30-ft and 40-ft roadways were essentially the same.

The results for M_x reported in Table 3 are for 20-ft roadway. The results for the 30-ft and 40-ft roadways are similar about the center line of the structure. The analytical comparison was made for the centerline influence ordinates only (Table 4), except in the case of the 30° skewed frame. Since it was desirable to check off center ordinates also, these calculations, which are quite laborious, were made for this frame and are reported in Table 4.

TABLE 4.—ANALYTICAL RESULTS FOR ABUTMENT MOMENTS, M_y

(These Results Are Applicable to 20-Ft, 30-Ft, and 40-Ft Roadways)

Skew angle	INFLUENCE POINTS ALONG CENTER LINE OF SLAB										
	Left	1L	2L	3L	4L	Center	4R	3R	2R	1R	Right
0	0	0	0	0	0	0	0	0		0	0
10	0	+0.002	+0.003	+0.008	−0.001	−0.011	−0.001	+0.008	+0.003	+0.002	0
20	0	+0.003	+0.010	+0.022	+0.003	−0.015	+0.003	+0.022	+0.010	+0.003	0
30	0	+0.005	+0.017	+0.038	+0.008	−0.025	+0.008	+0.038	+0.017	+0.005	0
40	0	+0.009	+0.041	+0.0005	−0.018	−0.039	−0.018	+0.0005	+0.041	+0.009	0
50	0	+0.016	+0.055	+0.008	+0.029	−0.063	−0.029	+0.008	+0.055	+0.016	0

VARIATION OF REACTIONS WITH SKEW ANGLE

The variation of rigid-frame reactions with skew angles as determined by theory and model study can now be determined. Tables 2(*a*) and 2(*b*) indicate that the horizontal reaction R_x is essentially the same for various skew angles, as is substantiated by both the equation study (Table 2(*b*)) and the model study (Table 2(*a*)). This condition has been previously stated by Richard M. Hodges and is embodied in his important simplification of the two-hinged rigid-frame bridge. He states:[7]

> "* * * The redundant reaction components H acting at the supports of a two-hinged rectangular rigid frame are so nearly identical with the corresponding redundant reaction components R_x acting at the supports of a two-hinged skewed rigid frame of the same right span, that for purposes of practical design, the difference is negligible. This fact has been attested by comparative analyses, regardless of skew, span, or deck curvature."

The variation of the reaction component, R_z, is indicated in Tables 2(*c*) and 2(*d*). For the flat-top bridge tested, the major part of R_z is equal to $R_x \tan \epsilon$, in which ϵ is the skew angle. The reactive moment M_x is not a part of R_z and the reactive moment M_y is practically negligible in its influence. Consequently, the reaction R_z varies almost directly with the R_x-ordinates multiplied by the tangent of the skew angle. This fact has been shown by the analytical solution and substantiated by the model study. Comparison shows that results do not vary more than 5%.

The variation of the reactive moment M_x is indicated in Table 3. It can be noted from Fig. 5 that the contour diagrams are warped surfaces. The moment M_x produced the most significant disagreement between analytical and model results. Along grid lines 4U and 4L the model results and the analytical results check well within the limits of model analyses. This fact was ascertained for the frame with a skew angle of 30° (Fig. 5(*b*)). However,

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[7] "Simplified Analysis of Skewed Reinforced Concrete Frames and Arches," by Richard M. Hodges *Transactions*, ASCE, Vol. 109, 1944, p. 915.

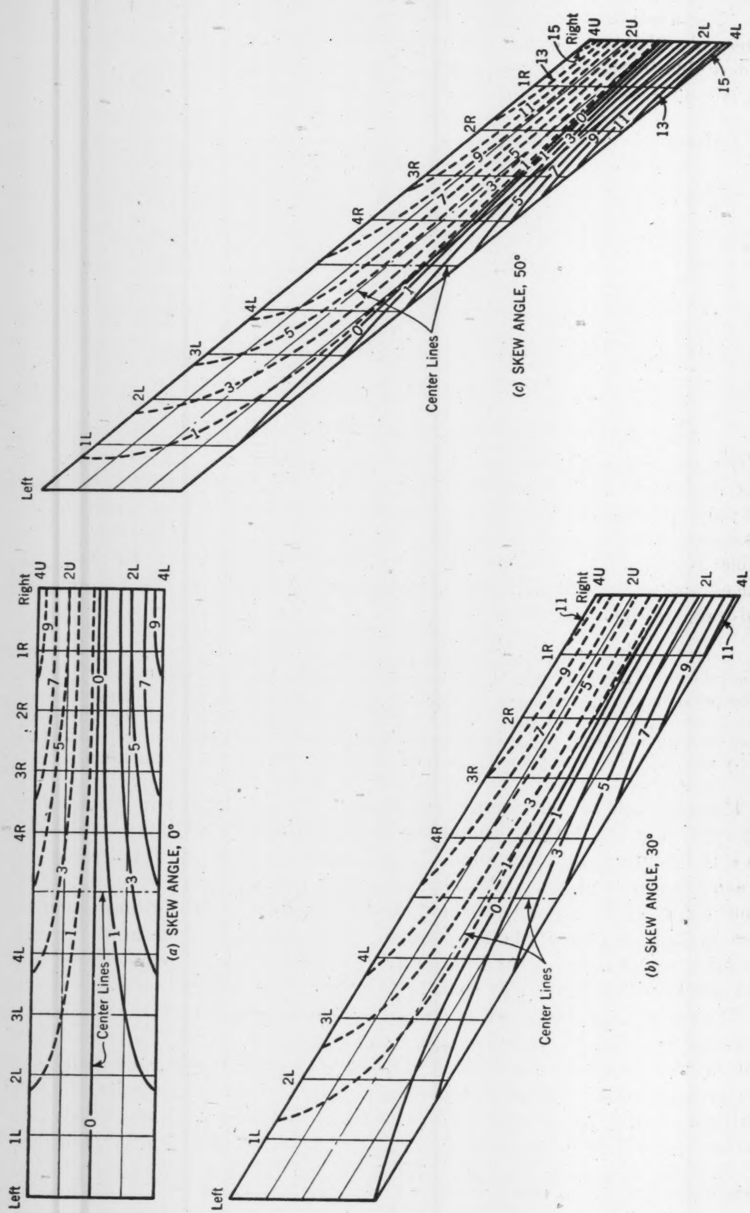

FIG. 5.—MOMENT INFLUENCE CONTOURS FOR FUNCTION M_x AT THE RIGHT

the center-line influence ordinates show the greatest variation between model and analytical results. Since the center-line influence ordinates are normally used in design, theory will tend to give results substantially below model results. This deviation becomes pronounced as the skew angle becomes large. In the writer's opinion, the factor that contributes most to the deviation of results is the torsional factor. The value used in computations was found for beams having width-to-depth ratios of 1 to 4, whereas rigid-frame proportions may vary as much as 20 to 1.

The variation of the reactive moment M_y was found to be almost negligible for the frame tested. The model surfaces did not produce enough deflection to make deflection readings practicable.

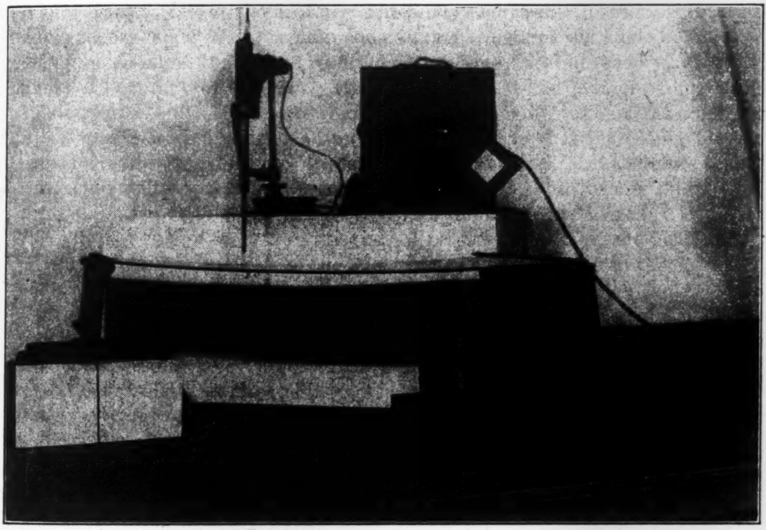

FIG. 6.—VIEW OF MODEL DEFLECTED IN THE x-DIRECTION

The test results indicate that for the two-hinged rigid-frame bridge the width of roadway does not influence the reactive elements, R_x and R_z. This fact is readily apparent when one considers that the deflected structure takes an unwarped, bowed shape for deflections along R_x and R_z. The model frame for reaction R_x is shown in Fig. 6. The model is deflected in the x-direction. The unwarped, bowed shape of the slab is typical of the results found for functions R_x and R_z. The magnitude of reactive element M_x varies almost directly with the roadway width and skew angle.

The model analysis method has proved to be very fruitful in checking the reactive elements in a skewed frame. In addition to yielding quantitative results, it has clearly indicated the qualitative trend of reactive elements.

Thus, Mr. Hodges' simplification may well be applied to such problems as the rigid frame with variable leg lengths.

ACKNOWLEDGMENT

The investigation on which this paper is based was reported in a thesis by the writer, entitled "The Effect of Skew Angle on Rigid Frame Reactions," which was presented to The Johns Hopkins University at Baltimore, Md., in June, 1947, in partial fulfilment of the requirements for the degree of Doctor of Engineering.

DISCUSSION

D. A. NETTLETON,[8] Assoc. M. ASCE.—An investigation such as this is interesting because it throws light upon a problem of stress analysis that has never been correctly solved by analytical methods.

Table 2 presents a comparison between model results and analytical results for reactions R_x and R_z. The model results for loads at points 1L, 2L, and 3L indicate an increase in R_x for a skew angle of 50° of from 8% to 9% over the values for zero skew. No increase is shown for a load at 4L, and a 1% increase is shown for a load at the center. The analytical results, based on Mr. Rathbun's theory, show R_x to remain constant regardless of the angle of skew. It might be thought that these results confirm the Rathbun theory, within the limits of accuracy of the test. However, the fixed-end moment for a slab of 100-ft span is only 7.5% more than the knee moment for the right-angle rigid frame under investigation. Many engineers believe that increasing the skew angle increases the stiffness of the leg relative to the deck, and therefore that the corner moment approaches the fixed-end moment with increasing skew. Since the maximum possible increase in the corner moment (and therefore R_x) is only 7.5%, it is evident that the variation in results of the test are such that no conclusion can be drawn as to the effect of skew on R_x, or as to the correctness of the analytical theory.

Table 3 presents a comparison between model results and analytical results for the reaction M_x. For a load at the center and for the various skew angles, the model results give values of the order of twenty times greater than the analytical results. It may be thought that the value of M_x is of little use in the design of a rigid-frame bridge, and therefore that this enormous variation is of no importance. However, such is not the case. Consider the eccentricity in vertical reaction represented by the model value of $-$ 4.64 for M_x due to a unit load at the center of the frame skewed at an angle of 50°. The vertical reaction is 0.5, and the eccentricity, therefore, is 9.28 ft toward the obtuse corner. Since the total width of the abutment for this skew is 31.12 ft, the vertical pressure on the foundation at the obtuse corner is more than three times the average pressure. Furthermore, the eccentricity of the vertical reaction increases for sections higher up the leg, due to the effect of R_z. At the top of the leg, the eccentricity is equal to $(-4.64 - 0.743 \times 22) \div 0.5 = 42$ ft. This eccentricity produces a maximum compressive force at the obtuse corner, from axial load, equal to nine times the average compression.

Since the axial force is considered along with bending moment in determining the stresses, this large variation in axial force is of considerable importance. Also, if the axial force varies so much along the width of the section, there is very good reason to believe that the bending moment also varies along the width of the section, probably increasing toward the obtuse corner. Neither the analytical Rathbun method nor the model tests presented by Mr. Boyer will yield any information as to the variation in moment across an internal section. Analytically, this variation could be determined only by some such method as that of "difference equations" and, experimentally, by strain gages along the section.

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[8] Supervising Urban Engr., State Highway Dept., Dallas, Tex.

An experimental determination of internal stresses in a model would be very valuable and should lead to a more correct method of stress analysis than the ones now in use.

MAURICE BARRON,[9] M. ASCE.—The ingenious apparatus and the method of deflection measurement for space structures as developed by the author is a worthy feature of this paper. Space structures have many characteristics that are not manifest in plane structures. The writer has demonstrated[10] that the elastic system about each of the three axes in space may be treated independently when determining the reactions and total stresses. Furthermore, the writer's demonstration shows that the rectangular elastic system is the predominating system and that the other two elastic systems (and the stresses usually associated with them) are of secondary order. Any quantity not determined by the rectangular analysis may be easily found by multiplying a primary value by a simple geometric or elastic constant. When this procedure is not followed and when the three elastic systems are analyzed simultaneously, the structural significance of most elements is obscured.

An examination of the author's original thesis (referred to under the heading "Acknowledgment") shows that the analytical values were determined by numerical solution of four simultaneous elastic equations for each position of load. For each skew angle (0°, 10°, 20°, 30°, 40°, 50°, and 60°) the analysis was repeated, and considerable time was spent on this numerical work. The time required for determining the theoretical values for off-center loading by this method was prohibitive, and, therefore, no comparison of analytical and test values was made.

It must be emphasized that the model chosen is an idealized rigid frame with vertical legs and horizontal deck and that the moment of inertia (I) is uniform for these component parts of the frame. It has been shown[11] that, for this idealized frame, almost all the complicated skew analysis disappears if the method of elastic centers and torsional elastic weights is used. Thus, for vertical loads on the idealized frame, it can be shown that

$$R_x = H \quad \text{(2a)}$$

in which H is the rectangular horizontal reaction on a rectangular frame with equal square span and with the same values of I, and

$$R_z = \epsilon\, R_x = \epsilon\, H \quad \text{(2b)}$$

For vertical loads symmetrically placed, transversely,

$$M_x = \epsilon\,(F - H\,h) \quad \text{(3a)}$$

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[9] Cons. Engr., Farkas and Barron, New York, N. Y.

[10] "Reinforced Concrete Skewed Rigid-Frame and Arch Bridges," by Maurice Barron, *Proceedings-Separate No. 13*, ASCE, Vol. 76, April, 1950.

[11] "The Rigid-Frame Bridge," by Arthur G. Hayden and Maurice Barron, John Wiley & Sons, Inc., 3d Ed., New York, N. Y., 1950, p. 149.

in which F is the average of the two fixed-end moments, considering an equivalent rectangular deck as a slab with fixed ends, and in which h is the deck height of the frame. For vertical loads not symmetrically placed, transversely,

$$M_z = \epsilon\,(F - H\,h) + P\,e \qquad (3b)$$

in which P is the vertical load and e is the transverse eccentricity. For all vertical loads

$$M_y = 0 \qquad (4)$$

Applied to skewed rigid frames with rising vaults and to skewed arches, Eqs. 2, 3, and 4 will be correct for all practical design purposes, but for academic accuracy (or for unusual structures) an adjustment of secondary nature will be required.

An examination of the analytical values shown in Tables 2, 3, and 4 will confirm the foregoing statements. Using Eq. 3b, analytical results for off-center loading in Table 3 can easily be determined in order to complete the comparison with model results. When the author determined the analytical results, he used the method[4] of partial summation and the numerical solution of simultaneous elastic equations. At that time no better published method was available. For his idealized rigid frame the elastic weight method yields

$$H = R_x = \frac{\displaystyle\int \frac{M\,m\,ds}{E\,I}}{\displaystyle\int \frac{m\,m\,ds}{E\,I}} \qquad (5)$$

The elastic weight is a uniform load, y/I $(= 22 \times 6/10 = 13.2$ units per linear foot). The numerator of Eq. 5 is the simple-span moment $[= 6.6\,(100\,x - x^2)]$ due to this uniform weight. The denominator may be evaluated as follows (E is constant and cancels out):

$$\int \frac{m\,m\,ds}{I} = \int \frac{y^2\,ds}{I} = 2\int_0^{22} \frac{y^2\,dy}{40/3} + \int_0^{100} \frac{22^2\,dl}{10/6} = 29{,}572\ \text{ft}^4$$

or if the steel angles (see Figs. 4 and 6) at the knees are considered to be infinitely stiff (author's assumption):

$$\int \frac{y^2\,ds}{I} = 2\int_0^{20} \frac{y^2\,dy}{40/3} + \int_0^{90} \frac{22^2\,dl}{10/6} = 26,\,536\ \text{ft}^4$$

The author's value for $\Sigma\,y^2/I$, using the mechanical integration (via blocks), is 26,480 ft^4. Therefore, substituting in Eq. 5

$$H = R_x = \frac{6.6\,(100\,x - x^2)}{26{,}480} \qquad (6)$$

Values for R_x $(= H)$ from Eq. 6 are presented in Table 5(a), and these values are the same for all skews. Values of R_z $(= \epsilon\,R_x)$ are shown for comparison with the values shown in Table 2.

TABLE 5.—ANALYTICAL VALUES OF CROSS SHEAR R_x AND MOMENT M_x FOR UNIT VERTICAL LOAD.

Skew angle Θ	$\epsilon = \tan\Theta$	(a) Cross Shear R_x;[a] Load at point:					(b) Values of M_x for Eq. 7; Load at point:				
		1	2	3	4	C[b]	1	2	3	4	C[b]
0°	0	0	0	0	0	0					
10°	0.176	0.040	0.070	0.092	0.106	0.110	0.079	0.141	0.185	0.211	0.220
20°	0.364	0.082	0.145	0.191	0.218	0.227	0.164	0.291	0.382	0.436	0.455
30°	0.577	0.130	0.231	0.303	0.346	0.360	0.260	0.461	0.605	0.692	0.721
40°	0.839	0.189	0.335	0.440	0.504	0.524	0.378	0.672	0.965	1.010	1.050
50°	1.192	0.268	0.478	0.626	0.716	0.746	0.537	0.954	1.250	1.430	1.490
R_x (all skews)[c]		0.225	0.400	0.525	0.600	0.625	0.225	0.400	0.525	0.600	0.625

[a] For comparison with the values in Table 2. [b] Center line. [c] By elastic weights.

Using the writer's methods, the following equation may be derived for vertical loads on the idealized rigid frame:

$$M_x = \epsilon \left[22\, R_x - \frac{x\,(L - x)}{2\,L} \right] \ldots\ldots\ldots\ldots\ldots\ldots (7a)$$

or

$$M_x = \epsilon\, x\,(L - x) \left(\frac{145.2}{26{,}480} - \frac{1}{200} \right) \ldots\ldots\ldots\ldots\ldots (7b)$$

Values for Eqs. 7 are shown in Table 5(*b*).

For all vertical loads

$$M_y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

The deviations from zero as shown in Table 4 are the results of small differences that are an inherent flaw in the method of using a numerical solution of simultaneous elastic equations as applied to skew structures.

The writer has repeatedly stressed a common source of error in model analysis. When model results are compared with analytical results and little agreement is shown, it is not proper to condemn either. Further tests and data are required to show which is incorrect. Even when exact agreement is shown, both series of data may be erroneous to the same degree. This warning is particularly suited to space structures. It can be shown that a torsional theory for skewed structures may be incorrect by several hundred percent, and yet an exact correlation will be shown for the rectangular elastic system.

Since the test data for M_x do not agree very well with the analytical results, everything should be suspect rather than merely the formula for F, the torsion factor. Slippage at the hinges or at the knee joints is just as likely. Furthermore, the assumption that the steel angles at the knee are infinitely stiff will bear close scrutiny. This assumption by the author is of little importance for the rectangular system, but the elastic block at the knee is very important for the torsional elastic system. This is easily verified by an examination of the effect the knee section has on the author's analytical results.

Future investigators should find the test apparatus and the method of measurement a great help. Models which would more nearly resemble the rigid-frame bridge could be built easily. Thus, the vault of the model could be cut from large diameter plastic tubes.

The bridges of the future will doubtless use curved-in-plan shapes. The variable skew frame and arch bridge is now coming into prominence as a curved-in-plan structure. Models for these unique, economical, and aesthetic structures could be built easily using a section of large diameter tubing for the curved-in-plan abutment walls. The author's apparatus and method of measurement would be the most logical procedure to adopt.

WALTER C. BOYER,[12] JUN. ASCE.—It is stimulating to observe the revived interest in the skew rigid-frame problem. Within the past few months, three papers about this subject have been published as *Proceedings-Separates* by ASCE.[10,13,14] Each paper has indicated or demonstrated simplified design procedures. The inordinate amount of work entailed in the analysis of skewed structures has been a positive deterrent to the consulting engineer. It is hoped that such recent developments as the simplified procedure of Maurice Barron will serve to countermand such objections.

Mr. Barron's discussion indicates pointedly the objections to the old analysis procedure involving simultaneous equations. Since the equations are quite unbalanced, residue values are evident in some cases (see Table 4), and other values which are normally small may show abnormalities. Values presented by Mr. Barron show closer agreement with model results than those presented by the writer, and since the elastic center permits the isolation of individual effects, these values are considered to be more valid as a comparison.

Mr. Barron has voiced a warning in interpreting the "model analysis" values. He feels that the knee joints or hinges may have been subject to slippage. This phase of the model study was carefully considered and no slippage of consequence could be measured. However, Mr. Barron has emphasized the apparent independency of the rectangular and torsional system. It is this condition that leads the writer to the conclusion that there is no real basis for confidence in the "torsional factor" as now conceived. This problem cannot be resolved, however, without a fundamental study and review of the torsional effect of plates with large width-to-thickness ratios.

Mr. Nettleton has contributed an interesting discussion, and although the writer agrees with his contentions in some respects, it is felt that the model investigation verifies the Rathbun theory in all essentials. The skew rigid frame, with a 100-ft span, was selected intentionally to isolate the effects peculiar to the skew problem. However, the qualitative trends developed indicate the general acceptance of the Rathbun theory for other leg-to-span ratios.

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[12] Asst. Prof., Civ. Eng. Dept., The Johns Hopkins Univ., Baltimore, Md.

[13] "Practical Design of Solid-Barrel, Reinforced-Concrete Skew Structures," by Bernard L. Weiner, *Proceedings-Separate No. 39*, ASCE, Vol. 76, October, 1950.

[14] "Laterally Loaded Plane Structures and Structures Curved in Space," by Frank Baron and James P. Michalos, *Proceedings-Separate No. 51*, ASCE, Vol. 77, January, 1951

It is to be emphasized that the influence of moment (M_x) is almost negligible on other reactive elements. However, this does not imply that its influence on stress distribution across a section is negligible. Mr. Nettleton has indicated that this effect can be quite pronounced in the skew frame. The writer wishes to point out, however, that the same condition is evident in the right frame for loads off the center line of the structure. Mr. Nettleton feels that study should be continued to indicate the variation in stress along the width of section, and the writer is in complete agreement. This is another phase of the skew problem and was not encompassed in the study reported in this paper.

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TRANSACTIONS

Paper No. 2466

SEDIMENTATION STUDIES AT CONCHAS RESERVOIR IN NEW MEXICO

By D. C. Bondurant,[1] Assoc. M. ASCE

With Discussion by Messrs. T. Blench, and D. C. Bondurant.

Synopsis

Sedimentation problems in reservoirs include not only the loss of storage but the distribution of storage loss in respect to the several functions of multiple purpose reservoirs and the possible damage resulting from aggradation above or degradation below the reservoir. Analytical procedures for the computation of anticipated form and extent of sediment deposits are not perfected; and there are, unfortunately, only meager published data available from existing reservoirs for comparative estimates. Data on the form, extent, and type of sediment deposits in the Conchas Reservoir are given together with a description of the reservoir, its use, and the contributing drainage.

Introduction

The Conchas Reservoir has provided an excellent subject for sedimentation studies inasmuch as the reservoir has remained at a relatively constant elevation since it first filled in 1941 and inasmuch as it has experienced four major floods since its completion in 1939. The constant water level has enabled observations of sedimentation without the disturbing influence of different water levels; and the floods have provided material for study. These data will also be valuable for correlation with future observations when the pool is expected to vary over a wide range of levels. Sediment accumulation has amounted to 24,000 acre-ft, about 80% of which is in delta formation and 20% of which is fairly evenly distributed along the reservoir bottom. An estimated 6,000 acre-ft to 15,000 acre-ft of sediment has been passed through the conduits and a total of 7,000 acre-ft of bed materials has been degraded from the river bed downstream.

The Conchas Dam and Reservoir Project was constructed by the Corps of Engineers, United States Army, on the South Canadian River in New

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Note.—Published in August, 1950, as *Proceedings-Separate No. 29.* Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Head, Sedimentation Section, Missouri River Div., Corps of Engrs., U. S. Army, Omaha, Nebr.; formerly Head, Hydrology and Sediment Section, Albuquerque Dist., Corps of Engrs., Albuquerque, N. Mex.

Mexico, immediately below the confluence of the Conchas River and about 30 miles northwest of Tucumcari, N. Mex. It is a dual purpose project for flood control and water conservation, with storage for each of these purposes allocated at separate levels. Certain space is to be maintained vacant and available for temporary flood storage at all times except during the course of a flood; whereas the conservation storage will be operated as requested by water users. Water stored in the conservation pool will be utilized primarily by the Tucumcari Irrigation Project of the Bureau of Reclamation.

Drainage Area

The total drainage area above the dam is 7,350 sq miles consisting of 3,420 sq miles of mountains, 2,840 sq miles of plateau, and 1,090 sq miles of plains. The South Canadian River drainage is predominantly from mountains and plateaus, whereas the Conchas River drainage is predominantly from plains. The South Canadian River emerges from the mountains into the plateau near Raton, N. Mex., descending into a deep canyon section near Taylor Springs, N. Mex. The river continues in this canyon to a point about 10 miles below the dam where the canyon is succeeded by a narrow valley bounded by gravelly hills. This valley extends for about 20 miles until the river again cuts deeper into the plains formation to a depth of from 300 ft to 400 ft below the elevation of the great "Staked Plains" at the New Mexico-Texas state line some 100 miles below the dam. All major tributaries above the mouth of the Conchas River head in the mountains and join the parent river in the plateau—either by surface channels entering above Taylor Springs or through deep canyons becoming confluent to the canyon section of the South Canadian River. The principal tributary (the Conchas River) heads in the plateau, emerges almost immediately through a short canyon, and flows across the plain to join the South Canadian River via a short canyon in the reservoir area immediately upstream from the dam.

The mountains are typically steep and rocky with a general forest cover which retards erosion and aids in reducing the precipitant runoff to some extent. These mountains are apparently composed of sedimentary types of rock, although adjacent mountain formations are volcanic and there are many minor volcanic cones in the vicinity. The plateau has a gentle flat slope eastward. Approximately one half of the plateau, adjacent to the mountains, has a fairly deep soil cover with numerous depressions and very poorly defined runoff; and the other half has a thin soil cover with numerous rock outcrops deeply eroded into rocky canyons which debouch into the parent canyon of the South Canadian River. The plains have a pronounced slope, deeper and more finely textured soil cover, and definite drainage showing evidence of considerable erosion of both sheet and gully types.

Reservoir Observations

The reservoir lies in the canyon section, is long and narrow in dimension with steep walls, and is V-shaped with the dam forming the apex of the V and the South Canadian and Conchas rivers, respectively, forming the arms. It has a total capacity of 800,000 acre-ft, divided as shown in Col. 3, Table 1. The dam, with a maximum height of 235 ft above the foundation, consists of a concrete gravity section across the canyon with contiguous earth dikes and with a 3,000-ft concrete emergency spillway at the north end. The service spillway, which is uncontrolled except by dimension, is located in the concrete dam, as are conduits for sluicing and low flow regulation, a small irrigation outlet, and a penstock for hydroelectric operating power. The conduits (Table 2(*a*)) are placed a few feet above the original river bed elevation. The main irrigation headworks (Table 2(*b*)) are at El. 4148 at the top of the dead storage pool and near the south end of the south dike. The length of the reservoir is 23 miles up the Canadian arm and 13 miles up the Conchas arm. Characteristics of the drainage area are given in Table 3.

TABLE 1.—CAPACITY DATA, CONCHAS DAM AND RESERVOIR IN NEW MEXICO

Flow	Acre-feet
Average annual flow	200,000
Inflow from January 1, 1939, to July, 1943	2,000,000

Purpose	Elevation (ft)	Original capacity (acre-ft)	Surface area (sq miles)	LENGTH OF POOL (MILES)		Spill-way net length (ft)	Spillway capacity (cu ft per sec)
				Cana-dian	Con-chas		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Dead storage	4050 to 4155	100,000	5.5	15.9	8.8	...	...
Conservation Storage (Crest of Service Spillway)— Water surface, El. 4218 Water surface, El. 4230	4155 to 4201	300,000	16.0	19.6	11.8	300	75,000 182,000
Flood Control Surface (Crest of Emergency Splilway)— Water surface, El. 4218 Water surface, El. 4230	4201 to 4218	200,000	21.7	21.7	13.1	3,000	0 478,000
Surcharge (maximum water surface)	4218 to 4230	200,000	25.5	23.0	13.8	...	...
Total		800,000					

The history of the reservoir from closure in 1939 through 1947 has included 2 years of exceptionally high flow, 3 years of normal flow, and 3 years of subnormal flow. Storage began behind the incompleted structure in the spring of 1939, rose slowly to El. 4147 in the fall of 1939 when the structure was essentially completed, and remained at this approximate level until March, 1941, when a minor flood raised the level to El. 4163. In April and May, 1941, a major flood filled the conservation pool and caused a maximum discharge of about 6,000 cu ft per sec over the spillway. Again in September, 1941, a moderately large flood caused a spill, followed by further spills from major floods in the spring and fall of 1942. Since 1942 the inflow has just about balanced losses and minor uses and the pool level has been relatively constant. The studies reported in this paper cover an operation period of 5 years from 1939 through 1944 in so far as actual surveys are concerned; but subsequent inflows have been insufficient to add appreciably to the sediment inflow. There is no adequate basis for comparing the observed period with probable frequency since the skew factor involved in a probability study of a 35-year period was so high as to indicate that the study was worthless. Insofar as the probability may be accepted, each of the 1942 floods was of a magnitude to be expected to be equaled or exceeded once in an average of 20 years. Since the other two floods were of intermediate magnitude and no other intermediate floods have been experienced, it is estimated that the period reported corresponds to a norm of about 10 years.

During the construction of the project, provision was made for sedimentation studies by the establishment of ranges which were surveyed prior to filling. These ranges were established by: (1) A hydrographic survey of the river 75 miles downstream from the dam and nearly 200 miles upstream; (2) an excellent topographic survey over the reservoir area; and (3) permanent monuments for future use. After the project was completed, the sediment ranges were resurveyed in October, 1940 (prior to any floods), in June, 1942 (subsequent to the first three floods experienced), and in October, 1942 (after the last of the floods). A further resurvey was made in October, 1944, and in 1943 the hydrographic traverse was relocated and resurveyed for a long enough distance to insure coverage of aggradation and degradation of the river. A program of chemical analyses of reservoir water, continued since 1939, may also offer aid in studies of sedimentation. To a limited extent the suspended sediment load has been sampled at river stations upstream from the reservoir; but such sampling is completely inadequate because the region is too inaccessible for regular visits and because it has been impossible to find observers living near by. Samples are obtained daily from the two rivers at stations some miles upstream from the reservoir; but the results as of 1948 do not include the period of experienced floods.

TABLE 2.—DIMENSIONS AND CAPACITY OF OUTLETS AT CONCHAS DAM IN NEW MEXICO

Purpose (1)	Invert elevation (ft) (2)	No. (3)	Size (ft) (4)	CAPACITY AT EL. 4201 (CU FT PER SEC) Each (5)	Total (6)
(a) CONDUITS					
Sluicing	4061.9	6	4 by 5	2,140	12,840
Regulating	4062.3	2	4	1,000	2,000
(b) OTHER OUTLETS					
Irrigation headworks	4148.0	1	11	...	...
Beell Ranch outlet	4151.25	1	1.5	...	...
Hydroelectric penstock	4129.0	1	2	...	...

Reservoir surveys were made by sounding along the established range lines, using a shallow, cone-shaped weight which had a reasonably rapid sinking rate but which did not sink in loose sediment deposits. Soundings were located by stadia distance or by intersection angles, or both; and each section was tied to two or more range markers at each end of the range. All range markers were located by closed traverse when first established, or were tied to such traverse by third-order triangulation.

TABLE 3.—TOPOGRAPHY OF DRAINAGE AREA, CONCHAS RESERVOIR IN NEW MEXICO

Stream (1)	Slope[a] (ft per mile) (2)	Mountains (sq miles) (3)	Plateau (sq miles) (4)	Plains (sq miles) (5)	Total (sq miles) (6)
South Canadian River—					
Near Logan, N. Mex. (75 miles downstream)	4.1	3,420	2,640	249	6,309
Below Conchas Dam	5.1				
Immediately upstream from Conchas Dam	5.6				
Head of reservoir	10.0				
Near Garmes, N. Mex. (23 miles upstream)	11.0				
Conchas River—					
Immediately upstream from confluence	13.1	0	200	841	1,041
Head of reservoir	13.1				
At Variadero, N. Mex. (30 miles upstream)	17.8				
Total (sq miles)		3,420	2,840	1,090	7,350

[a] River slopes prior to construction.

As of October, 1944, the total accumulation of sediment in the reservoir was 24,000 acre-ft, 15% of which was in the Conchas arm and 85% of which was in the Canadian arm (Table 4). The Conchas River has been by far the larger contributor in regard to sediment per unit of area drained and sediment per unit of inflow although, with the exception of the first flood (April–May, 1941), both normal and flood inflows of the Conchas River have been much smaller than those of the Canadian River.

TABLE 4.—RECORD OF SEDIMENT OBSERVATIONS, IN ACRE-FEET, AT CONCHAS RESERVOIR IN NEW MEXICO

Date of survey (1)	Conchas arm (2)	Canadian arm (3)	Total (4)	TOTAL, WITHIN PERIODS Acre-feet (5)	 Period (6)
May, 1940	800	600	1,400	1,400	Moderate inflow period ending in October, 1940
June, 1942	2,200	13,800	16,000	14,600	Period of first three floods
November, 1942	3,700	16,300	20,000	4,000	Flood of September, 1942
October, 1944	3,700	20,300	24,000	4,000	Moderate inflow period, November, 1942, to October, 1944
Total	(15%)	(85%)	(100%)	24,000	Total period

During the same period it is estimated that from 6,000 acre-ft to 15,000 acre-ft was passed through the conduits. This estimate is based on samples of conduit discharge showing sediment content up to 5% by weight of total sample during the flood of September, 1942. Degradation below the dam is recognizable for a distance of more than 40 miles, with appreciable lowering of the profile for 20 miles and a total of 7,000 acre-ft of material removed. Maximum depth of degradation below the dam is about 10 ft to rock outcrops and armored bed. There has been no aggradation above the reservoir limits; but a well-defined top-set bed is present within the limits of the backwater influence of the experienced floods. A reasonable correlation of suspended sediments in the river upstream and downstream against the deposit sediments in the reservoir, or a comparison of sediments before and after operation, cannot be made. Observers for sampling were simply not available in this sparsely settled area and the distances and terrain made sampling by operating personnel impracticable. Table 5 shows comparative data on sedimentation in other reservoirs.

TABLE 5.—COMPARATIVE SEDIMENTATION DATA

Reservoir and period	Area drained (sq miles)	Years	SEDIMENTATION RATE	
			Inflow[a]	Annual[b]
(1)	(2)	(3)	(4)	(5)
Elephant Butte, New Mexico—	26,312			
1916–1925		8.67	0.016	0.78
1925–1935		9.67	0.016	0.52
McMillan, New Mexico—	16,989			
1894–1915		21.42	0.006	0.11
1915–1932		17.00	0.001	0.02
Roosevelt, Arizona—	5,670			
1911–1925		14.00	0.009	1.25
1925–1935		10.00	0.001	0.14
Zuni, Arizona—	500			
1907–1922		15.7	0.033	1.26
1922–1927		5.8	0.023	0.47
1927–1932		4.7	0.0005	0.02
Conchas, New Mexico (1939–1943)—				
Conchas arm	1,041	3.75	0.021	0.87
Canadian arm	6,309	3.75	0.009	0.64

[a] Acre-feet of sediment per acre-feet of inflow. [b] Acre-feet of sediment per square mile per year.

The sediment beds in the reservoir show a fairly well-defined division into the typical categories of top-set, fore-set (or delta), and bottom-set beds—although the definition is much more apparent in the Canadian arm than in the Conchas arm. Fig. 1 shows the profile of the reservoir prior to construction and at the time of the 1944 survey. The top-set bed of the Canadian arm is sharply defined: It begins at the approximate location of the computed beginning of the backwater effect and intersects the fore-set bed at an elevation which is the maximum attained by the pool during the floods. The fore-set bed is equally well defined on both streams, with surface and face slopes that show little deviation throughout their length. The bottom-set bed, particularly in the Canadian arm, is notable for the even deposit throughout its length. Reservoir sections show that the surfaces of the top-set and fore-set deposits are approximately level across the section. The bottom deposits, on the other hand, show a tendency toward an even depth across the section despite the fact that they are always deeper in the deeper sections of the crossing. The narrow width and steep slopes of the reservoir wall make it difficult to define a tendency in this respect.

The variation in the lengths and slopes of the deposits in the two arms is apparent and is no doubt caused by the variation in the slopes of the river as well as by the magnitude of floods and quantity of material deposited. The slope of the river below the dam is 5.1 ft per mile. It increases progressively up the Canadian River to 10.0 ft per mile at the head of the reservoir and 11.0 ft per mile near Garmes, N. Mex., 23 miles upstream (Table 3). The Conchas River is much steeper with a slope of 13.1 ft per mile immediately above the confluence. This slope is retained to the head of the reservoir—but it increases to 17.8 ft per mile at Variadero, N. Mex., 18 miles above the head of the reservoir. Computations indicate that the backwater effect on the Conchas River from the experienced floods would have been appreciable only during the first flood when the pool was at a low level; and the top-set bed on that river is accordingly not well defined. The ratio of river slope to slopes of the top and face of the delta does show a rough correlation between the two arms—3.85 and 0.59 on the Canadian River against 4.78 and 0.45 on the Conchas River. Since a high flow on the Conchas River would still retain an appreciable velocity when reaching the break of the delta, this relation may not be valid. The maximum depth of deposit in the Canadian River is 42 ft; and, in the Conchas River, 32 ft.

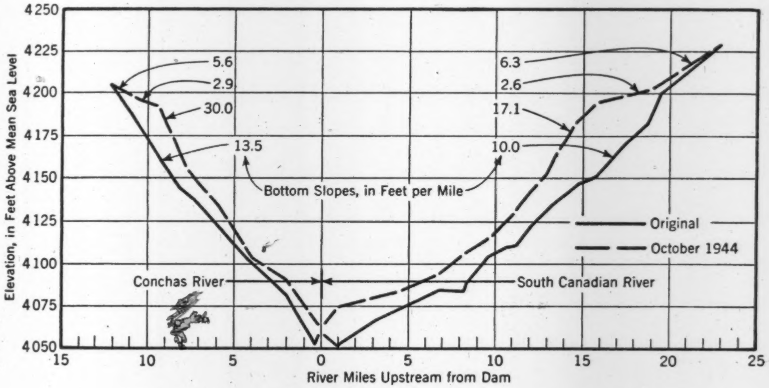

Fig. 1.—Bottom Profile Upstream from Conchas Reservoir

The bottom deposits contain somewhat less than 20% of the total accumulated sediment, and the longitudinal slope of these deposits appears to be almost completely a function of river slope with some tendency to level off in front of the dam. The sequence of pool level and flood magnitude indicates that the hump observable in the Conchas arm at mile 7 is actually the delta deposit of the flood of the spring of 1941. The bottom deposit is consolidated to the extent that a 16-lb sounding weight, dropped freely through the water, is stopped abruptly after penetrating a surface layer of soft material about 6 in. deep. The soft surface material will flow from an area about 2,000 yd in front of the dam when the conduits are opened; but, after this material has been removed, the flow becomes clear and soundings indicate that the material outside the 2,000-yd limit is undisturbed.

The grading of the material throughout the reservoir appears to be well defined according to location; but it will probably prove to be even better defined when sampling procedures can be improved. As of 1948, sampling of material was restricted to surface samples except for one location at the lower end of the exposed delta of the Canadian arm where sampling was continued to a depth of 3 ft before the saturated material caved into the hole. Samples from exposed surfaces were obtained with a "cookie cutter," which is a cylinder about 4 in. in diameter by 4 in. deep, pushed into the sediment. The material around the sides is shoveled away, and a plate is inserted under the sampler. Volume was determined by cutting the material off even with the ends of the sampler and the sample was then placed in a bag for transportation to the laboratory.

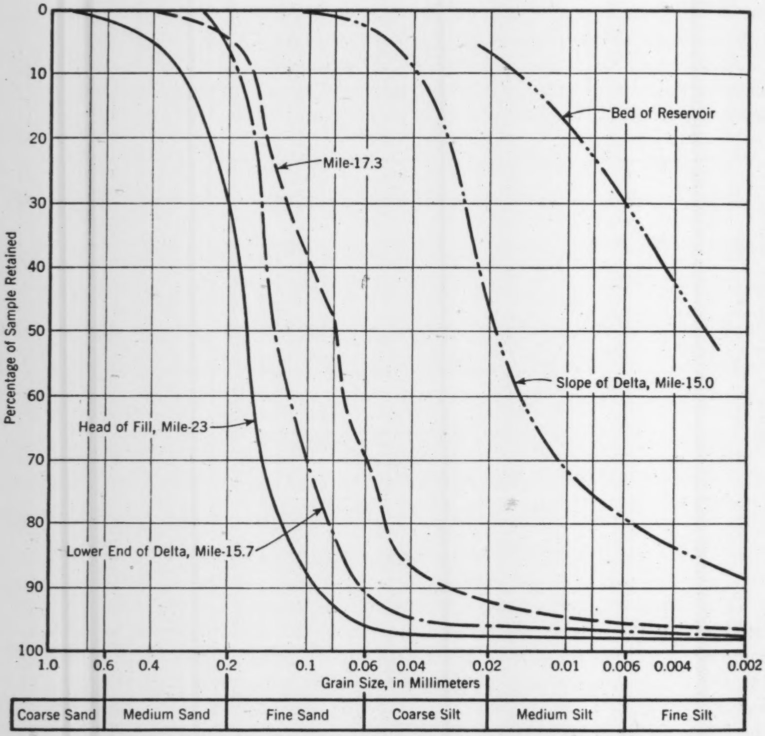

FIG. 2.—SAMPLES FROM CONCHAS RESERVOIR AT THE HEAD OF THE SOUTH CANADIAN ARM

Bottom samples were obtained by lowering a Foerst sampler to the bottom and tripping it after it penetrated the soft surface of the deposit. No density determinations could be made from the bottom samples, but the material was taken to the laboratory in a jar and was analyzed for particle size.

Fig. 2 shows the size analyses of samples from the surface at four locations extending from the lower end of the delta to the head of deposition. These analyses were made by the bottom withdrawal method. As might be expected, the sizes of the materials increase progressively upstream and the range of sizes in individual samples is more restricted. All these samples probably contain some material deposited by wind and by low flows. The deeper samples at the lower end of the delta show a more coarse material, which is restricted to a limited size range and which shows little variation with depth after the first foot. The curve of grain sizes corresponds closely with the curve for mile 17.3, Fig. 2. The soft surface material from the bottom deposit appears to have consistent characteristics throughout the length of the deposit as shown in Fig. 3.

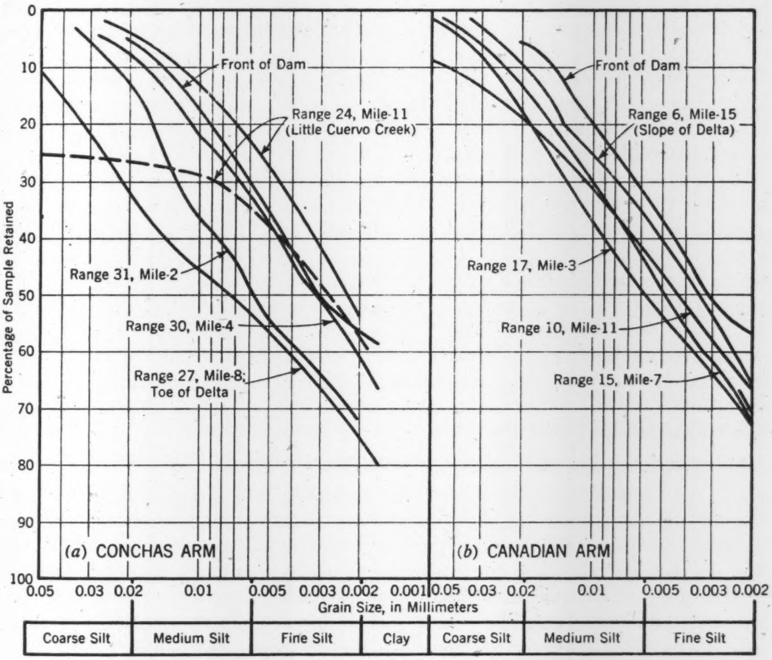

FIG. 3.—SEDIMENT FROM THE CONCHAS RESERVOIR, OCTOBER, 1946

If the analyses of the delta samples are truly characteristic of the delta, they seem to show a very abrupt change in material type. Practically no material smaller than 30 microns is stopped in the delta and no material larger than 30 microns goes beyond the delta.

No density samples of the bottom beds have been obtained because of the lack of sampling equipment adequate for such sampling. The density of the delta samples, however, varies from 62 lb (dry weight) per cu ft at the surface to 86 lb (dry weight) per cu ft at a depth of 3 ft. These weights correspond closely to those obtained by the writer and C. S. Howard, Aff. ASCE, at John

Martin Reservoir on the Arkansas River in Colorado and at Alamogordo Reservoir on the Pecos River in New Mexico.

Although the deposits of the delta offer problems such as length, volume, and slope, investigations to date indicate that these deposits contain materials that may be fairly simply analyzed and which follow some regular law of transport and deposit. The materials of the bottom beds, when compared to those from a similar location in Lake Mead (Arizona-Nevada), indicate the presence of factors which are not fully explained and which may be important in the design of reservoirs. The deposit in front of Hoover Dam[2] is quite soft, with a density of 27 lb per cu ft at depths of as much as 50 ft below the surface of the deposit. Such a density is less than one half of the sediment content of samples of material that was flowing at a high velocity down the channel of Rio Puerco in New Mexico. (Samples taken by dipping a milk bottle in a flood on Rio Puerco have contained 68% (dry weight) of sediment.) In contrast to the soft deposit in Lake Mead, the deposit at Conchas Reservoir is firm enough to stop a 16-lb pear-shaped weight abruptly.

Since the variation in the character of the deposits at Lake Mead and at Conchas Reservoir was so great, an extensive series of tests was made, running parallel tests with both Lake Mead sediment and Conchas sediment. The analyses were made by the bottom withdrawal tube method, with a few checks by the hydrometer method when the concentrations tested were adequate for the latter test. Each sediment was tested at various concentrations; each concentration was tested with Lake Mead water, Conchas water, and distilled water; and each water was used both with and without a deflocculating agent. In so far as these tests were concerned, the materials from the two sources did not show any differences that might indicate the reason for the variations in the natures of the deposit.

SUMMARY

It may appear that the bottom-set materials have been given undue consideration; but the erratic behavior of these materials may evolve more questions in the design of a reservoir project than does the more regular behavior of the delta material. The precise location of the sediment deposits, and the effect on the allocation of storage to separate functions, have been highly important questions in at least one reservoir with which the writer is familiar. The volume occupied by sediment will be an important consideration in all reservoirs; and this volume may vary to a large degree, as is evidenced by the respective densities at Lake Mead and at Conchas reservoirs. The value of conduits for sluicing would vary with the type of deposit also, since material of the low density found at Lake Mead should flow readily whereas the Conchas material can be passed through the conduits only while it is moving as a density current. In addition, the behavior of fine sediments, when tested at high concentrations, indicates the possibility that a highly concentrated inflow might result in deposition at the reservoir head regardless of the size of the material. It is believed that this phase of reservoir sedimentation is worthy of more general study.

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[2] "Stratified Flow in Reservoirs and Its Use in Prevention of Silting," by H. S. Bell, *Miscellaneous Publication No. 491*, U.S. Dept. of Agriculture, Washington, D. C., 1942, p. 25, Fig. 25.

DISCUSSION

T. BLENCH,[3] M. ASCE.—The useful information in this paper is appreciated, particularly the data in Table 5. It would be helpful if the author (in his closing statement) would show grading curves for the Lake Mead (Arizona-Nevada) sediment mentioned in the paragraph preceding the "Summary," for comparison with Fig. 3. Such curves should be for both lake water and for distilled water. The writer has some records of Lake Mead sediment from near the dam, and they show the material to be almost entirely of finer than silt size, unlike the material of Fig. 3.

The 68% dry weight cited in the second paragraph preceding the "Summary" seems physically possible only for a soil, and this anomaly needs further explanation.

D. C. BONDURANT,[4] ASSOC. M. ASCE.—Since preparation of the paper, the Conchas Reservoir on the South Canadian River in New Mexico has been resurveyed[5] and Table 6 is presented to include data from this survey. The total sediment accumulation as of February, 1949, was 35,600 acre-ft, with approximately 6,000 acre-ft (or 16.8% of the total) being deposited in the Conchas arm. Inflow from the Conchas River has been 6.6% of the total reservoir inflow. The average annual inflow into the reservoir for the 10 years of operation to 1949 was 266,500 acre-ft, somewhat exceeding the estimated 44-yr average flow of 213,400 acre-ft at the dam site. The great

TABLE 6.—CONCHAS RESERVOIR SEDIMENTATION SURVEYS

Date of Survey (Beginning January, 1939)	Period (years)	PERIOD INFLOW, IN ACRE-FEET[a]			PERIOD SEDIMENT ACCUMULATION, IN ACRE-FEET		
		Total	Mean Annual	Maximum Annual	Total	Mean Annual	Per sq miles per yr[b]
May, 1940	1.4	114,300	81,600	....	1,400	1,000	0.144
June, 1942	2.1	1,369,800	652,300	936,400	14,600	6,950	1.000
November, 1942	0.4	457,300	....	....	4,000	....	....
October, 1944	1.9	235,600	124,000	....	4,400	2,290	0.330
February, 1949	4.3	528,600	122,900	154,700	11,200	2,610	0.376
Total	10.1	2,705,600	266,536		35,600	3,520	0.506

[a] Average annual flow computed for 44-yr period—213,400 acre-ft. [b] Sediment contributing area redetermined as 6950, sq miles.

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[3] Cons. Engr.; Associate Prof., Civ. Eng., Univ. of Alberta, Edmonton, Alta., Canada.

[4] Head, Sedimentation Section, Missouri River Div., Corps of Engrs., U. S. Army, Omaha, Nebr.; formerly Head, Hydrology and Sediment Section, Albuquerque District, Corps of Engrs., Albuquerque, N. Mex.

[5] "Second Interim Report on Sedimentation in Conchas Reservoir, South Canadian River Watershed," Corps of Engrs., U. S. Army, Albuquerque District, Albuquerque, N. Mex.

spread in the data shown in Table 6 emphasizes the need for long term records for investigations of flash-type streams.

The level of storage in the revervoir has continued to remain relatively constant, and the minimum elevation attained since the first filling was El. 4192.05, or 9 ft below the service spillway crest. The top and face of the deltas have maintained essentially the same slopes as those previously quoted, although an added range located on the face of the delta of the Canadian arm shows the slope at that point to be 22.3 ft per mile instead of 17.1 ft per mile as shown in Fig. 1.

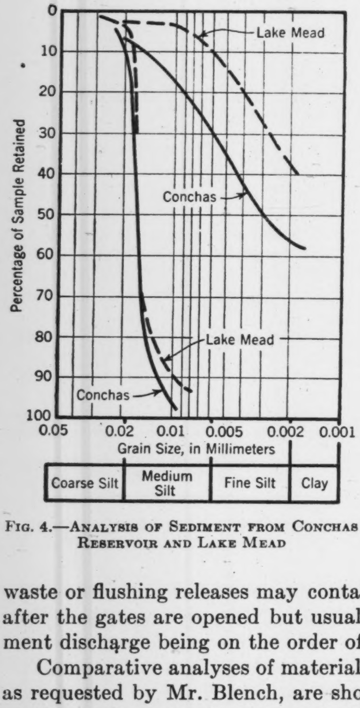

FIG. 4.—ANALYSIS OF SEDIMENT FROM CONCHAS RESERVOIR AND LAKE MEAD

Although the conduits in the dam are used only for flushing or for waste release, they have been operated at fairly frequent intervals for short periods. Samples of the conduit discharges indicate that approximately 230 acre-ft of sediment have been discharged since the survey of October, 1944. Most of this discharge has apparently occurred with density flows during minor flood inflows, as exemplified by the period from June 1 to June 4, 1948. On June 1 the mean daily inflow was 9,225 acre-ft, and sediment concentration in the release flow was less than 0.06% by weight. By June 3 the release flow contained 5.48% sediment that decreased to about 0.03% by the end of the fourth day. Total sediment discharge during this period was 94.8 acre-ft. In contrast to this discharge, normal waste or flushing releases may contain from 3% to 5% sediment immediately after the gates are opened but usually run clear within 15 min with the sediment discharge being on the order of 0.1 or 0.2 acre-ft.

Comparative analyses of material from Conchas Reservoir and Lake Mead, as requested by Mr. Blench, are shown in Fig. 4. Two sets of curves representing analyses, made on separate fractions of single samples, are included to illustrate the great spread of apparent particle sizes obtained by variations in the concentration of sediment and character of water used in the analyses. The analyses showing the finest particle sizes were made with sediment that had been continuously leached with distilled water for one week, although other tests indicated that the leaching had very little effect. These tests were made with distilled water, using the bottom withdrawal method at sediment concentrations in the tube of about 400 ppm. Altogether some 300 tests were made at varying concentrations in water from various sources, and about that many different results were obtained. As a result of these variations, an investigation of the activity of the clay minerals was initiated, and it is now recognized that the difference between the disperse analyses of the Conchas and Lake Mead samples, as shown in Fig. 4, is far more important than was previously known. Most of this difference is in the proportion of material smaller than 0.002 mm where the flocculating properties of the clay minerals become important.

The factor of particle size is also believed to explain the anomaly of the high concentrations of sediment in samples from Rio Puerco, N. Mex., on which Mr. Blench requested more information. The concentration of 68% is the highest sampled in that stream within the author's knowledge, but samples containing 30% to 40% are quite common. Even at 68% concentration the stream possessed all the characteristics of fully developed turbulence with numerous intermittent sand waves and high velocities. The author does not have ready access now to analyses of material from Rio Puerco, but he recalls that it was very fine and that samples remained quite turbid for several days. The ability of colloidal clay minerals to be transported at very high concentrations results from electro-chemical properties possessed by these very fine particles. The implications of the properties of the clay minerals are too extensive to be discussed in the space here available, but there is evidence that they are the controlling factors in the flocculation of sediments, formation of density currents, nature of the deposits formed, and other phenomena.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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TRANSACTIONS

Paper No. 2467

ORIGIN AND SIGNIFICANCE OF OPENWORK GRAVEL

By Allen S. Cary,[1] Assoc. M. ASCE

With Discussion by Messrs. Jacob Feld, A. Casagrande, A. Mayer, H. Cambefort, Hyde Forbes, L. F. Harza, and Allen S. Cary.

Synopsis

Openwork gravel, without interstitial sand, is found associated with stream deposits of well-graded sand and gravel in innumerable exposures throughout the Pacific Northwest. A study of the relationships suggests that openwork gravel is deposited by swift streams on the downstream face of a gravel bar or as sloping beds on a delta face. Vortex action may be partly responsible for the absence of sand in the deposit. Openwork gravel is of significance in engineering works involving water because of its common occurrence, the usual lenticular shape of a deposit, and its extremely high permeability.

Introduction

Openwork gravel differs markedly from what might be called "normal" gravel in that it is composed entirely of pebbles having a narrow range of sizes. In the absence of interstitial sand the pebbles are in mutual contact, leaving large open voids which form continuous passages for long distances. Normal gravel as considered in this paper is a fairly well-graded mixture of sand and gravel in which pebbles of a wide range of sizes are scattered throughout a sand matrix.

In conducting geologic studies for engineering works in the Northwest, the writer has seen hundreds of lenses of openwork gravel in natural and artificial cuts in gravel terrace remnants. Attention has been focused upon it with increasing intensity because it differs greatly from normal gravel in certain physical properties which are significant to the engineering geologist. The limited number of references to it in the literature suggests that it has been either overlooked or disregarded in geologic studies of gravel deposits.

The purpose of this paper is to discuss the mechanics of deposition of this special type of fluvial sediment and the significance of its origin to certain types of engineering problems.

Note.—Published in May, 1950, as *Proceedings-Separate No. 17*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Senior Geologist, Seattle Dist., Corps of Engrs., Dept. of the Army, Seattle, Wash.

CHARACTERISTICS

Figs. 1 and 2 illustrate the distinctive characteristics of openwork gravel—that is (*a*) the total absence of sand between the pebbles, and (*b*) the limited range of pebble sizes. Most of the pebbles in openwork gravel fall within the size range of the larger pebbles in the normal gravel with which they are associated. The contrast is seen clearly in Fig. 2, which includes seven mechanical gradation curves of openwork gravel samples from various occurrences in the Northwest, and in curve No. 8, which illustrates well-graded normal gravel. Most of the pebbles within a lens of openwork gravel might range from $\frac{1}{16}$ in. to $\frac{1}{4}$ in. in diameter, or from 3 in. to 6 in. in diameter; whereas in the associated normal gravel the particles might range downward from a corresponding maximum through sand to 0.001-in. diameter. There is, in fact, every gradation in the structure, from all one size of particle (with completely open voids, as in Fig. 3) to a relatively wide range of particle sizes, in which the voids become smaller with the additional small size particles (Fig. 4), approaching what is here termed normal gravel.

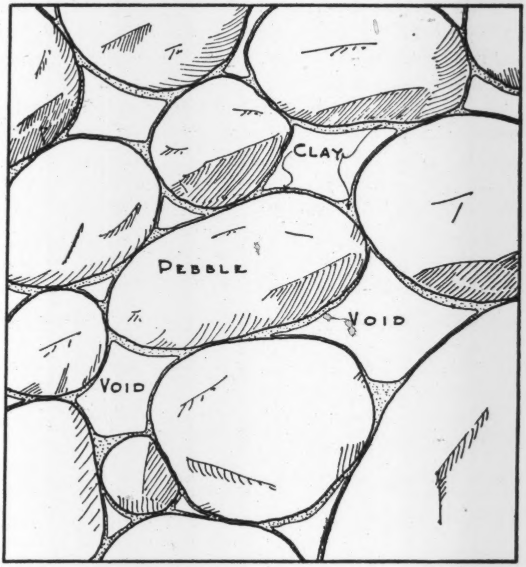

FIG. 1.—OPENWORK GRAVEL

OCCURRENCE

Most of the deposits of openwork gravel observed by the writer, together with the associated normal gravel, are in glacial stream and delta deposits. This is explained by the simple fact that the glacial history of the Northwest involves, during one or more stages, large-scale filling and trenching in the valleys and in the Puget Sound lowland in Washington. The terraces resulting from the filling and trenching are often directly or indirectly involved in engineering works. The presence of openwork gravel in river bars now in process of formation on the Columbia River in Washington, the Kootenai River in Montana, and the Skagway River in Alaska indicates that the structure is not peculiar to glacial outwash, or to aggradation as such, but may develop in any high-velocity stream carrying gravel bed load. Indeed, openwork pea gravel has been observed forming as sloping beds on a miniature delta face in a puddle after a rainstorm.

The most common occurrence of openwork gravel is in lenses associated with normal gravel and with sand. The openwork structure is best developed in lenses exhibiting torrential bedding, as illustrated in Fig. 5. The torrentially bedded openwork lenses usually range up to a maximum of 6 ft in thickness and a few tens of feet in length. The largest individual lens of this type so far observed is 120 ft long and evidently represents the filling of a scour channel in a stream bed by the downstream growth of a gravel bar. Although this and most other examples are parts of valley fills, it is certain that aggradation as such is only incidental; precisely the same scouring and filling processes are in operation in graded and slowly degrading streams.

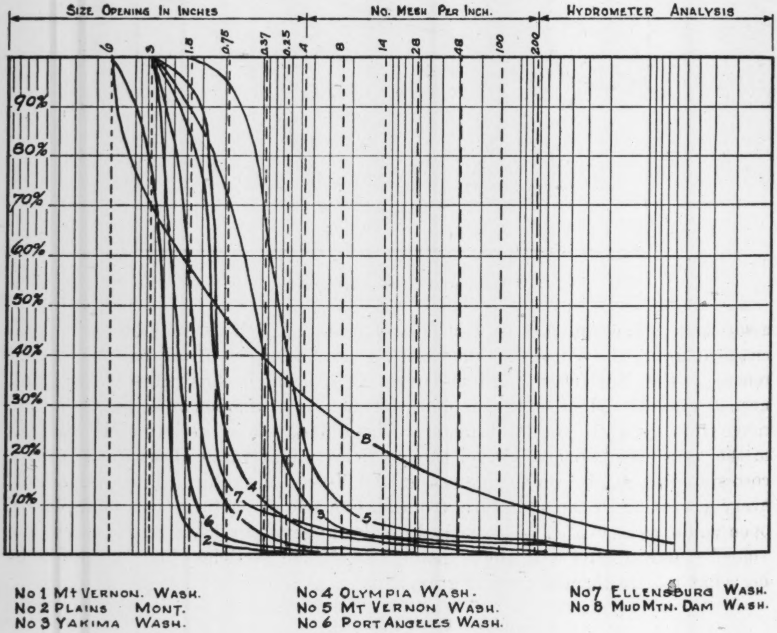

FIG. 2.—MECHANICAL ANALYSIS

In a second type of occurrence, the openwork lenses lie as sloping beds deposited on the forward faces of gravelly deltas during their formation. The fact that they are surrounded by sloping beds consisting of normal gravel with the same maximum pebble sizes indicates that the contrast must be due to some slight variation in conditions of deposition. Certain deposits which appear to be delta bedding may in fact be torrential bedding, deposited in very large glacial outwash streams.

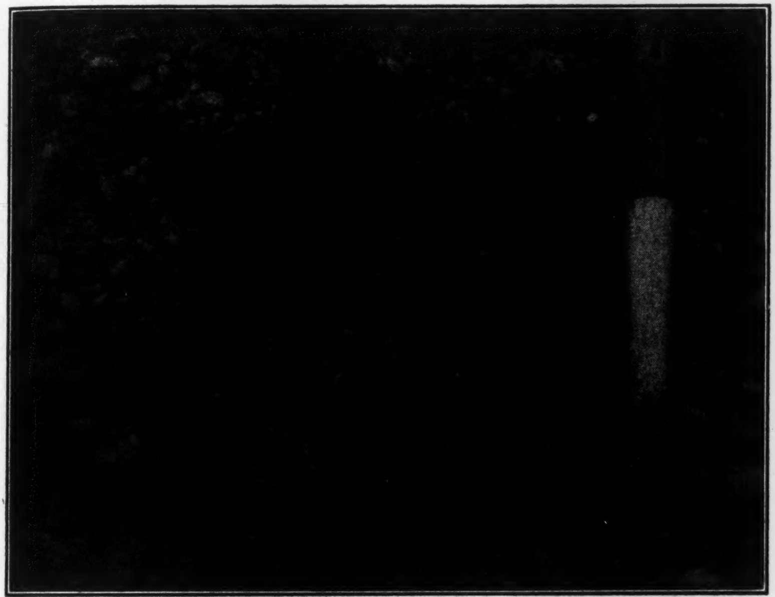

FIG. 3.—GRAVEL WITH OPEN VOIDS

In small exposures the torrentially bedded occurrences cannot be distinguished from delta beds. Under certain conditions in glaciated areas, as should be expected, one will grade into the other. The torrentially bedded structure may also grade into a third type, in which the pebble bedding is essentially parallel to the horizontal stratification of the associated normal gravel. This third type is relatively rare and usually not more than a few pebbles thick. The general relationships are shown in Fig. 5.

EARLY OBSERVATIONS

Haakon Wadel[2] (1936–1937) and William Morris Davis[3] (1890–1892) have each mentioned openwork gravel in connection with a special type of glacial stream deposit, but neither suggests the mechanics of deposition in the usual swift-flowing stream. A. W. Grabau[4] (1913) mentions the fact of separation of pebbles from sand as a beach process, but makes no mention of stream-deposited openwork gravel. This paper deals almost exclusively with openwork gravels which may be deposited in any swift-flowing stream, so that these special occurrences will not be discussed further.

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[2] "Volume, Shape and Shape Position of Rock Fragments in Openwork Gravel," by Haakon Wadel, *Geografiska Annaler*, 1936–1937, p. 99.

[3] "The Subglacial Origin of Certain Eskers," by W. M. Davis, *Proceedings*, Boston Soc. of Natural History, 1890–1892, p. 499.

FIG. 4.—WIDE RANGE OF PARTICLE SIZES WITH SMALL VOIDS

ORIGIN OF OPENWORK GRAVEL

The stratification of openwork gravel and its association with normal gravel indicate clearly that the most common occurrence is on the downstream face of a bar in a stream bed. Thus, the openwork gravel may be deposited as a result of the pebbles being rolled along the bar top by the stream current and coming to rest at their angle of repose underwater on the downstream face, whereas the finer material is carried on downstream. The ever-changing stream bed topography results in ever-changing current direction and velocity, so that sand or normal gravel is often found completely surrounding the openwork gravel in any exposure.

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[4] "Principles of Stratigraphy," by A. W. Grabau, A. G. Seiler & Co., New York, N. Y., 1913, p. 701.

Gerard H. Matthes,[5] Hon. M. ASCE (1947), attributes much stream erosion of bedrock to large-scale turbulence on the lee side of projections into a stream. Careful observation from a canoe on the Kootenai River, on the Clark Fork of the Columbia River in Montana, and on the Columbia River in Washington reveals analogous, although less vigorous, action at the downstream face of gravel bars in the stream beds. The simple case described in the preceding paragraph may not actually occur without the assistance of identical turbulence.

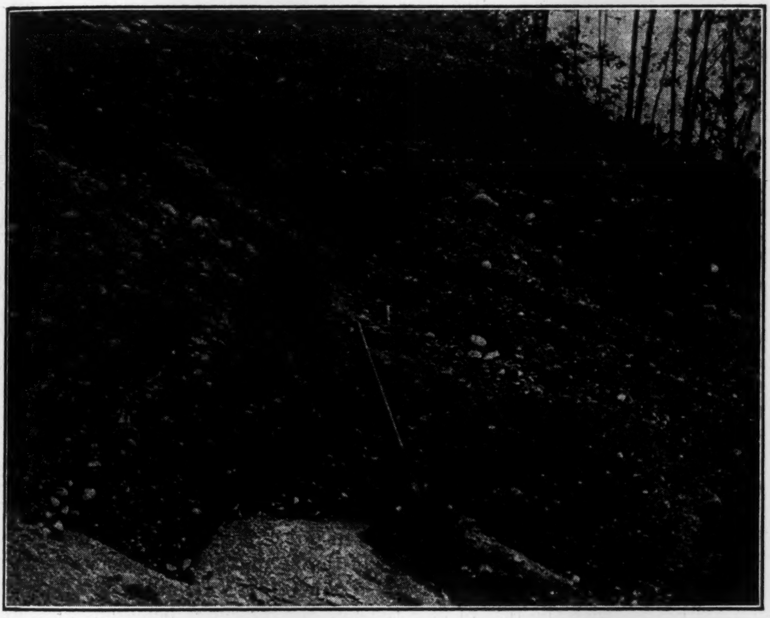

FIG. 5.—LENSES OF OPENWORK GRAVEL ASSOCIATED WITH NORMAL GRAVEL AND SAND

A vortex, forming at the downstream face of a bar, would lower the hydraulic pressure at the bar face, and would cause a movement of water outward from within the interstices of the gravel in the bar. Any sand that had been deposited on the bar face would tend to be removed and that sand which might have been deposited during the life of the vortex would be carried downstream. At the same time, the vortex might not be strong enough to prevent deposition of the pebbles that rolled over the bar face during that period. A nearly stable or continuous vortex, or a long series of vortices in succession, would keep the sand cleaned out of a bar face long enough for an indefinite length of openwork gravel to be deposited.

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[5] "Macroturbulence in Natural Stream Flow," by Gerard H. Matthes, *Transactions*, Am. Geophysical Union, April, 1947, p. 255.

Significance is attached to the observation that the pebble sizes found in openwork gravel are usually in the size range of the larger pebbles in the normal gravel with which the openwork gravel is associated. It suggests a self-evident but definite relationship between the current velocity (as evidenced by the caliber of its load) and the lifting power of the vortices (as evidenced by the caliber of load that remains after selective removal of the finer fractions).

Openwork gravel is considered to result from the ordinary processes of selective transportation and deposition, brought on by local variations in water velocity and by turbulence in rapid streams, and is considered to be one phase of the total deposit of any swift, gravel-bearing stream.

SECONDARY DEPOSITS WITHIN VOIDS

The clay coating (see Fig. 1) around each pebble in many openwork gravel lenses marks an entirely different phase of deposition. Water in the interstices, at any appreciable depth below a stream bottom, has little movement; but there will be enough circulation to carry clay, of colloidal dimensions, into the sands and gravels.

Water moving through gravel below stream level finds its way, by tortuous paths, among the sand grains, with low velocity. Silt particles and some colloids are filtered out in the process; but the balance of the colloids continues to move until the colloids are precipitated as a result of flocculation and a reduction in the rate of movement of water. When the water finds its way to an openwork gravel lens, its velocity is immediately checked, because of the very great increase in size and volume of the voids and the reduction in resistance to flow.

The openwork gravel is many times more permeable than the normal gravel which surrounds it; but the rate of movement through the entire deposit is generally governed by the permeability of the normal gravel rather than that of the openwork gravel—as if there were a series of large reservoirs in a body of normal gravel, with the rate of flow through the reservoirs being governed by the rate at which water can enter and leave through the surrounding sand and gravel. The colloids then may be precipitated out in the reservoir of openwork gravel voids and may tend to be concentrated on the bottom side of the pebbles by the downward component of movement of water.

When aggradation ceases and stream downcutting begins, the gravel deposit becomes subject to drainage. In the years or centuries after drainage of a deposit, downward circulating water may be partly, or almost wholly, responsible for the presence of the clay on the bottom of pebbles in greater quantity than on the top or sides. The downward circulation of water may also explain why some of the interstitial clay is found as partial void filling on the "floor" of a void, which frequently exhibits horizontal laminations. Whether the variations in thickness of clay are original depositional features, or secondary characteristics imposed by downward circulating water after drainage of the deposit, has no direct bearing on the origin of the gravel itself.

A special case of complete void filling will be discussed in another connection under Chief Joseph (formerly Foster Creek) dam site on the Columbia River in the State of Washington.

EXAMPLES OF SPECIAL INTEREST

(*a*) *Columbia River and Plateau.*—H. T. Harding[6] (1929) has indicated that the sudden lowering of glacial Lake Missoula played a dominant part in supplying the water to shape the erosional and depositional features of the channeled scablands of Washington described by J. Harlen Bretz[7] (1923). The writer has observed openwork gravel in bars from within the area once covered by glacial Lake Missoula, westward and southward throughout the channeled scablands, Moses Coulee, Grand Coulee, and the Washtucna, Priest Rapids, and Hanford areas in Washington. Many of these were observed in connection with geologic investigation for engineering structures.

(*b*) *Plains, Mont.*—J. J. Pardee[8] (1940) has described evidence of unusual currents in glacial Lake Missoula. About 2 miles downstream (west) from the town of Plains, on the Clark Fork of the Columbia River, an enormous gravel bar occurs, more than 300 ft thick and covering 5 sq miles, which is one additional current-built feature that bears further testimony as to the strength of the currents described by Mr. Pardee. The internal structure of the bar was revealed by trenching on the riverward slope and by drilling on the bar surface, under the supervision of the writer during exploration for a possible dam site. It consists wholly of openwork gravel and boulders, with some silt; but no clay coating or complete void filling was found. Evidently the entire bar accumulated in one extreme rush of water. The silt may have been entrapped within the gravel, or may have been carried downward from the surface by recent percolating water.

At the downstream end of this bar, the rock walls of the Clark Fork Valley close in and appear to present favorable conditions for an alternate dam site. Drilling in the valley bottom revealed more than 170 ft of openwork boulders and angular blocks of country rock ranging from 1 ft to 4 ft in diameter. The voids in the upper 50-ft stratum of boulders were filled or nearly filled with silt; but, at greater depth, the voids were absolutely open. Water level in the 6-in. drill casing could not be raised by pumping in at the rate of 75 gal per min. Indeed, it was also impossible to raise the water level, with the same pump, in a casing sunk by drilling on the Plains bar. Both dam sites were declared not feasible because of the extremely high permeability of the openwork materials.

(*c*) *Mount Vernon, Wash.*—Of special interest is a deposit near Mount Vernon, which has about 40% of the pebbles crushed and broken as a result of vertical loading. A deposit of Vashon (Wisconsin) glacial till overlying the gravel indicates that the area was covered by ice after the gravel was deposited. The occurrence of Vashon gravel and other ice marginal features high on adjacent mountain slopes indicates that no less than 4,000 ft of ice covered the area during the Vashon glacial stage. If the glacial ice is assumed to have had a specific gravity of 0.9, the pressure intensity under the ice would have been about 224,000 lb per sq ft. In the openwork gravel this load was distributed over a comparatively few points of contact on any single square foot of area. The unit pressures exceeded the ultimate strength of the gravel particles, with the resulting breakdown of the points of contact. The normal gravel in the same exposure was subject to the same pressure; but, because of the greater distribution and lesser intensity of pressure on the innumerable points of contact between the sand and gravel particles, the crushing did not occur. Examination of the gravel reveals that each pebble in the Mount Vernon deposit s deeply weathered, whereas many other gravels lying similarly under Vashon till are fresh and unweathered. There seems little doubt, therefore, that this deposit was weathered at the time of invasion by the ice which deposited the till and explains the crushing of the individual pebbles. Fresh openwork gravel lying directly under till near the international border, where the ice was at least 1,000 ft thicker, shows no crushing.

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[6] "Possible Water Supply for the Creation of Channeled Scab Lands," by H. T. Harding, *Science*, February 15, 1929, pp. 188–190.

[7] "The Channeled Scablands of the Columbia Plateau," by J. Harlen Bretz, *Journal of Geology*, November–December, 1923, p. 617.

[8] "Ripple Marks (?) in Glacial Lake Missoula, Montana," by J. T. Pardee. *Bulletin*, Geological Soc. of America, December, 1940, pp. 2028–2029 (abstracts).

Engineering Significance of Openwork Gravel

Filter Investigation.—During an experimental investigation of gravel filters, it was noted that water percolating downward through pit run gravel containing the clay mineral, montmorillonite (a bentonite type of clay), carried the clay into the clean filter material and deposited it on the underside of the individual pebbles in the filter. The mechanics of deposition of the clay within the filter appears to be identical to the deposition within openwork gravel in a natural deposit.

Earth Fill Materials.—The presence of clay in the voids of openwork gravel lenses affects the properties of an entire gravel deposit in such a manner that its value as a construction material may be rather limited. When a gravel pit is opened, vertical faces may stand for years with little raveling. A cut face of gravel, or of sand, exposed to the sun and protected from infiltrating water from above by an impervious till blanket, may appear very dry. A few inches inward from the cut face, however, the moisture content may be very high. When the material is excavated and thoroughly mixed for use as construction material, the mixture of clay, sand, and gravel may have the consistency of "cup grease and marbles." The explanation is simple: When the material is deposited by streams, there is good rock-to-rock contact, with an interlocking of rock particles. The clay, deposited later, is distributed everywhere as a coating around each pebble and sand particle, except at the mutual contacts. When the clay, sand, and gravel are thoroughly mixed, each pebble and sand grain becomes completely coated with wet clay and the entire mass may have a surprisingly low bearing capacity, especially if the water content of the clay is high. Since there is a very small percentage of the clay present, a slight change in moisture content has a very marked effect on the engineering properties of the material.

Water Jetting.—A jet pipe may penetrate vertically downward through sand or normal gravel without much difficulty, because the sand will usually wash up along the outside of the pipe. The removal of sand allows room for the pebbles to be displaced by the jet pipe. When a lens of openwork gravel is encountered, there is no sand to be washed out to make room for the jet pipe.

The pebbles, being in mutual contact and generally well interlocked, form an impassable barrier to jet pipes.

Permeability Characteristics.—The permeability characteristics of gravel deposits in their undisturbed condition are often among the most important questions confronting the engineer in the design and construction of dams, levees, drainage or irrigation canals, and other works involving water. Since openwork gravel usually occurs in lenses surrounded by normal gravel, one may easily be led to the erroneous conclusions that the permeability of a section of gravel is controlled by the normal gravel and that the openwork lenses constitute enlargements in the innumerable conduits that will act merely as reservoirs, and nothing more. Over very long distances this may be true; but the effective length of seepage path may be reduced an appreciable amount by the presence of strata that are particularly rich in openwork gravel lenses.

As stated, lenses have been measured which are 120 ft long and such lenses may be scattered through a deposit with only a foot or two of normal gravel (or of sand) between them. If the lenses are arranged in such a manner that their ends are separated by only a foot or two of sand and gravel it is conceivable that several hundred feet of gravel may prove to be 90% openwork gravel and only 10% normal gravel, in a horizontal path. Suppose that a dam is built abutting against a gravel bank so that water is ponded to a depth of 50 ft and the seepage path through the gravel is 500 ft. The hydraulic gradient would thus be 50/500, or 0.1. Suppose that in the 500 ft of gravel there are 450 ft of openwork gravel lenses and only 50 ft of effective normal gravel or sand. If this occurs near the bottom of the abutment, the effective gradient might be considerably higher than the measured gradient. Internal piping through sand lenses into openwork gravel might occur, thus creating an almost continuous path for water seepage with little frictional resistance.

Chief Joseph Dam Site.—The Chief Joseph dam site on the Columbia River, 51 miles downstream from the Grand Coulee Dam (Washington), is being studied by the Seattle District, Corps of Engineers, for development of 165 ft of available head for power. The foundation and left abutment are excellent granite for all the structures. The right abutment consists of 110 ft of gravel resting on the nearly level bedrock floor. The gravel is overlain by about 180 ft of extremely hard, impervious glacial till, forming a terrace into which the river has cut a 300-ft canyon to bedrock.

The very unusual nature of the gravel presents a major seepage problem both during construction, when the right abutment will be dewatered, and, after construction, for the control of seepage water in the zone of emergence

Test pits, drill holes, tunnels, and stopes have revealed that about 50% of the bulk of the stratum is openwork gravel. Indirect evidence strongly suggests that the permeability of this stratum is completely controlled by the openwork gravel and that the associated normal gravel occurs in isolated lenticular bodies that have no effect on the permeability except as they may reduce the effective cross-sectional area of the openwork stratum.

The evidence upon which this statement is based is the fact that the water surface in test pit 2A, which is about 1,100 ft from the river, lagged but 2 ft below the river during the rise of April-May, 1948. This constituted a time lag of from 2 days to 3 days.

All the voids in the openwork gravel above the ground-water table show evidence of having been completely and firmly filled with clay. Downcutting by the river to bedrock has lowered the ground-water table far below its position when the gravel was deposited. All the clay in the voids of the openwork gravel exhibits shrinkage cracks in all excavations as the material is uncovered. The fact that Portland cement grout was injected into the material and, upon excavation, was seen to fill all cracks firmly and completely proves that the cracking of the clay is not induced by the admittance of air through the excavations. Below the ground-water table, and particularly within 5 ft of bedrock, there are voids which are only partly filled with clay, but certain clay void fillings exhibited shrinkage cracks identical in appearance to those above the water table.

The moisture content of the clay above the ground-water table is now between 65% and 118% of the dry weight. Although some swelling of the clay may take place when the completed dam submerges the gravel, it cannot be depended upon to swell enough to close the cracks completely. The fact that clay below the ground-water table exhibits cracks indicates that the clay-water mixture was not an equilibrium condition when the clay was deposited.

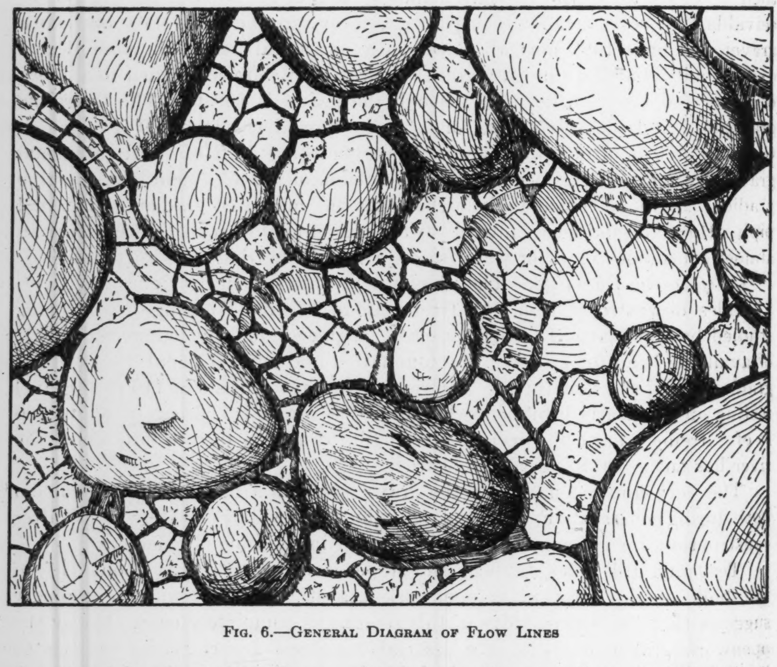

Fig. 6.—General Diagram of Flow Lines

Resubmerging the clay above the present ground-water table cannot be expected to reduce the cracks to less than the openings below the water table.

The origin of this clay filling again has no direct bearing on the origin of the gravel, but it does serve to emphasize the engineering significance. The clay could not have been deposited contemporaneously with the gravel since the openwork gravel requires very swift waters for its selective deposition and clay requires still water. It must then have been a secondary deposit in the voids. Two methods of introducing the clay suggest themselves: First, the clay may be introduced as a secondary deposit from nearly stagnant water in the gravel below river bed level. The mechanics in this case must necessarily have been a secondary wave of deposition following up from bedrock below the aggrading river that deposited the gravel. There is some question if the clay could have been deposited by this process in the dense state in which it is found. Allowing for swelling of the clay to fill the cracks and even allowing for some consolidation of the gravel mass by particle rearrangement when ice shove may have been active, the firmness of the clay seems incompatible with the depositional idea of origin.

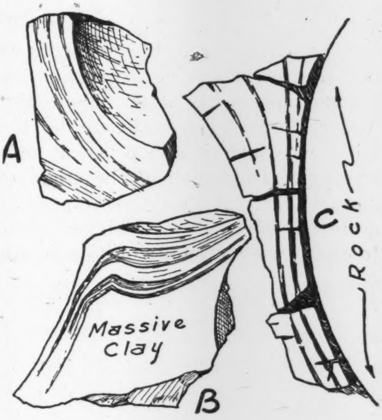

Fig. 7.—Flow Lines in Clay Blocks

A second method of introducing the clay requires a bed of clay and a bed of of gravel lying one above the other. Overriding ice, from 4,000 ft to 5,000 ft thick, could supply the pressure necessary to intrude the clay into all the voids in the gravel much as grout is intruded into aggregate to form "Prepakt" concrete. The presence of flow lines in individual blocks of clay, as shown in Figs. 6 and 7, suggests that it was intruded into the gravel. In Fig. 7, views A and B represent clay blocks sliced with a sharp knife and view C represents a clay block as it appeared upon exposure.

The presence of the clay introduces a lubricating element that could seriously affect the stability of the gravel under certain conditions. If ice shove has rearranged the pebbles in the deposit, then clay can be expected to be found between the pebble contacts. Running ground encountered in the tunnel suggests that there are zones in which the clay will act as a lubricant and this serves as a warning that seepage through the gravels may be a very dangerous thing.

Cedar River Failure in Washington.—The excessive leakage that led to the failure of the Cedar Reservoir on the Cedar River is attributed by J. Hoover Mackin[9] (1941) to "* * * the essential continuity of a sequence of open-textured (openwork) gravels from the pool to the steep north face of the embankment" where the failure occurred, at least 6,000 ft away. This failure emphasizes, as no other example possibly could, the significance of openwork gravel in engineering works.

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[9] "A Geologic Interpretation of the Failure of the Cedar Reservoir, Washington," by J. Hoover Mackin, *Bulletin No. 107*, Univ. of Washington Eng. Experiment Station, Seattle, Wash., 1941, p. 28.

ACKNOWLEDGMENT

The writer wishes to express his appreciation to Mr. Mackin, who, over a period of ten years, has listened patiently to innumerable discussions and read innumerable drafts of the manuscript. Thanks also are due J. S. Grygiel, Major, Corps of Engineers, and to Howard T. Harstad, civil engineer, for their criticism, which, it is hoped, will make the paper more understandable to an engineer. Discussion of the origin of the clay void filling at Chief Joseph dam site with Arthur Casagrande, M. ASCE, and with Edwin B. Burwell, geologist, Office of the Chief of Engineers, United States Army, has helped to clarify the mechanics of its deposition.

DISCUSSION

Jacob Feld,[10] **M. ASCE.**—Gravel formations of the type described by the author require special treatment in engineering design. The number of places on the eastern seaboard where material of this nature is found seems to indicate that openwork gravel is not an unusual soil condition. The engineering significance of three occurrences are described herewith showing the variety of nature and probable origin.

West of the Neversink River in the valley north of Port Jervis, N. Y., a very deep layer of clean, rather large-sized gravel is found at depths of from only 2 ft to 3 ft below the surface. The gravel runs from 1 in. to 6 in. deep, and is overlain with sand and with a very scant top soil cover. About 1925, two branches of the near-by Martin Creek were dammed up to provide recreational lakes. The bottoms of the lake areas were covered with local clay and river mud to form an impervious skin and were surfaced with sand. The dams themselves were built of local gravel blended with run-of-bank sand to reduce porosity; they were blanketed on the upstream side with a 12-in. layer of selected clay, protected by hand-laid riprap filled with sand. The top of the dam forms a roadway and, because the gravel is fairly well meshed, very little settlement has occurred; strangely enough, little seepage has been lost through the dams.

In this same area an attempt made to sink well points for developing a water system supply proved impossible because of the large void spaces, and actual wells had to be dug by excavation. In this vicinity, the gravel is apparently the result of flood deposition by the Neversink River and has been brought down from the Catskill Mountain areas. Individual stones are smooth and show no pitting.

In the immediate vicinity of Piney Branch of the Rock Creek near Sixteenth Street in Washington, D. C., there is a layer of openwork gravel extending from about 8 ft to 10 ft below street grade to a depth of at least 30 ft. This gravel varies in size from $\frac{1}{4}$ in. to 6 in., and individual stones tend to be flat. A rather thin coating of silt has been deposited in the void spaces. Again the gravel is evidently the result of deposition by river floods, and later flooding has left the thin coat of silt. Because of the rather large percentage of voids, the question arose as to the proper bearing capacity of this gravel for the support of a large building structure. Load tests on an area of 4 sq ft indicated that, up to a loading of 8 tons per sq ft (at the end of the tests), practically no settlement occurred beyond the elastic deformation of the testing frame. This gravel stands up vertically for several months even during considerable temperature changes and heavy rainfall. The individual stones are mostly limestone and sandstone in origin, thus explaining the tendency for flat shapes.

On the southern slope of the terminal moraine in northern Long Island (N. Y.), especially on the westerly end of that strip, there are numerous exposures of layers of openwork gravel, each layer consisting of different size ranges. The sizes range from very small, almost sand size, to a 1-in. gravel. These are definitely glacial depositions and the individual particles are completely clean of any later deposition. As a result the different layers are quite loose. The load test made on 4 sq ft of these materials indicates a continuous lateral flow and settlements of more than 1½in where 4 tons per sq ft is applied. This is the type of material that flows away from loads and causes rolling equipment to bog down very readily, both when wet and when dry.

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[10] Cons. Engr., New York, N. Y.

The foregoing examples indicate a variety of structural characteristics of open-gravel depositions and the discussion is written to point out that no general conclusions should be drawn for the necessary and proper use of these materials in engineering design.

A. Casagrande,[11] M. ASCE.—The writer recalls with pleasure the stimulating discussions which he had with the author and his colleagues on the mystery of the shrinkage cracks in the clay that fills the voids in the openwork gravel at the Chief Joseph dam site in Washington. While reading Mr. Cary's highly interesting paper the writer thought of still another explanation for these open fissures in the clay—namely, the possibility that they are due to ice segregation in the clay which developed while the gravel was covered by a glacier.

For several reasons the writer is now inclined toward the hypothesis that the clay was carried in and deposited by water seeping through the pores, rather than by intrusion of an overlying clay layer which was forced into the voids by the weight of a thick layer of ice. In order to admit the intrusion hypothesis from the standpoint of soil mechanics, one would have to assume that the clay layer existed in an extremely soft condition and that a heavy load of till and ice was deposited so quickly that the clay could not have had time to consolidate materially. At the slow normal rates of load application the clay would certainly have consolidated into a stiff consistency without flowing to any appreciable extent into the voids of the gravel. The writer believes that this could be proved by laboratory tests.

It is true that the clay deposit in the voids is fairly stiff; but the high water content indicates that it would be possible to deposit the clay by sedimentation from percolating water, with much of the strength developed after deposition by thixotropy.

The author suggests that it is difficult to visualize how the pores could have been filled by clay as a result of infiltration. However, the writer has seen stiff, highly plastic clay that completely filled solution channels in limestone; and there is no doubt that the clay had been deposited from circulating water.

In the case of the openwork gravel, it would seem possible also that the original deposit of fines in the voids of the gravel was of nonplastic character, and that it filled the voids only partly, but that particles of volcanic ash disintegrated in the course of time to form the present contents of montmorillonite. As a result of such chemical breakdown the material in the voids changed into a clay which could easily have filled the entire volume of the voids. Moreover, because of the limitation in volume expansion within the voids of the gravel, it is conceivable that such a clay filler could exist even with a stiffer consistency than the clay in the openwork gravel under discussion.

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[11] Prof. of Soil Mechanics and Foundation Eng., Harvard Univ., Cambridge, Mass.

The network of fissures and open spaces surrounding the pebbles resembles closely the network of ice which would form in the clay if the entire mass were frozen. Is it not conceivable that during the long period that a glacier covered this site, the entire stratum of openwork gravel was frozen? This would fully explain this network of "shrinkage cracks." Once a thixotropic clay is consolidated to a given stress (in this case the stresses resulting from ice segregation), removal of the stress (that is, melting of the network of ice in the clay) would not cause the clay to swell sufficiently to close the open fissures. Even in ordinary clays which have been frozen, one finds after thawing that the spaces which were filled with ice remain open and clearly visible for some time. However, in contrast to the clay in the openwork gravel, where the weight of the overlying material is carried entirely by the gravel, the open spaces in an ordinary clay layer are usually soon closed again due to the weight of overlying material.

The "flow lines" which Mr. Cary describes and illustrates in Fig. 7 can also be explained by frost action. Since frost penetrates much faster through the pebbles than through the clay (the freezing of water slows down the progress within the clay), freezing of the clay in the narrow passages would proceed chiefly from the walls of the pebbles toward the middle of the voids and would result in ice layers directly adjoining and parallel to the surface of the pebbles, with occasional transverse ice layers. In addition, the clay will form a laminated structure perpendicular to the direction of withdrawal of water by ice segregation—that is, in the narrow passages parallel to the surface of the pebbles. The formation of such a laminated pattern in clays due to consolidation has been described by M. Juul Hvorslev,[12] Assoc. M. ASCE, who attributed this phenomenon to orientation of the flaky particles perpendicular to the direction of the major principal stress. Later, L. Casagrande demonstrated that a laminated structure in clays can result from the formation of fissures caused by pore water flow. Such fissures develop perpendicular to the direction of flow and have been observed to form when very steep hydraulic or electro-osmotic gradients are created. Assuming the freezing hypothesis to be correct, steep hydraulic gradients must have developed in the clay adjacent to the growing ice lenses.

The question might be raised as to why effects of freezing of sediments underlying glaciers have not been observed in other cases. Probably the reason is that in the case of the openwork gravel thawing could take place without overlying weight recompressing the spaces which were occupied by ice layers, a condition which would normally not be fulfilled in other materials. Besides, it is possible that ice segregation in fine-grained soils, subjected to the very great load of an ice sheet, is much less pronounced and may even be entirely absent. The clay in the voids of the openwork gravel was protected from these high stresses because they were carried by the gravel itself.

The writer believes that his hypothesis could be checked by freezing tests on artificial specimens of an openwork gravel whose voids are filled with soft clay.

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[12] "Über die Festigkeitseigenschaften Gestörter Bindiger Böden," by M. Juul Hvorslev, *Ingeniørvidenskabelige Skrifter, A. No. 45,* Danmarks Naturvidenskabelige Samfund, København, 1937, pp. 11–14.

A. Mayer,[13] M. ASCE.—In connection with the subject of this paper, the writer recalls two instances in which openwork gravel was encountered. The first was that of a morainic deposit on the north bank of the Sautet reservoir in France. Its existence was well known from the very beginning of the dam studies. All the surveys had shown that the river had once followed a different course. The original valley had been filled by a morainic deposit, and the river had found its way through a fissured part of the limestone ridge to the lowlands. The question as to whether this sand and gravel mass would be watertight was investigated by several experts, who were all of the opinion that its thickness (approximately 1.5 miles) would suffice to prevent any appreciable percolation.

The dam was built and the reservoir was filled for the first time. By that time, everyone had forgotten about the gravelly layers, until several trees began to slide into a valley 1.5 miles away from the reservoir. Wells appeared which had never existed previously, and measurements showed that after a few days the total flow from them exceeded 1 cm per sec.

The head of the water in the reservoir did not exceed 100 ft above the level of the wells, corresponding to a hydraulic gradient of 1.5%. The filling of the reservoir was stopped and investigations were begun. Open pits sunk into the morainic material showed that it was composed of alternate layers of mixed sand and gravel and what Mr. Cary calls openwork gravel—that is, gravel layers without any binder. The trouble was that the original river bed, filled with these deposits, had a width of several hundred feet. Although the volume of the percolating water was not very important, the prevention of the leakage proved to be a difficult task. The first method was cement grouting. A large number of bore holes were sunk and a great volume of cement was pumped in, but without any result. The voids were much too large to be stopped with cement. Further investigations showed that the cement grout had passed through the largest voids and assembled in horizontal lenses. In some cases it had percolated into the reservoir.

Several attempts were made to silt the slope progressively. The tests were stopped as soon as the flow stopped decreasing. (In the writer's opinion, these tests would probably have given good results if they had been made more systematically and on a sufficiently large scale.) Grouting tests were then made with stabilized clay.[14] Preliminary tests, between a bore hole and an open pit, showed that a heavy grout of stabilized clay did not penetrate the sand and gravel layers; it percolated through the openwork gravel, and made it absolutely watertight when continued long enough. A number of holes were thus grouted and the total flow decreased 50%; but the cost of the borings was high, and the company that owned the dam decided to stop the work. Anyhow the experience showed that there was a way to prevent the seepage in case the total flow had increased; in fact, it remained constant. Since then, the percolation has continued at the same rate of from 1 cm per sec to 2 cm per sec according to the level in the reservoir. Some day, it may appear economical to close this gap; the technical means for doing so are available.

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[13] Ingénieur Général des Mines, Ministère de la Production Industrielle, Paris, France.

[14] "Dispositions les plus recentes pour empecher la formation des renards," by A. Mayer, Report to the Third Congress on Large Dams, Stockholm, June, 1948, Rept. 22, Question 10.

Another case of openwork gravel was met during the construction of the diversion canal at the Donzères-Mondragon dam site, in the Rhone Valley. This canal had to be dug through the alluvial deposits of the valley, for a distance of several miles. As the work began the gravel layers appeared to be composed of very distinct horizontal strata, some of them being quite characteristic of openwork gravel. The loss of water through these layers was very important. Tests were started to find an economical way to stop the flow. Excellent results could be obtained by simply plowing up and compacting the slopes of the cut, so as to have a graded mix covering up the openings of the gravel layers. This method also would very likely have been efficient in the previous case if applied on a sufficiently large surface and if the water had been silty enough to seal off the blanket progressively.

These two cases have been mentioned to emphasize the fact that openwork gravel is of very great significance to foundation engineers and its presence must be taken into account in all cases involving water flow. Therefore the writer is thankful to Mr. Cary for his very excellent paper. The flow through openwork gravel can be stopped, in all cases, by stabilized clay grouting. In some cases, simpler surface treatments may be sufficient to give quite sufficient results.

H. CAMBEFORT.[15]—Numerous studies in France have led engineers and geologists to believe that openwork gravel can be found in all normal alluvial deposits. Such gravel has been observed in the valleys of numerous large rivers—the Garonne River at Saint-Martory, the Rhône River at Donzère, and the Rhine River at Ottmarsheim—as well as in valleys of rivers of moderate size—the Durance River at Serre Poncon and Lamanon, the Drac River at Le Sautet and Monteynard, the Arc River at Aiguebelle, the Isère River at Aigueblanche, and the Fensch River at Hayange.

The type of alluvium in which openwork gravel is found is characterized by successive strata of highly distinctive grain size. For example, the sand strata will have grains smaller than 1 mm whereas the gravel beds will contain no pebbles smaller than 5 mm. Gravel beds generally occur in strata only a few centimeters thick. Each pebble is covered with a thin film of yellow when ferric oxide and of black when manganese oxide. The presence of oxides testifies that the pebble strata are the preferred channels for underground courses.

Sometimes (as at Ottmarsheim) the voids in the gravel are filled with fine sand identical with that in contiguous strata; but voids filled with clay as described by the author have never been reported in France. It is possible that the interstitial sand comes from a secondary deposition. For some reason, the entire openwork gravel stratum is never completely filled.

The engineer, however, is not concerned so much with the origin of these deposits as he is with their degree of heterogeneity. For this reason, observations on gravel size and the permeability of alluviums are most important. Each time that the writer was able to obtain some truly clean samples from the original deposits, the resulting grain-size curve, plotted on Gaussian logarithmic graph paper, was a straight line (see Fig. 8(*a*)). Thus, the gradation of these deposits conforms to Gibrat's law of the proportionality of pebble diameters. On the other hand, if the grain size of a mixture of the deposits is analyzed, the gradations also plot as straight lines, as shown in Fig. 8(*b*).

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[15] Civ. Engr., École Nationale des Ponts et Chaussées, and Head, Dept. for Studies of Constr., Entreprise de Fondations et Travaux Hydrauliques, Paris, France.

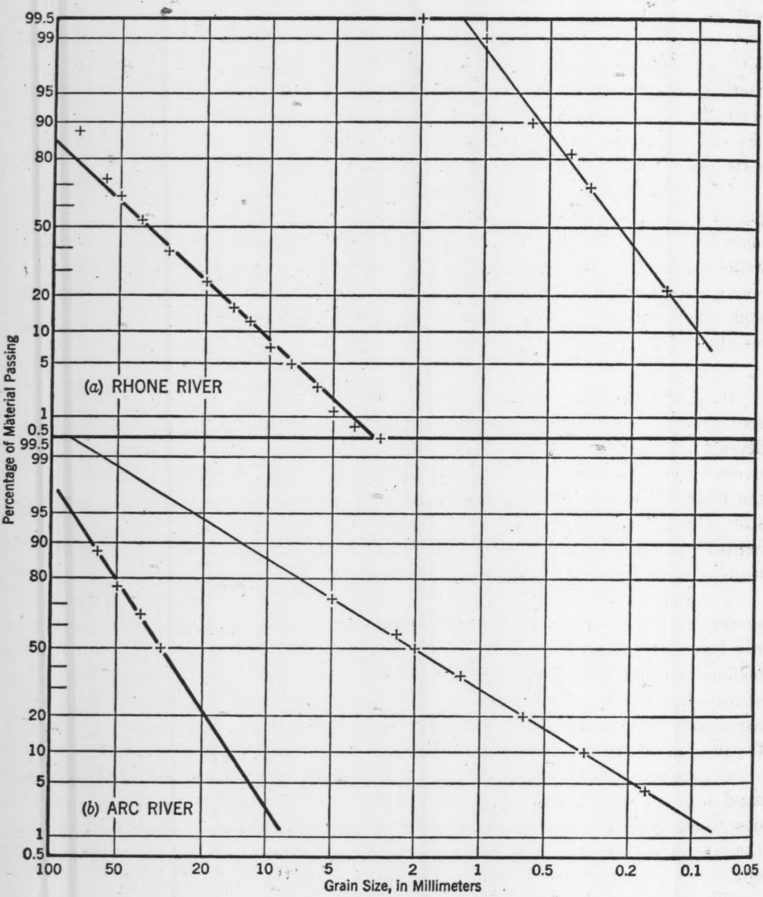

FIG. 8.—GRAIN SIZES OF ALLUVIUM SAMPLES

This fact permits engineers to judge the relative heterogeneity of the alluviums without recourse to extensive excavation. For a complete determination of heterogeneity, however, it is necessary to study the flow of ground water or to use "pumping-in" tests, and to decide whether the permeability measured "in place" corresponds to that estimated from the gradation curves. For example, observation of annual graphs for the Sautet reservoir on the Drac River presented the seepage from the neighboring valley of the Sézia River as a function of the lake level. After a lapse of several days the maximum and minimum lake levels were reflected in the seepage from the Drac River. Thus it was possible to compute the speed of the actual flow; and, from that flow, the permeability of the material in the ground-water passageway.

The path of percolation was about 1,200 m (about 4,000 ft) which indicates a permeability of about 2×10^{-2} m per sec (about 3.94 ft per min). (Contrary to a widely accepted opinion, the flows through these open structures are often laminar, as can be shown easily in the present case.) This value is obviously less than the permeability that can be estimated as logical under the circumstances—that is, a value of from 1×10^{-1} m per sec to 5×10^{-1} m per sec (from about 20 ft per min to 100 ft per min). Probably the disparity is explained by the distribution of the gravel in lenticular layers, which are more or less connected with each other. On the other hand, if the average permeability of the formation is determined, this time taking the total leakage into account, the flow is computed as approximately 3×10^{-3} m per sec (0.53 ft per min).

The foregoing comparison of ground flows shows that an average of 15% of the area exposed to percolation pressure offers a free channel for seepage. This ratio is plausible.

Even if openwork gravel deposits are not directly interconnected, they do increase the ease with which water can circulate through the alluviums. To reduce this circulation it is necessary to:

(1) Inject grout into all the large passages;

(2) Modify the direction of flow, forcing it in a direction perpendicular to the layers of the deposit; or

(3) Homogenize the entire deposit to a predetermined thickness.

In general, with a simple field test boring of large diameter (at least 400 mm or 16 in.) it is possible to determine the voids characteristics of alluviums; but the borings and water tests should be entrusted only to an engineer who is capable of interpreting the phenomena correctly during the course of the test.

Hyde Forbes,[16] M. ASCE.—The paper describes the occurrence of lenses of clean gravel in alluvial deposits, with the usual heterogeneous mixture of varying grain sizes, and suggests the way they were deposited. That such an occurrence is unusual is attested by the yield of wells bored in alluvial deposits of valleys. This phenomenon has not been encountered by the writer in the interior valleys or in the coastal plain of California, or in southwestern Arizona, although many penetration records of well bores and yield on pumping in these areas have been made available to him. However, such a deposit was encountered adjacent to the Columbia River channel in the vicinity of the Hanford area in Washington (mentioned by the author), and its effect on well yields may be of interest.

In connection with the development of a water supply for the construction camp at Hanford in 1944, about twenty-four test bores, 6 in. in diameter, were drilled between a bedrock promontory and the channel of the Columbia River. About half of these were developed into operating wells having larger diameter casings with depths varying from 54 ft to 121 ft. When the development was completed, pump capacity was installed on the basis of the completion tests, and thereafter records of discharge and drawdown were kept. Typical examples are given in Table 1(*a*), the wells being within from 1,500 ft to 5,000 ft from the river. The depths of the wells were dictated by the elevation at which the lacustrine clay beds were encountered under the flood plain alluvium. Specific yields in excess of 1,000 gal per foot of drawdown for such shallow wells have not been found at other locations within the writer's experience. In addition, such yields were not found for the wells drilled at, and north of, Richland, Wash., from 20 miles to 30 miles downstream, for the water supply o the town. There the Columbia River eroded well below its previously deposited alluvium into the lacustrine clays and the water table slopes toward, instead of from, the river. Typical data for the Richland wells are listed in Table 1(*b*). It is apparent that a special condition exists in the recent depositions of the Columbia River over its flood plain north and east of Gable Mountain near Hanford. The following were assumed necessary for the formation of large gravel bars in the stream channel to become part of the flood plain as the channel shifted in aggradation; namely,

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[16] Cons. Engr and Geologist, Palo Alto, Calif.

TABLE 1.—EXAMPLES OF DISCHARGE AND DRAWDOWN RECORDS IN OPENWORK GRAVEL

Diameter of casing (in.) (1)	Depths (Ft)		Safe yield (gal per min) (4)	Drawdown (ft) (5)	Yield (Gal per Min)	
	Total (2)	Below water table (3)			Per foot of drawdown (6)	Per square inch[a] (7)
(*a*) Hanford Area in Washington						
12	100	34	425	1.5	283	1.4
12	54	50	430	2.0	215	1.4
12	65	45	700	0.7	1,000	....
24	60	33	825	7.6	109	8.3
16	90	79	650	3.0	227	2.2
18	90	79	650	0.2	3,250	9.3
18	92	89	2,000	3.0	733	3.0
16	121	68	1,000	0.20	5,000	1.0
20	84	76	1,400	0.75	1,850	2.0
14	55	46	1,200	1.2	1,000	2.0
(*b*) Richland, Wash.						
12	62	31	470	18.9	25.0	1.4
18	72	63	2,266	33.3	68.0	3.2
18	72	58	2,575	30.2	85.0	4.1
14	78	60	1,500	27.8	54.0	2.7
14	80	60	920	25.8	36.0	2.1

[a] Yield, in gallons per minute per square inch of perforation.

(1) River stages and a velocity that would move the mass by bed rolling and keep it clear of fines;

(2) Sudden changes in river stage and velocity favoring deposition choking the channel and forcing the stream out of its banks allowing rapid burial of the masses of gravel;

(3) A source of abundant gravel composed of hard rock resistant to disintegration and decomposition, such as would be found in watersheds containing many glaciers; and

(4) Masses (condition No. 3) continuous with or directly connected with similar masses underlying the present stream channel.

L. F. HARZA,[17] M. ASCE.—This interesting paper does well to call attention to the possibility of a nearly continuous succession of lenticular bodies of open-void gravel (the term preferred by the writer) which might entail important leakage to a dam foundation or abutment if not intercepted by a cutoff. This is a possibility but a rather remote one in large dams where seepage distances are great and many variations in material along any seepage route are probable.

Normal gravel should never be a reason for alarm as it would be tighter than an equal volume of sand of the same character as that which fills the voids of the gravel. It is only the open-void gravel which might cause concern or require grouting.

The important problem to the engineer is how to detect such a condition of open voids in invisible locations and how to correct it. Under the heading "Engineering Significance of Openwork Gravel: Water Jetting," the author points out that a pipe can be jetted down through "normal" or what might be called "graded" or "void-filled" (preferred term) gravel, but will not penetrate open-void gravel which forms "an impassable barrier to jet pipes." This fact suggests a method of detecting its presence, but not its continuity—which is of far greater importance. Isolated lenses would not be harmful. Many jet pipe tests over the foundation area, revealing open-void gravel at consistent depths, might imply communication, but actual water tests would be necessary to confirm such a suspicion. The open-void gravel in the abutment of the Chief Joseph Dam site is under a great depth of glacial drift. This is an exceptional condition which would be difficult to explore from the surface.

Even where continuity of open-void gravel is indicated, it could probably be grouted with cement or clay grout. Void-filled gravels do not accept grout, except the so-called "chemical" type; but open-void gravel by its very nature should be susceptible to grouting as much so as the gravel used in the Prepakt method.

At one time a masonry cutoff to sound rock and a masonry core wall were considered essential for any earth dam. This requirement has given way by gradual stages to earth cutoff and core and, where rock is too deep to reach by excavation, to a sheet pile cutoff with earth core above, or, rarely, to other expedients. Within recent years high earth dams on deep foundations without any cutoff have proved successful. The outstanding example is the Davis Dam in Arizona-Nevada. The Fort Randall Dam (in South Dakota) will also rest on a deep sand bed of the Missouri River without cutoff, depending on a long path of travel of some 3,000 ft, or about 5% water gradient, through the sand to reduce the seepage to an acceptable amount. Sheet pile cutoffs may be assumed to be utilized even less frequently as experience accumulates and as confidence is gained in dams without such cutoffs. This trend, however, will emphasize the need to detect and grout any open-void gravel.

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[17] Harza Eng. Co., Cons. Engrs., Chicago, Ill.

ALLEN S. CARY,[18] Assoc. M. ASCE.—The wide distribution of openwork gravel is indicated by the geographical location of those who so kindly discussed the paper. The fact that each instance cited happens to be in a glaciated area further confirms the statement that openwork gravel is found more frequently in such areas than in unglaciated areas.

Mr. Feld's discussion regarding the wide range of bearing values of individual deposits is a worthwhile contribution.

The interesting discussion by Mr. Casagrande should serve as a warning, particularly to geologists, that the simplest, most obvious explanation for a natural phenomenon is not necessarily the actual one. The writer does not have sufficient familiarity with frost action in clay masses to discuss Mr. Casagrande's hypothesis intelligently. Laboratory tests should certainly resolve the question. Unfortunately, for the soils engineer and geologist the questions of origins are not of particular engineering significance, and funds for the necessary research are not readily available.

The discussion by Mr. Mayer concerning the Sautet Reservoir in France is interesting since this reservoir has its counterpart in a small reservoir in the State of Washington. However, the significance of the openwork gravel was not recognized in Washington until after the reservoir was filled when leakage downward into the gravel became excessive and an impervious blanket over the exposed gravel became necessary.

Mr. Cambefort has again emphasized the engineering significance of openwork gravel where water is involved and suggests means of determining the corrective steps necessary.

Mr. Forbes shows that openwork gravel has certain very distinct advantages if it lies below the ground-water table and one is looking for large quantities of well water.

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[18] Senior Geologist, Seattle Dist., Corps of Engrs., U. S. Army, Seattle, Wash.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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Paper No. 2468

STRENGTH OF I-BEAMS IN COMBINED BENDING AND TORSION

BY BASIL SOUROCHNIKOFF[1]

WITH DISCUSSION BY MESSRS. JACOB KAROL, MELVIN W. JACKSON, AND BASIL SOUROCHNIKOFF.

SYNOPSIS

In the design of a structural steel framework it is considered good practice to avoid subjecting I-beams to the combined effect of bending and torsion, because the torsional resistance of I-beams is usually low. It is often easy to determine a layout in which torsion is either entirely eliminated or greatly reduced. Sometimes, however, this is impracticable or impossible, and cases exist where beams carry eccentric loads and are therefore stressed in both bending and torsion. Laterally unsupported building spandrels offer a common example. Also, it is often impossible to avoid eccentricity of loads on beams in industrial structures because of the extreme complexity of the equipment they are designed to support. In these, and numerous other cases, the designer is called on to investigate stresses in I-beams loaded in combined bending and torsion.

Much has been written on the theory of torsion, but there appears to be little information on the design of I-beams in combined bending and torsion, apart from the method presented in the "Manual of Steel Construction."[2] In this method, the flange stresses due to vertical bending and those due to torsional bending are computed separately and superimposed to obtain the resultant bending stress. Although this method is probably the most complete of the various existing methods, one may question what allowable stress is to be used, and whether or not the allowable stress for torsional bending should be the same as that for vertical bending.

In this paper an attempt is made to develop allowable stress relations taking into account the interaction between the bending moment and the torque.

NOTE.—Published in September, 1950, as *Proceedings-Separate No. 33.* Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Eng. Dept., E. I. du Pont de Nemours & Co., Wilmington, Del.

[2] "Manual of Steel Construction," Bethlehem Steel Co., Bethlehem, Pa., 1934, p. 279 et seq.

The method given in the "Manual of Steel Construction" is followed closely and extended to include the influence of the deflections on the bending and torsional moments.

NOTATION

Letter symbols in this paper are defined where they first appear, in the text or illustrations, and are assembled alphabetically in the Appendix, for the guidance of readers and discussers.

INTRODUCTION

It is assumed that the beam rests on simple supports and that the ends are restrained against twist. The flanges, when considered as horizontal beams, are assumed to be simply supported at the ends. Therefore, there is no flange bending moment at the supports. The loads are parallel to the initial plane of the web of the beam. Their point of application is eccentric with respect to the centroid of the cross section by distances e_1 and e_2 in the directions normal to the web and parallel to the web, respectively. The deflections are assumed to be small compared to the original eccentricities, and the torque, although not negligible, is assumed to be small compared to the bending moment. This last condition amounts to assuming that the eccentricities are small compared to the span of the beam. It is thought that these assumptions represent the case most frequently encountered in practice. Obviously, other assumptions may be made as to the fixity of the cross section at the supports, which would lead to relations similar to the ones developed in this study except for the value of coefficients. If the loads were acting in the plane of the web ($e_1 = 0$), they would produce a bending moment about an axis normal to this plane. This moment will be termed "primary" bending moment. Because the loads are eccentric, a typical cross section of the beam twists through an angle β relative to its original position. In this case, the primary bending moment has a component in the lateral direction which produces a lateral deflection. Therefore, there is both a twist and a lateral deflection even before the loads have reached the critical value producing lateral instability.[3] The assumption that the torque is small compared to the bending moment leads to the conclusion that the lateral bending moment M_y is equal to the product of the primary bending moment M_x by the angle of twist, $M_y = \beta\, M_x$. If the torque were not small, a second term would have to be added to the expression for the lateral bending moment.[3]

As the beam deflects laterally and twists, the actual eccentricities are changed. As an illustration, consider the case of a beam under a concentrated load at midspan (Fig. 1). If the midsection of the beam has twisted by β_o and deflected laterally by y_o, the actual eccentricity of the load with respect to the supports becomes $e_1 + e_2\, \beta_o + y_o$, and the eccentricity with respect to the centroid of the cross section at midspan becomes $e_1 + e_2\, \beta_o$. Therefore the torque, both at the supports and at midspan, has increased as a result of the deflection and the twist. Thus, although the lateral deflection and the twist do not change the primary bending moment to a great extent, they produce a lateral bending moment and increase the torque.

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[3] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1936, p. 240, paragraph 46.

In the following sections, a detailed analysis is made of two cases of loading: (1) A concentrated load at midspan and (2) a uniformly distributed load. First, however, a preliminary case, involving torque alone, must be investigated.

1. Beams Subjected to Torque Alone

It is assumed that the torque is applied in the form of couples of forces in the planes of cross sections, thereby producing torsion and lateral bending of flanges due to torsion, but no bending of the beam as a whole. It is assumed, also, that the value of the torque is not affected by the twist it produces. This preliminary case helps to simplify the study of a more general case analyzed in Section 2, where it is assumed that the beam is subjected to the combined effect of bending and torsion, and that the values of moments are affected by the deformations they cause.

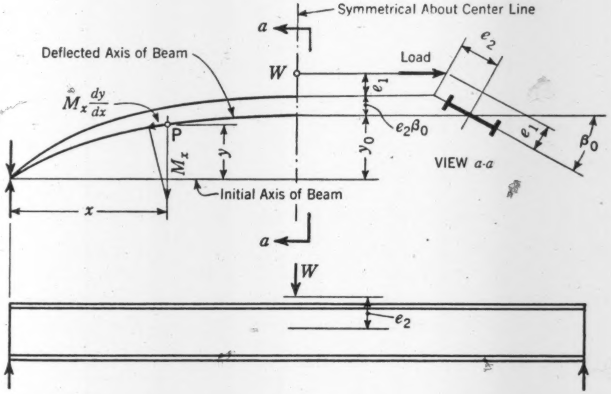

FIG. 1.—DISTORTION OF BEAM SUBJECTED TO A LOAD W AT MIDSPAN

The differential equation for the angle of twist ψ is[4,5]:

$$\frac{d\psi}{dx} - a^2 \frac{d^3\psi}{dx^3} = \frac{T_x}{E_s\,\kappa} \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

in which[2] a is a torsional bending constant $\left(= \frac{d}{2}\sqrt{\frac{E\,I_y}{E_s\,\kappa}}\right)$; T_x is the torque at the point of abscissa; E is the modulus of elasticity in tension; E_s is the modulus

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[4] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1936, p. 257, paragraph 49.

[5] "Structural Beams in Torsion," by Inge Lyse and Bruce G. Johnston, *Transactions*, ASCE, Vol. 101, 1936, p. 868 et seq.

of elasticity in shear; d is the distance between flange centroids; and κ is a torsional constant. In the various cases analyzed herein, the torque T_x is symmetrical with respect to the midspan of the beam, and therefore $\frac{d\psi}{dx} = 0$ for $x = \frac{l}{2}$. The ends of the beam are restrained against twist, but there is no restraining bending moment in the planes of the flanges at the ends. Therefore, $\psi = 0$ and $\frac{d^2\psi}{dx^2} = 0$ for $x = 0$. Thus, the boundary conditions are: For $x = 0$,

$$\psi = 0 \qquad \text{(2a)}$$

and

$$\frac{d^2\psi}{dx^2} = 0 \qquad \text{(2b)}$$

For $x = \frac{l}{2}$

$$\frac{d\psi}{dx} = 0 \qquad \text{(3)}$$

The general solution of Eq. 1 is[4,5]

$$\psi = A_1 \sinh\frac{x}{a} + B_1 \cosh\frac{x}{a} + C_1 + X_1 \qquad \text{(4)}$$

in which X_1 is a particular solution depending on the shape of the torque. The values of the integration constants A_1, B_1, and C_1 are determined from the conditions expressed in Eqs. 2 and 3. Assuming that the lateral moment of inertia of each flange is $I_y/2$, the flange bending moment is $-\frac{E\,I_y}{4}\,d\,\frac{d^2\psi}{dx^2}$ and the section modulus is $\frac{0.5\,I_y}{0.5\,b} = \frac{I_y}{b}$. Therefore, the flange stress f_f due to lateral flange bending is

$$f_f = -\frac{E\,I_y\,d}{4}\,\frac{b}{I_y}\,\frac{d^2\psi}{dx^2} = -\frac{E\,I_y\,d^2}{4\,E_s\,\kappa} \times \frac{b\,E_s\,\kappa}{I_y\,d}\,\frac{d^2\psi}{dx^2} = -\,a\,B\,E_s\,\kappa\,\frac{d^2\psi}{dx^2} \qquad \text{(5)}$$

The flange shear is

$$H = -\frac{E\,I_y}{2}\,\frac{d}{2}\,\frac{d^3\psi}{dx^3} = -\frac{a^2}{d}\,E_s\,\kappa\,\frac{d^3\psi}{dx^3} \qquad \text{(6)}$$

The parts of the applied torque resisted by the torsional constant T' and by the lateral flange bending T'' are, respectively,

$$T' = E_s\,\kappa\,\frac{d\psi}{dx} \qquad \text{(7a)}$$

and

$$T'' = -\frac{E\,I_y\,d^2}{4}\,\frac{d^3\psi}{dx^3} \qquad \text{(7b)}$$

Deflections and Stresses for Several Types of Torque Curves.—The mathematical expressions for several types of torque (see Fig. 2) are as follows: For case 1,

$$T_x = T_o \text{ (a constant)} \quad \text{(8a)}$$

for case 2,

$$T_x = T_o\left(1 - \frac{2\,x}{l}\right) \quad \text{(8b)}$$

for case 3,

$$T_x = T_o\left[1 - \left(\frac{2\,x}{l}\right)^2\right] \quad \text{(8c)}$$

and, for case 4,

$$T_x = T_o \cos\frac{\pi\,x}{l} \quad \text{(8d)}$$

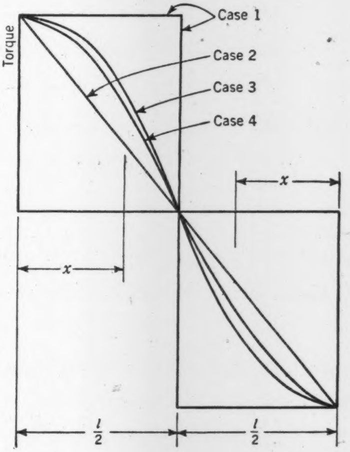

FIG. 2

The particular solutions X_1 are: For case 1,

$$X_1 = \frac{T_o\,l}{2\,E_s\,\kappa}\left(\frac{2\,x}{l}\right) \quad \text{(9a)}$$

for case 2,

$$X_1 = \frac{T_o\,l}{2\,E_s\,\kappa}\left[\left(\frac{2\,x}{l}\right) - \frac{1}{2}\left(\frac{2\,x}{l}\right)^2\right] \quad \text{(9b)}$$

for case 3,

$$X_1 = \frac{T_o\,l}{2\,E_s\,\kappa}\left\{-\frac{1}{3}\left(\frac{2\,x}{l}\right)^3 + \left[1 - 2\left(\frac{2\,a}{l}\right)^2\right]\left(\frac{2\,x}{l}\right)\right\} \quad \text{(9c)}$$

and, for case 4,

$$X_1 = \frac{T_o\,l}{\pi\,E_s\,\kappa\left(1 + \dfrac{\pi^2\,a^2}{l^2}\right)}\sin\frac{\pi\,x}{l} \quad \text{(9d)}$$

After determining the integration constants by the conditions defined as Eqs. 2 and 3, the values of the angle of twist become: For case 1,

$$\psi = T_o\,C\left[-\frac{\sinh\dfrac{x}{a}}{\cosh\dfrac{x}{a}} + \frac{l}{2\,a}\left(\frac{2\,x}{l}\right)\right] \quad \text{(10a)}$$

for case 2,

$$\psi = T_o\,C\left\{\frac{2\,a}{l}\left(\tanh\frac{l}{2\,a}\sinh\frac{x}{a} - \cosh\frac{x}{a} + 1\right)\right.$$
$$\left. + \frac{l}{2\,a}\left[\left(\frac{2\,x}{l}\right) - \frac{1}{2}\left(\frac{2\,x}{l}\right)^2\right]\right\} \quad \text{(10b)}$$

for case 3,

$$\psi = T_o\, C \left(2 \left(\frac{2\,a}{l} \right)^2 \frac{\sinh \frac{x}{a}}{\cosh \frac{l}{2\,a}} + \frac{l}{2\,a} \left\{ - \frac{1}{3} \left(\frac{2\,x}{l} \right)^3 + \left[1 - 2 \left(\frac{2\,a}{l} \right)^2 \right] \left(\frac{2\,x}{l} \right) \right\} \right) \ldots (10c)$$

and, for case 4,

$$\psi = T_o\, C \frac{1}{\frac{\pi\,a}{l} \left[1 + \left(\frac{\pi\,a}{l} \right)^2 \right]} \sin \frac{\pi\,x}{l} \ldots\ldots\ldots\ldots (10d)$$

in which[2] $C = \frac{a}{E_s\,\kappa}$ is the torsional twist constant. The maximum angle of twist occurs at midspan and may be expressed as

$$\psi_o = C\, R\, T_o \ldots\ldots\ldots\ldots (11)$$

in which the quantity R is defined as follows: For case 1,

$$R = R_c = -\tanh \frac{l}{2\,a} + \frac{l}{2\,a} \ldots\ldots\ldots\ldots (12a)$$

for case 2,

$$R = R_u = \frac{2\,a}{l} \left(\frac{1}{\cosh \frac{l}{2\,a}} - 1 \right) + \frac{1}{2} \frac{l}{2\,a} \ldots\ldots\ldots\ldots (12b)$$

for case 3,

$$R = R_p = 2 \left(\frac{l}{2\,a} \right) \left[\left(\frac{2\,a}{l} \right)^3 \tanh \frac{l}{2\,a} + \frac{1}{3} - \left(\frac{2\,a}{l} \right)^2 \right] \ldots\ldots (12c)$$

and, for case 4,

$$R = R_s = \frac{1}{\frac{\pi\,a}{l} \left[1 + \left(\frac{\pi\,a}{l} \right)^2 \right]} \ldots\ldots\ldots\ldots (12d)$$

The variations of R with $\frac{l}{2\,a}$ are shown in Fig. 3. The shape of the curve ψ (that is, the value of the ratio ψ/ψ_o) is given in Table 1 for various values of

TABLE 1.—VALUES OF ψ/ψ_o

$\underline{2\,x}$	$\frac{l}{2\,a} = 1.0$				$\frac{l}{2\,a} = 2.0$				$\frac{l}{2\,a} = 3.0$			
	Case 1	Case 2	Case 3	Case 4	Case 1	Case 2	Case 3	Case 4	Case 1	Case 2	Case 3	Case 4
0.25	0.365	0.390	0.375	0.384	0.350	0.393	0.379	0.384	0.334	0.422	0.376	0.384
0.50	0.686	0.713	0.703	0.707	0.665	0.717	0.705	0.707	0.641	0.765	0.703	0.707
0.75	0.918	0.925	0.915	0.925	0.905	0.925	0.925	0.925	0.893	0.985	0.921	0.925
1.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

$\frac{l}{2a}$ and $\frac{2x}{l}$. It is seen that all the curves ψ/ψ_o are nearly parabolic and that the shape varies little either with $\frac{l}{2a}$ or with the form of the torque. The actual ordinates of the curves ψ, however, vary considerably, as evidenced by Fig. 3.

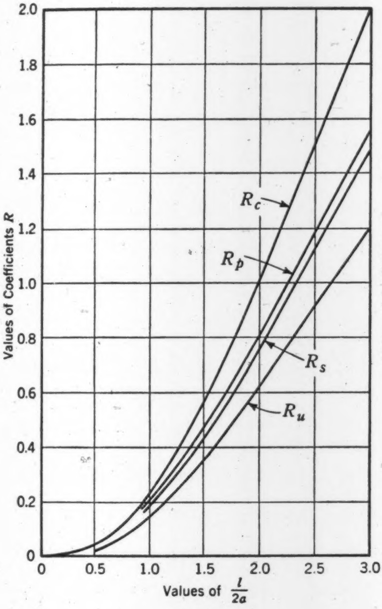

FIG. 3

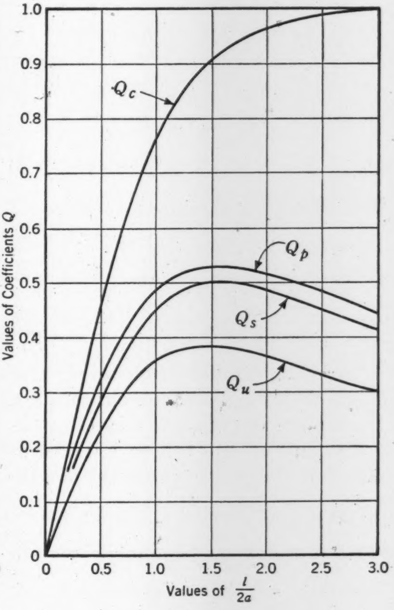

FIG. 4

The flange stress f_f is determined from Eqs. 5 and 10. Its value is found to be: For case 1,

$$f_f = B\,T_o \frac{\sinh x/a}{\cosh l/(2\,a)} \quad \text{(13a)}$$

for case 2,

$$f_f = B\,T_o \left(\frac{2\,a}{l}\right)\left(\tanh \frac{l}{2\,a} \sinh \frac{x}{a} - \cosh \frac{x}{a} + 1\right) \quad \text{(13b)}$$

for case 3,

$$f_f = 2\,B\,T_o \left(\frac{2\,a}{l}\right)\left(\frac{2\,a}{l} \frac{\sinh \frac{x}{a}}{\cosh \frac{l}{2\,a}} - \frac{2\,x}{l}\right) \quad \text{(13c)}$$

and, for case 4,

$$f_f = B\,T_o \frac{\pi\,a}{l} \frac{1}{1 + \left(\frac{\pi\,a}{l}\right)^2} \sin \frac{\pi\,x}{l} \quad \text{(13d)}$$

in which B is a torsional stress constant,[2] $\frac{a\,b}{I_y\,d}$. The maximum flange stress occurs at midspan. It may be expressed as

$$f_f = Q\,B\,T_o \qquad (14)$$

in which for case 1,

$$Q = Q_c = \tanh\frac{l}{2\,a} \qquad (15a)$$

for case 2,

$$Q = Q_u = \frac{2\,a}{l}\left[1 - \frac{1}{\cosh l/(2\,a)}\right] \qquad (15b)$$

for case 3,

$$Q = Q_p = 2\left(\frac{2\,a}{l}\right)\left[\left(\frac{2\,a}{l}\right)\tanh\frac{l}{2\,a} - 1\right] \qquad (15c)$$

and, for case 4,

$$Q = Q_s = \frac{\pi\,a}{l}\,\frac{1}{1 + \left(\frac{\pi\,a}{l}\right)^2} \qquad (15d)$$

The variation of Q with $\frac{l}{2\,a}$ is shown in Fig. 4.

A more detailed study of the torque resisted by the torsional constant shows that, in all cases except case 1, the effect of the shearing stress caused by the bending of flanges decreases as the span increases—that is, for large spans, practically all the torque is taken up by the torsional rigidity. This phenomenon is consistent with the fact that the flange stress coefficients Q decrease in value after having reached a maximum for a certain value of $\frac{l}{2\,a}$, since the effect of the flange bending decreases. In case 1, the applied torque is always taken up entirely by flange bending at midspan, regardless of the value of $\frac{l}{2\,a}$. Consistently, the coefficient Q_c increases with $\frac{l}{2\,a}$ and tends toward a finite value.

The results for case 1 are the same as those given in the "Manual of Steel Construction."[2] The results for case 2 are different. The similar case in the manual was apparently based on a different set of end conditions. Cases 3 and 4 are not found in the "Manual of Steel Construction."

2. Deflections of Beams Under Combined Effect of Bending and Torsion

An analysis of two cases of loading is presented in this section: (1) A concentrated load W at midspan and (2) a uniform load w per unit length of span. The assumptions are the same as those listed in the "Introduction."

It will be made apparent subsequently that the torque curves fall between those considered in Section 1. Therefore, β is nearly parabolic. Too much refinement in the determination of the values of β does not seem to be warranted as the actual angle between any section and the sections at the supports should include the initial degree to which the beam is "out of plane." The latter variable is always present and its magnitude is unknown. It will be assumed, therefore, that β is parabolic—that is,

$$\beta = \beta_o \left[2 \left(\frac{2\,x}{l} \right) - \left(\frac{2\,x}{l} \right)^2 \right] \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

The rotation β produces a moment M_y in the lateral direction which is, for a concentrated load,

$$M_y = \beta\, M_x = \frac{W\,l}{4}\,\frac{2\,x}{l}\,\beta_o \left[2 \left(\frac{2\,x}{l} \right) - \left(\frac{2\,x}{l} \right)^2 \right]$$

$$= \frac{W\,l}{4}\,\beta_o \left[2 \left(\frac{2\,x}{l} \right)^2 - \left(\frac{2\,x}{l} \right)^3 \right] \ldots (17a)$$

and, for a uniform load,

$$M_y = \beta\, M_x = \frac{w\,l^2}{8} \left[2 \left(\frac{2\,x}{l} \right) - \left(\frac{2\,x}{l} \right)^2 \right] \beta_o \left[2 \left(\frac{2\,x}{l} \right) - \left(\frac{2\,x}{l} \right)^2 \right]$$

$$= \frac{w\,l^2}{8}\,\beta_o \left[4 \left(\frac{2\,x}{l} \right)^2 - 4 \left(\frac{2\,x}{l} \right)^3 + \left(\frac{2\,x}{l} \right)^4 \right] \ldots (17b)$$

The lateral deflection is determined by $-\,E\,I_y \frac{d^2u}{dx^2} = M_y$, by double integration of the moment. The boundary conditions are $y = 0$ for $x = 0$, and $\frac{dy}{dx} = 0$ for $x = \frac{l}{2}$. Thus, for a concentrated load,

$$y = -\,\frac{W\,l^3}{16\,E\,I_y}\,\beta_o \left[\frac{1}{6} \left(\frac{2\,x}{l} \right)^4 - \frac{1}{20} \left(\frac{2\,x}{l} \right)^5 - \frac{5}{12} \left(\frac{2\,x}{l} \right) \right] \ldots (18a)$$

and, for a uniform load,

$$y = -\,\frac{w\,l^4}{32\,E\,I_y}\,\beta_o \left[\frac{1}{3} \left(\frac{2\,x}{l} \right)^4 - \frac{1}{5} \left(\frac{2\,x}{l} \right)^5 + \frac{1}{30} \left(\frac{2\,x}{l} \right)^6 - \frac{8}{15} \left(\frac{2\,x}{l} \right) \right] \ldots (18b)$$

The maximum deflection occurs at midspan in both cases: For a concentrated load,

$$y_o = 0.0187\,\frac{W\,l^3}{E\,I_y}\,\beta_o \ldots\ldots\ldots\ldots\ldots\ldots (19a)$$

and, for a uniform load,

$$y_o = 0.0115\,\frac{w\,l^4}{E\,I_y}\,\beta_o \ldots\ldots\ldots\ldots\ldots\ldots (19b)$$

In the case of the concentrated load, it is evident from Fig. 1 that the torque at any point p, of abscissa x, is

$$T_x = \frac{W}{2}\,(e_1 + e_2\,\beta_o + y_o - y) + M_x\,\frac{dy}{dx} \ldots\ldots\ldots (20a)$$

with $M_x = \frac{W}{4}\left(\frac{2\,x}{l}\right)$. Using this value of M_x, with Eqs. 18*a* and 19*a* for y, $\frac{dy}{dx}$, and y_o, the torque becomes

$$T_x = \frac{W}{2}\left[e_1 + e_2\,\beta_o\right] + \frac{W^2\,l^3}{32\,E\,I_y}\beta_o\left[\frac{3}{10} - \frac{1}{2}\left(\frac{2\,x}{l}\right)^4 + \frac{1}{5}\left(\frac{2\,x}{l}\right)^5\right] \quad (20b)$$

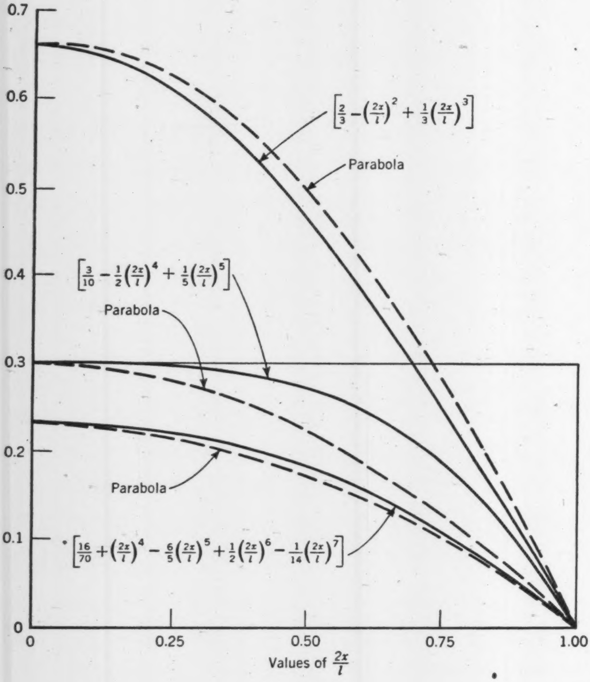

FIG. 5

The values of $\left[\frac{3}{10} - \frac{1}{2}\left(\frac{2\,x}{l}\right)^4 + \frac{1}{5}\left(\frac{2\,x}{l}\right)^5\right]$ are plotted in Fig. 5. Since the curve is quite convex and the lateral deflection was assumed to be small compared to the eccentricity e_1, there will be only a small error (on the safe side) if the torque is considered to be constant and equal to $\frac{W}{2}\left[e_1 + \left(e_2 + \frac{3}{160}\frac{W\,l^3}{E\,I_y}\right)\beta_o\right]$. On this assumption, the maximum angle of twist at midspan can be computed as follows: $\beta_o = C\,R_c\left[\frac{W}{2}(e_1 + e_2\,\beta_o)\right.$

$+\frac{3}{320}\frac{W^2 l^3}{E I_y}\beta_o\Big]$; from which

$$\beta_o = \frac{\psi_o}{1 - \frac{\psi_o}{e_1}\left(e_2 + 0.0188\,\frac{W l^3}{E I_y}\right)} \quad \text{. . . . .(21)}$$

Eq. 21 gives the maximum angle of twist under combined bending and torsion, computed on the assumption that the effect of the deflections on the moments is not negligible. The quantity ψ_o is the maximum angle of twist which would have been obtained if the torque alone were acting and the deflections were negligible—that is,

$$\psi_o = C\,R_c\,\frac{W e_1}{2} \quad \text{. . . . .(22)}$$

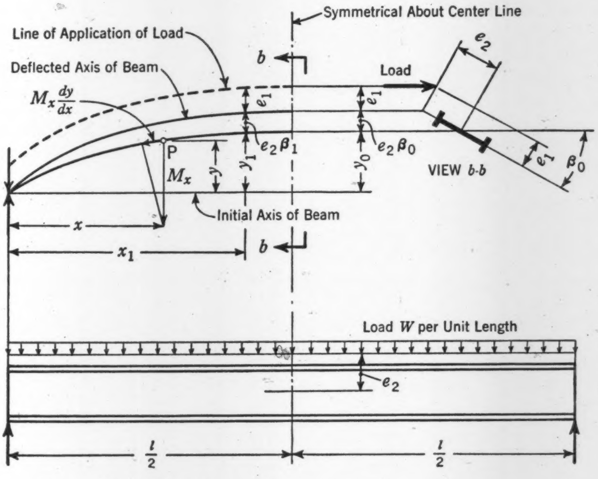
Symmetrical About Center Line
Line of Application of Load
Deflected Axis of Beam
$M_x \frac{dy}{dx}$
b
Load
e_2
e_1
e_1
$e_2\beta_1$
$e_2\beta_0$
P
M_x
y
y_1
y_0
VIEW b-b
e_1
β_0
Initial Axis of Beam
x
x_1
b
Load W per Unit Length
e_2
$\frac{l}{2}$
$\frac{l}{2}$

FIG. 6.—HALF-PLAN VIEW OF DEFLECTED BEAM

in which C is a torsional twist constant,[2] $\frac{a}{E_s\,\kappa}$. Considering now the case of the uniform load, it is evident from Fig. 6 that the torque at any point p of abscissa x is $T_x = \int_x^{l/2} w\,(e_1 + e_2\,\beta_1 + y_1 - y)\,dx + M_x\,\frac{dy}{dx}$; or

$$T_x = \frac{w\,l}{2}\,e_1\left(1 - \frac{2\,x}{l}\right) + w\,e_2\int_x^{l/2}\beta_1\,dx$$
$$+ w\int_o^{l/2} y_1\,dx - w\,y\,\frac{l}{2}\left(1 - \frac{2\,x}{l}\right) + M_x\,\frac{dy}{dx} \quad \text{. .(23a)}$$

Using Eqs. 18*b* and 19*b* and noting that $M_x = \frac{w\,l^2}{8}\left[2\left(\frac{2\,x}{l}\right) - \left(\frac{2\,x}{l}\right)^2\right]$, Eq. 23*a* may be written as

$$T_x = \frac{w\,l}{2}\,e_1\left(1 - \frac{2\,x}{l}\right) + w\,e_2\,\beta_o\,\frac{l}{2}\left[\frac{2}{3} - \left(\frac{2\,x}{l}\right)^2 + \frac{1}{3}\left(\frac{2\,x}{l}\right)^3\right] + \frac{w^2\,l^5}{64\,E\,I_y}$$
$$\times\,\beta_o\left[\frac{16}{70} + \left(\frac{2\,x}{l}\right)^4 - \frac{6}{5}\left(\frac{2\,x}{l}\right)^5 + \frac{1}{2}\left(\frac{2\,x}{l}\right)^6 - \frac{1}{14}\left(\frac{2\,x}{l}\right)^7\right] \quad\text{(23b)}$$

The values of $\left[\frac{2}{3} - \left(\frac{2\,x}{l}\right)^2 + \frac{1}{3}\left(\frac{2\,x}{l}\right)^3\right]$ and $\left[\frac{16}{70} + \left(\frac{2\,x}{l}\right)^4 - \frac{6}{5}\left(\frac{2\,x}{l}\right)^5 + \frac{1}{2}\left(\frac{2\,x}{l}\right)^6 - \frac{1}{14}\left(\frac{2\,x}{l}\right)^7\right]$ are plotted in Fig. 5. Both are close to a parabola. It may be assumed, therefore, that the torque consists of a component having a straight-line variation from a maximum of $\frac{w\,l}{2}\,e_1$ at the support to zero at midspan and a component having a parabolic variation from a maximum of $\frac{2}{3}\,w\,e_2\,\beta_o\,\frac{l}{2} + \frac{16}{70}\,\frac{w^2\,l^5}{64\,E\,I_y} = \frac{w\,l\,e_1}{2}\left(0.66\,e_2 + 0.00715\,\frac{w\,l^4}{E\,I_y}\right)\frac{\beta_o}{e_1}$ at the support to zero at midspan. The maximum angle of twist at midspan may then be computed as follows:

$$\beta_o = C\,R_u\,\frac{w\,l}{2}\,e_1 + C\,R_p\,\frac{w\,l}{2}\,e_1\left(0.66\,e_2 + 0.00715\,\frac{w\,l^4}{E\,I_y}\right)\frac{\beta_o}{e_1} = C\,R_u\,\frac{w\,l}{2}$$
$$\times\,e_1\left[1 + \frac{R_p}{R_u}\left(0.66\,e_2 + 0.00715\,\frac{w\,l^4}{E\,I_y}\right)\frac{\beta_o}{e_1}\right] \quad\text{(24a)}$$

Fig. 3 shows that the quantity $\frac{R_p}{R_u}$ is practically constant and equal to 1.27. Using this value, the expression for β_o may be written as

$$\beta_o = \frac{\psi_o}{1 - 0.85\,\frac{\psi_o}{e_1}\left(e_2 + 0.01075\,\frac{w\,l^4}{E\,I_y}\right)} \quad\text{(24b)}$$

Eq. 24*b* is similar to Eq. 21 and gives the maximum angle of twist under combined bending and torsion computed on the assumption that the effect of deflections on moments is not negligible. The quantity ψ_o is the maximum angle of twist that would have been obtained for uniform load if torque alone were acting and the deflections were negligible; that is,

$$\psi_o = C\,R_u\,\frac{w\,l\,e_1}{2} \quad\text{(25)}$$

3. NORMAL FIBER STRESSES UNDER COMBINED BENDING AND TORSION

The normal fiber stress is composed of three parts: (1) The primary bending stress under vertical loads neglecting torsion; (2) the stress due to lateral bending of the beam as a whole; and (3) the stress due to the bending of one flange relative to the other. The primary bending stress (part 1) is expressed as

$$f_b = \frac{M_x}{S_x} \quad \text{(26)}$$

The lateral bending moment (part 2) is $\beta\, M_x$ and its maximum $\left(\text{which occurs for } x = \frac{l}{2}\right)$ is $\beta_o\, M_x$. The bending stress is then $\beta_o \frac{M_x}{S_y}$. The sum of parts 1 and 2 is $\frac{M_x}{S_x} + \beta_o \frac{M_x}{S_y} = f_b \left(1 + \frac{S_x}{S_y} \beta_o\right)$. It was assumed that for a concentrated load at midspan the torque is constant and equal to $\frac{\beta_o}{C\, R_c} = \frac{\beta_o}{\psi_o} \frac{C\, R_c}{\psi_o} = \frac{W\, e_1}{2} \frac{\beta_o}{\psi_o}$. The flange stress is then $B\, Q_c \frac{W\, e_1}{2} \frac{\beta_o}{\psi_o}$, in which ψ_o is given by Eq. 22. In the case of a uniform load, it is assumed that the torque is divided into two parts. One part has a linear variation from $\frac{w\, l\, e_1}{2}$ at the support to zero at midspan. The second part has a parabolic variation from $\frac{w\, l\, e_1}{2} \left(0.66\, e_2 + 0.00715 \frac{w\, l^4}{E\, I_y}\right) \frac{\beta_o}{e_1}$ at the support to zero at midspan. The flange stress is then

$$f = \frac{w\, l\, e_1}{2} \cdot B\, Q_u \left[1 + \frac{1}{e_1} \frac{Q_p}{Q_u} \left(0.66\, e_2 + 0.00715 \frac{w\, l^4}{E\, I_y}\right) \beta_o\right] \quad \text{(27)}$$

As $\frac{Q_p}{Q_u}$ is nearly equal to $\frac{R_p}{R_u}$ (part 1), the quantity in brackets may be taken equal to $\frac{\beta_o}{\psi_o}$, and therefore the flange stress is $B\, Q_u \frac{w\, l\, e_1}{2} \frac{\beta_o}{\psi_o}$, in which ψ_o is given by Eq. 25. For more accuracy, $\frac{Q_p}{Q_u}$ may be assumed to be different from $\frac{R_p}{R_u}$. In this case, the quantity to be used instead of $\frac{\beta_o}{\psi_o}$ is $1 + \frac{Q_p}{Q_u} \frac{R_u}{R_p} \left(\frac{\beta_o}{\psi} - 1\right)$.

If the torque were considered alone, and if the effect of deflections were neglected, the flange bending stress would be: For a concentrated load,

$$f_f = B\, Q_c \frac{W\, e_1}{2} \quad \text{(28a)}$$

and, for a uniform load,

$$f_f = B\, Q_u \frac{w\, l\, e_1}{2} \quad \text{(28b)}$$

Using Eqs. 28, the flange bending stress (computed on the assumption that there is both bending and torque and that the effect of deflections is not negligible) may be expressed as $f_f \frac{\beta_o}{\psi}$ for both the concentrated and the uniform loads. The total normal stress may then be represented by the equation:

$$f = f_b \left(1 + \frac{S_x}{S_y} \beta_o\right) + f_f \frac{\beta_o}{\psi_o} \qquad (29)$$

for both cases of loading. The quantities f_b, f_f, ψ_o, and β_o, obviously, do not have the same value for the concentrated load as for the uniform load.

It should be noted that the total stress is not proportional to the load, since β_o as given by Eqs. 21 and 24*b* is not proportional to the load. Therefore, the factor of safety must be applied to the load rather than to the stress. Let n be the factor of safety with respect to the yield point stress f_y. The criterion for the safe load is then

$$f_y = n f_b \left(1 + \frac{S_x}{S_y} \beta_{on}\right) + n f_f \frac{\beta_{on}}{n \psi_o} \qquad (30)$$

in which β_{on} is computed from Eqs. 21 and 24*b* for a load equal to n times the design load; that is, for a concentrated load,

$$\beta_{on} = \frac{n \psi_o}{1 - \frac{n \psi_o}{e_1}\left(e_2 + 0.0188\, n \frac{W l^3}{E I_y}\right)} \qquad (31a)$$

and, for a uniform load,

$$\beta_{on} = \frac{n \psi_o}{1 - \frac{n \psi_o}{e_1} \times 0.85 \left(e_2 + 0.0175 \frac{n w l^4}{E I_y}\right)} \qquad (31b)$$

The angle of twist ψ is given by Eqs. 22 and 25 for the concentrated and the unform loads, respectively. Eq. 30 may be rewritten as follows:

$$\frac{f_b}{f'_b} + \frac{f_f}{f'_f} = 1 \qquad (32)$$

in which

$$f'_b = \frac{\frac{f_y}{n}}{1 + \frac{S_x}{S_y} \beta_{on}} \qquad (33a)$$

and

$$f'_f = f_y \frac{\psi_o}{\beta_{on}} \qquad (33b)$$

Eq. 32 is similar to the formula of the American Institute of Steel Construction defining the criterion for the design of members under combined compression and bending.[6]

Numerical Examples.—Investigate a 14WF74 beam, 20 ft long, simply supported and loaded at a point 2 in. from the plane of the web (e_1 = 2 in.).

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[6] "Specification for the Design, Fabrication and Erection of Steel for Buildings," A.I.S.C., New York, N. Y., February, 1946, p. 10.

Assume a factor of safety $n = 1.67$ with respect to the yield point stress of 33,400 lb per sq in. The following constants are taken from the "Manual of Steel Construction"[2]: $a = 63.04$; $B = 0.355$; $C = 0.00000144$; and $\frac{l}{2\,a} = 1.90$. From Figs. 3 and 4, $R_c = 0.943$; $Q_c = 0.956$; $R_u = 0.580$; and $Q_u = 0.370$.

Example 1.—Assume a concentrated load of 22,000 lb at midspan applied at the top flange ($e_2 = 7$ in.). Substituting in appropriate equations, the solution yields:

Equation	Quantity
22	$\psi_o = 0.0299$
31*a*	$\beta_{on} = 0.066$
33*a*	$f'_b = 15{,}600$
33*b*	$f'_f = 15{,}100$
26	$f_b = 11{,}750$
28*a*	$f_f = 7{,}450$

Testing the foregoing results by Eq. 32: $\frac{f_b}{f'_b} + \frac{f_f}{f'_f} = \frac{11{,}750}{15{,}600} + \frac{7{,}450}{15{,}100} = 1.250$.

Example 2.—Assume a uniform load of 2,200 lb per foot of span ($w\,l = 44{,}000$ lb) applied at the top flange ($e_2 = 7$ in.). As in example 1, substituting in the appropriate equations, the solution yields

Equation	Quantity
25	$\psi_o = 0.0367$
31*b*	$\beta_{on} = 0.082$
33*a*	$f'_b = 14{,}800$
33*b*	$f'_f = 14{,}900$
26	$f_b = 11{,}750$
28*b*	$f_f = 5{,}790$

Again testing results by Eq. 32: $\frac{11{,}750}{14{,}800} + \frac{5{,}700}{14{,}900} = 1.183$.

4. Critical Loads

The critical load is reached when the angle of twist and the lateral deflection become very large. Since the lateral deflection is proportional to the angle of twist, it is sufficient to consider the angle of twist only. Eqs. 21 and 24*b* show that the critical load is determined by: For a concentrated load,

$$C\,R_c\,\frac{W}{2}\left(e_2 + 0.0188\,\frac{W\,l^3}{E\,I_y}\right) = 1 \quad \ldots\ldots\ldots\ldots\ldots (34a)$$

and, for a uniform load,

$$0.85\,C\,R_u\,\frac{w\,l}{2}\left(e_2 + 0.01075\,\frac{w\,l^4}{E\,I_y}\right) = 1 \quad \ldots\ldots\ldots\ldots (34b)$$

In computing the stresses (Section 3), β_{on} was determined for n times the design load. The value of β_{on} thus found shows whether n times the design load approaches the critical value. Therefore, no additional calculations for critical loads are required.

If the eccentricity e_2 is zero, Eqs. 34 may be rewritten as follows: For a concentrated load,

$$0.0094\, R_c \frac{a}{l} \frac{l^4}{E_s\, \kappa\, E\, I_y} W^2 = 1 \ldots\ldots\ldots\ldots\ldots (35a)$$

and, for a distributed load,

$$0.00458\, R_u \frac{a}{l} \frac{l^4}{E_s\, \kappa\, E\, I_y} (w\, l^2) = 1 \ldots\ldots\ldots\ldots\ldots (35b)$$

Therefore, the critical load becomes: For a concentrated load,

$$W = m \frac{\sqrt{E_s\, \kappa\, E\, I_y}}{l^2} \ldots\ldots\ldots\ldots\ldots\ldots (36a)$$

and, for a distributed load,

$$w\, l = m \frac{\sqrt{E_s\, \kappa\, E\, I_y}}{l^2} \ldots\ldots\ldots\ldots\ldots\ldots (36b)$$

in which the coefficient m is: For a concentrated load,

$$m = 10.3 \sqrt{\frac{l}{a\, R_c}} \ldots (37a)$$

and, for a distributed load,

$$m = 14.8 \sqrt{\frac{l}{a\, R_u}} \ldots (37b)$$

Eqs. 36 are the same as those presented by S. Timoshenko in 1936.[7] The comparison of the numerical values of coefficients m in Table 2 shows that the difference between the two sets of coefficients is not large in spite of the fact that they were established by entirely different methods.

TABLE 2.—COMPARISON OF VALUES OF m

$\left(\frac{l}{a}\right)^2$	$\frac{l}{2\,a}$	CONCENTRATED LOAD		UNIFORM LOAD	
		Timoshenko	Eq. 37*a*	Timoshenko	Eq. 37*b*
0.4	0.316	86.4	82.5	143	....
4.0	1.000	31.9	30.2	53.0	54.3
8.0	1.415	25.6	24.0	42.6	43.7
16.0	2.000	21.8	20.4	36.3	37.2
24.0	2.450	20.3	19.0	33.8	34.3
32 0	2.828	19.6	18.3	32.6	33.5
48.0	3.465	18.8	17.4	31.5	32.2

CONCLUSIONS

Equations for normal stress in I-beams loaded in combined bending and torsion are developed in this paper. These equations take into account the interaction between bending and torsion and the effect of the deflections on moments. It is found that the allowable stress in bending is affected by the torque and that the primary bending affects the allowable stress for torsional flange bending. The numerical coefficients are computed for the values of $\frac{l}{2\,a}$ included between 0 and 3 since in most practical cases $\frac{l}{2\,a}$ falls between these limits.[2]

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[7] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1936, pp. 267 and 268, paragraph 52.

The shearing stresses can be obtained from Eqs. 7. The principal normal stress and the maximum shearing stress resulting from the combination of the normal fiber stress and shear will usually occur in a diagonal direction. The analysis of the principal stresses, however, is beyond the scope of this paper.

The main objective of this study is to analyze the stresses under the conditions listed in the "Introduction," rather than to develop practical design formulas. The expressions obtained, however, although somewhat lengthy, are of the type that can be rather conveniently used in design.

ACKNOWLEDGMENT

In preparing this paper the published works of several writers were found helpful, although they have not been cited directly. Among the writers included in this list are: D. H. Young[8] and George Winter,[9, 10] N. M. Newmark,[11] and C. R. Young,[12, 13] Members, ASCE.

APPENDIX. NOTATION

The following letter symbols conform essentially with "American Standard Letter Symbols for Structural Analysis" (ASA—Z10.8—1949) prepared by a sectional committee of the American Standards Association, with ASCE participation, and approved by the Association in 1949. Discussers are requested to adapt their comments to the symbols in the paper.

A_1 = integration constant (see B_1 and C_1);
a = torsional bending constant;
B = torsional stress constant;
B_1 = integration constant (see A_1 and C_1);
b = width of flange;
C = torsional twist constant;
C_1 = integration constant (see A_1 and B_1);
d = distance between flange centroids;
E = elasticity; modulus of elasticity in tension; E_s denotes the modulus of elasticity in shear;
e = eccentricity in relation to the plane of the web:
    e_1 = eccentricity normal to the web;
    e_2 = eccentricity parallel to the web;
= normal fiber stress:
    f' = allowable value of f;
    f_b = stress in bending;

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[8] "Rational Design of Steel Columns," by D. H. Young, *Transactions*, ASCE, Vol. 101, 1936, p. 422.

[9] "Lateral Stability of Unsymmetrical I-Beams and Trusses in Bending," by George Winter, *ibid.*, Vol. 108, 1943, p. 247.

[10] "Strength of Slender Beams," by George Winter, *ibid.*, Vol. 110, 1944, p. 1321.

[11] "Numerical Procedure for Computing Deflections, Moments, and Buckling Loads," by N. M. Newmark, *ibid.*, Vol. 108, 1943, p. 1161.

[12] *Engineering News-Record*, November 27, 1924, p. 882.

[13] *Ibid.*, December 25, 1924, p. 1047.

f_f = stress in torsional bending of one flange with respect to the other;
f_y = yield point stress;
H = total flange shear force;
I = moment of inertia about the principal axis:
  I_x = moment of inertia normal to the web;
  I_y = moment of inertia parallel to the web;
l = length of span;
M = bending moment about a principal axis:
  M_x = moment normal to the web;
  M_y = moment parallel to the web;
m = a coefficient defined by Eqs. 37;
n = a number (factor of safety with respect to yield point);
Q = normal stress function, subscripts c, u, p, and s denoting case 1, case 2, case 3, and case 4, respectively;
R = torsional twist function, subscripts c, u, p, and s denoting case 1 case 2, case 3, and case 4, respectively;
S = section modulus about the principal axis, subscripts x and y denoting axes normal to both and parallel to web, respectively;
T = torque:
  T' = part of the torque resisted by the torsional constant;
  T'' = part of the torque resisted by the lateral bending of flanges;
  T_x = torque applied at an abscissa distance x from the origin of the axes;
W = concentrated load (weight) at midspan;
w = uniform load per unit length of span;
X_1 = a parameter expressing a particular solution depending on the nature of the torque;
x = coordinate with y;
y = lateral deflection of a beam:
  y_o = maximum value of y;
  y_1 = value of y at a distance x_1;
β = angle of twist resulting from the combined bending and torsion computed on the assumption that the effect of deflection is not negligible:
  β_n = value of β computed n times the design load;
  β_o = maximum value of β;
κ = torsional constant; and
ψ = angle of twist resulting from applied torque alone, computed on the assumption that the effect of deflections is negligible; the maximum value of ψ is denoted by ψ_o.

DISCUSSION

Jacob Karol,[14] M. ASCE.—The theory of combined bending and torsion acting on I-beams is presented ably in this paper. In his conclusions, the author states that the main objective of the paper was to analyze stresses rather than to develop practical design formulas. It seems to the writer that an extension of the theory to include the effect of initial lateral bow and initial twist of the beam is all that is needed to make the method applicable to practical design. Moreover, these effects of bow and twist are present even on centrally loaded I-beams, and the designer should have some notion of their importance. This discussion presents the additional theory required to evaluate these effects.

Two additional types of torque curves will be considered: For case 5,

$$T_x = T_o \left[1 - 2\left(\frac{2\,x}{l}\right) + \left(\frac{2\,x}{l}\right)^2\right] \ldots\ldots\ldots\ldots\ldots (38a)$$

and, for case 6,

$$T_x = T_o \left[1 - 3\left(\frac{2\,x}{l}\right) + 3\left(\frac{2\,x}{l}\right)^2 - \left(\frac{2\,x}{l}\right)^3\right] \ldots\ldots\ldots (38b)$$

The particular solutions X_1 are: For case 5,

$$X_1 = \frac{T_o\, l}{2\,E_s\,\kappa}\left\{\frac{1}{3}\left(\frac{2\,x}{l}\right)^3 - \left(\frac{2\,x}{l}\right)^2 + \left(\frac{2\,x}{l}\right)\left[1 + 2\left(\frac{2\,a}{l}\right)^2\right]\right\} \ldots\ldots (39a)$$

and, for case 6,

$$\begin{aligned} X_1 = -\frac{T_o\, l}{8\,E_s\,\kappa}\Bigg\{\left(\frac{2\,x}{l}\right)^4 - 4\left(\frac{2\,x}{l}\right)^3 + 6\left(\frac{2\,x}{l}\right)^2 \left[1 + 2\left(\frac{2\,a}{l}\right)^2\right] \\ - 4\left(\frac{2\,x}{l}\right)\left[1 + 6\left(\frac{2\,a}{l}\right)^2\right]\Bigg\} \ldots (39b) \end{aligned}$$

After determining the integration constants from the boundary conditions, the values of the angle of twist become: For case 5,

$$\begin{aligned} \psi = T_o\, C\Bigg(2\left(\frac{2\,a}{l}\right)\left(\cosh\frac{x}{a} - 1\right) - 2\left(\frac{2\,a}{l}\right)^2\left(\frac{l}{2\,a}\sinh\frac{l}{2\,a} + 1\right)\frac{\sinh\dfrac{x}{a}}{\cosh\dfrac{l}{2\,a}} \\ + \frac{l}{2\,a}\left\{\frac{1}{3}\left(\frac{2\,x}{l}\right)^3 - \left(\frac{2\,x}{l}\right)^2 + \left(\frac{2\,x}{l}\right)\left[1 + 2\left(\frac{2\,a}{l}\right)^2\right]\right\}\Bigg) \ldots (40a) \end{aligned}$$

and, for case 6,

$$\begin{aligned} \psi = T_o\, C\Bigg(3\left(\frac{2\,a}{l}\right)\left[1 + 2\left(\frac{2\,a}{l}\right)^2\right]\left[\cosh\frac{x}{a} - 1 - \tanh\frac{l}{2\,a}\sinh\frac{x}{a}\right] \\ - \frac{1}{4}\left(\frac{l}{2\,a}\right)\Bigg\{\left(\frac{2\,x}{l}\right)^4 - 4\left(\frac{2\,x}{l}\right)^3 + 6\left(\frac{2\,x}{l}\right)^2\left[1 + 2\left(\frac{2\,a}{l}\right)^2\right] \\ - 4\left(\frac{2\,x}{l}\right)\left[1 + 6\left(\frac{2\,a}{l}\right)^2\right]\Bigg\}\Bigg) \ldots (40b) \end{aligned}$$

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[14] Kansas City, Mo.

in which $C = \frac{a}{E_s \kappa}$ is the torsional twist constant. The maximum angle of twist at midspan is given by Eq. 11 and the values of R are as follows: For case 5,

$$R = R_{ce} = \frac{l}{2a}\left[\frac{1}{3} + 2\left(\frac{2a}{l}\right)^2 \operatorname{sech}\frac{l}{2a} - 2\left(\frac{2a}{l}\right)^3 \tanh\frac{l}{2a}\right] \quad \text{(41a)}$$

and, for case 6,

$$R = R_{ue} = \frac{l}{2a}\left\{3\left(\frac{2a}{l}\right)^2\left[1 + 2\left(\frac{2a}{l}\right)^2\right]\left(\operatorname{sech}\frac{l}{2a} - 1\right) + \frac{1}{4} + 3\left(\frac{2a}{l}\right)^2\right\} \quad \text{(41b)}$$

The flange stress f_f is found from Eqs. 5 and 40. For case 5,

$$f_f = B\,T_o\left\{2\left(\frac{2a}{l}\right)\left[1 - \frac{2x}{l} - \cosh\frac{x}{a} + \tanh\frac{l}{2a}\sinh\frac{x}{a}\right] + 2\left(\frac{2a}{l}\right)^2 \operatorname{sech}\frac{l}{2a}\sinh\frac{x}{a}\right\} \quad \text{(42a)}$$

and, for case 6,

$$f_f = 3\,B\,T_o\left(\frac{2a}{l}\right)\left\{\left[1 + 2\left(\frac{2a}{l}\right)^2\right]\left[1 - \cosh\frac{x}{a} - \tanh\frac{l}{2a}\sinh\frac{x}{a}\right] - 2\left(\frac{2x}{l}\right) + \left(\frac{2x}{l}\right)^2\right\} \quad \text{(42b)}$$

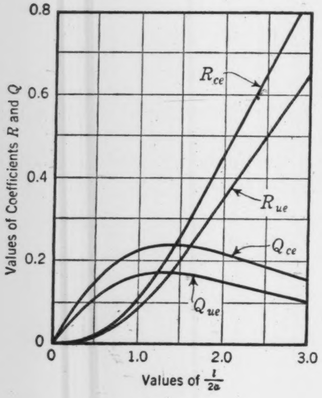

FIG. 7.

in which B is the torsional stress constant, $\frac{a\,b}{I_y\,d}$. The maximum flange stress, at midspan, is given by Eq. 14 and the values of Q are: For case 5,

$$Q = Q_{ce} = 2\left(\frac{2a}{l}\right)\left[\frac{2a}{l}\tanh\frac{l}{2a} - \operatorname{sech}\frac{l}{2a}\right] \quad \text{(43a)}$$

and, for case 6,

$$Q = Q_{ue} = 3\left(\frac{2a}{l}\right)\left\{2\left(\frac{2a}{l}\right)^2 - \left[1 + 2\left(\frac{2a}{l}\right)^2\right]\operatorname{sech}\frac{l}{2a}\right\} \quad \text{(43b)}$$

The variation of R and Q with $\frac{l}{2a}$ is shown in Fig. 7.

The initial lateral bow is assumed to be parabolic, so that

$$e = e_o\left[2\left(\frac{2x}{l}\right) - \left(\frac{2x}{l}\right)^2\right] \quad \text{(44)}$$

in which e_o is the bow at the center of the span. For simplicity, the initial twist in the beam is also assumed to be parabolic, hence

$$\phi = \phi_o \left[2 \left(\frac{2\,x}{l} \right) - \left(\frac{2\,x}{l} \right)^2 \right] \quad \ldots\ldots (45)$$

in which ϕ_o is the twist at the center of the beam. For the concentrated load, the added torque due to bow and twist is

$$\Delta T_x = \frac{W}{2} \left[1 - 2 \left(\frac{2\,x}{l} \right) + \left(\frac{2\,x}{l} \right)^2 \right] \left[e_o + e_2\,\phi_o \right] \quad \ldots\ldots (46)$$

and corresponds to case 5. For the uniform load, the added torque due to bow and twist is

$$\Delta T_x = \frac{w\,l}{3} \left[1 - 3 \left(\frac{2\,x}{l} \right) + 3 \left(\frac{2\,x}{l} \right)^2 - \left(\frac{2\,x}{l} \right)^3 \right] \left[e_o + e_2\,\phi_o \right] \quad \ldots (47)$$

corresponding to case 6. Hence, for the general case, including the effect of eccentricity, bow and twist, the maximum angle of twist at midspan is given by the expressions: For concentrated load—

$$\beta_o = \frac{C\,R_c \dfrac{W\,e_1}{2} + C\,R_{ce} \dfrac{W}{2} (e_2\,\phi_o + e_o\,) + \phi_o}{1 - C\,R_c \dfrac{W}{2} \left(e_2 + 0.0188 \dfrac{W\,l^3}{E\,I_y} \right)} \quad \ldots\ldots (48a)$$

and, for uniform load—

$$\beta_o = \frac{C\,R_u \dfrac{w\,l}{2} + C\,R_{ue} \dfrac{w\,l}{3} (e_2\,\phi_o + e_o) + \phi_o}{1 - C\,R_p \dfrac{w\,l}{3} \left(e_2 + 0.01075 \dfrac{w\,l^4}{E\,I_y} \right)} \quad \ldots\ldots (48b)$$

The added term ϕ_o in the numerators of Eqs. 48 accounts for the fact that the beam has an initial twist at the center.

General equations for the flange stress for the torque considered alone are: For concentrated load—

$$f_f = B\,Q_c \frac{W\,e_1}{2} + B\,Q_{ce} \frac{W}{2} (e_2\,\phi_o + e_o) \quad \ldots\ldots (49a)$$

and, for a uniform load—

$$f_f = B\,Q_u \frac{w\,l\,e_1}{2} + B\,Q_{ue} \frac{w\,l}{3} (e_2\,\phi_o + e_o) \quad \ldots\ldots (49b)$$

For the general case, and loads equal to n times the design load, the maximum angle of twist is: For concentrated load—

$$\beta_{on} = \frac{C\,R_c\,n \dfrac{W\,e_1}{2} + C\,R_{ce}\,n \dfrac{W}{2} (e_2\,\phi_o + e_o) + \phi_o}{1 - C\,R_c \dfrac{n\,W}{2} (e_2 + 0.0188) \dfrac{n\,W\,l^3}{E\,I_y}} \quad \ldots\ldots (50a)$$

and, for uniform load—

$$\beta_{on} = \frac{C\,R_u \frac{n\,w\,l\,e_1}{2} C\,R_{ue} \frac{n\,w\,l}{3} (e_2\,\phi_o + e_o) + \phi_o}{1 - C\,R_p \frac{n\,w\,l}{3} \left(e_2 + 0.01075 \frac{n\,w\,l^4}{E\,I_y} \right)} \qquad \text{(50b)}$$

If the initial bow and twist are zero, Eqs. 50 reduce to Eqs. 31. The stresses for design load are given by Eq. 29 and those for ultimate load are given by Eq. 30. Allowable loads for a given condition are determined by trial using either Eq. 30 or Eq. 32.

The foregoing general theory will now be applied to the author's examples. From the rolling tolerances indicated in the "Pocket Companion" (published by the Carnegie Steel Company), it may be assumed that $e_o = \frac{l}{1{,}000}$ and $\phi_o = 0.005$ The allowable loads on the 14WF74 beam for various combinations of eccentricity, bow, and twist are shown in Table 3. The detailed calculations are similar to those indicated by the author in his examples.

TABLE 3.—ALLOWABLE LOADS ON 14WF74 BEAM (SPAN, 20 FT)

Case	Deformation	Concentrated load, in pounds		Uniform loading, pounds per foot	
(1)	(2)	(3)	(4)	(5)	(6)
Eccentricity, e_1 in inches		0	2	0	2
Eccentricity, e_2 in inches		7	7	7	7
1	Neither bow nor twist	37,400	18,700	3,740	1,950
2	Bow, 0.24 in.	35,200	18,350	3,510	1,920
3	Bow, 0.24 in. Twist, 0.005	34,000	18,100	3,350	1,890

It is worth noting that, for no eccentricity of load, the effect of bow and twist is to reduce the allowable loads as determined by the author by approximately 10%. When the eccentricity of load is 2 in. from the plane of the web, the effect of bow and twist is to reduce the allowable loads by approximately 3%. It is evident then that the effect of bow and twist is large enough to be considered in design.

MELVIN W. JACKSON,[15] Assoc. M. ASCE.—This is a contribution to the solution of a knotty problem which is frequently glossed over in structural design practice because of its complexity. A simply supported beam with torsionally fixed ends, however, is rather uncommon. It is almost impossible to design a connection in riveted construction for this condition. It is also unusual in welded design, for if a beam is torsionally fixed, it is usually fixed against bending.

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[15] Asst. Prof. of Civ. Eng., Univ. of Colorado, Boulder, Colo.

The solution studied requires implicitly that the point of application of the load must remain fixed with respect to its original position. Any movement of the point of application of the load due to rotation will alter the solution. Laterally unsupported building spandrels were offered as a common type of problem involving bending and torsion; but where masonry or timber loads are supported by spandrels, there is frequently a shifting or redistribution of load when twisting occurs.

The basis for establishing the allowable normal unit stress in bending needs clarification. The conclusion that

> "* * * the allowable stress in bending is affected by the torque and that the primary bending affects the allowable stress for torsional flange bending."

is true, but it is not complete. It should not be implied that Eqs. 32 and 33*a* can be "conveniently used in design." It is true that, for a beam in bending, the maximum usable stress in flexure is the yield point stress. However, because of lateral instability, a beam may fail in bending below the yield point stress as computed by the flexure formula. If e_1 is zero, Eq. 33*a* reduces to $f'_b = \frac{f_y}{f_n}$, ignoring this effect of lateral buckling. The current design formula for compression in I-beams for building construction, based on the consideration of lateral instability,[16] is:

$$f_c = \frac{12{,}000{,}000}{l\,d/(b\,t)} \qquad (51)$$

in which f_c is the compression on the extreme fibers of I-beams when $\frac{l\,d}{b\,t}$ is greater than 600. Eq. 51 neglects the effect of torsion or accidental eccentricities. Probably both torsion and lateral buckling should be considered in establishing a satisfactory design formula for this problem.

The consideration of principal stress is excluded in the paper. This is consistent with the usual design of I-beams for bending in which the maximum flexural stress is the maximum principal stress. The exclusion of principal stresses does not seem satisfactory, however, if a formula for allowable stress in bending (Eq. 33*a*) is to be considered. In this case the maximum flexural stress is not the maximum principal stress. Failure of a beam under a long-time load (as distinguished from a load applied by the usual testing machine) will probably start when the maximum principal stress reaches the yield point stress of the material if lateral buckling is not critical. The principal stresses in cases of combined bending and torsion may be substantially larger than the normal stresses.

Much study needs to be done in the solution of structural design problems in combined bending and torsion. The solution for I-beams represents one of the simpler cases. Unsymmetrical spandrel beams supporting eccentric loads probably will continue to be designed for a long time by guess, intuition, and prayer.

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[16] "Steel Construction," 5th Ed., American Inst. of Steel Construction New York, N. Y., 1947, p. 286.

Basil Sourochnikoff[17].—The method presented in this paper has been extended by Mr. Karol to include the effect of initial bow and twist. This is an interesting contribution to the theory of torsion and lateral stability of beams.

One can readily agree with Mr. Jackson that much study needs to be done in the solution of problems in combined bending and torsion. Particularly, the determination of combined principal stresses should some day be undertaken. However, the writer finds it hard to agree with the remainder of the views advanced by Mr. Jackson and takes this opportunity to lay particular stress on three important points in which the paper was apparently misunderstood:

1. Contrary to the opinion of Mr. Jackson, the solution does not require that the point of application of the load remain fixed with respect to the original position. Figs. 1 and 6 and the accompanying text clearly show that the sideways movement of the load has been taken into account.

2. If e_1 is zero, Eq. 33*a* is not reduced to $f'_b = \frac{f_y}{n}$ when the load defined by Eqs. 34*a* or 34*b* is reached, since in this case the denominator of the expression for β_{on} becomes infinite. Therefore, the possibility of the beam failing laterally has been taken into account even in the case when no eccentricity is present. An advantage of Eq. 33*a* over the usual formulas for allowable compressive stress in beams is that it permits taking account of the eccentricity.

3. In the conventional steel framing, the girders carrying moderate torsional loads are connected to columns with light top and bottom angles in addition to the web connections. These angles are designed to be flexible in the longitudinal direction so an undesirable bending moment will not be transmitted to the columns. These angles, however, have enough lateral rigidity to prevent the girder from capsizing due to a torsional load. At the same time, being as a rule connected with only two rivets or bolts each to the girder flanges, they do not introduce an appreciable restraint against bending of the flange in its own plane.

The paper is an approximate analysis of the effect of combined bending and torsion in this widely used system of construction.

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[17] Eng. Dept., E. I. du Pont de Nemours & Co., Wilmington, Del.

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TRANSACTIONS

Paper No. 2469

SETTLEMENT CORRECTION AT LA GUARDIA FIELD

BY JOHN M. KYLE[1]

WITH DISCUSSION BY MESSRS. D. P. KRYNINE AND JOHN M. KYLE.

SYNOPSIS

A settlement of about 5 ft in the period prior to August, 1946, permitted sea water to cover parts of the runways at La Guardia Field (New York, N. Y.), thus putting the field out of operation on several occasions. This paper describes the steps, with the field in full operation, taken to prevent a recurrence of this condition.

INTRODUCTION

The City of New York, in August, 1946, approached The Port of New York Authority with the request that it review the airfield situation in the metropolitan area with a view to taking over the maintenance and operation of the city's airports. Among the serious problems involved in the review of the entire program was the maintenance of La Guardia Airport as an operating facility during the rehabilitation of the field. To explain this, it is necessary briefly to review the conditions as they existed at that time.

The geology of the site of La Guardia Airport is typical of glacial areas common to the north shore of Long Island.[2] The crystalline bedrock occurs at from 100 ft to 200 ft below sea level, sloping generally from the northwest to the southeast. Overlying the rock are clays, sands, and gravels of the Cretaceous period, followed by stratified glacial drift and till of the Pleistocene moraine, and, finally, by the recent soft organic silt and clay still being deposited.

Prior to 1928, the site was occupied by the North Beach Amusement Park. The upland area was confined to a relatively small area of about 55 acres north of the present hangars[3] 1, 3, and 5 (see Fig. 1) and rose to an elevation about 40

NOTE.—Published in November, 1950, *Proceedings-Separate No. 43.* Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Chf. Engr., The Port of New York Authority, New York, N. Y.

[2] "The Geology of Long Island, N. Y.," by Myron L. Fuller, U. S. Geological Survey, Washington, D. C., 1914.

[3] "New York Municipal Airport," by Brehon B. Somervell, *Civil Engineering,* April, 1940, p. 202, Fig. 1.

ft above sea level. In 1928–1929, the North Beach Airport was constructed. Rock bulkheads were built under the water, more or less parallel with the north shore line, and the area was extended by filling the intervening space with soil from the higher uplands to produce an airfield of about 105 acres. Construction of the present airfield was begun in late 1937, extending the area to about 560 acres by the addition of fill obtained from Rikers Island (in the East River), which was spread over the soft silty clay deposits on the north and east sides of the old airport, bordering Flushing Bay. During this period, the surface elevations in the area of the old airport were left at their original grade of about El. +14, whereas in the newly extended areas, runways and taxiways were generally built at about El. +13.5, grading to about El. +12.0 at the perimeter of the field.[4,5,6] El. 0 is mean low water at the Battery. The

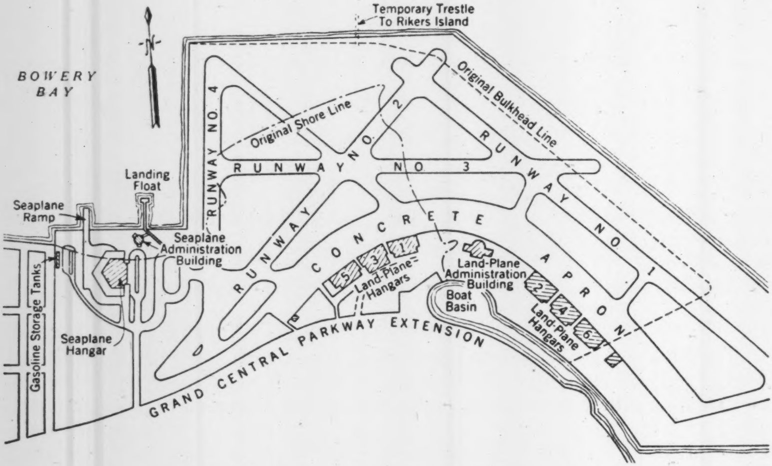
Temporary Trestle
To Rikers Island
N
BOWERY BAY
Original Bulkhead Line
Original Shore Line
RUNWAY NO. 4
RUNWAY NO. 2
RUNWAY NO 3
RUNWAY NO 1
Landing Float
Seaplane Ramp
Seaplane Administration Building
Gasoline Storage Tanks
Seaplane Hangar
CONCRETE APRON
5
3
1
Land-Plane Hangars
Land-Plane Administration Building
Boat Basin
2
4
6
Land-Plane Hangars
GRAND CENTRAL PARKWAY EXTENSION

FIG. 1.—PLAN OF LA GUARDIA FIELD, NEW YORK, N. Y.

Administration Building and the hangars in the southeasterly area (Nos. 2, 4, 6, and 8) were built on pile foundations. The piles were generally driven to bearing in the sand stratum underlying the soft silty clay, although in certain cases there is evidence that the penetration into the sand stratum was not accomplished, or that it was insufficient.

THE PROBLEM

With this background from the foundation standpoint, La Guardia Field may be divided into two parts: First, the part built on the original amusement park area plus the North Beach Airport area, which is founded on relatively

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[4] "Ash Fill on Muck Base Airport," *Engineering News-Record*, June, 1938, p. 779.

[5] "Soil Investigation and Suggested Method of Subsoil Stabilization—Municipal Airport No. 2, La Guardia Field," by Madigan-Hyland Engrs., report to the Dept. of Marine and Aviation, City of New York, N. Y., October 17, 1946.

"New York Municipal Airport," by Brehon B. Somervell, *Civil Engineering*, April, 1940, p. 201.

firm sand. (In these areas there has been practically no settlement of the runways, taxiways, or buildings.) The second part consists of extensions to the original airfield founded on a layer of miscellaneous fill about 20 ft thick on the average, overlying the soft silty clay stratum which varies from 0 ft to 60 ft in thickness. In these areas there was a settlement of about 5 ft in the 10 years since the airport was extended. This settlement has resulted in a series of problems which had to be solved if the airport was to continue in operation.

At the time the airfield was built, under the supervision of the United States Army Engineers, predictions were made that settlement would continue for an indefinite period, and that, to maintain the field as an operating facility, it would be necessary to raise the grade periodically by the placing of additional fill. Because of conditions imposed by World War II and the necessity for avoiding any interruptions in the use of the field, it was found expedient to defer this filling operation. On several occasions prior to August, 1946, northeast winds combined with a high tide had driven water over the field, inundating the runways and putting the entire facility out of operation.

The problem, then, was how to rehabilitate the field without interfering with flying operations. Review of air traffic in the metropolitan area indicated that, because of the traffic density, it would not be possible to suspend operations at La Guardia Airport for the time necessary to build the field up to the original grades by the simple expedient of placing fill and then repaving. In any case, preliminary investigation indicated that placing necessary fill on the field would probably result in mud waves in adjacent areas and might even endanger the safety of the entire field by developing lateral displacements in the underlying strata.

THE SOLUTION

The initial studies conducted before proposals were made to the City of New York encompassed engineering analysis of possible solutions to the settlement problem and design studies of the ultimate cost of the stabilization and rehabilitation of the field. A number of possibilities were studied and rejected. Among them were the pile-supported concrete runways, the flotation of parts of the runways, and the construction of wholly supported runways built of structural steel and subway grating. All these suggestions were rejected because of excessive cost and because they did not appear feasible if field operations were to be maintained.

When the New York Port Authority assumed jurisdiction of the field, tests were immediately inaugurated to determine the practicability, the time required, and the cost of field stabilization. The system which appeared to offer the greatest promise of coping with the problem of the continued settlement of La Guardia Field was the installation of sand drains to stabilize substrata.

Sand drains have been used for a number of years (particularly on the west coast of the United States) with varying degrees of success. In certain types of installations which appear unsuccessful, the fault probably lies in attempting to apply excessive loads before the substrata have consolidated. The resultant slippage of the underlying material may well have sheared the sand drains so that they were no longer effective. Sand drain technique, of course, lends itself particularly well to substrata such as that at La Guardia Field where the horizontal permeability is much higher than the vertical permeability. The installation of sand drains corrects this condition and allows the water to find its way up and out of the strata to be stabilized.

Sand drains consist of a vertical column of sand placed from the ground surface down into the layer to be stabilized. There are several satisfactory methods of installing these "piles" or drains. Briefly, the sequence of operations used at La Guardia Field was as follows:

1. The bottom of a mandrel is closed before driving by a removable steel plate.
2. A mandrel is pushed into the ground by the combined action of its own weight and a pile-driving hammer.
3. When the mandrel is "home" (in this case, driven approximately 80 ft), sand is loaded in the hopper.
4. The hopper is raised and the sand flows into the mandrel.
5. The plate on the bottom of the mandrel is released; air is then applied at the top. This action of the air, together with a lift on the mandrel by the hoist, removes the pipe from the ground, leaving a column of sand.
6. Piezometer tubes are inserted in the test section to record the pore pressures in the underlying strata.
7. Surcharged load (in this case, 12 ft of sand in 2 ft increments) is added in stages to the test area.
8. Under the squeezing action produced by the surcharged load, part of the water contained in the silty layer underlying the sand is gradually squeezed up through the sand drains and discharged into the bay. An indication of this squeezing action is given by a drop in pressure on the gages attached to the piezometer tubes. As this water is gradually squeezed out, the surface of the field subsides and the underlying stratum becomes more stable.

Three sand drain test areas, each 100 ft by 100 ft, were installed to the east of La Guardia Field, with the spacing of drains in each direction, as follows:

Segment	Spacing, in feet
I	14
II	11
III	8

Piezometer tubes were installed and continuous readings were taken of the pore pressure in the substrata as small increments of surcharge load were added to the test section. Eventually, a total surcharge load of 12 ft was imposed on the test section. Thereafter, under the total loading, the rates of settlement were as recorded in Table 1 and as shown in Fig. 2.

The rate of consolidation during the period of driving sand piles and placing the surcharge was greater than the values in Table 1, but no rate of consolidation can be given accurately because of the varying increments of load and the varying time intervals between loading.

Sand piles were driven in all segments between August 20, 1947, and September 23, 1947, and the surcharge was completed by December 3, 1947. The total settlements are as shown in Table 2.

The results of these tests indicated that it was possible to stabilize La Guardia Field by installing sand drains. The problem then facing the Port Authority was how it could best use the results of this test to consolidate and protect the field from high water and still permit operations. The installation of sand drains throughout the areas of all runways, at best, would have been a serious handicap to the operation of the field, and some doubt exists as to whether it would have been possible to continue uninterrupted flying operations if it had been decided to install drains throughout the field.

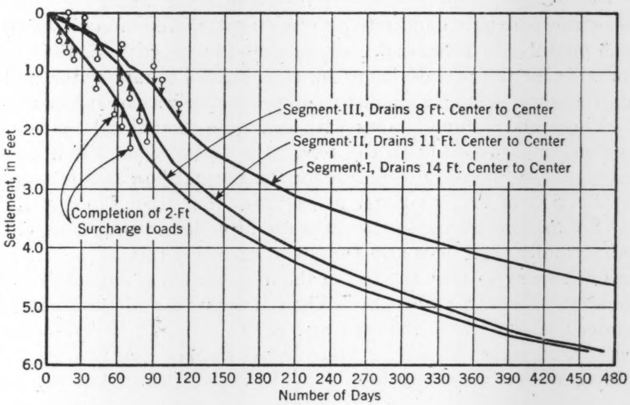

FIG. 2.—SETTLEMENT OBSERVATIONS ON SAND DRAINS FOR SUBSOIL STABLIZATION

TABLE 1.—SETTLEMENT, IN FEET

Elapsed time, in days	Segments:		
	I	II	III
Loading complete	Dec. 16, 1947	Dec. 4, 1947	Dec. 3, 1947
0 to 30	0.45	0.78	0.69
110 to 140	0.21	0.27	0.25
250 to 280	0.14	0.16	0.14

TABLE 2.—TOTAL SETTLEMENT, IN FEET

Segment	No. of Days After:		Total, in feet
	Driving	Loading	
LOADING COMPLETED JULY 17, 1950			
I	1,055	950	6.43
II	1,040	962	7.55
III	1,027	963	7.38

A better solution presented itself: Experience at various airfields in the United States and in other areas in the world has indicated that it is possible to operate a field inside a dike, at an elevation even lower than the adjacent waters, with perfect safety, provided that the glide paths to the runways are protected adequately. The solution was the installation of an earth dike around the perimeter of the low areas of La Guardia Airport where the maximum settlement had occurred; this dike was stabilized in such a manner that the erosion from the adjacent waters did not materially affect its integrity. In order to insure that this dike would not develop mud waves and itself be swept into the bay (particularly where the underlying mud was the deepest and the least stable), sand drains were installed in the area to be occupied by the dike before placing the dike fill. Work on this dike and the sand drains was begun on July 28, 1948, and was completed on August 15, 1949.

To provide for the field drainage, for the removal of rain water, and for the removal of water seeping through the dike, a ditch was built inside the dike proper. Three 36-in. runoff pipes with gravity-actuated tide gates were installed connecting the ditch to the bay. By draining the ditch at nonhigh tide periods, these pipes are sufficient for the normal flow due to seepage and for the collection of surplus water at periods other than those of maximum rainfall. Sumps and pumps in the ditch are provided to take care of drainage during abnormally high tides and periods of excessive rainfall.

It was necessary to raise the top of the dike by depositing additional fill at approximately 6-month intervals. When it has been determined that the strata under the dike itself are stable and will contain the field without danger of adjacent mud waves, it will be possible to raise the grade of the field by a simple filling and paving operation.

Summary

The entire operation to date (1950) has been conducted in such a manner that the field has been operable at all times. The high tides have not infiltrated through the dike.

The shore face of the dike is protected with stone macadam surfacing which resists erosion. Riprap rock fill has been installed along its outer face which protects it from the action of the high tides and the northeast winds. As heavy erosion and ice action are to be anticipated each winter in Flushing Bay, this facing is considered a necessity.

Because of the proximity of good fill on adjacent Port Authority property and by the simple expedient of erecting an earth-fill dike on a sand drain stabilized foundation, it has been possible to protect this vital air terminal and its airport operations have been maintained uninterruptedly.

Acknowledgment

The tests described in this paper were conducted under the direction of J. C. Evans, M. ASCE, chairman of the Engineering Board, Port of New York Authority, with the assistance of the following consultants: The late Frederic R. Harris, M. ASCE, and O. J. Porter and William Mueser, Members, ASCE.

DISCUSSION

D. P. Krynine,[7] M. ASCE.—Since the construction of La Guardia Airport (New York, N. Y.), American engineers have been extremely interested in developments at that field, especially concerning the large settlements in the area. This paper satisfies to a large degree the scientific curiosity of engineering circles, though some details require still further elucidation.

After giving some historical and geological information on the La Guardia Field area, the author presents, among other things, the following two basic items: (1) experimentation on the three large sand drain areas; and (2) construction of the dike surrounding that area of the airport in which the maximum settlement had occurred. The writer's comments refer to these two items only.

The general principle of the sand drain method consists of acceleration of the rate of settlement that may be expected due to the presence of a soft substratum. Since there is no possibility of eliminating such settlement, at least at the present (1951) state of the foundation technique, the sooner this settlement takes place, the better. With this criterion in mind the writer examined the comparative merits of different drain spacings used in the experimental areas (Segments I, II, and III). It appears from the data presented in Fig. 2 and Table 2 that during the whole time of experimentation there was a continuous lag in the behavior of Segment I with the largest drain spacing (14 ft) as compared with Segment II (11 ft drain spacing). Not only was the settlement of Segment I smaller than that of Segment II during the whole time of experimentation, but the final results differed by about 15% (6.43 ft settlement for Segment I and 7.55 ft settlement for Segment II). On the other hand, although in the first year of experimentation there was some difference in the behavior of Segments II and III (11-ft and 8-ft spacings of sand drains, respectively), close to the end of the first year this difference became negligible. The final settlement observed in Segment II was even a little larger than that in Segment III.

The writer's conclusion from the experimental results is that 14-ft spacing of drains (Segment I) under the conditions at La Guardia Field proved to be somewhat excessive, whereas the 8-ft spacing (Segment III) showed a wasteful crowding of drains. It is not clear from the paper what advantage was taken of the full-size experiment described, that is what was the actual drain spacing in the final design and construction of the dike. Another thing that interests the writer is whether some attempt was made to correlate the spacing of the drains and the results shown in Fig. 2 and Table 2 with the permeability (particularly with the ratio of the vertical and horizontal permeabilities) of the soil material or materials at La Guardia Field. Such information, if available, would be of extreme value to engineers who may find themselves facing a settlement problem similar to that described in this paper.

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[7] Cons. Engr., San Francisco, Calif.

The writer hopes that in his final closure Mr. Kyle will give some additional information with reference to the questions offered in this discussion. Some numerical data on the dimensions of the drains (length, diameter) and the cross section of the dike would also be of interest.

JOHN M. KYLE[8].—The discussion by Mr. Krynine has emphasized an important point, concerning the spacing of drains, that should clarified. By examination of Fig. 2 and Table 2 only, one could conclude that the 8-ft spacing was a wasteful crowding of drains.

Because of the limited space available at the airport, it was unfortunately necessary to have the segments of the test section adjacent to each other. Fig. 3 shows the layout of the test section and the settlement curve for each segment up to April, 1951. The drains in Segment I were spaced 14 ft on centers. Drain spacing in Segments II and III was 11 ft and 8 ft on centers, respectively. From the layout of the test section, it can be seen that Segment II, being surrounded on each side by one of the other segments, has a greater load acting on it than either Segment I or Segment III. By correcting for this additional load, a new settlement curve can be plotted for Segment II, and this curve is shown as a dashed line in Fig. 3. This corrected curve now takes a more proper place in relation to the other curves and would tend to prove that, under the same loading conditions, a quicker rate of settlement can be achieved by a smaller spacing of drains.

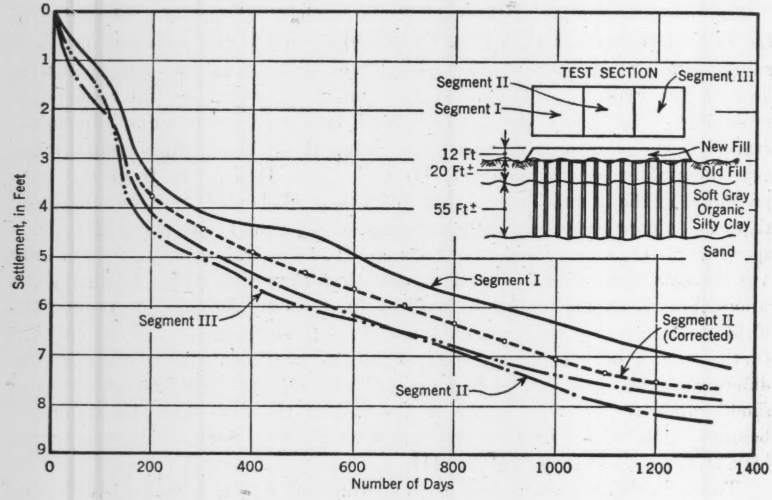

FIG. 3.—SETTLEMENT OBSERVATIONS ON SAND DRAINS FOR SUBSOIL STABILIZATION

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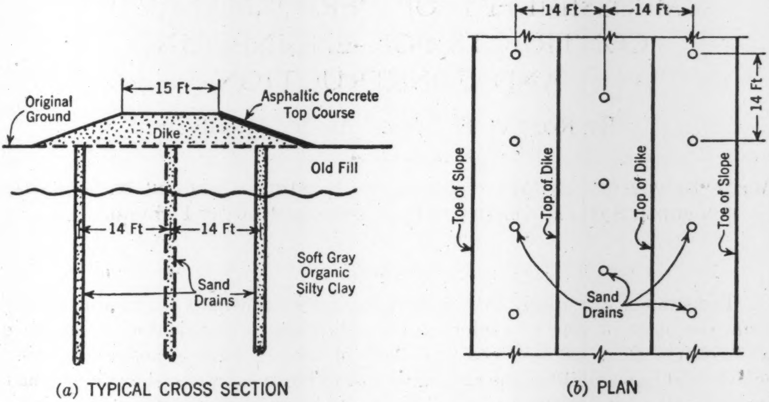
[8] Chf. Engr., The Port of New York Authority, New York, N. Y.

The ratio of horizontal permeability (K_h) to vertical permeability (K_v) was constant throughout the test section area, although it differed slightly throughout the airport. Because of this, no correlation between rates of settlement of different spacings of drains and different K_h/K_v ratios could be attempted. However, based on test section results and the theory of radial drainage, it is believed that any increase in K_h/K_v ratios would result in a relative increase in the rate of settlement, and that this increase would be constant for the different spacings of sand drains.


15 Ft
Original Ground
Asphaltic Concrete Top Course
Dike
Old Fill
14 Ft
14 Ft
Sand Drains
Soft Gray Organic Silty Clay
14 Ft
14 Ft
14 Ft
Toe of Slope
Top of Dike
Top of Dike
Toe of Slope
Sand Drains

(*a*) TYPICAL CROSS SECTION (*b*) PLAN

FIG. 4.—DIKE AT LA GUARDIA FIELD

Fig. 4 shows a typical cross section of the dike. Sand drains were installed in three rows at 14-ft spacing in the area to be occupied by the dike. A larger number of rows or a smaller interval of spacing of sand drains would have become unduly expensive and was not deemed necessary because of the nature and purpose of the dike. The sand drains were 18 in. in diameter and were approximately 80 ft long.

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Paper No. 2470

IMPOSSIBILITY OF PERFORMANCE IN CONTRACTS FOR ENGINEERING AND CONSTRUCTION

By Robert F. Borg,[1] Assoc. M. ASCE

With Discussion by Messrs. Samuel I. Sacks, Melvin W. Jackson, George J. Soffer, Abraham M. Aloff, and Robert F. Borg.

Synopsis

The subject of impossibility of performance of contracts is examined anew from the point of view of exploring its applications to the field of engineering and construction. A review is first made of the old Anglo-American doctrine which held impossibility of performance not to be an excuse to the one who had made an absolute promise for engineering or construction work. Fact situations of all description which have thus been decided are then grouped so as to give the reader the basis of the limits of this doctrine.

This is followed by an attempt to define and explain the exceptions to the rule, a subject which has been in a state of much confusion in recent years because of the efforts of the courts to increase the scope of the exceptions. These efforts have been created by the harshness of the old doctrine. This treatment has reflected the point of view of the contracting engineer. An effort has been made to define the exceptions not by iron-clad rules which have tended to stifle the development of this subject, but rather through the grouping of fact situations which present similar problems in engineering and construction.

Analysis of cases, from the earliest to the very latest (1948), reveals the present status and trend of the problem and indicates the directions in which the engineering profession can work with the courts in solving the problems presented by this difficult subject.

A. Impossibility No Excuse for Nonperformance of an Engineering Contract

1. It was more than three centuries ago, in 1647, that the rule became established in the Anglo-American law that the fact that a contract was

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Note.—Published in October, 1950, as *Proceedings-Separate No. 36*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Chf. Estimator, Psaty and Fuhrman, Inc., Contrs. and Engrs., New York, N. Y.

impossible to perform was no excuse for its breach by one who had undertaken it (1).[2] In 1796 the case of Bricknock versus Pritchard (2) made one of the earliest known applications of this rule to a construction contract. Certain contractors were sued upon their agreement to build a bridge in a substantial manner and to keep it in repair for a certain length of time. After erection, the bridge was destroyed by an extraordinary flood. The Court observed that this kind of loss must have been in the contemplation of all parties in this case, and held the builders liable on their covenant.

2. Today, no rule of law is more firmly established (3)(3*a*)(3*b*)(3*c*)(4)(4*a*)(5)(6)(7)(8)(8*a*), and it is uniformly held that where there is a contract to erect a building or structure, the fact that performance becomes impossible of accomplishment will not excuse the one who has made the promise, unless there has been a contrary stipulation by the parties.

Modern studies of the various aspects of the problem have made attempts to classify its different facets, for (9) "the law upon the matter is undoubtedly in process of evolution." It has been declared that (10)(4*b*) the word "impossible" must be given a practical rather than a scientifically exact meaning; thus, at the making of the contract or thereafter, the promised performance could become impracticable owing to some extreme or unreasonable difficulty, expense, injury, or loss involved.

(*a*) One basis for making a distinction between impossibility situations is that of subjective as opposed to objective impossibility (11). The case of objective impossibility presents itself because of the nature of the thing to be done which may exist at the time the contract was entered into, or at some later time. Subjective impossibility, on the other hand, is a result of the incapacity of the particular person who has undertaken the task; it results from the inability of the particular promisor (4*c*).

The cases holding that objective impossibility will not excuse performance in a contract for engineering and construction are legion. For example, a union rule did not excuse an agreement for hauling concrete materials (12). In another case, proof was made that the difficulties encountered in performing a written contract according to its terms (to make a machine automatically produce certain results) were insuperable (13); yet the Court could not relieve the promisor from the absolute bargain that he saw fit to make. In one early case (14) in England a builder contracted to perform certain work within a fixed time, including changes that might be ordered under the contract. He was held liable for not completing it within the time, although the changes ordered, without allowing an extension of time, rendered performance within the time impossible. The case of the contractor agreeing to keep a dam in repair has been mentioned (2). A later decision involved a contractor who agreed to construct a dam connecting two existing bulkheads (15). One of the bulkheads was washed away and the contractor refused to complete the work under the contract. In this case the Court stated:

> "Even if he had been right in contending that the additional work made necessary by the flood was 'extra,' within the meaning of the contract, he had no excuse for failing to perform his contract, unless the fact that he

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[2] For reference to numerals in parentheses, see "Court Citations and References" in the Appendix.

> could not perform it literally furnished the excuse. He could complete the dam to the place where the old west bulkhead had been located, but could not embrace such bulkhead in the construction work, because it was gone. The contract became impossible of literal performance on account of his breach of contract and negligence. Surely he was in no position to * * * be entitled to recover on 'quantum meruit' for the work done."

Likewise, delay is not excused because of weather, or conditions of soil, or the like, where the contractor promised to complete the structure by a certain day (16)(16*a*)(16*b*)(5)(17).

In another case the contractor undertook to execute work according to specifications prepared by the engineers of a corporation. An important part of the work could not be executed in the manner described by the specifications, and after fruitless attempts, in which the contractor incurred much expense, that part of the project had to be executed in a different way. It was held that no warranty could be implied on the part of the corporation, that the plans were such as to make the work in fact reasonably practicable, and that the contractor could not recover on such a warranty for the value of the work that had been wasted (18).

All the foregoing cases illustrate objective impossibility. Some examples of subjective impossibility can also be given.

Although substantial compliance, and not literal compliance, is sufficient to enable a builder to recover on a building contract, substantial compliance, even though literal compliance cannot be achieved, does not absolve a builder from responding in damages for his omissions and derelictions (19)(19*a*). Usually a building contract is not within that class which, as will be seen later, is deemed to be for personal services. Hence, upon the death of the builder, his personal representative is required to complete the contract (20)(21)(22). An inducement to a builder to fulfil a difficult contract by a promise of extra compensation is invalid since the contractor was already bound to do the work, and performance by him is no legal detriment (23)(4*d*).

A subway contractor cannot recover damages caused by overestimation if he was warned that an excessive estimate in the invitation to bid was approximate (24). Again, the contractor may find that it is impossible to continue work because his finances will not permit it. This can occur in the case of a building contract in which there is a damage clause for nonperformance by a certain time, with provisions for the payment of monthly estimates by the owner. Any delay caused by the wrongful withholding of such payments does not excuse performance within the stipulated time, unless such conduct prevents performance. Also, it does not relieve the builder from damages where the delay in completing the house is due to other and independent causes (25).

(*b*) Another basis of differentiation is that the promise may be impossible of performance at the time entered into or, it may become impossible after the start of performance. These are generally termed "existing" and "supervening" impossibility, respectively (4*d*)(4*e*).

An existing impossibility would be present if rock were to be encountered by a builder, making his contract more burdensome. This did not invalidate a contract whereby the promisor undertook to grade certain lots to street level in a 1942 Kentucky case (26). So also, the contractor assumed the risk of the nature of materials through which a sewer was to be driven, where the contract stated that the classification of borings on the plans would not form the basis of claims (27)(7*a*). The contractor undertaking to make an excavation assumes the risks of defects in the soil, unless such defects are hidden beyond his reasonable investigation thereof (28)(3)(7).

In Dermott versus Jones (28), the owner, in an action by a builder for his contract price, was allowed to recoup the expense of repairs to the building caused by its settlement into the soil. The Court, aided by the contract wording, placed the decision on the grounds that the builder undertook an absolute obligation and so was not relieved by defects in the soil. The Court failed to answer the contention of the builder that he was bound to follow the plans furnished by the owner (29).

A supervening impossibility occurred in a case where a dam was damaged during a contract to furnish electricity. Performance was not excused (30). In one early English case (31) an insurance company elected to repair insured buildings that had been damaged by fire. It was required to rebuild the buildings in entirety, however, when, subsequent to the election, the buildings had been regarded by public authorities as dangerous and ordered demolished.

B. Exceptions to the Rule

The harshness of applying the impossibility doctrine strictly was recognized at an early date by the courts. By the sixteenth and seventeenth centuries three exceptions came to be applied to the rule: (*a*) Death of one of the parties when the contract was personal (32); (*b*) destruction of the specific (as opposed to the general) subject matter of the contract (33); and (*c*) illegality owing to the operation of subsequent domestic law (34).

The difficulty, as has been stated (35), lies in finding a satisfactory stopping point in the recognition of hardship as a defense—

> "* * * a stopping point which will give due recognition to the privilege of a sane person to contract as he pleases without violating the law and which at the same time, will permit the court to prevent the extreme disappointment of expectations—[The difficulty] is greater * * * because of the freedom from blame of the plaintiff in the impossibility cases."

Various lists of the exceptions that should be permitted range them from four (36) to twelve (35), and the result has been confusion in the minds of the courts, the writers (17)(37)(38)(39), and in the engineering profession. It is perhaps the engineering profession more than any other group which has concern with the applications of the rule, for by the nature of the engineer's work, particularly the civil engineer, the problem is most often presented; yet the literature of the civil engineer is scant on the subject.

Many of the cases presented to the courts, which involve this subject, revolve upon fact situations resulting from engineering and construction difficulties. The judges, although cognizant of this circumstance, must and should rely on past decisions to aid them; but the essence of American law is its flexibility, and it is in the hope of bringing before the engineering profession an awareness of the problem that this paper has been written. The law on this point has been (9), and is, constantly changing. It is hoped that a discussion by engineers will be of help to the courts in the recognition and refinement of new decisions caused by impossibility in engineering and construction contracts.

C. Application of Exceptions

1. General.—The remainder of this paper will be devoted to a discussion of the exceptions to the rule of impossibility, in which excuse will be given. It is the belief of the writer that much of the present confusion on the subject has been the result of attempts to list as rules the situations which permit exception. For the most part, in the past students have had to strain to classify the new exceptions, as they were recognized, into one of the previously established rules. In this paper, however, the approach has been to group all the exceptions into various headings, under which are described cases in which the problem is of a nature that is familiar to the profession of engineering or construction. Therefore, a given situation or ruling may appear in more than one heading. It is by this method that it is hoped the various facets of this problem will be examined.

Treatment in this manner brings forth a rather unexpected result, for it reveals that many situations that could be treated as exceptions to the strict rule of impossibility are decided by courts under more conventional, older rules, probably in their reluctance to abandon the settled and well-defined rules in favor of increasing the number of less recognized exceptions to the impossibility doctrine.

2. Absolute Impossibility.—A building contract should be excused if its performance becomes factually impossible (40). As has been shown (17), enforcement of such a promise would be economically absurd since no amount of labor or materials could attain fulfilment. So also is an agreement factually impossible of performance when it is so vague and indefinite that the full intent of the parties cannot be garnered (41). In one case the written agreement was held void as being too vague and indefinite where it called for the erection of a building on a lot, and no dimensions or plans were specified or cited (42).

Faulty plans and specifications have caused a number of cases of factual impossibility. Where the plans and specifications are prepared by the contractor, generally he cannot escape responsibility for defectiveness of the work (43).

Wide divergence of view exists, however, where (as is general) the owner prepares the plans and specifications. In one case erroneous blueprints were held to be a misrepresentation rendering the city liable to the contractor for the extra expense of moving dirt to make a fill. The contractor's duty to examine the site and investigate the character of the soil, it was held, did not carry an assumption of risk of error in the blueprints. The contractor's failure to discover inaccuracies in the detail plans was excused by the general custom (as found in this case) of computing excavation from the detail data without reference to the general plan (44). The contractor who builds according to the owner's plans is also not responsible for the sufficiency of the work, even though the contractor is a specialist (45)(45*a*). In one case (46) in Michigan the collapse of a sewer, resulting from the design type and the specification which the city designated, was held to be a loss which had to be borne by the city. In United States versus Spearin (3), the Court stated:

> "This responsibility of the owner is not overcome by the usual clauses in the contract requiring the builders to visit the site, to check the plans, and to inform themselves of the requirements of the work***."

Some courts rest their decision on the contractor's reliance on the plans (47) and other courts, on the superior knowledge of the owner and his engineer or architect (at least when the defects in the plans are not patent to the ordinary mechanic, or discoverable by ordinary diligence) (48), although in the latter case weight is given to the factor of whether the owner exercises control over the details of the work (49). In cases in which the contractor claims to having had extensive experience, however, when in fact he has had little experience, he is held liable (48).

By far the greatest number of cases is decided in favor of the contractor in the faulty plans situation, on the theory of implied warranty of the owner (50)(51)(5*a*).

As to producing a desired result by following the plans and specifications—where the contract is only an agreement to comply with the plans and specifications—the contractor is not liable if he fails to produce the desired result (52)(52*a*). In the MacKnight case (52) specifications were prepared by the City of New York (N. Y.) containing a detailed plan of the construction of an underground room. The contractor agreed to make the room watertight. When it was built according to specifications and was not watertight it was held that the contractor had completed the contract and could recover the contract price. The Court stated:

> "If I agree to produce a certain result according to my own plan, I impliedly warrant its sufficiency, but if I agree to produce that result by strictly following the plan prepared by another party, he impliedly warrants the sufficiency."

Modern cases have also charged the owner with superior knowledge with regard to the adaptation of the plans to the site when the defects are not apparent on the face of the contract documents (53)(53*a*); but the contractor by his agreement can impliedly warrant that he can erect the building according to the specifications and hence is not entitled to excuse nonperformance in such a case on the grounds that the specifications are defective (54). However, if it is reasonable that he should not have known of the natural conditions involved, and, if the difficulty amounts to impossibility, the contractor is relieved of such warranty (55).

Should the contractor be given an appreciable degree of discretion as to the methods to be used for performance, then he assumes the risk of the sufficiency of the plans (52). Also, when the contractor agrees not only to follow the plans and specifications but also to produce a desired result, he is liable if he follows the plans and specifications (56).

In another type of situation there may be actual misrepresentations of existing conditions or facts by the owner. Here the modern view is that, where the contractor reasonably relies on such misrepresentations, he does not assume the risk of such misrepresentations, and is entitled to extra compensation for the expense caused thereby (57)(57*a*)(58)(5*b*). It is indicated that the owner is not liable on borings in the absence of bad faith, concealment of information at hand, or conscious misrepresentation (59).

The concept of "act of God" or "fortuitous event" has been used in many cases to excuse contractors from liability under their agreement (60). It has been pointed out by W. H. Page, however (38), that it is evident that a number of facts which cause impossiblity in the proper sense of the term may be the result of human agency. The courts have ignored the distinction between human agency and the "act of God" wherever this distinction would have produced any legal consequences. The destruction of the specific subject matter acts as a discharge on the theory of impossibility, whether such destruction is due to an "act of God" or not (40); on the other hand, the fact that performance is prevented by a so-called "act of God" is not of itself sufficient to show that excuse on account of impossibility should be given (61)(61*a*).

3. Practical Impossibility.—The most recent attempts have sought to define impossibility not only in the strict sense, but also as (4*b*) "impracticability because of extreme and unreasonable difficulty, expense, injury, or loss involved."

In one case where this viewpoint was followed (62) a certain contractor promised to take from the owner's land all the gravel needed to construct a bridge, and to pay for it on a unit basis. He extracted all the gravel available above the water level and then refused to proceed under the contract because dredging and drying the remaining gravel would involve ten to twelve times the customary expense. The Court held that the contract was at an end, and that the owner might recover only the agreed price for the gravel taken. The Court's language was as follows:

> "A thing is impossible in legal contemplation when it is not practicable; and a thing is impracticable when it can only be done at excessive and unreasonable cost * * * where the difference in cost is so great as here, and has the effect as found, of making performance impracticable, the situation is not different from that of total absence of earth and gravel."

This doctrine has by no means been accorded universal acceptance, but there is a sufficient number of situations where it may apply to warrant consideration.

Extreme expense or difficulty, or a promise so extremely burdensome as to involve risk which no sensible businessman would subject the promisor to, have excused performance, although it was not scientifically or actually impossible (63); but mere unanticipated difficulty that is not a supervening impossibility did not excuse one litigant (12).

In cases in which an accident merely causes a marked increase in the cost of construction, say, doubling (64)(64*a*), or even tripling, the cost (29), the contract is usually considered to be subsisting and damages are awarded for noncompletion. The difference in the degree of expense is usually the factor that distinguishes the cases in which relief is granted from those in which the contract is enforced.

On the other hand, where extra compensation is promised to a builder upon his refusal to complete the contract by reason of unforeseen difficulties, the promise has been held in one case to be supported by valid consideration, and to be enforceable (65).

The parties to the contract may also be mutually mistaken as to the facts attendant upon performance and this may serve as an excuse because of impracticability (53). In one instance, where a cellar was to be dug and foundations were to be underpinned for an existing building, the contractor discovered that the foundations rested on a crust overlying soft mud. Once the crust was penetrated, the building began to settle. Here the Court permitted relief to the contractor on the basis of a mistake of supposed facts (66).

4. Illegality.—It has long been recognized that illegality of the contract, owing to the operation of subsequent domestic law, will excuse performance (34). This is often held to be so on the grounds of public policy (67). A temporary impossibility of such a character that, if permanent, it would discharge the contractor's entire duty has been held to have that effect (68) if rendering the performance after the impossibility ceases would impose on the promisor a burden substantially greater than that which would have been imposed had there been no impossibility, but otherwise such temporary impossibility suspends the promisor's duty only while it exists.

Impossibility arising subsequent to the contract must be kept distinct from the situation where the law, governing the contract at the time it was entered into, interferes with performance. This latter fact usually makes the contract illegal and enforceable by neither party (11). For example, a civil engineer employed by the board of trustees of a town to design a water works could not recover damages if the board abandoned the project, since the board could not enter into the contract (69).

Some courts have deviated from this rule where the infraction was minor and the forfeiture after performance would have been great (70), or where the sole purpose of the statute was to obtain revenue (71)(71*a*). In one case (72) in Maryland the Court stated:

> "While ignorance of the law is not a valid excuse, contractors engaged in work all over the country cannot be supposed to keep familiar with every detail of municipal charters."

5. Death or Illness.—From an early period in the law (32) it was assumed that a contract which was personal in nature was discharged by the death of the party from whom performance was due. In building contracts a certificate of the architect is frequently, in express terms, made a condition precedent to the owner's duty to pay one or more installments. The courts usually disregard this condition when (as will be seen subsequently) the certificate is refused by the architect for some dishonest reason, and it has likewise been ignored if issuance of the certificate becomes impossible by the death or insanity of the architect (73).

Ordinarily a building contract is not brought within that class of contract which is deemed to be personal. Hence, upon the death of the contractor, his personal representative is bound to complete the contract (21)(22); but the services of a consulting engineer would be deemed personal and thus excused by his death (74).

6. Destruction of Specific Subject Matter.—Where there is a contract to repair or to do work on an existing building it is almost the universal rule that the builder cannot be held liable for his failure to perform under the contract by reason of its destruction (75)(75*a*). The same is true where the contractor had agreed to furnish part of the work and materials in a new building (76)(76*a*).

In the Krause case (75) a builder was to construct an addition to an existing building one wall of said existing building to furnish support to the new one. The old wall was weakened by fire and had to be taken down. The fact that the contractor had unnecessarily delayed the completion of the building until its completion was rendered impossible by the destruction of the old building did not render the contractor liable for failure to perform the contract.

In another case, destruction by fire of a building and uncompleted alcohol storage tanks (agreed to be furnished, fabricated, and erected within the building, and to be paid for when completed, tested, and accepted) released both parties from their obligations under the contract and each party was left to bear his own loss. The decision was held, notwithstanding a provision in the agreement that destruction of the property would not release the building owner, since "property" referred to two tanks—fabricated, erected, completed, tested, and accepted (77).

In the situation where the contractor is to do part of the work on a new building as part of several separate contracts, a clause in the contract requiring the separate contractors to work in harmony and coordination has been treated in one case as creating a single contract and thus the risk of destruction was felt to be on the general contractor (78).

When the contract to repair an existing building is discharged by its destruction, in most American courts, recovery is generally given for the work done after deducting progress payments (79)(76). Some courts reason that this is for benefits conferred on the owner (80); others seem to feel that the benefits must be affixed to the realty (81).

7. Destruction of General Subject Matter.—A contract may be broken by destruction of the building under erection; but such a contract is not discharged on any theory of impossibility (3*a*).

A certain building contract, for example, required the contractor to furnish all the labor and materials for a stated price and to protect the work and owner's property from loss in connection with the contract. It provided for installment payments but made no provision for payment of any part of the job. This agreement was held to be an indivisable contract, and the contractor had to bear the expense of damage to the building by flood in the course of its construction (82). When, under the terms of the building contract, the work is severable and the contract price is payable in installments upon completion of certain units, the contractor may recover on the amount due on the completed part of the job (83).

8. Deliberate Act of Owner.—If performance of a promise is made impossible by the deliberate or sufficiently culpable act of the owner, the promise is excused from performance (84)(84*a*). This has been held to be so in a large variety of situations, such as one causing delay (85), or the interference with the work by permitting others to occupy the site (86); but the federal government is not answerable for damages caused by delay unless it expressly assumes the duty of making the work available to the contractor in time to enable him to complete performance in accordance with the contract (87).

In the situation where the owner of premises delays a contractor in the performance of his contract, the election of the contractor to continue performance in spite of the delay does not deprive him of a right to sue for damages therefor (88).

In another case (89) the failure of the City of New York to determine the location of subway tracks did not excuse the city's failure to furnish drawings required by a subway construction contract. In another action on a subway contract (59) against the City of New York, however, the contractor could not recover damages for delay caused by a dangerous slide and by radical changes ordered in the work by the city engineer. Excessive delay by the city was not shown by the evidence and it was not shown that changes ordered by the engineer were unnecessary departures from the contract or that the work reasonably and safely could have been completed in accordance with the original contract drawings.

In the following situations delay caused by the owner has entitled the contractor to recovery: Delay because of owner's default (90); failure to approve shop drawings (91); delay in ordering additional work (92); or delay caused by the raising of disputes or objections (93).

ASCE Manual of Engineering Practice No. 8, in the section (7*b*), "Owner's Responsibility," bears out this point of view by stating that the owner should be charged with acquiring the right of way, and that where the owner delays the contractor, the contractor should be reimbursed for extra costs.

9. Act of Third Party.—A building contractor has been excused from complying with conditions of a contract requiring a certificate of the architect in order to be entitled to payment under his contract where the evidence has established that the architect in bad faith refused to give the contractor the certificate for further payment for extras (94). So too, under a contract for furnishing and placing a gravel filter layer for a dam, rejection of the work by the owner's engineer (having power of supervision and inspection under the contract) was held not to be conclusive if the engineer through collusion, conspiracy, or fraud, or obstinate persistence in demand beyond the fair intendment of the contract, arbitrarily rejected the work (95). On the other hand, a general contractor who had a government construction contract was held not chargeable with delays sustained by a subcontractor who had agreed to be bound by the terms of the prime contract (96). A strike will not excuse performance under the contract unless it is expressly provided against (12).

10. Failure of Contemplated Methods of Performance.—Where conditions essential to performance (as contemplated by the parties) either do not exist at the time of the contract or later, or, without the fault of the contractor, where they fail to exist long enough to permit performance (55), some courts have held the contract to be excused while others have not.

A case decided in Illinois illustrates the more liberal view (97)(4*f*). The furnishing of labor and materials by a river-straightening contractor for the construction of flumes and cofferdams, to avoid delay caused by an unusually high water level, was held authorized by a contract clause requiring the city to pay for additional labor and materials necessitated by unexpected contingencies.

It will be interesting to see whether this doctrine receives further attention from the courts.

11. Implied Condition.—Where so-called impossibility of performance is recognized as a defense to a contract it is in reality a condition implied by law which is precedent in fact but subsequent in form. For example, where a contract to do work on another's building is excused by its destruction (81) the condition in that case is a series of acts by the builder or his workmen; and they constitute the expected equivalent of the owner's payment.

On a railway construction project (98) a contract provided that contractors would not be permitted to subcontract any part of the work without the consent of the owner. In that case the general contractor sublet a part of the work under a stipulated rate without this consent. Both parties to the subcontract were aware of the provision in the contract with the railway company, and also of the fact that consent had not been given. The subcontractors performed part of the work under their contract but were prevented from completing it by the railway company. The Court held that the nonperformance was excusable and damages for breach of contract were denied. The Court gave as its reasoning that consent by the owner was an implied part of the contract.

12. Contemplated Happening Not Stipulated (Foreseeability).—When the impossibility was foreseeable at the time the contract was made, such impossibility cannot be pleaded as an excuse for nonperformance (99), as there should have been provision for it in the contract, and the absence of such provision gives rise to the inference that the risk was assumed (100). Then, too, the failure of the methods of performance which the contractor alone contemplated using will not excuse him (101).

Where the parties contract on the basis that the fact which is the subject of the contract is doubtful, and when the consequent risk that each is to encounter is taken into consideration in the stipulation assented to, the contract will be valid notwithstanding any mistake of one of the parties (102).

Under another contract (103) providing that a subcontractor would furnish sufficient trucks and do all the hauling from the contractor's excavation, the subcontractor was held obligated to do the hauling notwithstanding that during periods of wet weather hauling could not be done except with caterpillar trucks which the subcontractor did not have.

In a case decided in 1941 an addendum was made to the contract, whereby a contractor engaged a subcontractor to haul topsoil for the contractor's use in landscaping housing projects provided that the soil was to be removed from a certain tract on a designated farm. If the soil at the location was rejected by the local authority, the subcontractor agreed to haul it from any other location on the farm. When the housing authority rejected the soil from the entire farm, it was held that the prime contractor, in failing to provide for the contingency of rejection in terms sufficiently broad to cover the entire farm, "waived" rejection as a defense to the subcontractor's action on the contract after the contractor had used soil hauled by others (104).

13. Uncontemplated Happening Not Stipulated.—Where an unanticipated circumstance has made performance of a promise vitally different from what should reasonably have been within the contemplation of the parties when they entered into the contract, the risk should not be thrown on the promisor, and he should be excused from performance on the ground of impossibility (105). As has been pointed out (7*c*) in *ASCE Manual of Engineering Practice No. 8*, under the heading, "Compensation for Unforseen Conditions," general clauses charging the contractor with the entire responsibility for unforeseen conditions are unfair and increase costs. In the paper by Albert J. Himes (6*a*) it is also pointed out that a contract to assume a risk, such as an agreement to excavate a channel regardless of the character of the material, is binding; but the contractor is not bound to assume a risk not contemplated in the agreement.

In the Kinser case (55), in the course of the construction of a canal, natural conditions of the soil unexpectedly appeared. By its express terms the contract did not cover this contingency, which rendered the performance of the contract as planned impossible, and made necessary substantial changes in the nature and cost of the contract and substantially affected the work under the contract. The contractor was held not to have been guilty of a breach in refusing to complete the contract. This rule has by no means been fully adopted by the courts, as its complete acceptance would eventually force re-examination of a large part of the strict rule against impossibility furnishing an excuse.

14. Frustration.—The doctrine of frustration gives relief where the parties could not reasonably protect themselves by the terms of the contract against the occurrence of subsequent events. It is used primarily in England. In using this doctrine the courts have sometimes attempted to identify it with the theory of implied condition (see under the heading, "11. Implied Condition") of the thing or person (55). Under frustration, performance remains possible but the expected value of performance to the party seeking to be excused has been discharged by a fortuitous event which supervenes to cause actual but not literal failure of consideration (99). The English law applies the principle not only to cases where performance of the contract becomes impossible by cessation of the existence of the thing which is the subject matter of the contract, but also to cases where the event which renders the contract incapable of performance is the cessation or nonexistence of an express condition or state of things going to the root of the contract (106)

In August, 1940, a construction company and a village contracted for the construction of an electric power plant for the village, and an addendum to the specifications provided that the contract would be void if, in the village's litigation with a public utility company, it was determined that the utility's franchise was in effect until September, 1944. In February, 1942, the litigation was determined in favor of the village but, in the meantime, because of World War II, government restrictions prevented construction of the plant until at least October, 1945, when construction costs were much higher. The corporation was excused on the ground of temporary impossibility of performance from completing the contract (68).

A promise will not be discharged, however, because the performance promised in return has lost value on account of supervening fortuitous circumstances, unless they nearly, or quite completely, destroy the purposes both parties to the bargain had in mind. In one English case (107) a firm of contractors agreed, in July, 1914, to construct a reservoir for the water board within six years. Work was begun under the contract in 1914, but in February, 1916, the minister of munitions, acting under statutory powers, prohibited the continuance of the work for an indefinite time, and disposed of and sold the contractor's plant. The contractors claimed that the contract was at an end. The House of Lords held unanimously that, although the prohibition might be removed at some time, it was nevertheless of so permanent a character that any resumption after the prohibition would not be a performance of the old contract, but of a new contract that the parties did not make. It held, therefore, that as the effect of the statutory prohibition was to render performance of the real contract impossible, the parties were discharged. Actually this case rests upon a combination of the doctrines of illegality and frustration, but neither one alone would have been sufficient to decide it.

In building contracts where the principle has been applied, one of the most material factors considered by the courts has been that the price of the contract is fixed by the builder's estimate of the quantities of material and labor involved, and the prices of that material plus the price of wages during the period of time fixed by performance.

Courts also appear to be rather reluctant to hold that the usual extension-of-time clause in building contracts would prevent the operation of the principle of frustration. As was stated in the foregoing case (107),

> "Condition 32 provides for cases in which the contractor has in the opinion of the engineer, been unduly delayed or impeded in the completion of the contract by any of the causes therein enumerated or by any other causes, so that an extension of time was reasonable. Condition 32 does not cover the case in which the interruption is of such character and duration that it vitally and fundamentally changes the conditions of the contract, and could not possibly have been in the contemplation of the parties to the contract when it was made."

D. Conclusion

The doctrine of impossibility of performance is constantly undergoing a further series of refinements away from the old strict rule against excuse being given to the party who seeks to use it as a defense.

The earliest exceptions to the rule: (*a*) Those which gave relief in cases of death where the contract was personal; (*b*) destruction of the specific subject matter of the contract; and (*c*) illegality owing to subsequent domestic law; are now firmly entrenched. The limits of their application are fairly definite and well defined.

It is the more recent exceptions to the rule which are in a state of flux. Despite an increasing liberality of the courts to admit the newer concepts of impossibility, the tendency has been to use a varied number of definitions and terms, many of which should disappear as the new exceptions become crystallized.

It appears that the concept of defining impossibility as not only "absolute impossibility" but also "impracticability," owing to extreme or unreasonable difficulty, expense, injury, or loss, has opened a series of channels for more liberality in construing contracts; formerly the losses, because of uncontemplated happenings, were almost universally shifted to the contractor.

When some of the newer defenses of impossibility become more universally accepted, it may perhaps prove possible to embrace within these concepts a great number of fact situations now admitted as a defense under older theories of contracts. Logically, many of these cases appear to deserve classification (because of their fact situations) under "impossibility."

It is hoped that the engineering profession will take a contributing rôle with the courts of law in defining the further development of this subject as applied to contracts for engineering and construction. The facts that contribute to make the performance of these types of contracts impossible are usually of a highly technical nature, making it desirable for the courts to work hand in hand with the engineer in the solution of their problems.

APPENDIX

Court Citations and References

Abbreviations that appear in this Appendix are defined as follows:

A.	Atlantic Reporter.
A.L.R.	American Law Reports.
Am. St. Rep.	American State Reports.
App. Cas.	Appeal Cases, English Law Reports.
B. & S.	Best & Smith, English Queen's Bench Reports.
Cro. Eliz.	Croke, English King's Bench Reports *temp.* Elizabeth.
D. & C.	District and County Reports, Pennsylvania.
E. & E.	Ellis & Ellis, English Queen's Bench Reports.
ex. rel.	ex. relatione.
K. B.	King's Bench, English Law Reports.
L. Ed.	Lawyers' Edition, United States Supreme Court Reports.
L.R.	Law Reports.
L.R.A.	Lawyers' Reports, Annotated.
P.	Pacific Reporter.
Q.B.	Queen's Bench.
Restatement Contracts	Restatement of the law of contracts, as adopted and promulgated by the American Law Institute at Washington, D.C., May 6, 1932.
S.C.	Supreme Court.
T.R.	Term Reports (Durnford and East), English.
W. Jones	Sir William Jones, English King's Bench Reports.

(1) "Where the law creates a duty or charge, and the party is disabled to perform it without any default in him, and hath no remedy over, there the law will excuse him, * * * but when the party by his contract creates

a duty or charge upon himself, he is bound to make it good, if he may, * * * because he might have provided against it by his contract." Paradine *vs.* Jane, K. B. (1647) Aleyn 26, 82 Eng. Reprints 897.

(2) (1796) 6 T. R. 750.

(3) It is uniformly held that where there is a contract to erect a structure, not containing an express stipulation of the parties, the fact that performance becomes impossible will not excuse the one who has made the promise. United States *vs.* Spearin, 248 U. S. 132, 39 S. C. 59, 61, 63 L. Ed. 166 (1918). (*a*) Day *vs.* United States, 245 U. S. 159, 38 S. C. 57 (1917). (*b*) J. F. Fitzgerald Constr. Co. *vs.* Southbridge Water Supply Co., 304 Mass. 130, 23 N.E. (2d) 165 (1939). (*c*) O'Neill Constr. Co. *vs.* City of Philadelphia, 335 Pa. 359, 6 A. (2d) 525 (1939).

(4) Restatement of Contracts, 1932, Section 456. (*a*) Section 457. (*b*) Section 454. (*c*) Section 455. (*d*) Section 467. (*e*) Section 456, Illustration 2. (*f*) Sections 458–461.

(5) "The Risk of the Unexpected in Sub-Surface Construction Contracts," by Oren Clive Herwitz, *Transactions*, ASCE, Vol. 105, 1940, p. 454. (Mr. Herwitz makes a thorough study of the holdings in litigation resulting from disputes arising from subsoil conditions.) (*a*) p. 455. (*b*) p. 457.

(6) "The Position of the Constructing Engineer, and His Duties in Relation to Inspection and the Enforcement of Contracts," by Albert J. Himes, *ibid*, Vol. LVI, June, 1906, pp. 104–106. (Mr. Himes points out that a contract to assume a risk, such as an agreement to excavate a channel regardless of the character of the material is binding.) (*a*) p. 106.

(7) "Engineering and Contracting Procedure for Foundations," *ASCE Manual of Engineering Practice No. 8*, New York, N. Y., 1934, p. 14. (Under the heading, "Contractor's Responsibility," it is stated that variations in soil encountered during work should not relieve the contractor of his responsibility or entitle him to extra compensation.) (*a*) On page 12 it is stated: "Soil conditions between borings may be assumed by the contractor at his own risk." (*b*) p. 3. (*c*) p. 15.

(8) "The Law of Contracts," by S. Williston, Baker, Voorhis & Co., New York, N. Y., 1922–1930. Section 1931. (*a*) Sections 1935 and 1969.

(9) McCardie, J. [Justice] in Blackburn Bobbin Co. *vs.* Allen (1918) 1 K. B. 540.

(10) Vernon Lumber Corp. *vs.* Harcen Constr. Co., 60 Fed. Supp. 555 (D. C. N. Y., 1945).

(11) Rice *vs.* Smith Eng. Co., 307 U. S. 637. Two cases, 59 S. C. 1034, 83 L. Ed. 1519. Rehearing denied 308 U. S. 632. Two cases, 60 S. C. 68, 84 L. Ed. 527 (C.C.A. Mont., 1939).

(12) O'Dell *vs.* Criss & Shaver, 123 W. Va. 290, 14 S.E. (2d) 767 (1941).

(13) Beacon Tool, etc., Co. *vs.* National Prod. Mfg. Co., 252 Mass. 88, 147 N.E. 572 (1925).

(14) Jones *vs.* St. Johns College (1876) L. R. 6 Q. B. 115.

(15) American Sur. Co. *vs* Gonzales Water Power Co., 211 S.W. 251 (Tex. Civ. App., 1919).

(16) Ward *vs.* Hudson River Bldg. Co., 125 N. Y. 230, 26 N.E. 256 (1891). (*a*) Note in 53 A. L. R. 103, 105, 5 L. R. A. (N. S.) 1105. (*b*) Note in 59 Am. St. Rep. 277, 285.

(17) "Inevitable Accidents in Building Contracts, Allocation of Losses," *Harvard Law Review*, Vol. 54, p. 106. (It is pointed out that many accidents, although inevitable, might well have been expected (weather conditions that do not rise to the dignity of "acts of God") or else do not cause material damage sufficient to justify re-examination of the orthodox rules for measuring contract damages; such mishaps are within the risk assumed by the builder.)

(18) Thorn *vs.* Mayor of London (1876) 1 App. Cas. 120.

(19) Karlinski *vs.* P. R. & H. Lumber & Constr. Co., 68 N. D. 522, 281 N.W. 898 (1938). (*a*) Busboon *vs.* Smith, 191 P. (2d) 198 (Okla., 1948).

(20) "He no doubt expected to perform his contract through the labor and skill of others to a large extent, and he had the right to so perform the whole of his contract if he so desired. We have not had our attention called to any authorities holding that the obligation of an independent contractor under an agreement to build a building does not survive him, if the contract is not performed at the time of his death, while there is eminent authority to the contrary." MacDonald *vs.* O'Shea, 58 Wash. 169, 108 P. 436 (1910).

(21) In re Burke's Estate, 198 Calif. 163, 244 P. 340, 44 A. L. R. 1341 (1926) as cited in "Building Contracts, Effect of Death on Necessity of Performance," *Washington Law Review*, Vol. 2, 1926, p. 131.

(22) Quick *vs.* Ludborrow (1688) 81 Eng. Reprints 25. (Lord Coke, in 1615, stated: "If a man be bound to build a house for another before such a time, and he which is bound dies before the time, his executors are bound to perform them.")

(23) McGovern *vs.* City of New York, 234 N. Y. 377, 138 N.E. 26, 25 A. L. R. 1442 (1923).

(24) Litchfield Constr. Co. *vs.* City of New York, 216 App. Div. 517, 215 N. Y. 450. Modified 244 N. Y. 251, 155 N.E. 116. Motion denied 244 N. Y. 563, 155 N.E. 898 (1926). (The following excerpt is from Form of Contract, City of New York—Board of Transportation, New York City Transit System, dated May 17, 1948:

ARTICLE XIII. Quantities are Approximate. The contract drawings and the specifications show the amount and nature of the work to be done so far as the same are now apprehended, but neither the City nor the Board shall be held responsible that the said contract drawings and specifications are more than approximately correct, and the Contractor shall make no claim for damages or for loss of profit because of a difference between the quantities of the various classes of work actually required to be done and the indicated quantities on the said contract drawings or in the specifications.")

(25) Chamberlin *vs.* Booth & McLeroy, 135 Ga. 719, 70 S.E. 569, 35 L. R. A. (N.S.) 1223 Annotated (1911).

(26) Miller *vs.* Jones, 291 Ky. 126, 163 S.W. (2d) 9 (1942).

(27) United Constr. Co. *vs.* City of St. Louis, 334 Mo. 1006, 69 S.W. (2d) 639 (1934).

(28) Dermott *vs.* Jones, 2 Wall 1, 17 L. Ed. 762 (U. S. 1864).

(29) Stees *vs.* Leonard, 20 Minn. 494 (1874). (Dermott *vs.* Jones (see reference 28) is cited as authority that the furnishing of plans by the owner is immaterial where the loss is caused by defects in the soil.)

(30) Oakland Electric Co. *vs.* Union Gas, etc., Co., 107 Me. 279, 78 A. 288 (1910).

(31) Brown *vs.* Royal Insurance Co. (1859) 1 E. & E. 853.

(32) Hyde *vs.* Dean of Windsor (1597) Cro. Eliz. 552.

(33) Williams *vs.* Lloyd (1629) W. Jones 179.

(34) Abbot and Westminster *vs.* Clerke (1536–1537) Dyer 27 b. pl. 178, 28 b. pl. 186–188.

(35) "The Performance of Contracts, a Summary of Conditions in Contracts and Impossibility of Performance," by G. P. Costigan, *California Law Review*, Berkeley, Calif., 2d Ed., 1927.

(36) "Impossibility of Performance; A Treatise on the Law of Supervening Impossibility of Performance of Contract, Failure of Consideration and Frustration," by R. G. McElroy, The University Press, Cambridge, England, 1941.

(37) "Supervening Impossibility of Performance as a Defense," *Fordham Law Review*, Vol. 5, 1936, p. 322.

(38) "The Development of the Doctrine of Impossibility of Performance," by W. H. Page, *Michigan Law Review*, Vol. 18, 1920, p. 589.

(39) "Impossibility of Performance as an Excuse for Breach of Contract," by F. C. Woodward, *Columbia Law Review*, Vol. 1, 1901, p. 529.

(40) Taylor *vs.* Caldwell (1863) 3 B. & S. 826.

(41) People ex. rel. Heartburg *vs.* Interstate Eng. & Constr. Co., 58 Idaho 457, 75 P. (2d) 997 (1938).

(42) Cannaday *vs.* Martin, 89 S.W. (2d) 1009 (Tex. Civ. App., 1936).

(43) Barraque *vs.* Neff, 202 La. 360, 11 So. (2d) 697 (1943).

(44) Pitt Constr. Co. *vs.* City of Alliance, 12 Fed. (2d) 28 (C. C. A. Ohio, 1926).

(45) Canuso *vs.* City of Philadelphia, 326 Pa. 302, 192 A. 133 (1937). (*a*) Sandy Hites Co. *vs.* State Highway Commission, 347 Mo. 954, 149 S.W. (2d) 828. Transferred 128 S.W. (2d) 646 (1941).

(46) L. W. Kinnear, Inc., *vs.* City of Lincoln Park, 260 Mich. 250, 244 N.W. 463 (1932).

(47) City of Dallas *vs.* Shortall, 87 S.W. (2d) 844, error granted (Tex. Civ. App., 1935).

(48) Northern Pacific Ry. *vs.* Goss, 203 Fed. 904 (C. C. A. 8th, 1913).

(49) Montrose Constr. Co. *vs.* Westchester, 80 Fed. (2d) 841 (C. C. A. (2d) 1936). Certiorari denied, 298 U. S. 691, 56 S. C. 746 (1936).

(50) Culbertson *vs.* Ashland Cement and Constr. Co., 144 Ky. 614, 139 S.W. 792 (1911).

(51) "Liability of Building Contractor for Defects in Plans and Specifications Furnished by the Owner," by J. C. Brock, *Kentucky Law Journal*, Vol. 35, 1947, p. 232.

(52) MacKnight Flintic Stone Co. *vs.* City of New York, 160 N. Y. 72, 54 N.E. 661 at 663 (1899). (*a*) Filbert *vs.* Philadelphia, 181 Pa. 530, 37 A. 545 at 546 (1897).

(53) Hammaker *vs.* Schleigh, 157 Md. 652, 147 A. 790 (1929). (*a*) Faber *vs.* City of New York, 222 N. Y. 255, 118 N.E. 609 (1918). (In this case the parties made the contract with a common mistaken assumption as to the condition of a sprinkler system in the old part of the building.)

(54) McDaniel *vs.* City of Beaumont, 92 S.W. (2d) 552 (Tex. Civ. App., 1936).

(55) Building a lock in a canal became impracticable because of the nature of the soil. Kinser Constr. Co. *vs.* State of New York, 125 N. Y. Supp. 46 (C. C. 1910). Affirmed 145 App. Div. 41, 129 N. Y. Supp. 567 (1911); 204 N. Y. 381, 97 N.E. 871 (1912).

(56) Kuhs *vs.* Flower City Tissue Mills Co., 104 Misc. 243, 171 N. Y. 688 (1918). Modified 189 App. Div. 539, 179 N. Y. 450 (1919). Contra, Philadelphia Housing Authority *vs.* Turner Constr. Co., 343 Pa. 512, 23 A. (2d) 426 (1942). (*a*) pp. 514–516.

(57) Hollerbach *vs.* United States, 233 U. S. 165, 34 S. C. 553, 58 L. Ed. 898 (1914). (*a*) Foundation Co. *vs.* State of New York, 233 N. Y. 177, 135 N.E. 236 (1922).

(58) "Assumption of the Risk of Defects in Plans and Specifications," by J. E. Adamson, Jr., *Cornell Law Quarterly*, Vol. 24, 1938, p. 109. (A number of rules as to the reasonableness of the contractor's reliance are given).

(59) Spencer, White and Prentis *vs.* City of New York, 189 Misc. 616, 75 N.Y. (2d) 629. Affirmed 271 App. Div. 824, 66 N. Y. (2d) 619. Appeal denied 271 App. Div. 926, 67 N. Y. (2d) 701 (1948).

(60) Farnsworth *vs.* Sewerage and Water Board of New Orleans, 173 La. 1105, 139 So. 638 (1932).

(61) Meriwether *vs.* Lowndes County, 89 Ala. 362, 7 So. 198 (1890). (*a*) Runyan *vs.* Culver, 168 Ky. 45, 181 S.W. 640 (1916).

(62) Mineral Park Land Co. *vs.* Howard, 172 Calif. 289, 156 P. 458 (1916).

(63) Reading Co. *vs.* Allied Oil Co., 49 D. & C. 523 (Pa. Mun. 1944).

(64) School District No. 1 *vs.* Dauchy, 25 Conn. 530 (1857). (*a*) Tompkins *vs.* Dudley, 25 N. Y. 272 (1862).

(65) King *vs.* Duluth, M. & N. Ry. Co., 61 Minn. 482, 63 N.W. 1105 (1895).

(66) Linz *vs.* Schuck, 106 Md. 220, 67 A. 286, 11 L. R. A. (N.S.) 789 (1907).

(67) Takahashi *vs.* Pepper Tank and Cont. Co., 58 Wyo. 330, 131 P. (2d) 339 (1942).

(68) Village of Minneota *vs.* Fairbanks Morse & Co., 226 Minn. 1, 31 N.W. (2d) 920 (1948).

(69) Potts *vs.* Village of Haverstraw, 79 Fed. (2d) 102 (C. C. A. N. Y., 1935).

(70) Fischer Liemann Constr. Co. *vs.* Haase, 64 Ohio App. 413, 29 N.E. (2d) 46 (1940). (Violation of the Cincinnati (Ohio) building code with respect to minimum thickness of masonry veneer.)

(71) Sunflower Lumber Co. *vs.* Turner Supply Co., 158 Ala. 191, 48 So. 510 (1909). (*a*) Hughes *vs.* Snell, 28 Okla. 828, 115 P. 1105 (1911).

(72) Konig *vs.* Mayor, etc., of Baltimore, 128 Md. 456, 97 A. 837 (1916).

(73) Hebert *vs.* Dewey, 191 Mass. 403, 77 N.E. 822 (1906).

(74) Stubbs *vs.* Holywell Ry. Co. (1867) L. R. 2 Ex. 311.

(75) Krause *vs.* Board of Trustees, 162 Ind. 278, 70 N.E. 264 (1905). (*a*) Angus *vs.* Scully, 176 Mass. 357, 57 N. E. 674 (1906).

(76) Halsey *vs.* Waukesha Springs Sanitarium Co., 125 Wis. 311, 104 N.W. 94 (1905). (*a*) Teakle *vs.* Moore 131 Mich. 427, 91 N.W. 636 (1902).

(77) Rossville Alcohol & Chemical Corp. *vs.* Steel Constr. Co., 104 Ind. App. 515, 8 N.E. (2d) 1116 (1937).

(78) Commonwealth ex. rel. *vs.* Evans, 304 Pa. 445, 156 A. 139 (1931).

(79) Ganong & Chenovich *vs.* Brown, 88 Miss. 53, 40 So. 556 (1907).

(80) Dame *vs.* Wood, 75 N. H. 38, 70 A. 1081 (1908).

(81) Carroll *vs.* Bowersock, 100 Kans. 270, 164 P. 143 (1917).

(82) Clark *vs.* Till, 177 Miss. 891, 172 So. 133 (1937).

(83) Goin *vs.* Board of Education of City of Frankfort, 298 Ky. 645, 183 S. W. (2d) 819 (1945).

(84) Phoenix Bridge Co. *vs.* United States, 85 Ct. Cl. 603 (1937). (*a*) Edgemoor Iron Co. *vs.* United States, 61 Ct. Cl. 392 (1925).

(85) Levering & Garrigues *vs.* United States, 73 Ct. Cl. 566 (1941).

(86) Town and Country Eng. Corp. *vs.* State of New York, 46 N. Y. (2d) 792 (N. Y. Ct. Cl., 1944).

(87) United States *vs.* Foley, 329 U. S. 64, 67 S. C. 154, 91 L. Ed. 44 (1946).

(88) Louisville & Nashville Ry. Co. *vs.* Hollerbach, 105 Ind. 137, 5 N.E. 28 (1886).

(89) Rapid Transit Subway Constr. Co. *vs.* City of New York, 259 N. Y. 472, 182 N.E. 145, modifying 231 App. Div. 72, 246 N. Y. 869, which modified 129 Misc. 714, 223 N. Y. 24 (1932).

(90) Byrne *vs.* Bellingham Consol. School District No. 301, Whatcom County, Wash. (2d) 20, 108 P. (2d) 791 (1941).

(91) General Cont. Co. *vs.* Brennan Stone Co., 119 Conn. 296, 175 A. 921 (1935).

(92) Dodd *vs.* Churton (1897) 1. Q. B. 562.

(93) Russell *vs.* Bandiera (1862) 13 C. B. N. S. 149.

(94) Sikora *vs.* Hogan, 315 Mass. 66, 51 N.E. (2d) 970 (1944).

(95) Tobin Quarries *vs.* Central Nebraska Public Power and Irrig. Dist., 64 Fed. Supp. 200. Affirmed 157 Fed. (2d) 482 (C. C. A. 8th, 1946).

(96) United States ex. rel. Gillioz *vs.* John Kerns Constr. Co., 50 Fed. Supp. 692. Reversed 140 Fed. (2d) 792 (C. C. A. 8th, 1944).

(97) Great Lakes Dredge and Dry Dock Co. *vs.* City of Chicago, 353 Ill. 614, 188 N. E. 196, reversing 267 Ill. App. 46 (1934).

(98) Dolan *vs.* Rodgers, 149 N. Y. 489, 44 N.E. 167 (1896).

(99) Lloyd *vs.* Murphy, 66 Calif. App. (2d) 1020, 153 P. (2d) 47 (1944).

(100) Dubois *vs.* Gentry, 182 Tenn. 103, 184 S.W. (2d) 369 (1945).

(101) Summers *vs.* Hibbard, 153 Ill. 102, 38 N.E. 899 (1894).

(102) Fort Worth Sand & Gravel Co. *vs.* Peters, 103 S.W. (2d) 407, error dismissed (Tex. Civ. App., 1937).

(103) J. D. Harms, Inc., *vs.* Meade, 186 Wash. 287, 57 P. (2d) 1052 (1936).

(104) Bunch *vs.* Potter, 123 W. Va. 528, 17 S.E. (2d) 438 (1941).

(105) Transbay Constr. Co. *vs.* City and County of San Francisco, 35 Fed. Supp. 433. Reversed 134 Fed. (2d) 468. Certiorari denied 320 U. S. 749 (D. C. Calif., 1940).

(106) Nickoll *vs.* Ashton (1901) 2 K. B. 126.

(107) Metropolitan Water Board *vs.* Dick (1918) 2 K. B. 1, affirmed (1918) App. Cas. 119.

By Discussers

(108) "Law for Engineers and Architects," by Laurence P. Simpson and Essel R. Dillavou, West Publishing Co., St. Paul, Minn., 3d Ed., 1946, p. 77.

(109) "Law in the Service State: Freedom *versus* Equality," by Roscoe Pound, *Journal*, American Bar Association, Vol. 36, No. 12, December, 1950, p. 1051. (*a*) p. 980.

DISCUSSION

Samuel I. Sacks,[3] M. ASCE.—An outstanding classic, this paper assembles all the important cases and principles under one cover. The work involved must have been tremendous, and Mr. Borg is to be complimented on the result.

However, under the heading, "C. Application of Exceptions: 2. Absolute Impossibility," the author states:

> "Should the contractor be given an appreciable degree of discretion as to the methods to be used for performance, then he assumes the risk of the sufficiency of the plans (52).[3a] Also, when the contractor agrees not only to follow the plans and specifications but also to produce a desired result, he is liable if he follows the plans and specifications (56)."

A reading of the case of Philadelphia Housing Authority versus Turner Construction Company would appear to sustain the principle recited by Mr. Borg, namely,

> "* * * when the contractor agrees not only to follow the plans and specifications but also to produce a desired result, he is liable if he follows the plans and specifications (56)."

In the opinion of the writer, the Philadelphia Housing Authority case is a harsh one, but being the law, it must be followed. In that case, the Court held (56*a*):

> "Where a construction contract provided that 'by submitting a bid the bidder agrees that he has examined the site and the specification and drawings and, where the specification requires in any part of the work a given result to be produced, that the specification and drawings are adequate and the required result can be produced under the specification and drawings' and also provided, 'no claim for any extra work will be allowed because of alleged impossibilities in the production of the results specified or because of inadequate or improper plans and specifications and wherever a result is required, the successful bidder shall furnish any and all extras and make any changes needed to produce * * * the required result,' it was held that the contractor could not maintain a claim for the extra cost he was put to by the necessity of using another kind of material than that mentioned in the specification in order to produce the required result. (514–16)"

Melvin W. Jackson,[4] Assoc. M. ASCE.—According to the civil code of one state

> "It is essential to the existence of a contract that there should be: (1) Parties capable of contracting, (2) Their consent, (3) A lawful object, and (4) A sufficient consideration." (Civil Code of California)

Impossibility of performance is just one aspect of the third requirement of a valid contract. Regarding impossibility, this code states,

> "The object of a contract must be possible and ascertainable by the time the contract is to be performed. Everything is deemed possible except that

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[3] Attorney-at-Law and Cons. Civ. Engr., Sacks & Piwosky, Philadelphia, Pa.

[4] Asst. Prof. of Civ. Eng., Univ. of Colorado, Boulder, Colo.

> which is impossible in the nature of things. Where a contract with a single object is wholly impossible of performance or so vaguely expressed as to be wholly unascertainable, the entire contract is void."

Wherein, then, lies the difficulty in defining "impossibility"?

The study of impossibility by the author has represented a tremendous amount of work and study. It is a perfect lesson on the impossibility of defining "impossibility" exactly, a term which involves a wide range of legal and social relationships with an almost infinite number of mutations and permutations. For a satisfactory definition of impossibility, it would be necessary to compress the entire law of contracts, which has been developing for hundreds of years, into one sentence. It is at this precise point that the average engineer stumbles between law as he recognizes it and law as the legal profession recognizes it. An engineer studies physics, chemistry, and other branches of science in which a law may be stated briefly, possibly by an algebraic equation, such as $F = m\,a$. This law of science is a relation between cause and effect which, so far as existing experience goes, has been proved valid in all cases.

Law, in the sense of public writings, is an entirely different thing. It is theoretically possible to present a short statement of law such as is found in the civil codes existing in many states. Applying the law is not as simple as the statement in the code would lead one to believe. It is necessary to take cognizance of, and to apply, the hundreds and thousands of qualifying statements located elsewhere in both organic and ordinary law, both written and unwritten. This explains the difficulty of presenting a precise definition of impossibility and the necessity for citing cases involving a specific point. The author should be commended for his work and analysis in bringing together these cases involving a certain principle applied to engineering work.

The paper should not lead any engineer to attempt to solve his own legal problems. There is a legal maxim to the effect that a man who is his own lawyer has a fool for a client. To a lawyer preparing a specific case, this summary of impossibility as related to engineering contracts may be valuable and time saving. Laws are so manifold that a lawyer can no more hope to be thoroughly informed on all branches of the law than an engineer can hope to be an expert in all the various specialized phases of engineering. Situations arise where the engineer may be better prepared to understand legal cases involving his specialty than the lawyer in general practice. This is one reason why it is desirable for the courts to work hand in hand with the engineer where legal problems of engineering are in conflict.

One important class of laws of the land consists of judicial decisions, whether classed officially as written or classed as unwritten if they have no certain repository. Most of the cases cited by the author regarding impossibility cover the past one hundred years or more. There is no reason to expect that a clear cut definition of impossibility will be developed, but that, under the legal system of the United States, the courts will continue for centuries to modify rules of law in their attempt to render justice. A fluid system of laws such as this seems desirable for effecting justice under changing conditions.

George J. Soffer,[5] **Jun. ASCE.**—A stimulating challenge to the engineer is presented by the author who emphasizes the necessity of the engineer's having some understanding of the law. The paper substantiates the theory that the engineer should have a liberal education. Many schools now require a course in law for undergraduate students.

In his article, however, Mr. Borg has included cases that apply to entirely different rules of law. For example, the case of King versus Duluth (65) was decided on the rule of consideration in contracts which states "* * * that a promise to do what one is already legally bound to do is not a consideration for the promise of another." (108)

Courts of law are provided to administer justice and also to settle inequities. Cases brought to trial represent situations in which the contracting parties cannot agree as to the terms of their contract. It seems that the duties of the engineer are (1) To avoid such disagreements by the wording of the original contract and (2) to negotiate with the parties in order to reach a compromise, since neither party recovers 100% after legal fees.

The law, written and unwritten, goes back many centuries. When a rule of law is followed for hundreds of years, it means that this law represents reason, logic, and justice. Excuse from a contract, because of impossibility of performance, is limited to: (*a*) Destruction of the subject matter; (*b*) death of a party in a personal service contract; (*c*) conditions essential to performance that do not exist; and (*d*) a change in the law. Such limitations can be appreciated because otherwise contracts would be broken at will by one party and would cause considerable damage to the other contracting party. However the law of contracts is profuse with rules, and when relief cannot be found under one particular rule, it may be sought under another. In such cases the lawyer's services are needed. The engineer cannot replace the lawyer, nor can the lawyer replace the engineer. Each one plays his part in society.

Abraham M. Aloff,[6] **Assoc. M. ASCE.**—One who undertakes to expound a legal topic to engineers runs the risk of treating his subject in a manner that is too obscure to be of value to engineers and too superficial to be of value to lawyers. The author has evidently attempted to avoid both these risks, and his paper is well worth the scrutiny of both professions. The fact that most of the references supporting his statements are to the original case reports rather than to treatises or law-review articles enhances the authoritativeness of the discussion from the lawyer's point of view. The engineer will appreciate the simplicity and absence of legal technicalities in the text.

The engineer-reader, however, should be cautioned that many of the rules set forth, although supported by actual decisions in one or more jurisdictions, are not necessarily of universal application. Thus a particular rule may have been modified or may not have been followed at all in a similar case in another state. What is even more frequent is that the rule governing a set of facts already adjudicated in states A, B and C may not have been announced in state D because such a case has not arisen there. In such a situation, the lawyer must use his technical knowledge to predict the rule that state D will follow, a prediction that is sometimes extremely difficult to make. Thus the relative simplicity of the author's treatment, helpful as it is as a point of departure, should not mislead the reader into considering the rules as definitive for all states. As the author indicates, the rules may change even in a given jurisdiction over a period of time.

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[5] Draftsman, Burns and Roe, Inc., New York, N.Y.

[6] Examiner, Massachusetts Div. of Civ. Service, Boston, Mass.

Of particular interest is the author's forecast as to the probable trend of practice in the future. In suggesting that the courts will be increasingly liberal in admitting the newer concepts of impossibility, the author is in accord with the weight of opinion. The jurist Roscoe Pound goes even further, expressing the opinion(109) that

> "* * * belief in the obligatory force of contracts and respect for the given word are going, if not actually gone, in the law of today."

But instead of relying on such a broad generalization, contractors will be well advised to be guided by the principles set forth in the author's paper.

ROBERT F. BORG,[7] ASSOC. M. ASCE.—The several discussions reveal that an awareness of the problems dealt with in the paper exists in the profession. The writer is indeed indebted to Mr. Sacks for his perceptive comment on the holding that a contractor is liable when he agrees not only to produce a desired result but also to follow the plans and specifications. He is correct when he states that the decision in the Philadelphia Housing Authority case (56) rather than that in the Kuhs case (56) applies this principle. The Kuhs case applies the more liberal view set forth in the MacKnight case (52) which is to the contrary. The cases as cited in the Appendix (56) should be listed, therefore, in reverse order, with "contra" before the Kuhs case.

That the rule in the Philadelphia Housing Authority case is a harsh one, there can be no doubt. It is submitted that the decision of this case can be avoided if the engineer as the specification writer will be realistic about his requirements. To insist upon literal compliance and a guaranteed result, as Mr. Sacks states, is to invite a harsh application of law. Only incidentally might it be mentioned that a specification with such an onerous clause also invites higher construction costs because the contractor is certain to provide for such a contingency in his estimate.

In his extremely thoughtful discussion, Mr. Jackson raises the interesting question of legislative statutes or codes and their effect on decisions in the cases involving impossibility.

It was originally contemplated by the writer that the paper would include a section devoted to the treatment of impossibility in those jurisdictions governed by civil law. By definition, these jurisdictions would rule in particular cases on the basis of legislative statute rather than (as is the method followed in most states of the United States) on the basis of past decisions or common law. Space limitations prevented inclusion of the subject at the time.

Under the common law, a promise (as was stated at the beginning of the paper) is binding in accordance with its terms, even when it later proves to be impossible of performance, unless the promisor is able to exempt himself by proof that he falls within one of the recognized exceptions. The civil law, however, usually regards impossibility as an excuse unless the promisee is able to prove that the other party assumed the risk of impossibility. Strange to state, although the two systems of law appear to be at opposite poles, the decisions in actual cases do not very greatly. Some of the jurisdictions governed by civil codes include: Louisiana in the United States, France, Italy, Spain, Portugal, The Netherlands, and Chile.

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[7] Chf. Estimator, G. L. Rugo & Sons, Inc., General Contractors, Boston, Mass.

Thus, as Mr. Jackson states, although the civil law may attempt in a code to present a short statement of the law, applying the law after such a statement is at least as difficult as the system of common law. As was observed by Mr. Jackson, the common law is the more fluid of the two systems, and rules of law are thus modified over the years to render more effective justice.

This point is admirably stated by Mr. Aloff in his caution against using the cases decided in one jurisdiction as being the law at every other place. Indeed, on some points, the forty-eight state jurisdictions may be equally divided on their interpretation of the law. Moreover, there may be three, or perhaps four, different recognized rules for a particular point of law, each one of these rules having states adhering to them; but (as previously emphasized in connection with the civil law view of impossibility), although the rules may be diametrically opposed, the decision in a particular set of facts may be the same in the various jurisdictions.

The advice of Mr. Soffer appears to offer the soundest solution to this apparent paradox. The rules of contracts are many and technical. It remains for the lawyer and engineer to work together in providing the relief which is sought. When, as Mr. Soffer states, relief cannot be found under one particular rule, it may be sought in another. The writer is particularly indebted to Mr. Soffer for his reasoned discussion of adaptation of the rules of law toward giving the relief sought. He is quite correct, of course, in his comment on the King case (65).

The writer should have made it plain that it was the dicta in this case which gave rise to the comments about extra compensation. The words of the court were:

> "But where the party refusing to complete his contract does so by reason of some unforeseen and substantial difficulties in the performance of the contract, which were not known or anticipated by the parties when the contract was entered into, and which cast upon him additional burden not contemplated by the parties, and the opposite party promises him extra pay or benefits if he will complete his contract, and he so promises, the promise to pay is supported by a valid consideration. In such a case the natural inference arising from the transaction, if unmodified by any equitable considerations, is rebutted, and the presumption arises that by the voluntary and mutual promises of the parties their respective rights and obligations under the original contract are waived, and those of the new or modified contract substituted for them. Cases of this character form an exception to the general rule that a promise to do that which a party is already legally bound to do is not a sufficient consideration to support a promise by the other party to the contract to give the former an additional compensation or benefit."

The fact that Laurence P. Simpson and Essel R. Dillavou saw fit to include the foregoing case in the section of their book (108) on "Consideration," rather than farther back under their treatment of "Impossibility," was no accident. The paragraph quoted, however, could just as well have come from any number of cases decided on the grounds of impossibility.

The duties of the engineer are succinctly stated by Mr. Soffer as (1) avoiding disputes by his wording of the original contract and (2) negotiating for a compromise. Stated in a few words, this is the key to the relationship of the engineer to the entire subject at hand.

Mr. Aloff's comments on the trend of the law and his quotation of Roscoe Pound (109) are extremely stimulating in their illustration of the dangers of carrying useful progression of the law to extremes. Mr. Pound can be more fully appreciated, however, when the sentence quoted is read together with those which immediately precede it (109):

> "* * * Partly, French lawyers tell us, there is a moral idea here. Contracts might be improvident or changes in the economic situation might affect the value of the promised performance or of the given or promised equivalent. This is akin to an idea we see at work in the law of legal liability everywhere. It is a humanitarian idea of lifting or shifting burdens and losses so as to put them upon those better able to bear them. Belief in the obligatory force of contracts and respect for the given word are going, if not actually gone, in the law of today."

Elsewhere in the same article, however, Mr. Pound presents the solution to this difficult problem of achieving desirable checks and balances in modern society with these words (109*a*):

> "* * * An organized profession does not seek to advance the money-making feature of professional activity. It seeks rather to make as effective as possible its primary character of a public service. An engineer may patent his inventions. A manufacturer may get legal protection for his trade secret. What a member of a profession invents or discovers as to the art of his profession is not his property. It is at the service of the public. A tradition of duty of the physician to the patient, to the medical profession and to the public, a tradition of the duty of the lawyer to the client, to the profession, to the court and to the public, authoritatively declared in codes of professional ethics, taught by precept and example, and made effective by an organized profession, makes for effective service to the public such as could not be had from individual practitioners not bred to the tradition and motivated as in a trade primarily if not solely by quest of pecuniary gain. * * *"

It is hoped that the paper will aid in opening the way for a unity of action between law and engineering on the problems of practice involving the technical difficulties presented.

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TRANSACTIONS

Paper No. 2471

WIND-LOAD STANDARDS IN EUROPE

By John W. T. Van Erp[1]

With Discussion by Messrs. Paul Rogers; Carl H. Walther; Francis L. Castleman, Jr., and Aron L. Mirsky; S. K. Ghaswala; Herbert S. Saffir; Theodore B. Rights; and John W. T. Van Erp.

Synopsis

As a result of a continued drive toward greater economy in structural design, wind-load standards have been more and more refined so that they agree almost exactly with actually occurring conditions. The task of the engineer is to proportion his design to fit the stipulated purpose of the structure with the utmost degree of economy while retaining the required factor of safety. The factor of safety covers a certain number of unknown quantities and it has been the constant endeavor of engineers to acquire a better understanding of these unknowns. Thus, on the one hand, analysis of the structure will have to yield stresses agreeing as closely as possible with those actually occurring whereas, on the other hand, the loads should be the expected ones with the least degree of exaggeration. It is imperative, therefore, that the designer have the best possible knowledge of loading conditions—the least information being available about wind loads. The fundamental dynamic character of wind loads must be translated into static equivalents to give simple data for the use of the designing engineer.

Introduction

1. *History.*—For a number of decades, attempts have been made to formulate the forces exerted on structures by wind. From theoretical considerations, but mostly from tests, an often bewildering variety of data has been obtained. To make such data applicable to purposes of practical design, simplification and condensation into a concise set of rules have been attempted in different countries. These rules, in turn, have been the subject of various publications, and finally have emerged in new standards. It is worthwhile to review their most salient points. In most building codes in the United States,

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Note.—Published in November, 1950, as *Proceedings-Separate No. 42.* Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Structural Engr., George A. Fuller Co., New York, N. Y.

obsolete conceptions of wind load still prevail; and cases of damage from wind recur far too frequently. Although windstorm damage can never be eliminated entirely, a better understanding of all the aspects of the phenomenon of wind load will certainly help to reduce the regular manifestation of its disastrous effects.

One of the first standards, complete with detailed shape coefficients, was fixed in an explicit and practical form in the Netherlands.[2] That this standard should have originated in that fuel-scarce and flat country is understandable, for handling the wind has been a tradition for centuries. In former times, the wind was utilized as a propellent force for ships, or as the motive power for windmills; in modern days the emphasis has been toward a more thorough understanding of its nonbeneficial effect as a loading condition on buildings.

Older codes, although differing in detail, were essentially the same in that they assumed that wind load caused only pressures on the windward side. They specified the following formulas for the calculation of those pressure forces, perpendicular to the surface: For plane surfaces normal to the wind,

$$W_n = p\,A \quad \text{(1)}$$

in which p is the pressure per unit of area; A is the area; and W is the total wind force, normal to the exposed surface. For plane surfaces, inclined at an angle α against the wind,

$$W_n = p \sin \alpha\, A \quad \text{(2a)}$$

according to other codes,

$$W_n = p \sin^2 \alpha\, A \quad \text{(2b)}$$

or, in America, the widely used Duchemin formula,

$$W_n = p \frac{2 \sin \alpha}{1 + \sin^2 \alpha} A \quad \text{(2c)}$$

Neither Eq. 1 nor Eqs. 2 gave results that could be corroborated in actual experience. According to these formulas, for instance, a horizontal roof would receive no pressure; and no suction was indicated on any surface, under any condition. Modern analysis requires an approach to the problem from quite a different angle.

In view of the complicated pattern of pressure distribution on an arbitrary structure, the first simplification one must make is the division of the total force into a single force for each separate surface or plane. Assume a uniform pressure per unit of area for a given surface. This pressure is determined by two factors—velocity and shape coefficient.

2. Velocity Pressure.—Let q be the pressure exerted on the center of a large plate by the moving air stream, perpendicular to its direction, where the velocity is zero. This "velocity pressure" varies with the following factors: t equal to time; y equal to the height aboveground; and β equal to the direction of wind. The velocity pressure q can be expressed as a general function of these factors, thus:

$$q = f(t, y, \beta) \quad \text{(3)}$$

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[2] "Windbelasting op Bouwwerken," by R. L. A. Schoemaker and I. Wouters, *Het Bouwbedrijf*, No. 22, 1932, p. 3.

3. Shape Coefficient.—The symbol c denotes a dimensionless number, being a function of: K equal to a symbolic parameter, dependent on the shape of the structure; D equal to a scalar dimension of the structure; k/D equal to a roughness factor, when k is the average height of a single protuberance of the surface; v equal to the wind velocity; ρ equal to the density of the air; and μ equal to the viscosity of the air. Using these symbols the Reynoids number can be expressed as

$$\mathbf{R} = \frac{D\,v\,\rho}{\mu} \qquad (4)$$

and c can be expressed as a general function of these factors; thus:

$$c = f\left(R, \frac{k}{D}, K\right) \qquad (5)$$

By application of Eqs. 3 and 5, the wind force on each surface—

$$W_n = c\,q\,A \qquad (6)$$

—can be determined. The total wind force on the structure is the summation of the forces on each plane; therefore the key formula for the final determination of wind load is

$$W = \Sigma W_n = \Sigma(c\,q\,A) \qquad (7)$$

VELOCITY PRESSURE

4. Velocity-Time Relation.—The main characteristic of the wind in nature is that it does not occur like a steady stream of air in a wind tunnel, but in gusts, which could be defined as the ratio of the maximum air velocity to the average air velocity. The air moves with an ever-changing velocity and direction, which cannot be reproduced accurately in model tests. Thus, the velocity of the average main stream of the natural wind is not uniform. The accelerating and decelerating movement of air therefore must be translated into a statically equivalent load.

Gusts can be classified into three types: (1) Those with very small-scale turbulence in the finer structure of the wind; (2) medium-sized frictional eddies, caused by roughness of surface of objects; and (3) large-scale gusts. Of these three, the large-scale gusts are the only ones that need to be regarded as exerting forces of significant magnitudes, and among these it is particularly the maximum gust that must be considered. The basic velocity pressures which are of importance for present purposes, therefore, will be quite different from those obtained from metereologically observed velocities. Such observations yield values only as transcribed by instruments, which generally have a considerable factor of inertia, so that the actual instantaneous maxima are not indicated; nor is the extent of the gust, vertically and horizontally. The absolute maximum velocities in wind gusts increase with height, but less than does the average wind velocity. This fact is recognized in the decreasing gust factor specifically mentioned and incorporated in the design wind pressures sponsored by the American Standards Association (ASA).[3]

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[3] "Building Code Requirements for Minimum Design Loads in Buildings and Other Structures," *ASA–A58.1–1945*, p. 20, 5-1(*c*).

Wind gusts are caused by the unevenness of the surface of the earth and of the objects on it, or by the instability of the streaming air, demonstrating itself in line squalls or superadiabatic gradients. They are also created when air of greater elevations penetrates lower altitudes, retaining to a great extent its higher velocity. These are some of the reasons for the discrepancy between the shapes of the different curves shown in Fig. 1. In other words, each curve represents at least a threefold compromise between the metereologically observed maximum velocities, the velocity-height ratio, and the "gust factor" considered necessary by the authority from which the curve originates. When superadiabatic gradients occur, vertical movement of air will result; but its velocity is not high enough to demand consideration, and the direction of the wind can be taken as horizontal for most purposes. Where this condition does not hold true (as, for instance, for open sheds without walls), the exception should be considered as a different possible case of wind load, as in Table 1.

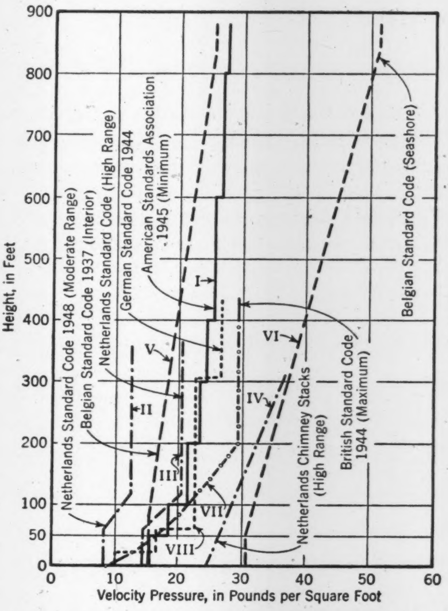

FIG. 1.—VELOCITY PRESSURES AT DIFFERENT HEIGHTS

Large-scale gusts could be particularly destructive if they occurred in a certain range, synchronous (or nearly so) with the periodicity of the structure.[4] To reach an appreciable state of resonance the wind gust would have to recur during, and synchronously with, at least a few subsequent oscillations of the structure. Such an occurrence is so improbable that it need not be considered. The periodicity of vibration of most structures is usually less than that of wind gusts (if, indeed, vibration exists at any periodicity) thus indicating the desirability of rigid types of construction with low periods of vibration.

5. Velocity-Height Relation.—Wind velocities depend on the geology of the locality under consideration, and so, vary in different countries; but their magnitude as a function of height above grade is the same everywhere. This basic relation between velocity (or pressure) and height has been the subject of theoretical investigation[5] as well as of experimental research. The net experience, however, as expressed in the standards of different countries varies considerably. This variation is also partly the result of simplification into formulas for practical use and partly the result of the reasons mentioned in section 4.

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[4] "Vibration in Tall Stacks Solved by Aerodynamics," by W. Watters Pagon, *Engineering News-Record*, July 12, 1934, p. 41.

[5] "Wind Velocity in Relation to Height Above Ground," by W. Watters Pagon, *ibid.*, May 23, 1935, p. 742.

Velocities are expressed as pressures by the Bernoulli relation (in pound-second-feet units):

$$q = \frac{1}{2}\frac{\rho v^2}{g} \qquad (8)$$

in which g is the gravity constant in feet per second.[2]

Theoretically, velocity has been found to be an exponential function of height, particularly in investigations by American and German engineers. On the other hand, designers in Belgium and the Netherlands assume a linear increase with height (especially for the design of chimney stacks). For slender structures like chimney stacks the length-width ratio reaches quite an extreme, and experimental tests support the conclusion that the shape coefficient should be increased. For simplicity in adapting the code the shape coefficients in Germany and the Netherlands are maintained in their simple form without any consideration of the length-width ratio; for these slender structures a special ratio of velocity pressures, increasing more rapidly with height than meteorological data alone would warrant, has been established (see curve IV, Fig. 1).

TABLE 1.—SHAPE COEFFICIENTS FOR OPEN SHEDS (NO SIDE WALLS)

Condition (1)	Slope of roof (2)	Windward slope (3)	Leeward slope (4)
(a) GABLE ROOFS			
1	0° to 20°......	−1.2	−0.4
	30° and more...	−0.8	−0.8
2	0°............	+1.2	+0.4
	10° to 20°......	+0.8	0.0
	30° and more...	+0.8	−0.4
(b) SINGLE SLOPE ROOFS[a]			
....	0° to 10°......	+1.2 or −1.2	+0.4 or −0.4
....	40° and more...	+1.8 or −1.8	+1.0 or −1.0

[a] Use the positive value when the wind is from the left and the negative value when the wind is from the right. For intermediate slopes, coefficients have to be obtained by linear interpolation.

A new principle—frequency differentiation—has been introduced in the specified basic velocity pressures in the Netherlands standard, resulting in a still greater possible degree of economy in construction, mainly for steel structures. Two ranges of basic velocity pressures have been established—a moderate range and a high range. Each range has been subgraduated according to geographic locality, which is not concerned with frequency differentiation. Experience has shown that the velocity pressures formerly accepted as standard were too high. Such pressures did occur very seldom and then only in relatively small gusts, so that the average wind load on a larger area is always less than the former standards. Therefore, a new basic moderate range of velocity pressures has been established, which is 60% of the range of velocity pressures formerly considered normal. The former standard pressures are now established in the high range of pressures. The moderate range indicates a recurring and more or less stable loading condition (like live loads without impact) as compared with the fluctuating or dynamic load exerted by the velocity pressures in the high range. The reduced wind loads are applied in the design, retaining the normally allowed stresses (that is, tensile and bending stresses of 20,000 lb per sq in. in structural steel). At the same time the structure is designed to withstand the high range of wind load (whereby the allowable stresses for steel, for instance, are raised by a factor of 1.15). This rise is considered permissible in view of the incidental character ascribed to the high range of velocity pressures and of the relatively few times that they occur. A similar attitude, in rare combinations of loading conditions, is that of American designers who increase allowable stresses one third for the relatively rare combination of dead load, live load, snow load, and wind load.

The moderate range of wind loads is also applicable to reinforced concrete construction with the normally allowed stresses. The structure need not be designed for the high range of loads, in view of the stiffness that is characteristic of this type of construction.

For wood and masonry structures such refinement in design is not attainable and does not warrant the frequency differentiation of velocity pressures. Therefore, the old (that is, the high) range of velocity pressures has been maintained without an increase in the allowable stresses.

6. *Velocity and Direction of the Wind.*—Velocities of wind arriving from some directions usually exert greater pressures than those from other directions. For instance, in the Netherlands the following classification for velocity pressures can be made:

Wind from	Design pressure
Northeast or east	70%
North or southeast	90%
All other directions	100%

In most standards, however, differentiation in velocity pressure according to direction is not customary.

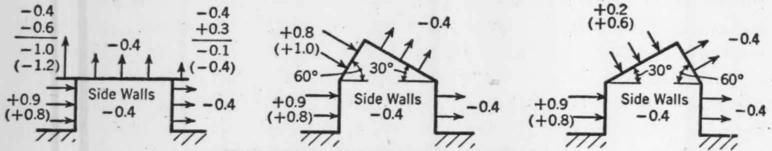
Therefore the factors mentioned in sections 4 and 5 are the only ones that will be considered, and Eq. 3 becomes

$$q = f(t,y) \qquad (9)$$

The influences of t and y are usually taken as combined in one table or graph of velocity pressures for different heights above ground.

7. *Wind and Snow Load.*—Maximum snow loads and wind loads in many European standards do not need to be regarded as simultaneous. The maximum snow load may be expected with the maximum wind velocities only if a condition of solidification by thawing and subsequent freezing has made the heavy layer of snow impervious to a hurricane. On the other hand, metereological observations show that with frost the wind directions to be expected never produce maximum velocities, so that maximum snow loads and wind loads do not have to be considered simultaneously. Evidently this condition is typical only of regions with certain climatological characteristics and it is

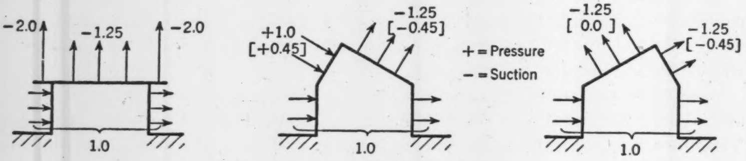
logical therefore, that a ruling to this effect appears in some countries: Belgium, the Netherlands, and Germany. The Belgian code, however, acknowledges the possibility of a combined load of snow and 40% of the maximum wind load,



−0.4
−0.6
−1.0
(−1.2)
−0.4
−0.4
+0.3
−0.1
(−0.4)
+0.9
(+0.8)
Side Walls
−0.4
−0.4
+0.8
(+1.0)
−0.4
60°
30°
+0.9
(+0.8)
Side Walls
−0.4
−0.4
+0.2
(+0.6)
−0.4
30°
60°
+0.9
(+0.8)
Side Walls
−0.4
−0.4

(a) STANDARDS FOR GERMANY AND THE NETHERLANDS

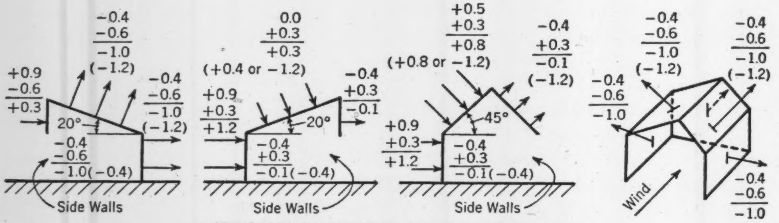
Note: Values in Parentheses ( ) Denote German Standards When They Differ From Netherlands Standards
Values in Brackets [ ] Denote ASA Standards



−2.0
−1.25
−2.0
1.0
+1.0
[+0.45]
−1.25
[−0.45]
1.0
+ = Pressure
− = Suction
−1.25
[0.0]
−1.25
[−0.45]
1.0

(b) AMERICAN STANDARDS ASSOCIATION

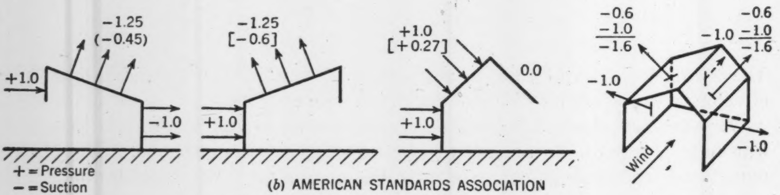
FIG. 2.—SHAPE COEFFICIENTS FOR CLOSED BUILDINGS, WIND FROM THE LEFT



+0.9
−0.6
+0.3
−0.4
−0.6
−1.0
(−1.2)
20°
−0.4
−0.6
−1.0
(−1.2)
−0.4
−0.6
−1.0(−0.4)
Side Walls
0.0
+0.3
+0.3
(+0.4 or −1.2)
+0.9
+0.3
+1.2
20°
−0.4
+0.3
−0.1
−0.4
+0.3
−0.1(−0.4)
Side Walls
+0.5
+0.3
+0.8
(+0.8 or −1.2)
−0.4
+0.3
−0.1
(−1.2)
45°
+0.9
+0.3
+1.2
−0.4
+0.3
−0.1(−0.4)
Side Walls
−0.4
−0.6
−1.0
(−1.2)
−0.4
−0.6
−1.0
(−1.2)
−0.4
−0.6
−1.0
−0.4
−0.6
−1.0
Wind

(a) STANDARDS FOR GERMANY AND THE NETHERLANDS

Note: Values in Parentheses ( ) Denote German Standards When They Differ From Netherlands Standards
Values in Brackets [ ] Denote ASA Standards



−1.25
(−0.45)
+1.0
−1.0
−1.25
[−0.6]
+1.0
+1.0
[+0.27]
0.0
+1.0
−0.6
−1.0
−1.6
−0.6
−1.0
−1.0
−1.6
−1.0
−1.0
Wind
+ = Pressure
− = Suction

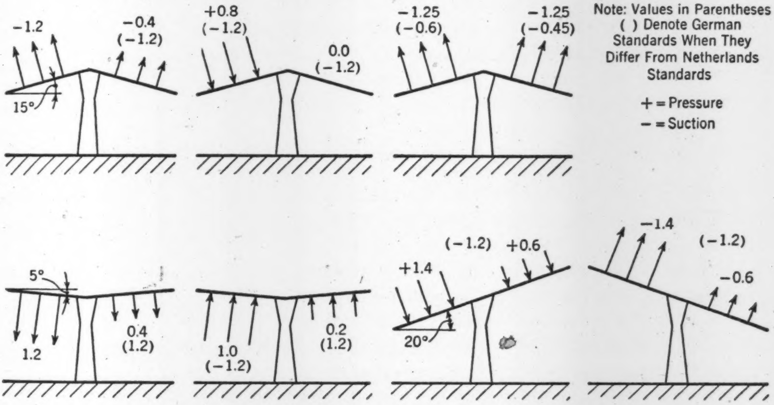
(b) AMERICAN STANDARDS ASSOCIATION

FIG. 3.—SHAPE COEFFICIENTS FOR BUILDINGS CLOSED ON ONE SIDE, USED FROM THE LEFT

whereas the German code recognizes a simultaneous existence of snow loads on roofs as steep as 45°, with maximum wind velocity. The simultaneous occurrence of full wind load and snow load is considered possible in Great Britain and the United States.

SHAPE COEFFICIENTS

8. Scale Effect, Reynolds Number, and Roughness of Surface.—In 1914, G. Eiffel[6] showed that the distribution of pressures observed from model tests on sharp-edged bodies can be transferred directly to any larger scale regardless of velocity. In other words, there is no scale effect and the Reynolds number is 1. This is not the case for rounded surfaces (like gasholders, chimney stacks, pipes, wires, etc.), the test results of which cannot be transferred simply to larger scales. For most buildings and structures, there is no appreciable scale effect. However, a much longer time was required before it was universally understood that on a prismatic structure only the windward side undergoes pressure, whereas on all other sides suctions are exerted. By that time, also, designers had recognized the universal occurrence of vortex layers, resulting from differences between kinetic and frictional forces, in which the roughness of the surface plays a minor role. Thus, in the application of test results the influence of vortex layers can be disregarded, particularly in the design of angular bodies. Therefore, Eq. 5 can be simplified into



Note: Values in Parentheses ( ) Denote German Standards When They Differ From Netherlands Standards

+ = Pressure
− = Suction

FIG. 4.—SHAPE COEFFICIENTS FOR OPEN SHEDS, GERMANY AND THE NETHERLANDS, WIND FROM THE LEFT

$$c = f(K) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

in which the shape coefficients c are identical with K for various shapes of structures.

9. Shape Coefficients.—The most complete set of wind-load coefficients for different shapes of buildings is to be found in standards of the Netherlands and of Germany, both arrived at independently as the result of extensive wind-tunnel tests on models. In graphic form the coefficients have been presented in Figs. 2(*a*), 3(*a*), and 4 compared with those of the ASA standards. Contrary to most standards (including those used in the United States) the German and Dutch specifications distinguish clearly between forces on windward and leeward surfaces, thereby localizing wind loads conveniently for structural design. As a rule the Netherlands standard is simplest in form, simplest to handle, and, at the same time, consistent with test results.

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[6] "Nouvelles recherches sur la résistance de l'air et l'aviation faites au laboratoire d'Auteuil," by G. Eiffel, Dunod et Pinat, Paris, 1914.

In the following tabulation, as in Figs. 2(*a*), 3(*a*), and 4, values in parentheses are the German coefficients wherever they do not agree with the Netherlands coefficients:

Description	Coefficient
1. Closed Buildings (see Fig. 2(*a*)):	
a. Windward Side—	
Surfaces with angles of inclination ranging from 90° to 65° with the horizontal....................	$+0.9\ (+0.8)$
Surfaces with angles of inclination less than 65° if the approach angle α, of the wind, is at that angle.........	$+0.02\,\alpha-0.4\ (1.2\sin\alpha-0.4)$
For all angles of roof, the Belgian standard requires...............	$\sin(\alpha+10)$
b. Leeward side for all roof slopes........	-0.4
c. Surfaces parallel to the wind direction...	-0.4
2. Buildings Open on One Side, for at Least One Third of the Wall Area (see Fig. 3(*a*)).—When the coefficients for the exterior of closed buildings apply, the interior is subject to the following shape coefficients:	
a. If the open side is facing the direction of the wind, to a pressure........	$+0.6\ (+0.8)$
b. If the open side is away from the wind, or in the direction parallel to the wind, to a vacuum of...........	$-0.3\ (-0.4)$
3. Sheds Without Walls (See Fig. 4):	
a. For gable roofs, two loading conditions, 1 or 2 applied alternately, with the wind in the same direction, must be considered, giving the shape coefficients in Table 1(*a*), referring as usual to the upper side of the roof	....
b. For inverted gable roofs (that is, **V**-roofs sloping toward a central valley) the same coefficients apply as for normal gable roofs (item *a*), but now referring to the under side of the roof. In both cases, the two extreme loading conditions must be taken into account to allow for	

Distribution	Coefficient
any possible deviation of the wind direction from the horizontal (see section 4)	....
c. For roofs with a single slope, the coefficients in Table 1(*b*) (applied to the roof section only) are used, selecting the negative or the positive value depending on the direction of the wind	....
4. Separate Walls:	
a. For pressure and vacuum together (suction)	1.4 (1.2)
5. Open Trusses and Latticed Structures:	
a. For pressure and vacuum together (suction), applied to the area of the structure projected on the plane through the axis of the members	1.6

For comparison, Figs. 2(*b*) and 3(*b*) denote the requirements of the ASA Standards for conditions similar to those in Figs. 2(*a*) and 3(*a*), respectively.[7]

Conclusions

Although wind is a complex phenomenon, causing variable and complicated sets of loads, it has been shown possible to establish rules that are practical for actual design purposes—rules with a precision sufficient for most conditions. A general comparison between established standards in Europe and America would indicate that in some European standards a much greater refinement in the definition of loading conditions (for instance, in shape coefficients) has been sought and attained, leading to a reduction in the factor of safety and ultimately to the attainment of a greater degree of economy. This increased refinement becomes more desirable in view of the trend toward the construction of less dead load such as: Thin wall construction, light metal (aluminum) construction, prestressed concrete construction, and prefabricated construction.

However in actual practice the very common case of a structure among neighboring and partly shielded buildings often prevents an accurate forecast of loading conditions. In such cases only an investigation with actual model tests can supply definite answers; and, although the size of the project will often justify the costs of such tests, it is the time involved which prevents the undertaking of extensive tests. Available test results[8] seem to indicate that on structures consisting of a multiple repetition of units, an adjacent unit will often receive an increased rather than a reduced wind load compared to the exposed unit (particularly as the result of a vacuum). Therefore the rule that no reductions shall be allowed for so-called shielded locations often does not coincide with actual conditions, which even show an increased load particularly where suction occurs.

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[7] "Building Code Requirements for Minimum Design Loads in Buildings and Other Structures," *ASA–A58.1–1945*, Appendix 5-3.

[8] "Wind-Pressure on Buildings Including Effects of Adjacent Buildings," by Alfred Bailey and Noel D. G. Vincent, *Journal*, Inst. of Civ. Engrs., Vol. 20, 1943, p. 243.

Among phenomena about which relatively little is known are wind gusts. This wide field of research still lies unexplored. The outcome of such research might quite possibly influence and alter some of the fundamental concepts of wind loads on structures. Knowledge is still too limited on such specific details as: Size of wind gusts, horizontal as well as vertical; their spacing in the direction of the wind as well as perpendicular to it; and the velocity with which they propagate in the direction of the wind, a velocity possibly differing from that of the wind.

DISCUSSION

PAUL ROGERS,[9] M. ASCE.—It has been known to engineers for a long time that the customary wind pressures applied in the design of structures are not in accord with the actual wind-forces. By presenting several wind-load standards of various European countries, the author gives a worthwhile contribution to the engineering literature.

The writer is at variance, however, with the author's opinion that the possibility of resonance between the frequencies of wind and structure need not be investigated. The author himself refers to a paper[4] that deals with exactly such occurrences.

A few years ago the writer was called upon to investigate the strange behavior of a large smokestack, second in a row of four stacks. The stack in question underwent considerable deflections, perpendicular to the wind direction, during moderate wind velocities, whereas it was perfectly stable at high wind velocities. No faults could be discovered with the statical computations or structural design of the stacks. Subsequent analysis has proved the possibility of resonance between the periodicity of the eddies shed in the wake of the first stack and of the natural period of vibration of the second stack, when the wind was blowing from a certain direction at critical wind velocity. The stack vibrations were finally eliminated after the installation of the lining, which substantially changed the natural frequency of the stack. As a result of investigations, it was deemed advisable to adopt, as a design standard, the stipulation that the spacing of smokestacks should not fall within the critical range, that is, between 3½ to 5½ times the stack diameter.

CARL H. WALTHER,[10] ASSOC. M. ASCE.—A useful service has been rendered in presenting this summary of the pressures and coefficients for wind loadings in use in various European standard codes. It is all too true that, as the author states (under the heading "Introduction"), "In most building codes in the United States obsolete conceptions of wind load still prevail * * *." Even among practicing engineers similar primitive conceptions of the magnitude and distribution of wind pressures remain common in spite of the numerous publications that have dealt with the subject over the past twenty-five years. Any attempt to bring the well-established principles governing wind pressure to the attention of a large part of the profession is to be applauded.

The writer notes that the present paper does not include information on the location of the resultant wind force for wind directions oblique to the walls of a building. It is well known that the center of pressure on a plate oblique to the direction of the wind does not coincide with the geometrical centroid of the area but is displaced toward the leading edge of the plate.[11] A similar effect is observed in wind-tunnel tests of models of buildings with the wind at oblique angles of incidence. The resultant wind force does not, in general, pass through the center of twist of the structure as viewed in the plan. Therefore, the structure is subjected to a torsional moment about a vertical axis through its center of twist. The amount of the torque depends on the height-to-width ratio and length-to-width ratio of the building, as well as on the angle of incidence of the wind, but tests have shown that the torque is not negligible even for structures of quite ordinary proportions.[12,13,14] For long and narrow buildings the twisting moment may be disastrously large. For example, it has been suggested that the mode of failure of the Meyer-Kiser Building in the Miami (Fla.) hurricane of 1926 confirms the existence of such moments.[15,16]

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[9] Structural Design Engr., Sargent & Lundy, Engrs., Chicago, Ill.

[10] Prof. of Civ. Eng., George Washington Univ., Washington, D. C.

[11] "What Aerodynamics Can Teach the Civil Engineer," by W. Watters Pagon, *Engineering News-Record*, Vol. 112, March 15, 1934, p. 348.

Torsional moments of considerable magnitude must certainly be considered in the design of tall buildings. These moments will vitally affect the design of the wind bracing in mill or industrial buildings. Consequently it is rather difficult to understand why the writers of standards have ignored the condition of oblique incidence of wind.

It appears to the writer that an opportunity exists for the author to present a summary of provisions specifying the location of the resultant wind force as dealt with in various standards. The information should prove to be of great value to engineers concerned with the design of structures. It is impossible to estimate the waste of material on one hand and the possible reduction in presumed factors of safety on the other that may be caused by the use of unrealistic standards for wind loading. The annual cost is certainly heavy. When there is a shortage of building materials, any information that contributes to more rational design of structures is welcome.

It may be that available data are too meager to permit the formulation of standard provisions for twisting moments caused by oblique wind. If such is the case, it would appear that this is a fit subject for experiment on a well-chosen series of model shapes. Some use could be made of published results that are presented in the form of pressure distributions, for pressure distributions may be translated into twisting moments by numerical integration methods.[17,18]

The investigation of a limited number of shapes should provide the information needed to permit the formulation of standard provisions for buildings of the usual types and proportions. It is always expected that special model tests will be required for structures of unusual shape or dimensions.[19]

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[12] "Wind Pressure on Buildings," by J. O. V. Irminger and Chr. Nøkkentved, *Ingeniørvidenskabelige Skrifter*, A Nr. 23, Copenhagen, Denmark, 1930.

[13] *Ibid.*, A Nr. 42, 1936.

[14] "Wind Pressures on Structures," by Hugh L. Dryden and George C. Hill, *Scientific Papers of the U. S. Bureau of Standards*, Vol. 20, 1926, p. 696.

[15] "Wind-Tunnel Studies Reveal Pressure Distribution on Buildings," by W. Watters Pagon, *Engineering News-Record*, Vol. 113, 1934, p. 814.

[16] "Elementary Stress Analysis," by Phillip E. Soneson, Pitman Publishing Corp., New York, N.Y., 1949, p. 142.

[17] "Wind Pressures on Structures," by Hugh L. Dryden and George C. Hill, *Scientific Papers of the U. S. Bureau of Standards*, Vol. 20, 1926, p. 714.

[18] "Wind Pressure on a Model of a Mill Building," by Hugh L. Dryden and George C. Hill, *Journal of Research*, U. S. Bureau of Standards, Vol. 6, 1931, p. 735.

[19] "Aerodynamics of the Perisphere and Trylon at World's Fair," by A. Klemin, E. B. Schaefer, and J. G. Beerer, Jr., *Transactions*, ASCE, Vol. 104, 1939, p. 1449.

Francis L. Castleman, Jr.,[20] M. ASCE, and Aron L. Mirsky,[21] Assoc. M. ASCE.—The School of Engineering of the University of Connecticut (Storrs, Conn.), in the course of setting up a research program on the action and effect of the wind on structures, made a critical survey of literature on the subject. Although experimental results of this research project are not available, it is believed some of the results of the literature search might prove of interest and emphasize some of the differences of opinion encountered.

There appears to be no definite agreement among the various writers on scale effect (Reynolds number). (In this connection it is pertinent to mention that the absence of scale effect is equivalent to zero viscosity. This results in a Reynolds number of infinity and not unity as stated by the author in the section on "Shape Coefficients.") Early investigators tended to discount scale effect in the case of structures with sharp edges, such as ordinary buildings, bridges, and other common engineering structures, as either absent entirely or negligible. However, later investigations seemed to indicate that scale effect may be of some importance in such cases, but the diversity of opinion is still great, and as yet there appears to be no critical standard (other than personal opinion) to indicate whether scale effect may be neglected or not.

G. Eiffel,[22] incidental to his studies on hangars, investigated three geometrically similar models with peaked roofs and with dimensions in the ratios of 400:50:8. Within the limits of precision of his measurements he reported identical pressure distributions on all three models and concluded there was no scale effect; he then applied that conclusion to hangars with rounded roofs. Apparently, also, he did not attempt to verify his conclusions by varying the wind velocity (hence, the Reynolds number) during the tests on any one model.

C. Nøkkentved[23] and J. Irminger[13] have shown that the relationship between the slope of the leading (windward) vortex sheet and the slope of the windward roof determines whether there is pressure or suction on the roof surface. They have also shown that the slope of the windward vortex sheet, in turn, is a function of the roughness of the ground surface to the windward of the structure (prototype or model). Consequently, the slope is a function of the Reynolds number as well as of the height of the structure. A consequence of this is that each structure or model has a critical height. In the case of Mr. Eiffel's models the critical height was so much larger than the height of the largest model that the effect was apparently not evident. Had a larger range of model ratios been investigated, the effect would undoubtedly have been recognized.

In an earlier publication,[12] however, Messrs. Nøkkentved and Irminger were of the opposite opinion, namely, that the extent of the vortex region in the case of sharp-edged bodies is independent of wind velocity, that is, that scale effect does not exist for such bodies.

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[20] Dean, School of Eng.; Prof. of Civ. Eng., Univ. of Connecticut, Storrs, Conn.

[21] Instr. in Civ. Eng., Univ. of Connecticut, Storrs, Conn.

[22] "Nouvelles recherches sur la résistance de l'air et l'aviation faites au laboratoire d'Auteuil," by G. Eiffel, Dunod et Pinat, Paris, 1914, pp. 283–302.

[23] "Wind Pressure on Buildings. An Investigation into the Influence of the Size of the Models," by Chr. Nøkkentved, *Publications*, International Assn. for Bridge and Structural Eng., Vol. 2, 1933, pp. 257–264.

The conclusions of Hugh L. Dryden and George C. Hill[14,24,25,26] follow the same pattern as those of Messrs. Irminger and Nøkkentved. At first Messrs. Dryden and Hill discounted scale effect, but later they reached the conclusion that errors due to neglecting it were small and commented that until such belief has been confirmed by full-scale experiments on actual structures, results from model tests could not command the full confidence of engineers engaged in the design of buildings. To date, the writers have been able to find no such verification in the literature.

In regard to the influence of neighboring structures C. L. Harris[27] found a slightly larger (but still minor) scale effect when including models of neighboring buildings in the experiment. A. Bailey and N. Vincent[28] found a definite scale effect, such as Mr. Bailey[29] himself had found in earlier experiments. In both sets of experiments it was noted that the scale effect was particularly evident on the leeward sides of the structures. Such suction effect of the wind, although found in experiments as far back as the 1890's, has only recently been recognized by the engineering profession.

In 1950 M. R. Horne[30] concluded that sufficient work had been done on models to predict the steady wind pressure on any normally shaped structure provided that some unspecified allowance be made for scale effect. The nature of this scale effect requires further investigation, but sufficient data are at hand (he does not specify where) to indicate what allowance should be made for it.

From a review of the experimental data, the theoretical analyses, and the speculations recorded in the literature, only a minor fraction of which has been cited, it may be seen that as yet there is no pat answer to the question of the existence of a scale effect and whether it is of practical importance in the case of ordinary engineering structures. The writers do not pretend to know. From the conflicting evidence so far presented, one of the writers tends to the opinion that for conventional sharp-edged structures, scale effect is probably not a critical factor, and the other is highly doubtful. Both writers, however, are firmly of the opinion that the existence and importance of scale effect remains to be proved by experiment.

S. K. Ghaswala,[31] Assoc. M. ASCE.—An interesting sidelight on wind-load analysis has been presented in this paper which represents one of the few attempts to chart out the subject from international standards. It is for this reason that the paper should be studied by many workers in the field.

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[24] "Wind Pressure on Circular Cylinders and Chimneys," by H. L. Dryden and George C. Hill, *Journal of Research*, U. S. Bureau of Standards, Vol. 5, 1930, pp. 657–658.

[25] "Wind Pressure on a Model of a Mill Building," by Hugh L. Dryden and George C. Hill, *Journal of Research*, U. S. Bureau of Standards, Vol. 6, 1931, p. 755.

[26] "Wind Pressure on a Model of the Empire State Building," by Hugh L. Dryden and George C. Hill, *Journal of Research*, U. S. Bureau of Standards, Vol. 10, 1933, p. 495.

[27] "Influence of Neighboring Structures on the Wind Pressure on Tall Buildings," by C. L. Harris, *Journal of Research*, U. S. Bureau of Standards, Vol. 12, 1934, pp. 107–108.

[28] "Wind-Pressure on Buildings Including Effects of Adjacent Buildings," by Alfred Bailey and Noel D. G. Vincent, *Journal*, Inst. of Civ. Engrs., Vol. 20, 1943, pp. 264 and 269.

[29] "Wind-Pressures on Buildings," by Alfred Bailey, *Paper 139*, Inst. of Civ. Engrs., 1933.

[30] "Wind Loads on Structures," by Michael Rex Horne, *Journal*, Inst. of Civ. Engrs., Vol. 33, 1950, pp. 158–159.

[31] Chartered Engr., "Edena," Queen's Road, Fort., Bombay, India.

Wind is motion of the air caused by gravity, by deflective forces due to the earth's rotation (termed Coriolis forces), and by centrifugal forces due to curvature of the wind path. It is evident that an appraisal of its exact effect is not simple as this effect is dependent on several factors that in themselves are further functions of other variables, such as friction and viscosity. The flow of air at any period is never smooth but is subject to fluctuations. When sudden and relatively brief, these fluctuations represent gusts. Gustiness or atmospheric turbulence is of great importance in structural design and is accompanied by so many variations of velocity, temperature, and humidity that no completely satisfactory theory exists as to why it arises and how it proceeds.[32] It is believed, however, that gusts are caused mainly by the growth of eddies due to mechanical obstacles, irregularities of ground, local pressure differences, deflection around objects, and vertical thermodynamic interchange, or a combination of the above causes.[33] The most complete and accurate observations and records of meterorological factors to be used in appraising wind velocity and gusts have been given by H. Ferrington in England[34] and R. H. Sherlock, M. ASCE, in the United States.[35] These studies indicate that the maximum hourly mean wind is the most appropriate value to be considered in design and should be assumed as the steady pressure acting on a structure. On this force should be superimposed a periodic pressure produced by gusts and recorded over the same period as the hourly mean wind. An examination of these results would have added much to the value of the paper.

It was Sir Isaac Newton who first evolved the algebraic expression for wind pressure, giving it as equivalent to the square of the wind velocity. This expression was used by many French engineers for designing structures at the beginning of the twentieth century. These expressions were later replaced by the now familiar concept of E. N. Duchemin and J. S. Hutton. Unfortunately these formulas are not accurate, as demonstrated by recent aerodynamic research and wind-tunnel experiments[36] that clearly reveal the lack of consideration for wind pressure on the leeward side of the structure. Pioneering work on wind-tunnel analysis that threw new light on the subject was carried out by J. O. V. Irminger in Copenhagen. His work was continued by many other prominent workers in the field.[33]

The writer fully agrees with the author that the subject of gusts is still in a nebular state of development. The science of wind-pressure analysis and design can be brought to perfection only when the fundamental dynamic character of wind loads is expressed as static equivalents to give simple and reliable data for the designing engineer.

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[32] "Physical and Dynamical Meteorology," by David Brunt, Cambridge University Press, Cambridge, England, 1939.

[33] "Aerodynamic Aspects of Civil Engineering," by S. K. Ghaswala, *Civil Engineering and Public Works Review*, London, England, September, 1950, pp. 586–589; October, 1950, pp. 646–649; November, 1950, pp. 725–727.

[34] "Wind Loads on Buildings," by H. Ferrington, *The Structural Engineer*, December, 1943, p. 497; January, 1944, p. 15.

[35] "Gust Factors for the Design of Buildings," by R. H. Sherlock, *International Association for Bridge and Structural Engineering*, Vol. 8, 1947, p. 207.

[36] "The Function of Aerodynamics in Civil Engineering," by S. K. Ghaswala, *Journal of the Inst. of Engineers*, India (publication pending).

Herbert S. Saffir,[37] Assoc. M. ASCE.—Detailed description of wind-load standards in European countries should prove of value in formulating wind-load requirements for building codes in the United States. The compilation of velocity pressures at different heights and wind-load coefficients for different shapes of buildings can serve as a useful guide in revising building codes and for use in structural design. Because of the dynamic character of the loading and the probability of occurrence of such loading, it is doubtful if wind-load standards can ever be refined enough to agree with actual conditions.

The idea of frequency differential, introduced in the specified basic velocity pressures in the Netherlands standard, is mentioned. It is stated that each range has been subgraduated according to geographic locality. The writer believes that a similar plan should be established for different regions in the United States. Various wind loadings could be specified in standard building codes and standard design specifications for varying conditions throughout the entire country. The writer has in mind a map comparable to the velocity pressure map prepared by the United States Bureau of Standards from United States Weather Bureau data.[38] A map of this type could be used for structural design in localities not covered by building codes or by local building code committees writing and revising building codes. The shape and exposure of the structure and the probability of gust velocities would be other factors entering into the choice of a design wind load for a particular structure.

Shape coefficients, such as those shown for open sheds in Fig. 4, are of value from a designer's viewpoint. Specifications have been drafted to account for the shape effect, and a typical provision is quoted here:[39]

> "The effect of shape of irregular or unusual roofs may be determined by wind tunnel or equivalent tests. In determining the effect of shape, the assumed wind velocity shall be the maximum average for five (5) minute period shown in the records of the U. S. Weather Bureau during the past fifteen (15) years, and the prescribed wind forces shall be modified accordingly."

The writer notes that coefficients for circular arc roofs are not mentioned in the paper. Are shape coefficients for such types of roofs omitted because of the scale effect in evaluating test results?

Theodore B. Rights,[40] Assoc. M. ASCE.—The author is to be congratulated for bringing a greater realism into a subject that some engineers consistently refuse to study objectively. Many engineers concern themselves only in what the "code" says, with no further analysis. The subject of wind loads is a serious matter, as many who experienced the wind storm along the eastern seaboard on November 25, 1950, can testify.

In the introduction the author speaks of "* * * cases of damage from wind recur far too frequently." There should be an "autopsy" of wind failures for bridges and buildings. Only in cases of large structures, such as the Tacoma Narrows Bridge (Wash.) or the Mississippi River Bridge at Chester, Ill., are investigations made and the findings published. The small structure that blows down without killing anyone is only reported in the local paper by such comment as: "The high winds of yesterday blew down a section of the walls of the new A Manufacturing Company building." Even this small item is omitted from the paper if A Manufacturing Company has a publicity agent who sees that unfavorable news is omitted from the papers. Building inspectors should be compelled by law to submit investigation reports of failures and pertinent data to a board for further analysis. Since the engineering profession has no money to set up such a board, and since it also dislikes to advertise its mistakes, this board should be set up by the state. The main purpose of such a board would be to study the submitted reports and to make recommendations that would enable engineers to design their structures realistically to resist wind pressure.

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[37] Asst. County Engr., Dade County Eng. Dept., Miami, Fla.

[38] "American Standard Building Code Requirements for Minimum Design Loads in Buildings and Other Structures," *Miscellaneous Publication M179*, U. S. Bureau of Standards, Washington, D. C., 1945.

[39] "Basic Building Code," Building Officials' Conference of America, Inc., New York, N. Y., 1950 Ed., Section 715.4, p. 125.

[40] Cons. Engr., Roselle, N. J.

As stated by the author, there are few codes in the United States that specify drag or suction; yet only a short time ago, a case was brought to the writer's attention of a building in Canada that had part of its steel roof deck ripped off. The suction, or the inside pressure, was so great that steel sheets tore around the welds which fastened them to the purlins. In the same building, a rolling steel door was blown out and rolled down the tracks. Isn't it about time that officials realize this is a serious problem and try to do something about it?

In June, 1946, J. C. Early asked about the analysis of slender angle girts.[41] So far as the writer knows there has been no answer to his questions. Perhaps the reason that girts do not fail is that structural engineers do not find that the actual wind load is as great as the design load. If this is so, why doesn't the designer use lighter loads for the design of wind bracing and uplift?

Since there are many more low buildings designed than the high, or skyscraper, type, engineers should study and have the data of wind effects on small buildings. Several excellent studies have been made on tall buildings. Perhaps someone will study the small, or warehouse type, building less than 60 ft high.

It seems peculiar that the author referred to excellent articles[4,5] published as long ago as 1934 and 1935. Are there no later articles on this subject? In succeeding years the engineering profession should have collected more data on this difficult subject.

John W. T. Van Erp.[42]—The various discussions have introduced many more aspects of the subject than those covered in the paper, notably those aspects relating to observed data and phenomena and their interpretation into a set of design specifications.

Mr. Rogers is correct in his statement that resonance is a factor of prime importance, especially for slender structures of little mass and rigidity, such as the one mentioned in his example. Even single wind gusts without any cumulative resonance effect cause a greater deflection on these structures than on structures of great mass and rigidity, such as a building. For these latter, however, the period of vibration is many times as large as that of any pulsating wind load, so the writer still maintains that any chance of resonance is extremely remote. Although an exact determination of the period of such a structure by theoretical methods can hardly be arrived at, empirical formulas that give an answer exist. A pertinent paper,[43] for instance, gives:

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[41] "Analysis of Slender Angle Grits," by J. C. Early, *Civil Engineering*, Vol. 16, 1946, p. 267.

[42] Associate Engr., Trygve Hoff and Associates, Cleveland, Ohio.

$$T = c \frac{H}{\sqrt{b}} \qquad (11)$$

in which T is the period in seconds; H is the height in feet; b is the width in feet in the direction of vibration; and c is a coefficient varying between .05 and .06 for most buildings. Another element that enters into the picture is that slender structures presenting only a small cross-sectional area perpendicular to the direction of the wind (whose area, in other words, is equal to or smaller than that of the wind gust) are more sensitive to a pulsating wind load. A design for such a structure, based on the standard velocity-pressure loads alone, might be too light. In connection with this subject reference is invited to the section on "Velocity Pressure" and also to curve IV in Fig. 1. On this basis it is recommended that for structures of small cross-sectional area perpendicular to the wind, say 200 sq ft or less, the design wind load should be increased by 10%.

Mr. Walther refers to the possibility of a phenomenon, the existence of which has not been conclusively proven, to the best knowledge of the writer. Here another subject for further research is clearly indicated. In case, however, the definite existence of such an eccentricity of the wind load resultant at certain angles of obliquity in the horizontal plane would be proven, it seems appropriate that this factor should be taken into consideration in conjunction with the specification of maximum velocity pressures for certain orientations, depending on geographical location. A velocity pressure map[44] should, therefore, be complemented by compass cards, indicating orientation of the maximum velocity pressures at key locations.

In their discussion Mr. Castleman and Mr. Mirsky present a comprehensive review of experimental and theoretical analyses on the existence of scale effect. The general conclusion still seems to be that the existence of scale effect is, to say the least, doubtful. Thus, there seems to be no reason why data derived from small-scale tests should not be incorporated into design specifications, the exposition of which has been the chief purpose of the paper.

It is evident from Mr. Ghaswala's discussion that the paper has only partially covered the available knowledge in this field. Apparently the engineering knowledge available in print is already much more extensive than that covered by the writer, apart from the sphere outside of his necessarily limited confinements.

The aim to strive for in a perfect set of wind-load standards should be a comprehensive coverage of all possible loading conditions, and the writer is optimistic enough to believe that this can be achieved, using a not-too-unwieldy set of standards. The hurricane of November 25, 1950, on the east coastal area provided a very tangible demonstration of the general lack of proper knowledge of the effects of wind load. The recorded wind velocities indicated that maximum velocity pressures of about 25 lb per sq ft were reached, an excess of only 25% over specifications of the local code. The damage to one of the newest tall buildings, running well into five figures, probably indicates the existence of such a phenomenon as discussed by Mr. Walther. The eccentricity of the wind load, resultant at oblique angles of wind direction, should be particularly noticeable with an oblong-shaped plan, such as the one of this building. Furthermore the ubiquitous occurrence of suction on the leeward sides was demonstrated by many broken shop windows on those sides.

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[43] "Lateral Forces of Earthquake and Wind," by a Joint Committee of the San Francisco, Calif. Section, ASCE, and the Structural Engineers Association of Northern California, *Proceedings-Separate No. 66*, ASCE, Vol. 77, April, 1951.

[44] "Building Code Requirements for Minimum Design Loads in Buildings and Other Structures," *A58, 1-1945*, American Standards Association, New York, N. Y., 1945, Fig. 2, p. 20.

For wind loads on arched roofs, the shape coefficients given in the paper are of sufficient accuracy to govern design loads without consideration of any scale effect. The given shape coefficients would indicate considerable suctions over the entire roof area for flat curvatures of the arch, corroborated, for instance, by early wind-tunnel tests on models of gas holders with a dome top. The writer fully agrees with the suggestion made in the discussion by Mr. Rights that wind loads up to now have definitely been treated as an unimportant factor in design load. Consequently, not enough importance is being attached to the multitude of observations of the smaller everyday storm damages that could otherwise, by proper coordination and classification, furnish an elaborate basis for the perfection of wind-load standards. Finally, it may be stated that, in general, a more articulate and specific set of wind-load standards than those available is desired. The writer hopes that the cited examples indicate the way in which these standards can be developed, on the basis of abundant experimental and theoretical data, into a versatile expedient at the service of the designing engineer.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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TRANSACTIONS

Paper No. 2472

THE PROBLEM OF WAVE ACTION ON EARTH SLOPES

By Martin A. Mason,[1] Assoc. M. ASCE

With Discussion by Messrs. Robert Y. Hudson; Fred C. Walker; Henry H. Jewell; C. L. Bretschneider and R. R. Putz; and Martin A. Mason.

Synopsis

A brief resumé of current knowledge and research on wave action on earth slopes and the possible application of World War II developments, particularly in regard to methods for predicting wave action, is discussed. The Sverdrup-Munk method of predicting wave height and period from synoptic weather maps is recommended as a substitute for the older purely empirical methods. The Iribarren formula for the calculation of sizes of stone required on various slopes for protection against wave action is recommended as the only available satisfactory method of design. An outline of necessary further study and research is presented.

Current Knowledge

Wave action is the primary factor to be considered in the design of slope protection for earth dams. It is precisely this factor which is least known or understood by the engineering profession. Students of the subject have concentrated their attention on the empirical formulas for predicting wave action which have been used in the past; and have been criticized, perhaps justly, for the profession's failure to have made use of wartime advancements in the knowledge of waves.

In support of the profession's apparent neglect of wartime developments of wave knowledge, it must be noted that much of the knowledge was cloaked in military secrecy for a long time. Relatively little information on the engineering applications is available even now (1950). The only confirmation of the relationships developed by H. U. Sverdrup and W. H. Munk,[2] which can be

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Note.—Published in November, 1950, as *Proceedings-Separate No. 44.* Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Chf. Engr., Beach Erosion Board, Washington, D. C.

[2] "Wind Waves and Swell Principles in Forecasting," by H. U. Sverdrup and W. H. Munk, *Miscellaneous Publication No. 11275*, U. S. Hydrographic Office, Washington, D. C., 1946.

cited in support of their applicability to reservoir situations lies in the lagoon tests made by J. W. Johnson, M. ASCE, at Abbotts Lagoon (Calif.), on which a complete report has been published.[3]

The Sverdrup-Munk relationships were developed for application to oceanic situations, where clearly defined fetches of hundreds of miles, wind velocities of 60 knots, and wind duration times of many hours represent the magnitudes involved. In the development of the relationships certain theoretical considerations were used to develop equations that involved constants. These constants were then evaluated through the use of previously observed characteristics of ocean waves. Continued observations of waves have been made and they have been used to verify the values of the constants. Under these circumstances the applicability of the Sverdrup-Munk relationships to the radically different reservoir situation appears questionable until positive evidence is available to demonstrate the applicability. Additional tests to confirm the applicability of these relationships to various reservoir conditions are needed and until such confirmation is available the use of the relationships must be tempered with judgment.

The Corps of Engineers, United States Army, has been studying wave action in reservoirs by means of instruments installed in the reservoirs created by Fort Peck (Montana) and Denison (Texas) dams and in Lake Okeechobee (Florida) to obtain reliable data on winds, waves, wind set-up, and wave run-up. This ambitious and extensive program of carefully planned study undoubtedly will add to current knowledge a large body of factual data which can lead to a satisfactory solution (at least in part) of the more pressing problems of wave action on dam slopes. The Lake Okeechobee installation has been in operation since the fall of 1949 and the hurricane sequence has been experienced. Insufficient data have been accumulated to permit drawing conclusions and even tentative confirmation of present practices must await additional evidence.

FURTHER STUDY

Further research and study of reservoir waves are important to the engineering profession. Although the exercise of judgment and the use of the existing formulas, including the Sverdrup-Munk development, represent good engineering practice in the present state of knowledge, additional and extensive study is badly needed on the following subjects:

1. Methods for determining in advance of construction the characteristics of the wave action to be expected in reservoirs created by dams;
2. Details of wave action on dam slopes protected by riprap, paving, or other covers; and
3. Formulation of design requirements and criteria for earth-dam slope protection to withstand the effects of wave action, weather, and reservoir drawdown.

Research on items 1 and 2 can be done concurrently to good advantage; but item 3 can be studied best only after some authoritative results are derived from studying the details of wave action on slope protection.

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[3] "Relationships Between Wind and Waves, Abbotts Lagoon, California," by J. W. Johnson, *Transactions*, Am. Geophysical Union, Vol. 31, 1950, p. 386.

The known factors of concern in the problem, as applied to a proposed reservoir location, will be as follows:

a. The elevations of the pool water surface under operating conditions;
b. The geography of the pool surface at the operating elevations—that is, the configuration, depths, length, and azimuths of the pool;
c. The topography of the area contiguous to the pool and of the area that has an influence on the wind structure over the pool; and
d. The location, orientation, character of material, and geometry of the upstream face of the earth dam creating the pool.

Generally, information on the velocity, direction, and duration of the wind at or near the pool level (factor *c*) will not be available; but reasonably accurate, historical, synoptic weather charts should be obtainable for any locations within the United States.

WAVE ACTION IN RESERVOIRS

The first problem is to determine the characteristics of the wave action to be expected in the proposed reservoir. For this purpose various purely empirical expressions are available, such as the well-known equations proposed by T. Stevenson, the late D. A. Molitor, M. ASCE, and W. P. Creager, Hon. M. ASCE, and the partly empirical expressions developed by Messrs. Sverdrup and Munk. The former expressions purport to define the maxima to be expected; the latter claim to define a "significant height and period" of a fictitious wave which is representative of the average of the highest one third of the waves to be expected. There is little conclusive evidence confirming the accuracy of any of the various formulas when applied to reservoir situations, and there appears to be sufficient doubt regarding their accuracy to justify study to confirm, reject, or modify existing controversial formulas.

In this connection it is believed that the waves of most importance in the design of slope protection are the maximum waves to be expected, assuming that these maxima will occur with sufficient frequency to be considered significant. This belief is based on the necessity for specifying maximum wave run-up as well as other wave effects. It is obvious then that the Sverdrup-Munk relations must be modified to provide information on the maximum waves to be expected rather than information on an average of the one-third highest waves.

During the planning stage the only factors on which knowledge is available for computing expected maximum wave action in the reservoir are: (*a*) The planned operating pool levels; (*b*) the geography of the pool surface at operating levels; (*c*) the topography of the pool boundaries; and (*d*) the daily synoptic weather maps. The problem is to relate these factors in such a way as to arrive at a definition of maximum wave action to be expected.

Experience has shown that the purely empirical approach to a solution can be ignored. A rigorous theoretical approach appears to be practical. Research on ocean waves has proved that, with adequate weather maps, wind waves on water surfaces can be predicted if accurate relationships between the wave characteristics and the fetch, velocity, and duration of the wind are known. These relationships can be determined theoretically from considerations of wave energy, and they have been formulated for certain oceanic situations by Messrs. Sverdrup and Munk.[2] The procedure involves the consideration of a generating area in which the waves are formed by wind action. Expressions are established for the rate of energy transfer from the wind and the rate at which wave energy advances from the generating area. From these expressions it is possible to form a differential equation from which the relationships sought are obtained as special solutions. The solutions contain three numerical constants which must be evaluated. The principal uncertainty attached to the use of the Sverdrup-Munk relationships for reservoir situations arises in evaluating these constants, since the presently assigned values were derived from oceanic situations. Relatively few observations for reservoir situations should permit the re-evaluation of the constants to fit those situations and should establish the Sverdrup-Munk relations, or a modification, as an acceptable method of predicting wave action.

Unfortunately this is not all that must be done. There must be established methods of deriving wind velocity, fetch, and duration, at or near the operating pool level from the daily synoptic weather maps. In this aspect the reservoir situation is much more complicated than the open ocean because of the effect of surrounding topography in modifying the wind structure, and the configuration of the reservoir in controlling fetch. Reservoirs can be considered as bodies of water of short fetch, with the fetch determined by the shore lines and the wind direction. Under these conditions the wave action in the reservoir depends on the velocity and direction of the wind, with wind duration usually not being a limiting factor. The possibility of simplifying the Sverdrup-Munk relationships is apparent in this case. It appears reasonably certain that observed wind data from near-by weather stations have little or no value for use in predicting formulas since local effects control the observed values, and there is no reason to believe that the winds at the observation station are effective in generating waves in a distant reservoir.

Meteorologists must be approached for the solution to this phase of the problem, which solution may be difficult to express in a generally applicable form. This part of the problem is the controlling element in a successful prediction of reservoir waves and it will be the most difficult to handle.

Wave Action on Dam Slopes

The second problem of concern is the action of waves on protected slopes. Little is known of the details of wave action on any but impermeable vertical slopes. Most knowledge of the details of wave action on protected slopes has been derived by incidental observation in nature, with little having been derived from experiments made for the purpose. There are several items of interest and importance—

1. The extent of wave uprush and backwash, for various slopes and cover material;
2. The disruptive and transporting forces associated directly with wave uprush and backwash;

3. The impact forces resulting from wave action for various slopes, slope combinations, and cover material; and
4. Secondary phenomena, such as the alternate compression and expansion of air trapped in cavities or interstices of the protective cover.

To solve all the design problems of protective cover information on these items is necessary, and it would appear to be desirable to plan any study of the two problems (wave action in reservoirs and on protected slopes) concurrently.

Engineers who have had practical experience with wave action on structures can help by supplying information on protection failures. Data on specific causes of failure, methods of failure in detail, character of wave action, and secondary effects contributing to failure are particularly needed.

The only extensive studies of these problems, known to the writer, which have resulted in generalized criteria applicable to many situations, were made by Ramon Iribarren Cavanilles.[4] Perhaps the best statement of the current knowledge of these problems lies in a quotation by Mr. Iribarren of a statement made in 1933 by E. de Castro (translated version):

> "* * * Authors of universal renown hesitate when the moment arrives to determine the slope of a rock-fill dam beaten by the waves, and justify their doubts and hesitation in the great difficulties which the problem holds. * * * The unknown force of the waves, the anomalies of storms and the heavy and grievous replies with which the sea responds to the science (sic) and prudence of the builders show that they are right. * * * At the conclusion of these lines one will have the impression that the problem is very obscure and only the ignorant can believe that it is solved. In spite of all, and until the definite truth is known there need not be lacking provisional truths, or better said, approximations to the truth."

Mr. Iribarren presents, as an example of such an approximation to the truth, a formula which fixes the slope and size of stone necessary to resist the violent beatings of the sea:

$$W = \frac{K\,h^3\,\gamma}{(\cos\alpha - \sin\alpha)^3\,(\gamma - 1)^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which W is the weight of stone, in kilograms; K is a coefficient, with a value of 15 for natural rock fill and a value of 19 for artificially placed stone; h is the total height of the wave, through to crest, in meters; γ is the specific weight of stone, in tons per cubic meter; and α is the slope of the dike, measured as the angle with the horizontal. In the development of the formula, consideration has been given to the details of wave action previously noted, and logical arguments introduced to support the evaluation of the various factors in the formula. An important part of the arguments is a contention that, regardless of the slope of the structure, the waves strike at approximate normality to the slope.

Proceeding further, Mr. Iribarren states that the height of the structure should be greater than the still water level height by an amount equal to the wave height, or 1.25 times the wave height. These heights are stated to be sufficient to prevent overtopping.

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[4] "A Formula for the Calculation of Rock-Fill Dikes," by Ramon Iribarren Cavanilles, *Bulletin*, Beach Erosion Board, Washington, D. C., January, 1949.

Mr. Iribarren's work has been accorded some importance in this paper because of the great interest it has occasioned among harbor and shore control engineers concerned with the design of protective structures, and because of its obvious import to designers of protective works. In the opinion of the writer, it represents the most logical solution available for designing slope protection cover.

SUMMARY

It is apparent that much remains to be done. Equally apparent is the fact that progress toward a closer approximation to the truth must depend on scientific studies of the intimate details of wave action on slopes rather than on empirical correlations of observational data. Much of the required research will not have an immediately discernible practical value. The studies will be difficult and expensive. Nevertheless, study and research are the only acceptable methods of obtaining the required knowledge. Perhaps nowhere in the engineering field can one find a less logical, less technical approach to an engineering problem, or larger expenditures made on the basis of well-intentioned but faulty estimates of a situation, than in the field of slope protection. Research efforts are long overdue and should be prosecuted with utmost vigor.

ACKNOWLEDGMENT

The substance of this paper formed part of the 1949 progress report of the Committee on Earth Dams of the Soil Mechanics and Foundations Division, through its Subcommittee on Slope Protection.[5]

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[5] *Civil Engineering,* Vol. 19, 1949, p. 98.

DISCUSSION

Robert Y. Hudson,[6] M. ASCE.—The use of the Iribarren formula for the design of rubble-mound structures to insure the stability of these structures during wave attack is suggested in this paper. The writer concurs in this recommendation.

The factor γ in equation $\left(W = \dfrac{K\,h^3\,\gamma}{(\cos\alpha - \sin\alpha)^3\,(\gamma - 1)^3}\right)$ is worthy of further consideration, however. Although γ was defined as specific weight in tons per cubic meter, it would have been better, perhaps, to have stated that γ was mass density. Metric tons per cubic meter reduces to grams per cubic centimeter, which is mass density in the centimeter-gram-second system of units. In this system, specific weight has the dimensions of dynes per cubic centimeter. Thus, in the centimeter-gram-second system, the density of fresh water is approximately one, whereas the specific weight of fresh water is about 980. In the American system of units, the density of fresh water at 60° F is 1.938 slugs per cu ft, and the specific weight is 62.4 lb per cu ft.

The Iribarren formula is not dimensionally homogeneous, and this fact precludes its general verification by experimental tests. If the formula is derived in a manner similar to that originally employed, using the same assumptions and force diagram as Mr. Iribarren used in his derivation, the following more complete forms of the equation can be obtained:

$$W = \frac{K'\,\gamma^3_f\,\gamma_r\,\mu^3\,h^3}{(\mu\cos\alpha - \sin\alpha)^3\,(\gamma_r - \gamma_f)^3} \dots\dots\dots\dots\dots (2a)$$

and

$$W = \frac{K'\,\gamma_w\,S^3_f\,S_r\,\mu^3\,h^3}{(\mu\cos\alpha - \sin\alpha)^3\,(S_r - S_f)^3} \dots\dots\dots\dots\dots (2b)$$

or if $S_f = 1.00$

$$W = \frac{K'\,\gamma_w\,S_r\,\mu^3\,h^3}{(\mu\cos\alpha - \sin\alpha)^3\,(S_r - 1)^3} \dots\dots\dots\dots\dots (2c)$$

In Eqs. 2, W is the weight of individual cap rock; K' is an undetermined dimensionless coefficient; γ_w is the specific weight of fresh water; γ_f is the specific weight ($\gamma_w\,S_f$) of the fluid in which the cap rock is submerged; γ_r is the specific weight ($\gamma_w\,S_r$) of the cap rock; S_f is the specific gravity (γ_f/γ_w) of the fluid in which the cap rock, is submerged; S_r is the specific gravity (γ_r/γ_w) of the cap rock; μ is the effective coefficient of friction, rock on rock; h is the height of wave at the position of the structure before the structure was constructed (selected design wave); and α is the angle of the seaside slope of the rubble mound, measured from the horizontal.

Eqs. 2 are dimensionally homogeneous and are valid in any consistent system of units. By comparing Eq. 1 with Eqs. 2 or by dimensional reasoning, it can be shown that $K = \gamma_w\,K'$. Therefore, the values of K' corresponding to $K = 15$ and $K = 19$ are $K' = 0.015$ and $K' = 0.019$, respectively. These

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[6] Chf., Wave Action Section, Waterways Experiment Station, Corps of Engrs., U. S. Army, Vicksburg, Miss.

K-values were determined by Mr. Iribarren from observations of a limited number of full-scale structures after wave attack. When the difficulty of obtaining accurate measurements of the seaside slope of rubble mounds after construction is taken into consideration, as well as the difficulty of determining the height of the largest wave that did not displace cap rock from the slope of the structure, it can be seen that the values of K as recommended by Mr. Iribarren could be in serious error. Data obtained from small-scale experiments at the Waterways Experiment Station, Vicksburg, Miss., show that K' is a function of the ratio of water depth at the structure to wave length (d/λ) and the slope of the rubble mound (α). Also, K' is believed to be a function of the ratio of wave height to wave length (h/λ), the percentage of voids in the rubble mound, the shape factor of the rock, and the width of the structure near the still-water level. The importance of these variables, as they affect the design of rubble-mound breakwaters is being studied at the Waterways Experiment Station at the present time. The results of tests completed to date indicate that the breakwater slope (α) is the most important variable contained in K'. It has been found that the average value of K' is about 0.005 for a slope of 1 on 1 and about 0.03 for a slope of 1 on 3. Also, K' tends to increase with decreasing values of the ratio d/λ. Mr. Iribarren assumed a value of unity for the coefficient of friction ($\mu = 1$). If this were true, a rubble mound with a slope of 1 on 1 would fail when attacked by a wave of infinitesimal height. Actually, rubble mounds of 1 on 1 slope will withstand the attack of waves of finite height. The effective coefficient of friction of the rubble used in the small-scale tests at the Waterways Experiment station varied from about 1.01 to 1.10.

Until more accurate and more detailed data are available, it is recommended that the more complete and dimensionally homogeneous form of Iribarren's formula (Eqs. 2) be adopted for design of rubble-mound breakwaters and slope-protection cover, and that values of 0.02 for K' and 1.05 for μ be used for seaside slopes ranging in slope from 1 on 1-½ to 1 on 3.

FRED C. WALKER,[7] M. ASCE.—The engineer faced with the problem of designing protective surfaces for earth dams is equally as concerned about the deterioration caused by wave action as he is about the forceful displacement of the protective cover by maximum waves. The continous action of wetting and drying and freezing and thawing, the continuous tapping produced by wave action, and the grinding produced by sand in the wave-borne water will, if given sufficient time, tear down the strongest protective cover. This is the day to day action of the waves in which magnitude of the waves is relatively unimportant as compared to durability of the protective cover material. It is not difficult to visualize a protective cover designed and constructed to resist maximum wave action according to the most scientific studies, only to find that sufficient deterioration to render the cover completely ineffective has occurred long before the maximum wave action is experienced. This action will, of course, be very different for smooth-surfaced covers than it will be for riprap, just as the effect of maximum waves is different.

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[7] Head, Earth Dams Section, Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo.

In the field in which rock riprap is considered, a very real need exists for a method of effectively evaluating the relative durability of different kinds of rock. When rock riprap and other types of protective cover are potentially competitive, a method by which such surfaces may be precisely compared is essential. At present (1951) the means of solving problems of this nature is very inadequate, and the experience and judgment of the designing engineer must be relied upon far more than is desirable.

Since 1945 engineers of the Bureau of Reclamation (USBR) of the United States Department of the Interior have been faced increasingly with the necessity of choosing between a somewhat questionable rock found close at hand or a better, more costly rock from a distant source. In some instances the thickness of the protective layer has been increased, in some replacement has been anticipated, and in others top quality material has been used at considerable expense. There have been instances in which larger-sized rock than would be required by any known formula has been specified, not because the adequacy of the formula was questioned but rather because breakdown of the rock by weathering was anticipated. Such factors must be considered both in the creation of a design and in the development of experience data.

As the cost of providing adequate rock-riprap slope protection rises, there is always increased pressure for the adoption of some alternative method of protection in spite of the comparatively poor showing of such methods in the past. Anticipating a vital need for such knowledge, the USBR in March, 1951, let a contract for the construction of a test section in which several design variations, using either asphalt or portland cement, will be subjected to the deteriorating processes of nature. The test section is located on the south side of Bonny Reservoir in eastern Colorado where conditions of greater than average severity are believed to prevail. Local soils or aggregates, comparable to those generally found in the Missouri River Basin, will be used. Although several years are expected to pass before an appraisal can be made of the adequacy of the various methods being tested, studies are in process to determine the nature of the design modifications that will have to be made to the supporting embankment itself. The use of an impervious surfacing and the smooth surface provided by these materials are expected to require extensive modification to present design practices that apply where rock riprap is the slope protection material.

A second element in the design of rock-riprap slope protection that has not received sufficient emphasis concerns both the shape of the stones and size variation in the mass. In the past highly rounded stones have proved to be unsatisfactory on some minor irrigation structures, and their use is now commonly avoided. Flat, slabby rocks are also avoided because of the unsatisfactory surface developed by practical construction methods and the ease with which breakdown is accomplished. Generally, experienced engineers believe that a well-graded rock mass provides better protection against waves than that obtained when a single size of rock is used. Moreover, it will normally be much more economical to provide a graded rock mass than to provide one consisting of a single size. Neither the Iribarren formula nor the Ishbash formula makes any provision for shape or gradation. The utility of these formulas would be improved if the effect of these factors were evaluated.

Like many of the factors that must be considered in earth dam design, the forces that attack the protective covering are of two different types, both of which have a time relationship. These factors are: (1) the attacking forces whose magnitude increases as the frequency with which they are experienced decreases and (2) the resistive forces that deteriorate at an accelerated rate with the passage of time. The author has described the principal attacking forces and concentrated upon the promotion of a theoretical approach for defence against them. However, the element of judgment based on experience cannot be eliminated. It is equally as dangerous to use the result of a theoretical analysis without reference to experience of the limitations of the theory as it is to use a purely empirical approach.

Within the decade 1940 to 1950 a very pronounced change was introduced in the requirements of many earth dams. The multipurpose reservoir is very much a fact, and carry-over is also an important element in the design of dam structures. The reservoirs of the future will be subjected to a very different relationship between attacking and resistive forces than has been true in the past. Large portions of the embankment surfaces will be either above the water surface or below it for long periods of time. Therefore the laws of probability must be used to establish the chance of finding the reservoir at any given level at the same time as anticipated wave phenomena occur, with the likelihood that sufficient deterioration will have been effected to make the protective cover useless. There is a very real need for the establishment of a guiding principle outlining the degree of risk that may properly be assumed in the design of protective surfacing under such circumstances.

Surprisingly, reservoirs in which protection from wave action is least important are often located where ample supplies of good riprap material are readily available. With modern excavating equipment such material can be placed quite economically, and as a result ample protection is provided in the designs for such structures without much study. In areas in which wave action is severe, the chance that parts of the dam will be threatened is so remote that somewhat questionable devices for wave protection are being used. For example, several dams have been provided with a flattened slope, unprotected by any riprap, on the lower portion of the dam. The same device has been used in cases where low dikes, either in the upper flood-control storage or freeboard area, are required. In some cases, the upper dam sections that are infrequently used for flood control have been protected with inferior rock. In others, size is reduced, and the thickness of the protecting layer is also reduced.

Inland reservoirs may be grouped conveniently into three classes: (*a*) The small sheltered reservoirs in which the thinnest practical cover would be adequate against any anticipated wave; (*b*) the normal storage reservoir in which waves between 4 ft and 6 ft high may be expected occasionally; and (*c*) the very large reservoirs (located principally along the major tributaries of the Mississippi-Missouri River system or the Texas coastal plains) upon which major wave action will be experienced. The first group of structures obviously will not be greatly benefited by the results of extensive wave study. The second group in which deterioration is a major problem can be treated adequately by existing empirical methods, without appreciable benefits evident from refinements in maximum wave analysis. The third group will benefit very materially from any knowledge developed either in the field of maximum wave action or in the field of durability of protective materials.

Henry H. Jewell,[8] M. ASCE.—A resumé of current knowledge and research on wave action on earth slopes is given in this paper. It also recommends the Iribarren formula for the design of stone protection for earth dams.

Although it may be possible to solve problems of wave characteristics by a rigorous theoretical approach as the author states, the design of protection against wave action on earth dams ought to be accomplished only by the customary combination of theory and practice. This is particularly true of protection by rock riprap.

Some further examination and explanation of the applicability of the Iribarren formula appears necessary. The translation referred to by the author in the section on wave action on dam slopes[4] shows that the formula was derived for use in the design of rock-fill dikes. Under the heading "Wave Action on Dam Slopes," the author states that "* * * it represents the most logical solution available for designing slope protection cover." If the quoted statement refers to the top layer of rock in an abnormally thick cover of riprap over a filter layer, the formula may be of some value. Alone, it is not likely to offer a practical and economical solution for the design of riprap on earth dams.

Using this formula W is found to be 3,680 lb, assuming the height of wave is equal to 3.5 m (about the maximum for design on the largest reservoirs), the specific weight of stone is 160 lb per cu ft, the slope of the dam is equal to 1 on 2.5, and the value of K is 15. As it would seem that such large stones for a cover layer would be "artificially placed stone," the coefficient K would be 19, and W would then be 4,650 lb. Actual experience with rock riprap has shown that these stones would be heavier than necessary under the conditions stated. Does the author mean that stones of such magnitude should be used for rock riprap on earth dams under the assumption that such protection is the same as rock fill in the formula for design of rock-fill dikes?

Although large stones in the exposed face of the cover are essential, it must be recognized that, if riprap is to be used, the size and weight of stones for protection of earth dams must be within the practical limits of capacity of suitable equipment for quarrying, transporting, and placing the stones. These operations must be done at reasonable cost, under conditions quite different from those governing dike construction in harbors. Although it would be possible to construct riprap of stones weighing 3,000 lb to 5,000 lb, the cost including an adequate filter layer would be prohibitive. Using such large stones for the cover layer would surely require additional quantities of smaller stones both in the top layer and in the underlying filter layer in order to prevent damage caused by washing out the filter layer through the openings between the large cover stones.

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[8] Chf., Construction Branch, Division of Community Facilities and Special Operations, Housing and Home Finance Agency, Washington, D. C.

Of equal or of greater importance than the weight of stones for earth dam protection is the design of the filter layer that must be placed under the rock cover. The unsuitability of large blocks without a filter layer was shown in the failure of this type of protection on the Belle Fourche Dam many years ago. The importance of a filter layer was emphasized in experience with Kingsley (Nebr.) and Santee-Cooper Dams (S. C.) described in ASCE reports.[9,10,11]

C. L. BRETSCHNEIDER,[12] JUN. ASCE, AND R. R. PUTZ.[13]—One of the foremost problems concerning wave action on earth slopes is the prediction of the heights of waves in relatively small bodies of water, such as small lakes, bays, and reservoirs. Although a method for forecasting ocean waves has been developed,[2] little has been done to extend the theory and method to inland lakes and reservoirs. In the Sverdrup-Munk original work on wave forecasting, dimensionless quantities $g\,H/U^2$ and C/U versus $g\,F/U^2$ were determined for a number of observations at sea, and relationships were also found to exist between $g\,H/U^2$ versus $g\,F/U^2$ and C/U versus $g\,F/U^2$. In the above, g is the acceleration due to gravity; U is the wind velocity; F is the fetch length, defined as the distance over which the wind is blowing; C is the wave velocity given by $C = g\,T/2\pi$ for deep water (in which T is the wave period in seconds); and H is the significant wave height, that for statistical purposes is defined as the average height of the highest one-third of all waves present at a given place and time.

It has been a long-standing question whether the same relationships hold for small bodies of water. Observations made at Clear Lake,[14] Calif., and at Abbotts Lagoon,[3] Calif., presented much valuable data that were used to extend the dimensionless relationships for relatively small bodies of water. Further studies of wind waves[15] have been made in the wave channel at the University of California, Berkeley, Calif. Additional data have become available from other sources, and it is now possible to revise the Sverdrup-Munk curve, that relates the above dimensionless variables, and to extend the relationship to include all values of $g\,F/U^2$ from 0.1 to 200,000.[16]

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[9] "Protecting Upstream Slope of Kingsley Dam," by Henry H. Jewell, *Civil Engineering*, Vol. 15, November, 1945, p. 493.

[10] "Rock Riprap Replaces Porous Concrete Slope Protection at Santee-Cooper Project," by Henry H. Jewell, *Civil Engineering*, Vol. 18, January, 1948, p. 14.

[11] "Review of Slope Protection Methods," Report of the Subcommittee on Slope Protection of the Committee on Earth Dams of the Soil Mechanics and Foundations Division, *Proceedings*, ASCE, Vol. 74, 1948, p. 845.

[12] Research Engr., Inst. of Engineering Research, Univ. of California, Berkeley, Calif.

[13] Mathematician, Inst. of Engineering Research, Univ. of California, Berkeley, Calif.

[14] "The Characteristics of Wind Waves on Lakes and Protected Bays," by J. W. Johnson, *Transactions*, Am. Geophysical Union, Vol. 29, 1948, pp. 671–681.

[15] "An Experimental Investigation of Wind-Generated Waves," by J. W. Johnson and E. K. Rice, Series 3, *Issue 321*, Inst. of Engineering Research, Univ. of California, Berkeley, Calif. (unpublished report).

[16] "Revised Wave-Forecasting Curves and Procedures," by C. L. Bretschneider, *Report No. 155-47*, Inst. of Engineering Research, Univ. of California, Berkeley, Calif., 1951 (unpublished report).

With unlimited wind duration and for values of $g\,F/U^2$ below 5,000, the relationships for $g\,H/U^2$ versus $g\,F/U^2$ and C/U versus $g\,F/U^2$ on log-log scale paper can be represented conveniently by straight lines, the equations of which are

$$g\,H/U^2 = 0.0035\,(g\,F/U^2)^{0.423} \qquad (3)$$

and

$$C/U = 0.064\,(g\,F/U^2)^{0.313} \qquad (4)$$

For $g = 32.16$ ft per sec per sec, $F =$ statute miles, $U =$ miles per hour, $C = g\,T/2\,\pi$, $T =$ seconds, and $H =$ feet, the above equations reduce to

$$H = 0.0313\,F^{0.423}\,U^{1.154} \qquad (5)$$

and

$$T = 0.624\,F^{0.313}\,U^{0.374} \qquad (6)$$

Another variable, not yet mentioned, that determines the wave height and period is the duration of the wind. For relatively long fetches the duration of the wind is the limiting factor, whereas for relatively short fetches, the wind usually is of such a duration that the fetch is the limiting condition. For any fetch length and for a certain wind velocity the wave height and period increase with time until a certain duration is reached; then no matter how long the wind continues to blow, the wave height and period will remain essentially constant; that is, the waves have reached a stable condition. This duration of the wind is called "minimum duration." For waves to reach stable conditions the relationship between wind velocity (U), duration of wind (t_d), and fetch length (F) has the form of

$$t_d\,U/F = 30\,(U^2/g\,F)^{0.222} \qquad (7)$$

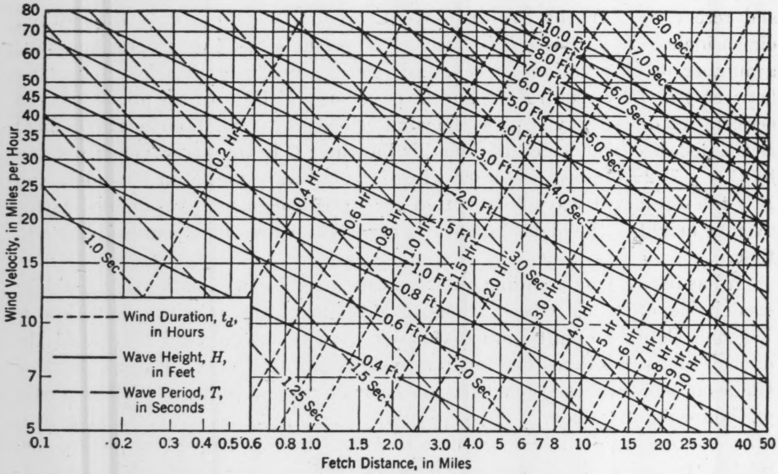

FIG. 1.—WAVE HEIGHT AND PERIOD AS A FUNCTION OF WIND VELOCITY AND FETCH FOR SHORT FETCHES

that for t_d = hours, F = statute miles, and U = miles per hour reduces to

$$t_d = 2.45\, F^{0.778}/U^{0.556} \qquad (8)$$

The solutions of Eqs. 4, 6, and 8 are given in Fig. 1. Knowing the wind velocity in miles per hour, the fetch length in miles, and the duration of the wind, one enters the right of Fig. 1, proceeds until either the duration or the fetch length is reached first, and reads off the appropriate wave height and period.

The author has pointed out the necessity of relating the maximum wave height to the significant wave height. Some work on this has been done[17,18] in the analysis of data from ocean wave recorders. The variability in the heights of consecutive waves from a storm may be examined from several points of view. For example, the heights of waves produced and propagated under completely specified conditions vary with time, if observed at a fixed point, and vary in space, if observed at a fixed time. Moreover, a wave observed at a given time and place, arising from incompletely specified conditions (for instance, a storm for which the wind velocity is known only as an average value or whose characteristics are derived from synoptic weather maps), will vary with the exact nature of the generating and propagating conditions. Thus a weather-based wave forecast may be expected to give only a statistical average of individual wave heights. The wave heights themselves are describable only by probabilities, the particular heights depending on the exact conditions. In this way the individual wave heights may be thought of as having statistical probability distributions.

Studies have indicated a rather high degree of regularity in the shape and the relative dispersion of the statistical frequency distribution of wave heights observed in the ocean, in lakes, and in the laboratory channel. These regularities seem to hold with only minor changes in the constants occurring in the relations describing them, whether the waves are in the area of generation or in the area of decay, in deep water or in the zone of breaking. The regularities also appear to be of value for predicting other statistical features of wave-height patterns when the significant wave height (H) is known.

A theoretical model,[19] found to provide a satisfactory fit to data that consist of the heights of successive waves occurring over a given length of time at a given point, involves two parameters. These are, first, the relative dispersion, measured by the ratio (σ/μ) of the standard deviation to the arithmetic mean of the set of successive wave heights and, second, the degree of asymmetry, measured by the skewness coefficient (α_3) for the frequency distribution of the consecutive wave heights. Observed values of the ratio σ/μ and of the coefficient α_3 show relatively little fluctuation, averaging between + 0.3 and + 0.7 and between + 0.0 and + 0.8, respectively, for various groups of data. Typical

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[17] "An Analysis of Data from Wave Recorders on the Pacific Coast of the United States," by R. L. Wiegel, *Transactions*, Am. Geophysical Union, Vol. 30, October, 1949, pp. 700–704.

[18] "Results on Research on Surface Waves of the Western North Atlantic," by H. R. Seiwell, *Papers*, Physical Oceanography and Meteorology, Massachusetts Inst. of Technology, Cambridge, Mass., and Woods Hole Oceanographic Inst., Woods Hole, Mass., Vol. 10, 1948, p. 56.

[19] "Wave Height Variability; Prediction of the Distribution Function," by R. R. Putz, Series 3, *Issue 318*, Inst. of Engineering Research, Univ. of California, Berkeley, Calif., December, 1950 (unpublished report).

values of these parameters for wind waves on Abbotts Lagoon[3] were found to be $\sigma/\mu = 0.585$ and $\alpha_3 = 0.6$. Based on these values, the family of curves in Fig. 2 shows the proportion of waves that may be expected to be above any given height when the significant height (H) is known, the wave system being assumed to be in a steady state. The letter symbols used in Fig. 2 can be defined as follows: t_d = the duration of the storm, in hours; $t = t_d$ minus the build-up time; T = the period of the wave, in seconds; h = the height in feet above which a given percentage of wave heights occur or the height of the long-run-average maximum wave. It is seen from the curves that 13.0% of the wave heights are above the significant wave height. For example, for a significant wave height of 3 ft, the corresponding curve yields a value of 0.01% for the long-run expected proportion of waves higher than 7 ft in height. In this case, if waves above 7 ft in height are of interest, such a statement implies the necessity, on the average, of a long wait for waves of such height. During an extended period of time, weather and wave conditions may undergo marked changes, creating a situation significantly different from that upon which the earlier probability of occurrence was based.

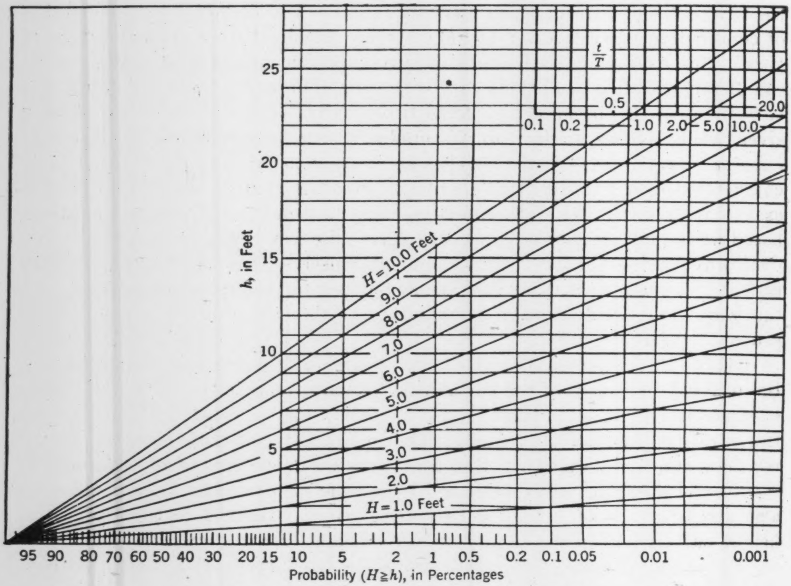

FIG. 2.—WAVE HEIGHT DISTRIBUTION AND LONG-RUN AVERAGE MAXIMUM WAVE HEIGHT

If the wave-height system may be assumed to be in a steady state (the distribution of wave heights is independent of the time), Fig. 2 will provide an indication of the maximum wave height to be expected for a system of waves lasting a known length of time (t, measured in hours), and of known average period (T, measured in seconds), when the significant wave height is known. For example, let $H = 4.0$ ft, $t = 1.0$ hr, and $T = 4.0$ sec; then, referring to the proper H-curve, it is found that the value of h corresponding to the value $t/T = 0.25$ is 8.4 ft. Under these conditions, then, our model gives 8.4 ft as the average value of the maximum wave height. Since the observed maximum value is not determined by the conditions given in such an example, it should be expected to fluctuate about its average value from one storm of this type to the next, being higher, in fact, than the average of its values approximately 45% of the time.

Although much observational work on long sequences of consecutive waves remains to be done, the results obtained may be of value in ascertaining the effects of certain weather conditions. It would be desirable to know much more about the temporal homogeneity of wave processes produced by various wind conditions as well as about the degree and nature of the dependence between successive observed waves.

Further application of Figs. 1 and 2 may be illustrated by using the Santee-Cooper Dam[10] as an example. In this case the fetch is limited to about 15 miles. For a typical condition, assume a wind velocity of 30 miles per hr lasting for 12 hr. From Fig. 1, the significant wave height $H = 5.0$ ft, the average wave period $T = 5.3$ sec, and the duration of wind necessary to reach steady-state $t_d = 3$ hr. At the point of observation for the additional 9 hr ($12 - 3 = 9$), from Fig. 2 for $t/T = 9/5.3 = 1.70$, it is determined that the average value of H_{max}, the maximum wave height to be expected over the 9-hr. steady-state period, is 12 ft. Of the individual wave heights, 13% are at least 5.0 ft, and 87% are less than 5.0 ft.

When waves approach the face of the dam, they will tend to peak-up and break at a height slightly greater than that given by Fig. 1. For periods, T, between 4 and 6 sec, the waves having the significant wave height in deep water will break[20] with a height increased by 10 to 15%, and they will break when the depth of water is 1.3 to 1.5 times the wave height. In the preceding example, the 5.0-ft, 5.3-sec deep-water wave would peak-up to 5.4 ft and break when the depth at normal water level is 6.9 ft. Also, the average maximum breaker height to be expected in the long run would be 13.0 ft (from Fig. 2). In this particular example, since the freeboard is 18 ft, it would be expected that in the long run the average storm (having a wind velocity of 30 miles per hr duration of wind of 12 hr, and fetch length of 15 statute miles) would not generate a wave that would go over the top of the dam. However, for stronger winds and longer durations, greater significant wave heights would be developed, and higher maximum wave heights would be expected.

When waves approach perpendicular to the face of the dam, they will tend to spill or splash the water against the face of the dam, and the returning water under the influence of gravity would cause rip currents. Waves approaching at an angle to the face of the dam would produce a littoral current parallel to the face of the dam and also a return rip current. A theoretical relationship for the littoral current velocity as a function of wave height, period, beach slope, and the angle of approach of the wave can be applied to field and laboratory observations.[21] The investigation that established this relationship covered beach slopes between 0.016 and 0.26. The slope of the Santee North Dam (North Carolina) is about 0.36, and the waves may behave somewhat differently. The waves would tend to splash or slap against the face of the dam before or at the point of breaking, and their damaging effect would be due both to the momentum of the water and the return current. The impact and the current both depend on the maximum wave height, period, and the slope of the face of the dam. The effect of the riprapping in place of porous concrete for the upstream face would be to diminish the above forces.

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[20] "Breakers and Surf, Principles in Forecasting," Hydrographic Office, U. S. Navy Dept., *Publication No. 234*, Washington, D. C., November, 1949.

It should be noted that the equations for the wind-generated waves can be used for larger fetches than are represented in Fig. 1, provided they do not exceed $g\,F/U^2 = 5{,}000$. Since $g\,F/U^2$ decreases as the wind velocity squared increases, the limit of the equation is greater for higher wind velocity with increasing fetches. Of particular significance, Eqs. 4, 6, and 8 are dimensionally correct as seen from Eqs. 3, 5, and 7. Fig. 2, with the aid of Fig. 1, can be used to advantage for any particular storm to indicate the maximum wave height to be expected. Although more information on the behavior of waves for particular cases is necessary, the above discussion, with the aid of the references given, completes certain aspects of the study relating to wind-generated waves.

MARTIN A. MASON,[22] Assoc. M. ASCE.—The purpose of the paper was to present a resumé of current knowledge and research (as known to the writer) on wave action on earth slopes. It was hoped that, by discussion, the profession would supply deficiencies attributable to the writer's shortcomings and contribute additional knowledge or hypotheses pertinent to the problem. In this respect the results obtained are perhaps somewhat disappointing. The four discussions contributed are valuable beyond question. They offer critical comment and contribute useful knowledge on this poorly known subject. However, if the lack of a wider response on the controversial subjects of the paper and on the calculated failure to mention the importance of filter layers may be interpreted as an indication of complacency and satisfaction of the profession with present conditions, then the situation is one which the engineering profession cannot view with pride.

The writer can offer little valuable comment on the excellent discussions so kindly contributed, except to concur in the comments, and to clarify a statement apparently misunderstood by Mr. Jewell. The Iribarren formula is not recommended by the writer for the design of slope protection but is recommended as the most logical available design formula for determining the size of the cover stone of slope protection.

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[21] "The Prediction of Longshore Currents," by J. A. Putnam, W. H. Munk, and M. A. Traylor, *Transactions*, Am. Geophysical Union, Vol. 30, 1949, pp. 337-345.

[22] Chf. Engr., Beach Erosion Board, Washington, D. C.

This opportunity is taken also to call attention to the May, 1950, publication of a generalization of the Iribarren formula.[23]

The additional data on prediction of waves should find much application to reservoir situations, although the problem of determining applicable fetch and wind velocity is vexatious still.

Continued study and research on slope protection are indicated certainly; existing efforts in this field should be encouraged and expanded with vigor.

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[23] "Generalization of the Formula for Calculation of Rock Fill Dikes and Verification of its Coefficients." by R. Iribarren Cavanilles and Casto Nogales y Olano (translation), *Bulletin*, Beach Erosion Board, Vol. 5, 1951, p. 4.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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Paper No. 2473

COMPREHENSIVE PLAN FOR THE COLUMBIA BASIN

BY WILLIAM WHIPPLE, JR.,[1] M. ASCE

WITH DISCUSSION BY MR. L. E. RYDELL.

SYNOPSIS

A comprehensive plan for the rapidly growing Columbia Basin (in Oregon and Washington), to supplement Grand Coulee, Bonneville, McNary, and other developments, was completed shortly after the disastrous flood of 1948. The planning was done mainly by the Corps of Engineers (United States Army) and the Bureau of Reclamation (United States Department of the Interior), with the participation of other federal and state agencies through a variety of special arrangements, and with general coordination established through the Columbia Basin Interagency Committee. Congress has authorized the construction of most of the projects recommended. The paper describes the planning process (including formulation of objectives of water resource development), site investigation and selection, over-all basin studies, and reconciliation of conflicting aims in the choice of projects.

The resulting plan is impressive in its magnitude, providing for more than 10,000,000 kw of firm power, 25,000,000 acre-ft of storage usable for flood control, 800 miles of navigable inland waterways, and, ultimately, irrigation of millions of additional acres of land. The over-all mechanical efficiency of the plan is so much better than that of the individual power-producing projects considered singly that a 25% increase in the 2-mill rates established for Bonneville and Grand Coulee dams will pay for the entire power plan, in spite of a doubling of unit construction costs since those projects were built. Requirements of an operation plan are developed, including an adaptation of a forecast basis for obtaining joint use of storage for power and flood control and criteria for unified control of certain elements of the plan, as against advantages of decentralization both functionally and geographically.

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*NOTE.—Published in November, 1950, as *Proceedings-Separate No. 45*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Col., Corps of Engrs., Washington, D. C.; formerly Dist. Engr., Walla Walla, Wash.

Relationships between the main objectives of water resource development and other related factors, such as conservation of soil and forest cover and measures for the preservation of Columbia River salmon, are discussed.

I. Comprehensive Water Resource Development Needed

The life and economy of the people in the Northwest are much more dependent on sound water resource planning than in most areas of the United States, and these needs are typical of a situation requiring comprehensive planning. There are now (1950) 4,000,000 acres of irrigated land in the Columbia Basin, with a potentiality for much greater expansion. Damaging floods occur not only locally but also generally, as great catastrophes, like the flood of June, 1948, which cost more than sixty lives and an estimated $102,000.000 in property damage (see Fig. 1). The growth in sales of power,

Fig. 1.—Vanport, North Portland, Ore., and Vancouver, Wash., Along the Columbia River During the Flood of 1948

largely from hydroelectric generation, has been phenomenal; output from Bonneville and Grand Coulee dams, sold at the low 2-mill demand rate, returns more than $20,000,000 annually in cash. This return is sufficient to pay all power costs, including amortization, and return a net annual increment to the United States Treasury greater than $2,000,000 for Bonneville Dam power alone.

On the lower Columbia River commercial navigation is great, and above Bonneville Dam it amounts to more than a million tons annually, despite strong currents and rapids which require towboat power ratios up to ten times those adequate for slack water navigation. Scenic and recreational values of the waterways in this area are outstanding; and the preservation of spawning grounds for the salmon industry is most important. Since these factors are all interrelated, the need for comphehensive planning is apparent.

II. PREPARATION OF THE COLUMBIA RIVER PLAN

Agency Cooperation.—The new comprehensive plan for the Columbia Basin was initiated in 1943. A congressional directive to the Corps of Engineers (USED) required a general review of the previous comprehensive plan published in 1932. (This was a "308" Report, so called after *House Document No. 308,* which originally listed those first authorized.) Other agencies were soon drawn into a general planning effort, which lasted five years. The Bureau of Reclamation (USBR) was especially active, and in 1946 published in tentative form a general plan, popularly known as the "Blue Book," which was of great assistance. This report contained an authoritative survey of irrigation possibilities and allied developments throughout the basin, and a more general treatment of other types of projects.

The planning effort of the USED included an exceptionably widespread and thorough study of major power, flood control, and navigation possibilities, prepared with elaborate procedures for coordination with local interests and other federal agencies, ending in action through the Columbia Basin Interagency Committee (formed from the main federal agencies concerned with water resource development and the governors of the states in the basin). After a review of the USBR findings as to irrigation, the USED accepted them as valid; and after various supplementary studies and the publication of the USED report, the USBR accepted the flood control, navigation, and power projects recommended by the USED.

Thus the final plan is embodied in two reports which supplement one another, each report advocating the same final list of recommended projects. Some of the major projects, particularly the one known as Hell's Canyon, were studied by both agencies, but physical data were pooled. After complete discussion, the comprehensive plan has won general acceptance from all the states and federal agencies concerned.

Reconciliation of Conflicting Views.—The processes of coordination, reconciliation, and, ultimately, choice between various conflicting views are essential parts of the planning process. As tentative plans are explored, various reactions develop, often unexpectedly, and the interests concerned must be convinced of the worth of the plan. The plan itself must be readjusted, or an irreconcilable divergency will become a disadvantage of that particular project or feature, to be weighed ultimately against its advantages.

During the course of the planning process the USED held numerous public hearings to obtain the views of local interests. Some of these hearings were very lively. Preliminary plans were presented to the public, advantages and disadvantages were discussed, and modifications were made to arrive at a balanced general plan supported by consensus. Flood control is a form of stream development that has been almost uniformly favored by the public, although interest has varied greatly depending on the number of years since the last flood. Irrigation also has been generally advocated strongly; yet, in some cases, landowners whose lands were being profitably dry-farmed during the current cycle of more abundant rainfall opposed the extension of irrigation. Power projects were opposed in certain areas by private power companies, which maintained that they could meet the local power demand themselves.

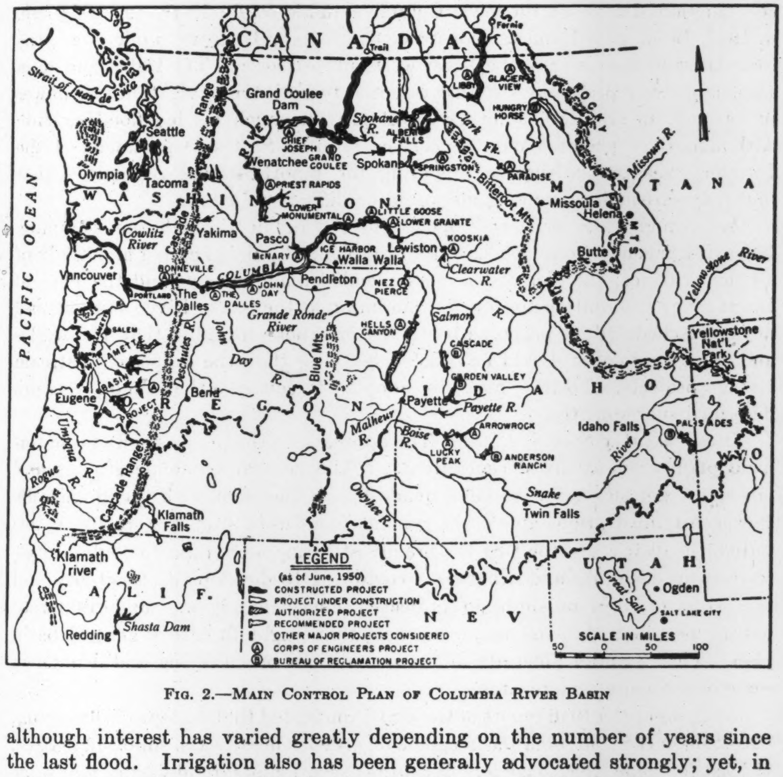
CANADA
Fernie
Trail
Libby
Glacier View
Hungry Horse
Strait of Juan de Fuca
Grand Coulee Dam
Seattle
Chief Joseph
Grand Coulee
Spokane R.
Albeni Falls
Spokane
Springston
Clark Fk.
Rocky Mts.
Missouri R.
Olympia
Tacoma
Wenatchee
Priest Rapids
Paradise
WASHINGTON
MONTANA
Missoula
Helena
PACIFIC OCEAN
Lower Monumental
Little Goose
Lower Granite
Bitterroot Mts.
Cowlitz River
Cascade Range
Yakima
Pasco
Ice Harbor
Kooskia
McNary
Walla Walla
Lewiston
Butte
Vancouver
Bonneville
Columbia
Clearwater R.
Yellowstone River
Portland
The Dalles
John Day
Pendleton
Nez Pierce
Salem
Willamette Basin Project
Grande Ronde River
Hells Canyon
Salmon R.
Yellowstone Nat'l. Park
Deschutes R.
John Day R.
Blue Mts.
Cascade
Garden Valley
Eugene
Bend
OREGON
IDAHO
Payette
Payette R.
Umpqua R.
Malheur R.
Boise R.
Arrowrock
Lucky Peak
Anderson Ranch
Idaho Falls
Palisades
River
WYO
Rogue R.
Owyhee R.
Snake
Twin Falls
Klamath Falls
Klamath river
CALIF.
UTAH
Great Salt
Ogden
Salt Lake City
NEV
Redding
Shasta Dam
LEGEND
(as of June, 1950)
CONSTRUCTED PROJECT
PROJECT UNDER CONSTRUCTION
AUTHORIZED PROJECT
RECOMMENDED PROJECT
OTHER MAJOR PROJECTS CONSIDERED
Ⓐ CORPS OF ENGINEERS PROJECT
Ⓑ BUREAU OF RECLAMATION PROJECT
SCALE IN MILES
50 0 50 100

FIG. 2.—MAIN CONTROL PLAN OF COLUMBIA RIVER BASIN

Several of the major reservoirs (see Fig. 2) presented some grave local problems and at least potential local opposition. The Glacier View project had to be held in abeyance because it would flood park lands used as winter feeding grounds for wild game; the Boundary and Springston projects were indefinitely postponed because of local opposition based on commercially important mineral deposits; and the Kooskia and Paradise projects had to be dropped from consideration because of the small towns, farm lands, and communications involved. However, the John Day project won acceptance, in spite of the planned relocation of virtually an entire town, as did the Hell's Canyon project, even though it drowned out an existing power plant.

Unquestionably the most difficult conflict of interests to be resolved lay in the attempt by fishery interests to defer (or, if possible, prevent entirely) all construction of dams on the lower Columbia and lower Snake rivers. Of these, in 1947, Bonneville Dam was constructed; McNary Dam was under way; the four dams on the lower Snake River were authorized; and The Dalles and John Day dams were proposed. All are essential to both the navigation and power plans, and, in addition, John Day will provide much needed flood control. Although the Fish and Wildlife Service (United States Department of the Interior) was cooperating in preparing the report, it was recognized that mutually satisfactory conclusions would be difficult to attain.

Accordingly, the issue was opened to the public through the Columbia Basin Interagency Committee. After a prolonged public hearing and study of all the evidence, the committee recommended unanimously (and the Federal Interagency Committee approved) a finding that the Snake River dams should not be rescheduled, as proposed by the fishery spokesmen, but that The Dalles and John Day dams should be constructed later than the proposed upstream storage projects. Positive measures adopted to conserve the salmon runs are described subsequently.

Development of Specific Plans.—The approach to the preparation of a basin-wide plan was initially decentralized. While general economic studies and power market surveys were being made by arrangement with the Bonneville Power Administration, and other general basin-wide studies were being initiated, the districts of the USED were investigating more than 1,000 dam sites, restudying old data, and otherwise attempting to develop the feasibility and the scope of local or subbasin projects. Especially, it was undertaken to outline pertinent data on major projects that might fit into a general basin plan. Figs. 3 and 4 illustrate difficult conditions of surveying and drilling in some of the more remote sites.

Analysis of the 1946 report of the USBR indicated that, except for financing, irrigation in the Columbia Basin can be regarded generally as a subbasin matter, at least for some time to come. With the exception of Grand Coulee and some irrigation diversions from other projects already authorized, irrigation is not an essential part of the major dams required for flood control, navigation, and power. Such projects as Libby, Hungry Horse, and Hell's Canyon (see Fig. 2) are located where direct irrigation is impractical and the use of their storage for irrigation downstream is unnecessary.

Moreover, the water supply of the Columbia River is so vast relative to the possible use for irrigation that, within the economic life of projects now con-

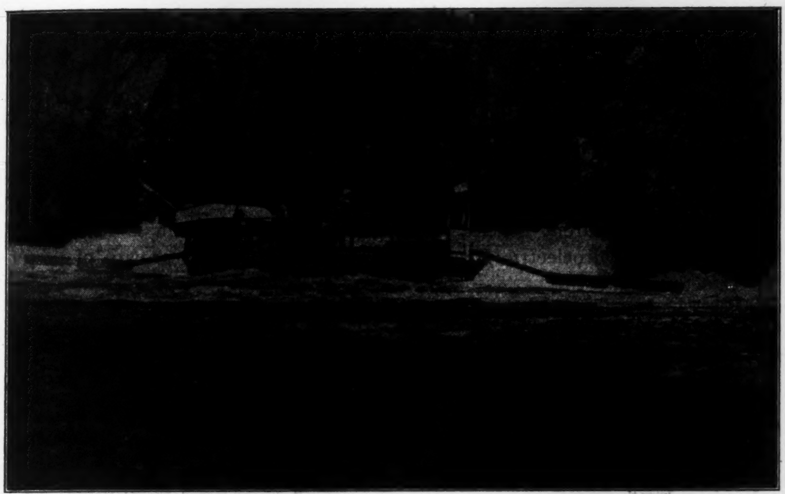

FIG. 3.—SURVEYING IN THE SALMON RIVER, IDAHO; UNITED STATES CORPS OF ENGINEERS

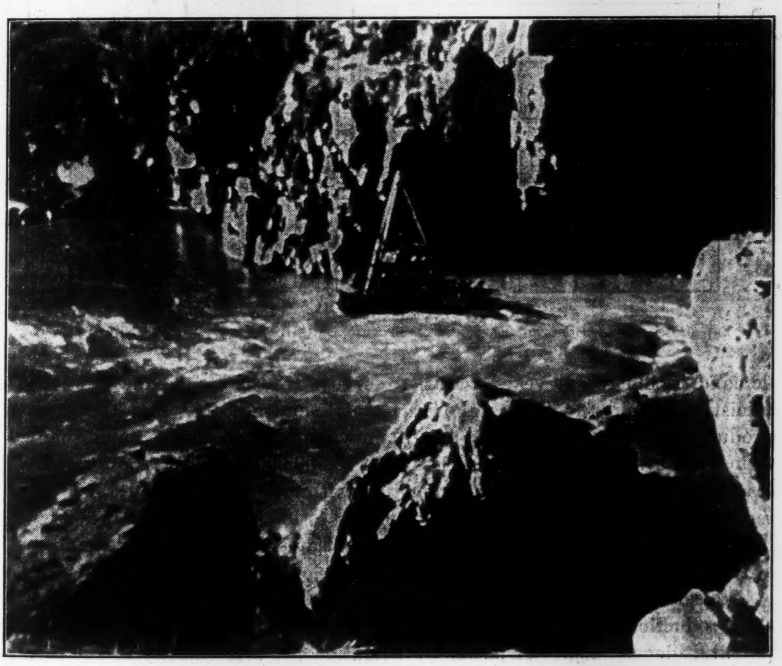

FIG. 4.—DRILL BARGE IN HELL'S CANYON, SNAKE RIVER, IDAHO

sidered, irrigation can be given full priority over all other downstream needs without disadvantage to the recommended federal projects, other than a minor reduction in power output. There remain some difficult questions of water priority within certain subbasins, particularly the upper and middle area of the Snake River Basin; but over-all planning indicates how these conflicts can be laid aside for localized determination entirely in the interests of the subbasins immediately concerned. This conclusion greatly simplified the engineering approach in this basin.

The most apparent general need in the Columbia Basin is for greater production of hydroelectric power. Fig. 5 shows the great growth of power demand within recent years. Market surveys by the Bonneville Power Administration and the Federal Power Commission indicated that the present production from Grand Coulee and Bonneville dams should be supplemented to insure a system capacity of no less than 10,500,000 kw of firm power at 75% load factor from federal projects some time within the decade 1960–1970. The most rapid load growth within this range of estimates is illustrated in Fig. 6. The more conservative alternate estimate indicates the same total demand, but a date of 1970 for its full development.

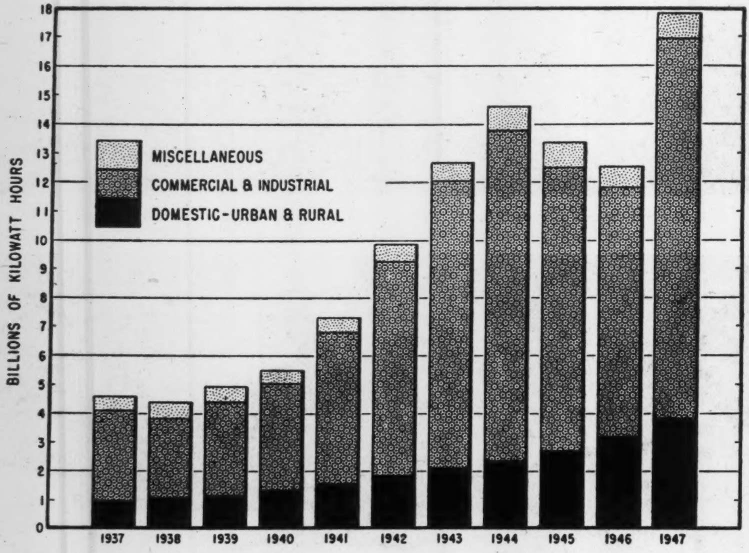

Fig. 5.—Sales of Electric Energy by Electric Utilities in the Pacific Northwest

No satisfactory alternative energy base of coal or oil exists in this area, and the production of hydroelectric power is relatively inexpensive. The total potential developments of power by irrigation projects (mostly in the Snake

River Valley) and by local flood control projects (mostly in the Willamette Basin) will produce less than half a million kilowatts of firm power. Potential expansion of private and municipal developments is also relatively small if low rates are to continue. The main output of 10,000,000 kw must therefore be provided by the central system of major federal projects, of which Bonneville, Grand Coulee, McNary, Hungry Horse, and Chief Joseph (formerly Foster Creek) (see Fig. 2) are the initial units. Fig. 7 shows the extent of dependance upon the relatively few large projects, and the future load variation during a drought year.

After studying many individual projects and groups of projects, the most advantageous system for power production was found to be a series of run-of-the-river dams on the main Columbia and lower Snake rivers, with very large storage reservoirs above them to "firm up" power during winter months and drought periods. In such a system, balance is essential. Consecutive run-of-the-river dams added to the system supplement one another indefinitely in proportion to their head. The new storage dams, however, will supplement the run-of-the-river dams in accordance with a law of diminishing returns. The first 10,000,000 acre-ft of storage to be added will be highly productive, provided the run-of-the-river dams are all built. The second 10,000,000 acre-ft will be about two thirds as productive as the first; but the next 20,000,000 acre-ft thereafter would only add as much as the second 10,000,000—and adding another 50,000,000 acre-ft would produce a still smaller increment.

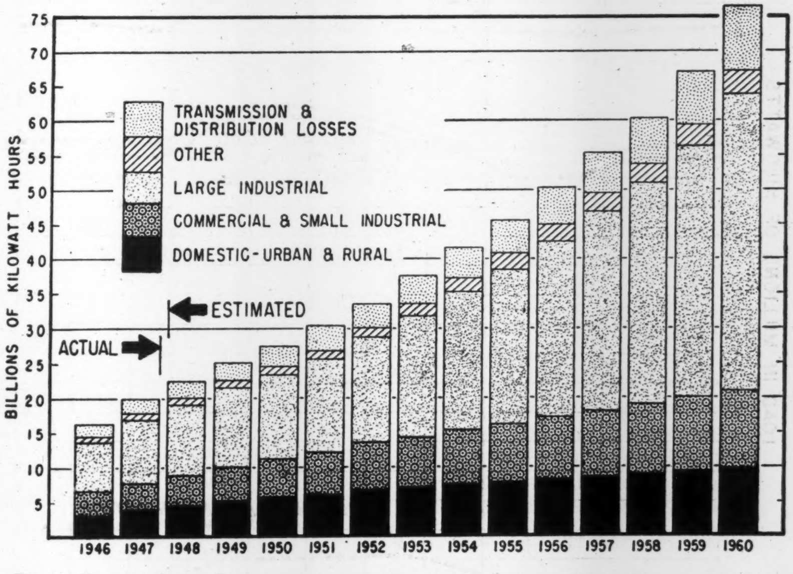

Fig. 6.—Electric Power Requirements in the Pacific Northwest by Major Class of Customer

These values are approximate, but they indicate the rapidly decreasing economy of extremely large storage increments. Ultimately, additional storage projects would provide practically no system benefits except for power produced at those sites. One of the interesting side lights resulting from this interrelationship of storage projects is that it is highly misleading to evaluate the economic feasibility of storage projects in terms of a definite order of construction. In one case, the indicated economic power benefits of a given storage project would be more than doubled if considered as constructed first rather than last in the series of recommended projects, because of the much shorter critical power period over which storage would be utilized in the initial system.

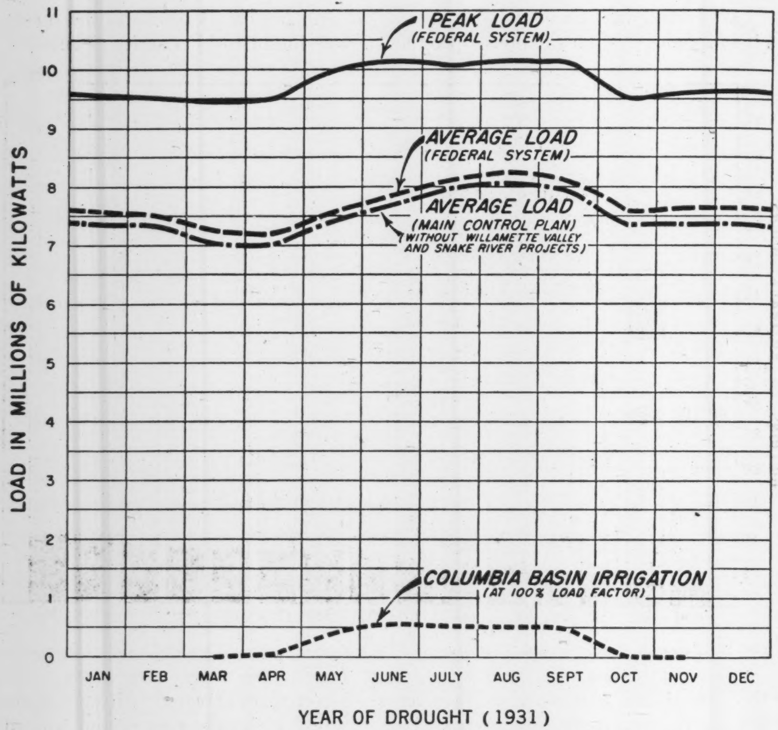

Fig. 7.—Estimated Loads for Major Projects; Phase C

Economic Feasibility.—The entire system of power-producing dams by itself was found to be economically feasible even if no multiple-purpose features were introduced. The relationship of this conclusion to inland waterway navigation was immediately apparent. In brief, the lower river dams required for the power system are also those most effective for navigation. Although a system of dams solely for navigation would be economically unfeasible by a wide margin, navigation benefits are considerably more than the additional annual cost of locks; and the surplus increases the net benefit of the main project. Therefore, navigation features, which had always been desired for the lower Columbia and Snake rivers, were demonstrated to be economically justified in the new system.

The economic feasibility of including flood control was initially considered much more questionable than to include navigation. In view of the immense volume and duration of the Columbia River floods, which have reached a peak of 1,240,000 cu ft per sec, and which last for weeks, general reservoir flood control had not been considered economically feasible, even in conjunction with other uses. A serious Columbia River flood had not occurred since 1894 and there was little public appreciation of a need for general flood control on the main branch of the Columbia River, although a large number of levees had been constructed and held successfully through minor floods.

However, the new power-producing storage reservoirs, even if operated solely for effective power output, would draw down most of the available storage by April 1 in dry years, and a similar drawdown in other years would not reduce prime power. Storage could be refilled during the high water season, which results largely from snow melt, with a crest between May 20 and July 10 each year. Since snow melt and precipitation can be correlated to provide an advance prediction of the approximate volume of such floods, a fairly promising opportunity seemed available for incidental flood control benefits.

After more specific study, it was found that Albeni Falls (Fig. 2) would make no substantial contribution to general flood control because of certain of its hydraulic characteristics; and Grand Coulee, because of its inherent design, would inevitably be filled prior to the arrival of a major flood crest, which would thereafter pass unimpeded over the spillway. Hungry Horse reservoir, which was then about to be built by the USBR, was studied by the USED; it was found that it could contribute materially to Columbia River flood control by increasing its outlet capacity by 50%, which was soon agreed to. Joint-use storage was possible at other projects but the actual attainment of economical flood control was by no means apparent.

Final System Plan.—During the study of general flood control as the final major step in arriving at a balanced plan, the serious May-June, 1948, flood occurred. It added one more to the major floods of the frequency table; it indicated an unexpectedly great amount of damage for a flood of its height; and it dramatically confirmed the assumption that the major floods, such as those prior to 1900, should be expected periodically in the future. Moreover, it changed the public lethargy toward flood control to the demand (more of less as a matter of course) that such a plan be produced, although the final "308" Report was then due in only 4 months, and in spite of the fact that two of the best storage projects had just been dropped because of the objections of local interests.

Anyone familiar with the many details and complications of the planning process will sympathize with the planners at this critical juncture. After almost five years of continuous study and after eliminating literally 1,000 dam sites from consideration, they were finally left with this short time, much of it required for final editing and reproduction, and with only one new storage project (Hell's Canyon) which could definitely be recommended as contributing materially to flood control. It was fortunate that engineering studies and preliminary plans of such wide scope were available at this time.

The final selection of projects, completed in about three months of intensive work from the date the flood crested, represents no compromise with the basic standards of economic feasibility and confirmed engineering soundness. The first major "break" came when the board of Canadian and American engineers, which had been formed to study the international aspects of the Columbia Basin development, advanced its studies sufficiently to indorse the Libby project (Fig. 2). This project on the Kootenai River, an international tributary of the Columbia, is unique in its location; there lie in Canada one third of the reservoir and also one third of the intensively cultivated alluvial valley downstream, which was flooded from bluff to bluff by the 1948 flood. Engineering studies of this excellent project were already practically complete. It could be included in the plan without time-consuming surveys, and thus provide more than 4,000,000 acre-ft of usable storage.

TABLE 1.—RESERVOIRS OF THE MAIN CONTROL PLAN, COLUMBIA RIVER BASIN

Item (1)	Reservoir (see Fig. 2) (2)	Status as of July, 1950 (3)	Construction cost (thousand dollars) (4)	Usable storage[a] (acre-feet) (5)	Normal head of dam (feet) (6)	Installed capacity (kilowatts) (7)
1	Hungry Horse.....	Under construction (USBR)	107,000	2,100,000 (J) 880,000 (P)	485	300,000
2	Glacier View......	Alternate to be recommended	94,962	3,160,000 (J)	402	210,000
3	Libby............	Authorized (USED)	239,077	4,250,000 (J)	353	588,000
4	Albeni Falls.......	Authorized (USED)	31,070	1,140,000 (P)	25.5	42,600
5	Grand Coulee.....	In operation (USBR)[b]	273,000	5,120,000 (J)	348	1,944,000
6	Chief Joseph......	Under construction (USED)	225,000	Pondage	170	1,280,000
7	Priest Rapids......	Authorized (USED)	326,124	2,100,000 (FC)	146	1,219,000
8	Hell's Canyon.....	Recommended	342,076	3,280,000 (J)	575	980,000
9	Lower Snake River dams...........	Authorized (USED)	359,000	Pondage	80 to 100	980,000
10	McNary..........	Under construction (USED)	227,000	Pondage	90	980,000
11	John Day.........	Authorized (USED)	379,826	2,000,000 (FC)	95	1,105,000
12	The Dalles........	Authorized (USED)	286,286	Pondage	88	980,000
13	Bonneville........	In operation (USED)	85,000	Pondage	65	518,000

[a] Abbreviations: "J" denotes joint use of storage; "P" denotes power storage; and "FC" denotes flood control storage. [b] Flood control use recommended.

Other alternates were reconsidered but, one by one, they had to be regretfully discarded or deferred again because they were too expensive or impracticable. Finally a way was found by the USED to utilize Grand Coulee for flood control, despite its inherent design limitations. This is made possible by providing a total of more than 4,000,000 acre-ft of storage for flood control alone, usable only during major floods, at Priest Rapids and John Day reservoirs, on the lower main stem of the Columbia River. Adding the storage of

Hell's Canyon, almost 7,500,000 acre-ft can be held in reserve during major floods until after Grand Coulee is completely filled. Storage in the lower reservoirs can then be used to eliminate the peak flows and the latter half of the flood. Under this plan, with some relatively inexpensive outlet changes at Grand Coulee, that project can operate with great effectiveness in a reservoir system to control flood damages on the lower Columbia. Flood flows at The Dalles project will be reduced to less than 800,000 cu ft per sec, which can be controlled by levees. In Fig. 2 are shown the reservoirs authorized as of June, 1950, the projects recommended since that time, and the principal alternates studied.

Major reservoirs operated in this general flood control plan, listed in Table 1, will have 22,010,000 acre-ft of effective storage usable for general flood control as a primary purpose. In addition, Albeni Falls and the upper and central Snake River Valley projects, at times, will have important general flood control effects incidental to their operation for subbasin and power purposes. Their storage will total more than 5,000,000 acre-ft in addition to the 22,010,000 acre-ft.

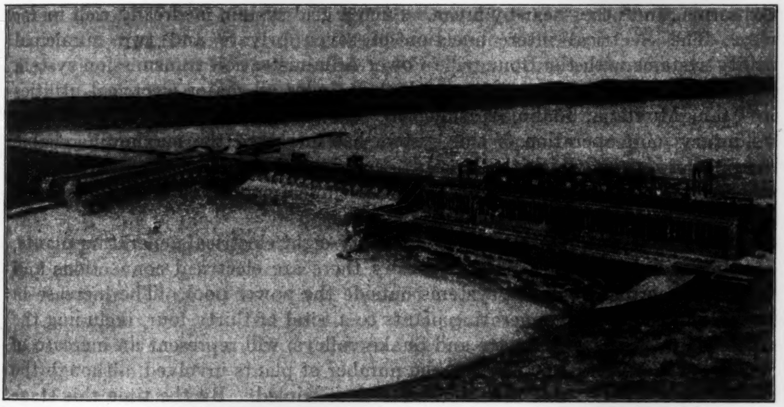

FIG. 8.—MCNARY DAM, ON THE COLUMBIA RIVER

Main Projects.—In the Columbia River system, the principal projects, called the Main Control Plan, include the existing major dams, Grand Coulee and Bonneville; the three under construction (August, 1950)—McNary, Hungry Horse, and Chief Joseph; the previously authorized lower Snake River dams; and other recommended projects (all described in Table 1).

These projects are all of large size, particularly as to installed hydroelectric capacity. The three great projects, Bonneville, Grand Coulee, and McNary, which were once considered as isolated "super-projects," now fall into their proper perspective, as units of the same order of magnitude as the remainder of the system. One of the largest projects is shown in Fig. 8.

In introducing flood control into the extremely effective hydroelectric power system of the preliminary plan, really careful study had to be given to the effect on power output. For example, when considered individually, the drawing down of Grand Coulee, in order to vacate the desired flood control storage, would reduce its peak power output by more than 500,000 kw from that otherwise obtainable at full pool. It was at first thought that reliable flood control on the Columbia River would be obtainable only by major sacrifices in power output. It was found, however, that, on a system basis, practically no sacrifice of firm power need be entailed since the higher run-of-the-river dams can provide surplus capacity at such times if plans are properly coordinated.

III. System Structure

From an engineering point of view, the power from these new dams (which are few in number but large in size) will presumably be distributed by interconnected power grids connecting all parts of the Pacific Northwest with Canada, California, and other near-by areas. Such a grid system is already well under way. The electrical interconnection of seven private and two municipal utility systems with the Bonneville Power Administration transmission system forms the Northwest Power Pool, which includes all major electrical utilities in Utah, Montana, Idaho, Washington, and most of Oregon.[2] The entirely voluntary joint operation of this system has increased the maximum reliable firm power output which the members could reliably produce individually by 600,000 kw in the aggregate, or 18%. Technical coordination of the system has been good and is creditable to all concerned.

The system now (1949) consists of eighty-eight electrical generating plants, of which seventeen are steam. Already there are electrical connections and technical coordination with systems outside the power pool. The increase in the number of federal generating plants to a total of thirty four, including the minor dams in the Willamette and Snake valleys, will represent an increase of not much more than one third in the number of plants involved, although the total output of the system will be more than tripled. By the time this stage occurs, it appears probable that all electric systems on the Pacific Coast and in western Canada will be interconnected and closely coordinated, with additional connections to still other areas. The Northwest Power Pool already covers a market area much more extensive than the Columbia Basin, but coincidence of power market areas with drainage basin boundaries is not generally found now and will be still less characteristic of the future. Some policy questions in the power distribution field remain to be resolved, but the practicality and utility of a backbone distribution network, interconnecting private as well as public systems over a wide market area, appear to be established incontrovertibly.

As a second conclusion regarding system structure, certain major subbasins can be set aside for multiple-purpose development mainly in accordance with subbasin needs. The Willamette Basin is an outstanding example. Comprehensive planning for the entire Columbia Basin has resulted in developing a multiple-purpose plan for the Willamette Basin, the initial projects of which are already constructed; but this subbasin plan will have little operating relationship with problems of the Columbia River Valley, other than the exchange of waterborne commerce between the two rivers at Portland, Ore., the sale of power through a common transmission network, and the conduct of flood fights in the Portland area. Flows from the Willamette River are downstream from all Columbia River reservoirs and cannot affect them, and floods on the Willamette River occur at such times that they contribute only infrequently to the Columbia River floods. The projects in this subbasin should be operated in accordance with a carefully integrated plan, primarily for subbasin needs, especially flood control.

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[2] *Civil Engineering*, August, 1948, p. 20.

A similar situation prevails generally as regards the Snake River Valley in southern Idaho and Wyoming. Here irrigation is of primary importance, and the main Columbia flood control plan has been so developed that irregularities in controlled flood flows resulting from irrigation storage at Idaho and Wyoming can be fully ironed out by other main stem storage above the main areas of flood damage. Any suggestion that the valuable waters of the Snake River be disposed of for any consideration other than the direct needs of the Snake River Valley would be unacceptable to the upstream states concerned. Other uses have not been requested by downstream states and are unnecessary for the development of the basin as a whole under plans outlined in this paper.

The large present and future irrigation depletions in certain areas of the Columbia Basin have only a minor influence on the discharge of the lower Snake River and the main stem of the Columbia River. Plans for constructing and operating projects in the upper Snake River and other similar areas can be based on subbasin needs without detriment to the optimum development of the Columbia River system as a whole. Similarly, many other subbasin and local projects, including all levee and channel improvements, navigation features of dams, and the proposed irrigation developments throughout the basin can be localized for operation, or operated in accordance with specialized functional control.

The relationship between irrigation pumping and firm power output is of comparatively little importance now but will be of increasing significance as future irrigation plans for the Columbia Basin are developed. As far as presently authorized and recommended projects are concerned, there is no substantial conflict. Even under severest drought conditions prime power from projects previously authorized would not be reduced by irrigation pumping needs because the critical storage period extends only from October through April. In the recommended comprehensive plan, it is recognized that an occasional conflict will occur not so much because of an increase in irrigation storage pumping as by the extension of the critical storage period.

In May, August, and September of very dry years prime energy must be taken to maintain the irrigation pumping load, although the frequency of such occurrences will not be great. However, the very large potential extensions of pump irrigation to be ultimetely included in future projects, not yet recommended for immediate construction, will cause much larger use of prime power for irrigation with a frequency that cannot be disregarded. For such projects, it will become impossible to base irrigation pumping any longer on purely secondary power at nominal cost. As far as the present plan is concerned, however, the irrigation pumping loads, superimposed on the lower power requirements for other purposes, will balance fairly closely with the available system energy, including available secondary energy in dry years.

IV. Flood Control and Power Operations

The only part of the plan where requirements are difficult to resolve, from an operational point of view, lies in the joint use of the major reservoirs for power output and Columbia River flood control purposes. The operating plan to be outlined here has been developed far enough to demonstrate the practicality of reliable flood control and the general lines of an operating plan. Refinements will be made as the system is constructed and more data become available for forecasts.

Seven main reservoirs, with a usable joint storage capacity of 22,010,000 acre-ft, are designed for main stem Columbia River flood control. Operated during most of the year mainly for power purposes, they must be capable of insuring reliable control of floods, of magnitudes up to 1,240,000 cu ft per sec at The Dalles, which ordinarily occur with a peak between May 20 and July 10. Other reservoirs will contribute incidentally to the control of major floods, but will be operated for other primary purposes. Most reliance must be placed on these seven reservoirs to reduce all main Columbia River floods to flows that can be contained by practicable levees.

During the fall and winter months the flood control plan will not require reservation of storage primarily for flood control except at John Day and Priest Rapids, where emergency flood control storage will always be kept available. By April 1 all joint-use storage will be available, except at Grand Coulee where only 3,000,000 of the 5,100,000 would be drawn down. Such drawdowns, of course, would be made by operating-rule curves, such as those shown in Fig. 9, through normal operating procedures. As long as the drawdown is effected by April 1, the flood control criterion will be met and operation up to that date can consider primarily power output.

Operation after April 1 will be in accordance with an initial flood forecast. This forecast will be generally similar to those made for the mainstem Columbia River by the United States Weather Bureau, the Soil Conservation Service (United States Department of Agriculture), and the USED, but an expanded meteorological network will be required.

Although floods on the Columbia River can be predicted much more accurately than those of most major rivers, there will remain certain inherent potential errors in the forecast. Columbia River floods occur primarily from melting of the snow pack, the water content of which is measurable with great accuracy if sufficient stations are installed. There is a normal relationship between the water content of snow on the ground and the peak flood discharge, depending on the usual temperature sequences and a normal amount of spring precipitation; and variations from this relationship are usually quite small. However, the greater floods are likely to be caused by a combination of events in which a heavy snow pack is exposed to unusually cold weather during April and May, and to heavy rains in May, accompanied by sudden prolonged warm weather. A further major factor is a retardation in the normally earlier

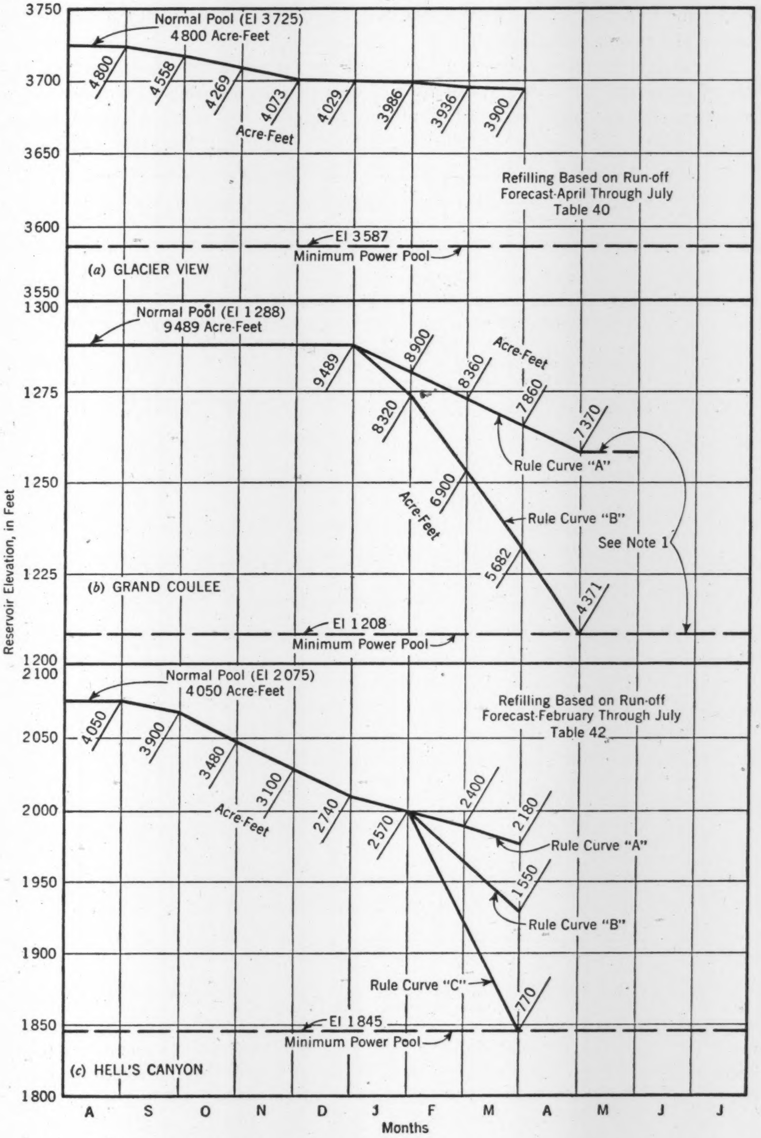

FIG. 9.—OPERATING-RULE CURVES FOR POWER AND FLOOD RESERVATION; PHASE C DEVELOPMENT, COLUMBIA RIVER AND TRIBUTARIES

flood runoff of the Snake River, resulting in a coincidence of flood peak with that of the upper Columbia.

These factors may cause very unusual deviations from the norm, on the side of danger. Such deviations do not occur in accordance with unweighted frequency curves but are most likely to occur during years of great floods, toward which they are a contributing factor. Although the data are insufficient for accurate correlation, the fact that such conditions occurred during 1894 and 1948 (the only two very large floods of which there is accurate knowledge) indicates the necessity of allowing a sufficient safety factor to provide for such "unusual" occurrences during any major floods, despite the increase in forecast reliability expected in the future.

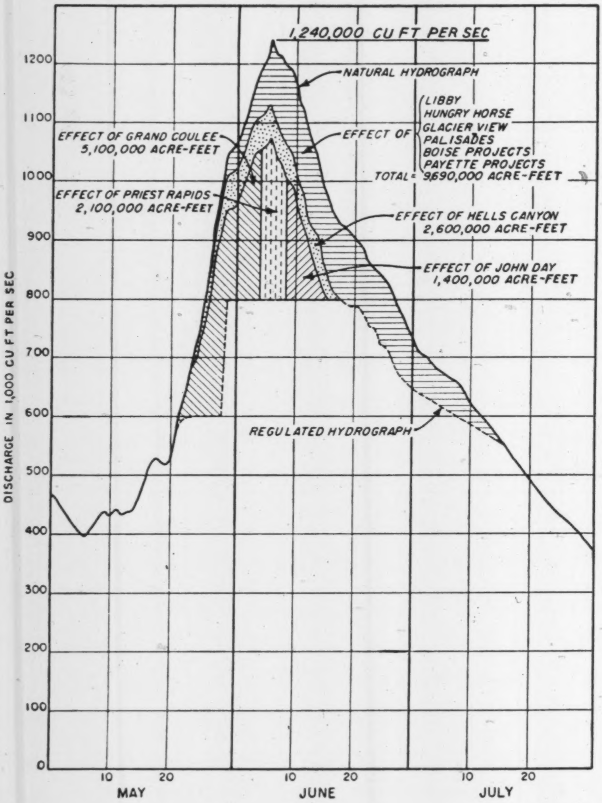

FIG. 10.—HYDROGRAPH OF THE 1894 FLOOD, REGULATED BY THE MAIN CONTROL PLAN: COLUMBIA RIVER AT THE DALLES, ORE.

If the forecast on April 1 is for a flood of 665,000 cu ft per sec or more, all available joint-use storage will be held empty, and Grand Coulee (initially providing 3,000,000 acre-ft of storage) will be further drawn down to provide its full 5,100,000 acre-ft by May 1. This system storage is adequate to control a repetition of the 1894 flood, with an operating margin of 2,900,000 acre-ft to allow for possible inaccuracies in reservoir operation and discharge data during early flood stages. Fig. 10 illustrates results of applying these procedures in the case of the 1894 flood.

The system storage to be provided is so great that, if the 1894 flood were repeated, with all reservoirs drawn down, and with use of the aforementioned operating margin, more than 6,000,000 acre-ft of storage would remain unused at the end of the flood. Excess storage probably not worth reserving for flood control use would remain at Hungry Horse and Glacier View, and on the Boise and Payette rivers (where main stem flood control is only incidental). A usable reserve of 600,000 acre-ft would remain at John Day.

During years of drought, which constitute the critical power period, there is no need to hold all these great reservoirs empty. When the prediction on April 1 is that the peak discharge will not exceed 450,000 cu ft per sec (about one year in four), all joint-use storage will be allowed to be filled as rapidly as power production requires. The levee capacity to hold flows of 800,000 sec-ft, plus the reserve capacity available in the two lower river reservoirs, is ample protection against any flood that may occur. On May 1, another prediction will be made. When floods with peaks greater than 750,000 cu ft per sec are anticipated, all storage will be held down until the flood arrives. If a flood between 465,000 cu ft per sec and 750,000 cu ft per sec is anticipated, proportionate reservations of joint-use storage will be made.

In the later stages of floods, minor deviations and inequalities will develop, particularly as a result of unanticipated discharges from minor tributaries, such as the Yakima, and the time elapsing before releases in the upper Flathead and the Kootenai arrive in the lower Columbia. However, Hell's Canyon and Grand Coulee will eliminate most of these irregularities, and Priest Rapids and John Day can handle the remainder. For example, an unexpected discharge of 50,000 cu ft per sec extra from the Yakima River, menacing the levees on the lower Columbia, could be balanced by the retention of storage for 48 hours in the Priest Rapids reservoir, amounting to 200,000 acre-ft, or 10% of its capacity. Meanwhile a faster rate of storage in Grand Coulee or Hell's Canyon could become effective and thus allow the quick release of storage in Priest Rapids prior to the arrival of other unexpected crests.

During actual passage of a major flood, day-to-day releases and even hour-to-hour releases from at least the four lowest storage reservoirs must be controlled by one operating center. Because of the paramount importance of flood control at such a time, and the necessity for quick action, truly unified control with full authority and responsibility is highly desirable. For this part of the system, ordinary coordination at such times might not be sufficient.

As regards power production during a flood, the seven major storage reservoirs can continue to have their normal operating relationships with the much more numerous municipal and private power projects comprising the balance of the Northwest Power Pool. Rapid changes in head and unusual amounts of drift during the flood will entail operating problems rather more

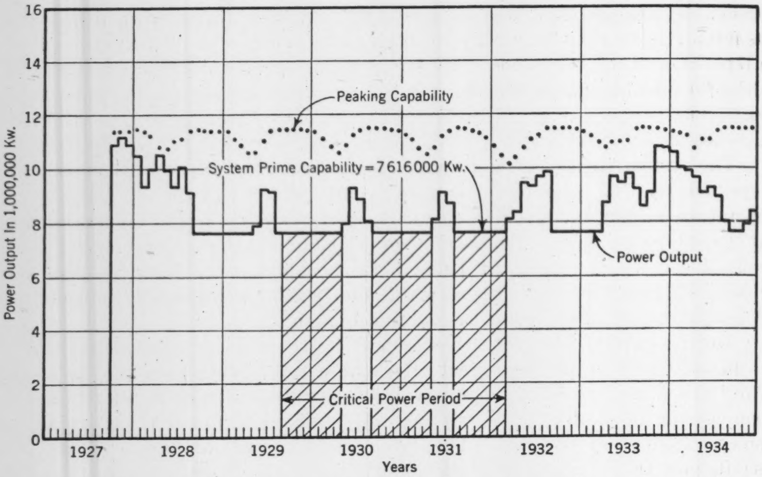

FIG. 11.—POWER OUTPUT AND PEAKING CAPABILITY; PHASE C SYSTEM, COLUMBIA RIVER AND TRIBUTARIES

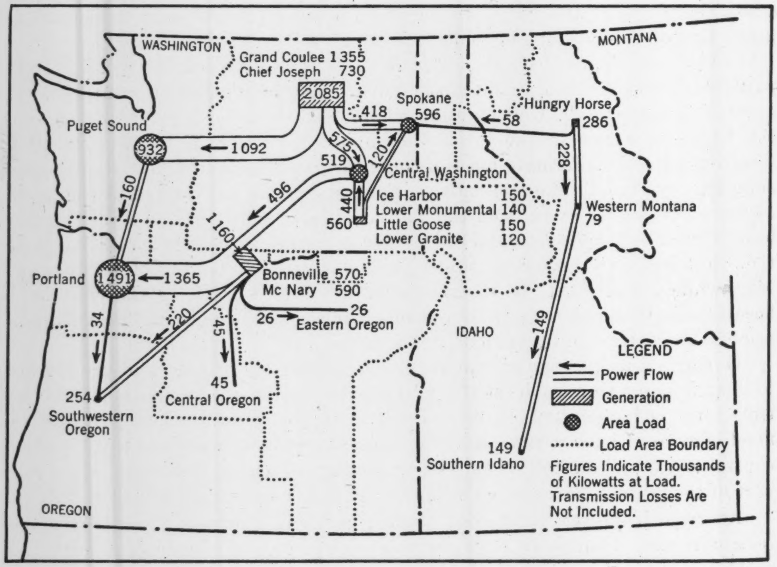
WASHINGTON
MONTANA
Grand Coulee 1 355
Chief Joseph 730
2 085
418
Spokane
596
58
Hungry Horse
286
Puget Sound
932
1 092
575
519
120
Central Washington
228
Ice Harbor 150
Lower Monumental 140
Little Goose 150
Lower Granite 120
440
560
496
1 160
Western Montana
79
160
Portland
1491
1 365
Bonneville 570
Mc Nary 590
26
26
Eastern Oregon
34
220
45
45
Central Oregon
IDAHO
149
254
Southwestern Oregon
149
Southern Idaho
OREGON
LEGEND
Power Flow
Generation
Area Load
Load Area Boundary
Figures Indicate Thousands of Kilowatts at Load.
Transmission Losses Are Not Included.

FIG. 12.—POWER FLOWS AT THE TIME OF THE DECEMBER PEAK LOAD
(FIRM POWER FROM PRESENT AND AUTHORIZED PLANTS)

difficult than those normally faced, but similar in kind. Except in the very largest floods (even larger than that of 1948) there need be no directly conflicting interest between flood control and power except as regards production of secondary power; and at such times flood control operation should have overriding priority, subject only to emergency provisions to assure safety of each structure, if reliable flood control is to be obtained.

Power Output.—The operating-rule curves proposed would be modified by additional experience; but they appear to present no real difficulty in reconciling interests of flood control and power. System capability during an assumed critical drought period of the future and a system flow chart for major groups of projects and, as recommended, are shown in Figs. 11 and 12. Only secondary energy is sacrificed by this plan for flood control operations of Main Control Plan reservoirs; and this secondary energy, on account of the small proportion of thermal power output in the region, has been assumed to be of no value in determining economic benefits.

V. Associated Plans of Other Agencies

Transmission System.—In the Columbia Basin, although the USBR markets power to a limited extent, the main distribution and marketing agency for federal power is the Bonneville Power Administration. For projects included in the "308" Report, the Bonneville Power Administration made plans and cost estimates for a transmission system which would distribute power from the projects, link all public and private utilities in the region, and connect by interregional ties with four other major power areas, including western Canada. The scope of this proposed network may be visualized from the estimated first cost of $462,000,000. Costs of this extensive system were included in studying the economics of the Main Control Plan.

Lower Columbia River Fisheries Plan.—The Columbia River poses a unique problem in that it is necessary to conserve great runs of salmon while developing the stream for other purposes. The salmon spawn in bars of main rivers and tributary streams, the young fingerlings swim down to the sea, and 4 to 6 years later the mature fish return to repeat the cycle. Many natural spawning grounds have been completely blocked over a period of years by dams built without proper provision for passage of the fish; and overfishing and pollution of waters have contributed to reduce the original runs. In 1938 the Bonneville Dam was constructed, with elaborate and well-designed fish-passing devices which have proved highly successful. In the case of McNary Dam, the fish-passing devices will cost $14,000,000 and will incorporate improvements over those at Bonneville.

However, since there is undoubtedly some cumulative adverse effect of a large number of dams, even when carefully designed to aid passage of fish, the comprehensive plan calls for a compensatory program of $20,000,000, known as the Lower Columbia Fisheries Plan, to clear natural obstacles from lower tributary streams and to build hatcheries. This plan was developed by the Fish and Wildlife Service. Although the plan as a whole has not yet been authorized by Congress, some money has already been appropriated, and the

State of Washington has taken legislative action to prevent encroachment upon the streams concerned in that state.

Watershed Improvement Plans.—The programs of soil conservation and reforestation are of great importance in their own right, in addition to their effect on floods and runoff. In the Columbia Basin, as elsewhere, they aid flood control and general water conservation. Although the great floods of 1876 and 1894 in this watershed, before forests were generally cut over, are ample warning that such floods are not basically man made, there is no doubt that in certain local areas, improper farming and cutting or overgrazing of natural cover have contributed to the flood problem and increased the silt carried in streams. In certain parts of the Willamette watershed, and (even more strikingly) in the Walla Walla, Touchet, and other loessal areas, flood control by reservoirs and channel works must be very closely integrated with plans for watershed treatment.

Fortunately, the sediment content of the main Columbia River is relatively low; and the same problems are not so acute for the major reservoirs. However, if the present soil and forest cover loses capacity to absorb runoff, so that when the most severe climatic conditions recur they produce larger floods than previously, it will be necessary to provide additional reservoir storage to compensate. To this extent the over-all plans for reservoir flood control and for improvement of soil and forest cover are complementary.

Financing Irrigation Development.—In the later stages of preparing the comprehensive plan a basic lack of economic balance was noted in certain important benefits. In particular, practically all the irrigation potentiality lies east of the Cascades, whereas more than two thirds of all the potential power demand in the Northwest is expected to develop in the area west of the Cascades. Although the plan contained liberal provisions for expansion of irrigation, based mainly on USBR plans, great difficulty had been found in financing such projects.

Water users have generally been able to repay only a part of the irrigation costs. Most potential irrigation is physically widely separated from the power-producing projects; and the inclusion of irrigation projects as a part of a "comprehensive plan" would not invalidate general provisions of law requiring repayment. The Portland and Puget Sound areas, which use most of the power, are also interested in irrigation development, because the irrigable areas of the "Inland Empire" will form a great addition to the trade hinterland of the port cities.

As one solution, it was suggested in the division engineer's report that a small amount, which would be immaterial to the average power user, be added to power rates in the Northwest in order to cover irrigation costs not chargeable to water users. Subsequently, as part of an agreement signed on April 11, 1949, the secretaries of the Army and the Interior agreed to recommend to Congress that financial assistance from all power revenue producing projects in the Pacific Northwest should be pooled and utilized to aid irrigation under principles consistent with those embodied in reclamation law. The cost of recommended irrigation development was found to be relatively small compared to anticipated power revenues.

VI. GENERAL CONSIDERATIONS

As has been indicated, the comprehensive planning process has as its major objective the delineation of general basin-wide plans and criteria. These are not general ideas or principles but highly specific engineering conclusions and relationships which vary in accordance with the stage of development. The main concept or structure of the optimum basin plan and the degree of integration necessary for operational control cannot be predetermined from general principles but are empirically derived from engineering relationships and physical and economic conditions. If the comprehensive planning process is omitted or incomplete it is likely that serious mistakes will be made, not only by developing projects in the wrong priority but, more seriously, by adopting conflicting or overlapping plans, or by basic deficiencies in project design impossible to detect other than by a basin-wide, multifunctional approach.

Once a comprehensive plan has been developed, many more detailed supplementary plans can be prepared within its terms of reference; and these plans can often be approached on the basis of local or unifunctional needs, and properly coordinated with the main plan. When sound plans are made, it does not necessarily follow that the geographical basis for construction, operation, and maintenance will be the same as for planning.

Construction is always best decentralized; distribution of electrical power is usually best accomplished through the widest possible interconnected system, not related to watershed boundaries; and project operation may be best centralized into a single basin-wide system, decentralized into subbasin components, separated into functional groupings, or operated as a combination of the three, depending on actual conditions. Review of plans on a comprehensive basis will be necessary from time to time as conditions change and new techniques are evolved.

As regards many other aspects of national resource development, Congressional policies are implemented throughout the United States, so that, as regards each function, such as irrigation, flood control, and soil conservation, similar procedures are applied in adjacent major watersheds. This is true in technical as well as administrative matters.

Important specific problems sometimes arise concerning interbasin waterways or diversions. Although possibilities are physically limited, new proposals, advanced from time to time, are often the occasion for serious conflicts of interest between the respective localities and interests concerned. When such problems arise, the approach must be shifted to a broader geographical basis, including the entire area concerned, at least for the period of the investigation.

It is a major task of comprehensive planners to minimize possible conflicts of interest and provide a system with optimum public benefits. Ordinarily separate storage must be provided for power and flood control in order for both to be fully reliable. It is only where fairly accurate flood predictions are possible that true joint-use storage can be developed. The same is true in cases of joint-use storage for flood control and irrigation. Moreover, the requirement that the irrigable land must be located so that it can be served from the controlling reservoir often precludes successful development of irrigation in multiple-purpose projects.

Where the heads of dams are not too great, navigation use of reservoirs can usually be provided without material detriment to other purposes. There is often a potential conflict between water supply for irrigation purposes and for other downstream uses; and in the Columbia Basin it is possible to avoid such a major difficulty only by careful planning to provide adequate storage intermediate between the major areas concerned, and a properly balanced general operation plan.

On the Columbia River with its numerous possibilities for power production exceeding 30,000,000 kw in the ultimate, certain general results were obtained which might not be so apparent in lesser water sheds. First, the large projects proved generally to be more economical sources of hydroelectric power than the small projects, and the balanced series of dams, more economical than a number of projects on separate tributaries. For various cases, especially railroad and community relocations, and because of the necessity of restricted heights for passage of anadromous fish, projects often could not be planned to the full size which would otherwise have been most economical. All studies on the replacement of large dams by a greater number of small dams showed an increase in cost, usually very material. Also, despite criticism of the amounts of valley land required by large reservoirs, systems of small reservoirs with the same effective storage required much greater land areas.

With few exceptions, multiple-purpose use of major reservoirs gave the maximum benefits and best all-around plan. However, the optimum combination of functions varied widely.

Although the most economical sites are obviously generally developed first, the entire system of reservoirs is much more effective from an engineering point of view than individual reservoirs separately. Power from the complete system, it is estimated, need be sold at only about 25% more per kw-yr to repay costs, as compared to present rates for Bonneville and Grand Coulee power, although unit construction costs have doubled since those projects were built. This favorable result is possible only because of a reservoir system wherein head and storage development and multiple-purpose use are exceptionally well balanced.

New head developments on the main stem can advantageously be added to the system as long as sites remain. On the other hand, storage developments confer system benefits in accordance with a law of rapidly diminishing returns, initially very advantageous, if sufficient head is developed downstream, but with much less return from later projects.

Lastly, experience at Bonneville and Grand Coulee has proved that properly conceived projects in the Northwest can produce power at very low rates and can make an important contribution to the regional economy, at the same time returning to the United States Treasury a surplus over and above all power costs. Such a combination of effective engineering, fiscal and economic soundness, and general public benefit is the end to be sought in comprehensive water resource development.

DISCUSSION

L. E. Rydell,[3] Assoc. M. ASCE.—A clear and concise exposition of the principles of multipurpose planning as applied to one of the greatest rivers in the United States is presented. This paper is especially significant because of the wide range of functions and river resources directly concerned (including navigation, flood control, irrigation, power, fisheries, recreation, and others) and for the logical manner in which these functions, some of them divergent and conflicting, were effectively coordinated into a unified plan. It is significant, also, for the many interrelated interests that are treated, directly or indirectly, requiring exhaustive consideration of such factors as regional resource development in relation to future power markets. The paper reflects the author's decisive contribution in directing and coordinating the final formulation of this comprehensive plan for the Columbia Basin in the States of Oregon, Washington, Idaho, and Montana.

Hydroelectric power is probably the outstanding resource of the Columbia River and its tributaries. Yet it is only since 1930 that the people of the Pacific Northwest have become aware of its magnitude and potentialities for regional development. The economy of the region over most of its history has been one based primarily on the wealth of timber, mineral, and agricultural resources. The potentialities of the Columbia River were first brought to public attention on a comprehensive scale by the original "308" Report prepared by the Corps of Engineers, United States Army, in the early 1930's. At that time the development of power on the order of 15 to 20 million kilowatts, comparable to the total hydroelectric capacity of the entire nation at that time, seemed extremely remote, considering existing power markets and demand. However, the plan outlined in that report was well adapted to development in stages and soon became the basis for authorization and construction of the Bonneville and Grand Coulee projects. Even as late as 1937 the significance of these projects to the region was not generally appreciated; Grand Coulee was referred to as a white elephant, and many informed people questioned whether the power from the first two units (108,000 KW), then installed at Bonneville Dam, could be absorbed in the Northwest power market in the immediately foreseeable future.

This situation was changed prior to World War II, when the needs for defense production, coinciding with the availability of low-cost federal power, suddenly focused attention on this Pacific Northwest resource. Power demand for domestic, commercial, and small industrial purposes was already developing at an increasing rate in 1940, when establishment of a large aluminum plant at Vancouver signified the advent of large-scale industrial use of power in this region. Rapid establishment of other large industries followed, principally in the electrochemical and electrometallurgical fields (that are especially dependent on low-cost power) with a consequent phenomenal increase in power demand. This continued to the limit of expanding capacity throughout the war years. Contrary to the expectations of many, the growth in power demand was slowed only temporarily by the ending of the war, and in the succeeding years has continued with even greater vigor.

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[3] Chf., Eng. Planning Branch, Corps of Engrs., Walla Walla, Wash.

The increased emphasis on production of power was a major consideration in planning the key project of the main control plan, and the need for flood control was dramatically emphasized by the great flood of 1948. It is with respect to these two functions that the plan outlined by the author differs most significantly from the comprehensive plan of the original "308" Report, and represents a logical evolution from it.

The history of water resource planning in the Columbia Basin illustrates the problem facing all planners—to decide realistically between the apparently conflicting alternatives of meeting the immediately foreseeable needs or of planning for the long-range future requirements. Either course is beset with pitfalls. If planning is too shortsighted, the plan may soon prove inadequate and require revisions, and often it may then be too late or too costly to meet the larger objectives. On the other hand, if planning is too ambitious and perhaps visionary, no matter how desirable the objectives in themselves may be, the plan may arouse justified opposition from those adversely affected or face such long delay in accomplishment as to fail in its basic purposes. Sound and effective planning must achieve a judicious combination of the two apparently divergent but actually parallel objectives. This has been accomplished in the plan for the Columbia River Basin.

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PRACTICAL DESIGN OF SOLID-BARREL, REINFORCED-CONCRETE SKEW STRUCTURES

By Bernard L. Weiner,[1] M. ASCE

With Discussion by Messrs. Maurice Barron, Arthur G. Hayden, and Bernard L. Weiner.

Synopsis

No new theories are presented in this paper. Its purpose is rather to show the simplifications that are possible in the application of existing theory. These simplifications have been tested in practice. A solid-barrel skew frame can be designed with little more work than that required for the design of a similar right-angle frame. Theoretically, it is true that a skew frame must be analyzed in three dimensions, whereas a right-angle structure can be analyzed in a single plane. It is to be expected, therefore, that there will be stresses in the former type of frame which either do not exist in the latter type or exist only to a very moderate extent. Although these additional stresses (often referred to as "skew stresses") are important, provision can be made for them, for practical design purposes, with very little additional work.

Introduction

It is the purpose of this paper to show that by taking advantage of certain characteristics of skew frames and of certain other factors that follow directly from considerations of static equilibrium, permissible approximations in existing theories may be made which reduce the labor involved in the design of a skew frame to little more than that required for a similar right-angle structure.

That these approximations are possible should not be very surprising. It happens often that "practical design" differs considerably from "theoretical analysis" for research purposes. Where "accurate" formulas are too complex and cumbersome for everyday use, approximate methods are developed by or for the practicing engineer; in other words, "engineering design" often becomes merely "conventional design."

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[1] New York, N. Y.

For "small" skews, the expedient of designing on the skew span has often been used. A consideration of the arguments in this paper will show that this expedient is entirely incorrect; it leads to results that either are on the side of danger or are very wasteful. The former is true in the case of members which are essentially horizontal; the latter is true for members which are essentially vertical. Only in the case of a member that is essentially horizontal and that is subjected to uniformly distributed vertical loads are the results correct; for the same loading, the results are wasteful for essentially vertical members. With the simplifications described in this paper, this expedient, besides being incorrect, also becomes unnecessary.

The methods described have been used since about 1930 for the design of numerous structures (such as those on the Westchester County Parkway System in New York and on the Belt Parkway System in New York, N. Y.) with skew angles as great as 50°, with soffits of various shapes, and of one, two, and three spans. The thickness of the members and the amount of reinforcing steel do not appear excessive or unusual compared to that required for a right-angle structure of span equal approximately to the skew span. Such skew structures have been built and have been in service from 6 years to 20 years without showing signs of distress.

In recent years, a few splayed bridges, with faces not parallel (such as those designed for the Cross-Bronx Expressway in New York City) and bridges with curved faces (such as those on the Saw Mill River Parkway in Westchester County and on route 4 Parkway in New Jersey), have been built. An extension of the principles presented in this paper leads to a practical procedure for the design of this type of structure also.

CHARACTERISTICS OF SKEW FRAMES

The practical design of most structures—as distinct from the theoretical analysis for research purposes—may be divided into two steps:

1. The computation of the total moments, thrusts, and shears; and
2. The design of the sections to resist the total stresses computed in step 1.

It should be remembered that, in a very real sense, the total stresses—especially the moments—are mere mathematical abstractions; that is, the effects of the loads on a structure are resisted by unit stresses and not by moments and total forces. In some types of structures such as narrow beams, it is possible to express the sum, or resultants, of the unit stresses in terms of couples, about some more or less arbitrary axis, and in terms of total forces. In the case of wide structures such as slabs, total stresses have little significance and are practically useless for design purposes. In the latter case, it is stated that the distribution must be known—which merely begs the question.

As a matter of fact, in the accurate theory of slabs, the unit stresses are first found from the elastic equilibrium and elastic deformation equations of the mathematical theory of elasticity, stated in terms of partial differential equations—and then, working backward, the moments and shears and their so-called "distribution," are found. The meaning of "distribution," in this case, is that the unit stresses can be summed up in one direction (the depth of the slab), resulting in moments and shears per unit length in the other two directions. An analogous situation exists in the case of the accurate theory for the bending of narrow beams, although, in this instance, summing up over the depth gives more complete information and the resultant moments and the total thrusts and shears have more real significance.

Fortunately, since the "run-of-the-mine" structure is of the narrow beam type, it is possible, in most cases of importance, to sum up the unit stresses in terms of total moments and forces—"packaged" quite neatly for the purpose of discussion and design. Furthermore, in such cases it is also possible to compute the resultant stresses directly. In other words, engineers usually do designs backward. The method is convenient, however, where it can be used.

Failure to remember the real nature of everyday design procedure has often resulted in confusion, and this has been particularly true of the approach to the analysis of skew structures of the solid-barrel type.

The analysis and the design of a rectangular frame or arch, for example, are really based on the assumption that the frame or arch can be divided into narrow strips, each one of which may be designed as a narrow beam—because of symmetry along the width of the structure. Although it is not possible—in abstract theory—to follow this procedure for skew structures, a method for finding the total stresses, based partly on analogy, has nevertheless been developed. There is no doubt, however, as to the correctness of the method for finding the total stresses since the computed results have been checked by tests.

It must be remembered that the computation of the total stresses presupposes unit stresses of which the former are merely the resultants; that is, the designer must work backward to find the underlying unit stresses. The direct method, of course, would be to find the unit stresses from the equilibrium equations of the mathematical theory of elasticity, but the complexity of the problem makes the indirect method a necessity. Even this indirect method, in its full unabridged form, is rather complex, long, and tedious. Fortunately, extensive simplifications can be made for practical design purposes—so much so that the design of a skew structure need not require much more work than that required for a similar rectangular one.

In a narrow, beam type of structure or in a rectangular wide one treated as a series of narrow beams indentically loaded, the resultant forces on any section are a bending moment, a thrust, and a shear. Similar total stresses exist in a skew structure; but, in addition, there are also a torsion moment, another bending moment, and another shear, as shown in Fig. 1—that is, six total stresses altogether. Theoretically, the six stresses cannot be separated except in at least one special case where the median or "neutral" surface is straight, truly vertical, and truly horizontal. For this special case, the writer has shown, elsewhere,[2] that the stresses which exist in a rectangular frame and those which exist only in a skew frame (loaded symmetrically with respect to their widths) are separable and may therefore be treated as two separate elastic systems. Simplifications result from this consideration for, instead of having six unknowns for each span, there are now two sets of three unknowns.

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[2] "Design of a Reinforced Concrete Skew Arch," by Bernard L. Weiner, *Transactions*, ASCE, Vol. 96, 1932, p. 1281, part III.

It has been found in the general case, however, that, where the neutral surfaces of the several members are neither straight nor vertical and horizontal, there is little error in considering the two sets of stresses independently and in analyzing them as two separate elastic systems. The closeness of the approximation has been checked for a single-span symmetrical rigid frame by actual computation, and by comparison of the stresses obtained from the "exact" theory with those obtained from the approximate analyses.[3] Similar results were later obtained by Richard M. Hodges[4] and by others. The approximation holds good for any number of spans and for any shape of frame or arch that would be encountered in practical design.

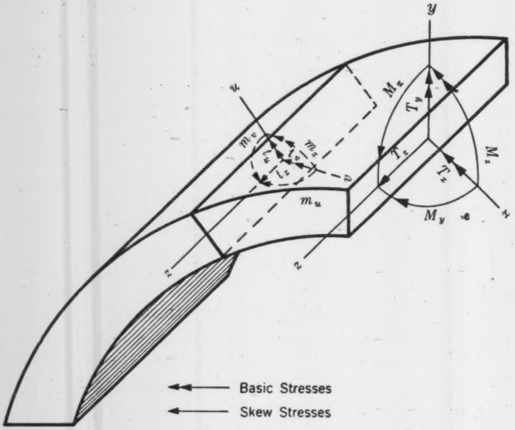

FIG. 1.—TOTAL STRESSES

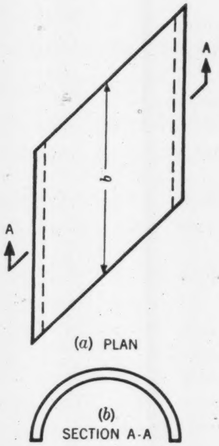

FIG. 2.—REPRESENTATIVE PLAN AND SECTION

One important result from this approximation is that, except for temperature and shrinkage, the set of three equations for those total stresses which exist also in a right frame takes the same form as in a right frame. In other words, these stresses—hereafter to be referred to as the "basic stresses"—are equal to those in a rectangular frame of the same square span and of the same elastic and geometric properties. Therefore, it is possible to obtain the basic stresses from a right-frame analysis, with exceptions to be noted.

The basic stresses for temperature and shrinkage present no great difficulty. Analysis shows that they are equal to those in a right frame multiplied by the square of the secant of the skew angle.[2] Although these approximations reduce the work considerably, study of the internal stress distribution leads to further simplification. It will be shown that, in most cases, the "skew stresses" need not be computed at all for design purposes. Designs that are adequate for both safety and economy can be made by applying a rather simple correction to a right-frame design.

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[3] "Design of a Reinforced Concrete Skew Arch," by Bernard L. Weiner, *Transactions*, ASCE, Vol. 96, 1932, p. 1286.

[4] "Simplified Analysis of Skewed Reinforced Concrete Frames and Arches," by Richard M. Hodges, *Transactions*, Vol. 109, 1944, p. 913.

The right-frame analysis for the total stresses can be made by any of the methods in common use and this phase of the problem, except for one detail, therefore, needs no further discussion. Theoretically, a skew frame cannot be divided into narrow strips and must therefore be treated in three dimensions. It is found, however, that the skew width b, measured parallel to the abutments, can be factored out of all the terms of the equations but one, the so-called "factor of torsion." This term is equal, however, to a constant multiplied by a factor $\frac{1}{b^3} + \frac{1}{t^3}$; and, since $\frac{1}{b^3}$ is small compared to $\frac{1}{t^3}$ (t being the total depth of any section), there is practically no loss of accuracy if the former is neglected even in the complete skew-frame analysis. Thus, the resulting stress computations made on this basis—with b factored out—give the total stresses divided by b, which may or may not be the stresses per unit width, depending on whether or not the stresses are uniformly distributed. In most cases, however, this point is merely of academic importance, as will be shown.

In theory, then, the skew frame is treated as a whole in three dimensions, and dividing by b is merely an arithmetical convenience. Nevertheless, it is possible to use a strip of unit width provided a proper "sample" is taken. Since in the unabridged theory, the square span is used, it must also be used in the strip. The proper section for the substitute right frame is illustrated in Fig. 2. Projecting a skew section on to a plane perpendicular to the abutments results in a representative "sample" for analyzing an equivalent right frame with the same square span as the skew frame and with the same elastic and geometric properties.

Stress Distribution and Boundary Conditions

Considerable information about the action of a structure may be obtained from a qualitative analysis of the boundary conditions. It is well known for instance that unit shears (normal to the same axis), on planes perpendicular to each other, must be equal—which leads to the important corollary that, at the intersection of a section with a free surface at right angles to it, the unit shears must be zero. The same reasoning leads to the more general corollary that, if the section and the free surface are not perpendicular to each other, the unit shears cannot be zero. Stated in another way, at the intersection of a plane with a free surface oblique to it, the shears must have such a value that the resultant of all stresses will be parallel to the free surface. This conclusion follows from the fact that any other resultant will have components that can be kept in equilibrium only by stresses on the free surface, which, of course, is impossible—or is a contradiction in terms. It follows, therefore, that at the boundary, if the direct stresses are known, the shears are also known—because the direction of the resultant is known.

Since the boundary values of the unit shears are independent of the total shears and torsion moments, and are dependent only on the direct stresses, only the latter need to be computed for the design of the fascia sections. It is implicitly assumed in the unabridged theory for the computation of the total stresses that the ordinary bending moment and the direct thrust are uniformly distributed; and, since this theory has been checked by tests, the same assumption can logically be made for the design of the sections. This assumption has

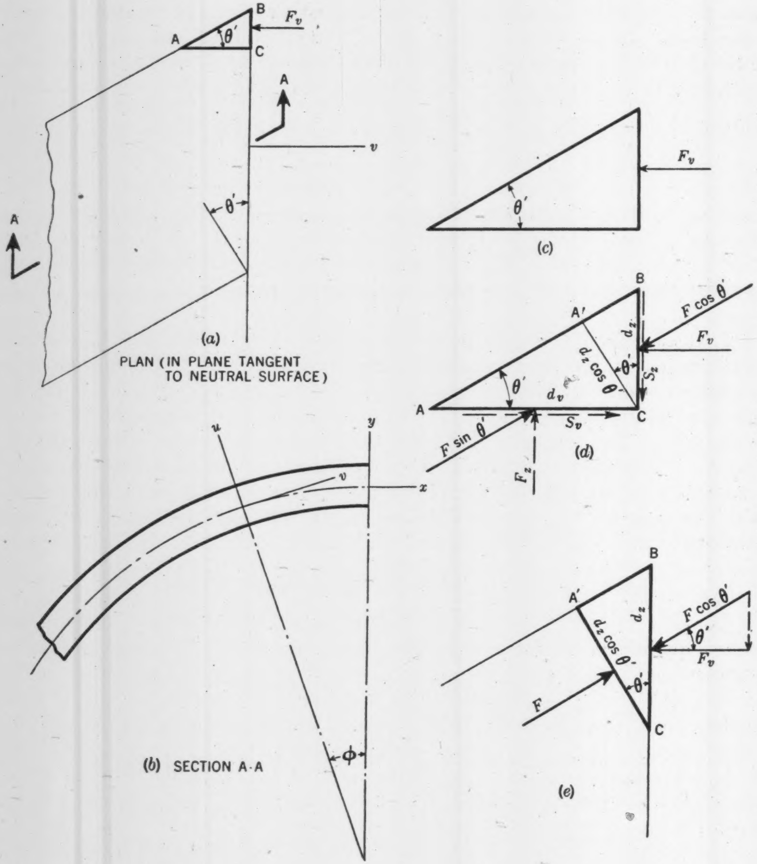
B
F_v
A
θ'
C
A
v
θ'
A
(a)
PLAN (IN PLANE TANGENT TO NEUTRAL SURFACE)
F_v
θ'
(c)
B
A'
$F \cos \theta'$
d_z
F_v
$d_z \cos \theta'$
θ'
S_z
θ'
d_v
A
C
S_v
$F \sin \theta'$
(d)
F_z
u
y
v
x
B
A'
$F \cos \theta'$
d_z
$d_z \cos \theta'$
θ'
F_v
θ'
F
C
(b) SECTION A-A
ϕ
(e)

Fig. 3

been made in all skew-arch design theory (and also for right arches). Until a study of this point is made, based on the mathematical theory of elasticity and substantiated by tests, no exact answer can be obtained. The total basic stresses are nearly equal, however, to those in a rectangular frame of the same square span, as previously explained. Hence, it follows that, as far as the fascia sections are concerned, only a right-frame analysis need be made to obtain the total stresses. Although it is true that the second bending moment, M_y at the crown (or m_u elsewhere), produces the maximum stresses at the fascia, the latter are so small that they are neglected even in an unabridged design.

RESULTANT MOMENT AND THRUST PER UNIT WIDTH AT FASCIA

The force shown in Fig. 3(*c*) is assumed to be known, as previously explained. For equilibrium, this force, F_v per unit width, must be accompanied by the other three forces per unit width shown as dashed-line vectors in Fig. 3(*d*). The value of each of these forces in terms of F_v and of the modified skew angle, θ' (the value of which in terms of the skew angle, θ, will be determined later) is easily found from considerations of equilibrium; thus, because $\Sigma V = 0$:

$$S_v\, dv = F_v\, dz \qquad (1)$$

and

$$S_v = F_v \frac{dz}{dv} = F_v \tan \theta' = \epsilon' F_v \qquad (2)$$

Since $\Sigma M = 0$, writing ϵ' for $\tan \theta'$, taking moments about point C, for convenience,

$$F_z \frac{\overline{dv^2}}{2} = F_v \frac{\overline{dz^2}}{2} \qquad (3)$$

and

$$F_z = F_v \left(\frac{dz}{dv}\right)^2 = F_v \tan^2 \theta' = (\epsilon')^2 F_v \qquad (4)$$

For $\Sigma Z = 0$,

$$S_z\, dz = F_z\, dv \qquad (5)$$

$$S_z \frac{dz}{dv} = F_z \qquad (6)$$

$$\epsilon' S_z = (\epsilon')^2 F_v \qquad (7)$$

and

$$S_z = \epsilon' F_v = S_v \qquad (8)$$

Eq. 8 proves the well-known fundamental relationship that the unit shears (normal to the same axis) on planes perpendicular to each other are equal.

Also,

$$\frac{S_z}{F_v} = \frac{\epsilon' F_v}{F_v} = \epsilon' \qquad (9)$$

The resultant, therefore, is parallel to the boundary (face of structure), as it should be. To find the resultant stress on surface BC:

$$F\, dz \cos \theta' = \sqrt{(F_v\, dz)^2 + (S_z\, dz)^2} = dz \sqrt{F^2{}_v + (\epsilon')^2 F^2{}_v} \qquad (10)$$

$$F = F_v \sec^2 \theta' \qquad (11)$$

Also,

$$\frac{F_z}{S_v} = \frac{(\epsilon')^2 F_v}{\epsilon' F_v} = \epsilon' \ldots\ldots\ldots\ldots \quad (12)$$

The resultant on surface AC, Fig. 3(*d*), is therefore equal and opposite to that on line BC—as it should be, since surface AB is a free surface or boundary.

To find the resultant stress on surface AC, Fig. 3(*d*):

$$F\,dv \sin\theta' = \sqrt{(S_v\,dv)^2 + (F_z\,dv)^2} = dv\sqrt{S^2_v + F^2_z} \ldots\ldots \quad (13)$$

$$F \sin\theta' = \sqrt{(\epsilon')^2 F^2_v + (\epsilon')^4 F^2_v} = \epsilon' \sec\theta' F_v \ldots\ldots \quad (14)$$

and

$$F = F_v \sec^2\theta' \ldots\ldots\ldots\ldots \quad (15)$$

The foregoing derivation is necessary for an unabridged design and is repeated for completeness. For the present purpose, only Eq. 11 is necessary and it can be derived more directly.

In Fig. 3(*e*), for equilibrium (since surface A′B is a boundary and can have no stresses on it, and, since there can also be no horizontal shear on line A′C which is normal to it),

$$F\,dz \cos\theta' = F_v\,dz \sec\theta' \ldots\ldots\ldots\ldots \quad (16)$$

and

$$F = F_v \sec^2\theta' \ldots\ldots\ldots\ldots \quad (17)$$

Attention is called to the fact that the quantity F_v can be either a moment or a thrust per unit width.

STRESSES IN THE INTERIOR PARTS OF A SECTION

The stresses in the interior parts of a section differ from those at the fascia and the question is whether safety and economy permit the use of the fascia design for the entire width of structure. If this is possible, then it becomes unnecessary to use the skew-arch analysis for the computation of the total stresses—that is, neither the "exact" nor the approximate skew-arch analysis is necessary for design purposes. Its use then becomes largely restricted to analysis for research purposes only. Elsewhere,[5] the writer has given typical examples of a complete design, leading to the conclusion that the design of the fascia, only, is sufficient. Later designs, for single-span and multiple-span frames and full centered "arches" (frames with curved soffits) and for skew angles up to 50°, also showed that adequate designs can be made without recourse to complete skew-arch analysis.

In the "accurate" design of the sections, the total stresses must first be reduced to stresses per unit width; and then, by a process exactly the same as that used in the ellipse of stress for unit stresses, the maximum unit bending moments on principal planes, as well as the maximum unit torsion moments, are found. Although the maximum unit thrusts and shears may also be found, their computation usually is not necessary since the normal practice of determining maximum unit stresses from the moments is also applicable to this case.

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[5] "Design of a Reinforced Concrete Skew Arch," by Bernard L. Weiner, *Transactions*, ASCE, Vol. 96, 1932, pp. 1273–1281.

Formulas for computing the various moments and thrusts per unit width (see Fig. 4), which were developed elsewhere,[6] are

$$t_s = \frac{3}{2}\left(\frac{t_z - \epsilon' \, t_v}{b}\right)\left(1 - \frac{4\,z^2}{b^2}\right) + \epsilon' \, \frac{t_v}{b} + 12\,\frac{\epsilon' \, m_u \, z}{b^3} \quad \ldots\ldots\ldots (18)$$

$$t'_n = \epsilon' \, t_s \quad \ldots\ldots\ldots (19)$$

At the face of bridge,

$$m_s = \frac{\epsilon' \, m_z}{b} \quad \ldots\ldots\ldots (20a)$$

and, at any other point of the section,

$$m_s = \frac{m_v + \epsilon' \, m_z}{2\,b} \quad \ldots\ldots\ldots (20b)$$

At the face of bridge,

$$m' = \frac{(\epsilon')^2 \, m_z}{b} \quad \ldots\ldots\ldots (21a)$$

and, at any point of the section,

$$m' = \frac{\epsilon' \, m_v}{b} \quad \ldots\ldots\ldots (21b)$$

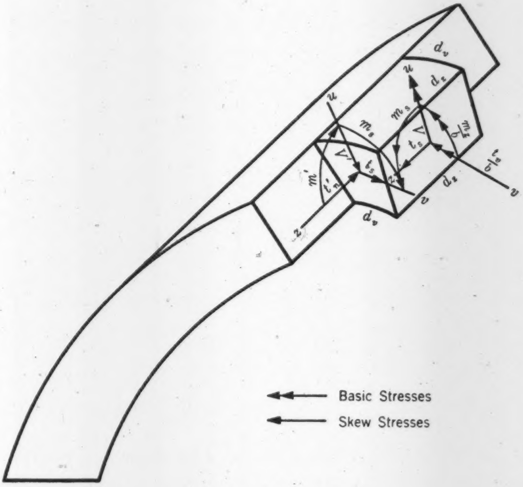

FIG. 4.—STRESSES PER UNIT WIDTH OF FRAME

At the boundary (face of bridge) only the basic stresses need be known, and all these are special cases of the generalized stress, F_v, in Eqs. 2, 4, 8, and 11. Equations for the radial shear (parallel to the u-axis or the y-axis) are not given as these shears are of such a magnitude that they do not govern the design in either a right or a skew solid-barrel frame.

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[6] "Design of a Reinforced Concrete Skew Arch," by Bernard L. Weiner, *Transactions*, ASCE, Vol. 96, 1932, pp. 1251–1260 and p. 1297.

All the formulas thus far developed are in terms of θ', the modified skew angle, which is the angle of skew in a plane tangent to the neutral surface at any section. The value of θ' at any section may be easily computed from the known skew angle, θ in the horizontal plane.

From Fig. 5 (writing $\epsilon = \tan\theta$ and $\epsilon' = \tan\theta$ for brevity), and

$$\epsilon' = \tan\theta' = \tan\theta\cos\phi \qquad (22)$$

$$\tan\phi' = \tan\phi\cos\phi \qquad (23)$$

The complete designs showed that for moderate skews (less than 35°) there is little difference at the crown between the fascia and the interior parts. Furthermore, at the haunches or springing lines of the frames there was practically no difference for all skew angles up to 50°, the stresses at the fascia always being slightly larger. For skew angles of from 35° to 50°, the crown stresses at the fascia were considerably larger than those in the interior, the ratio increasing with the higher skews. However, the loss in economy is not so great as would appear at first glance.

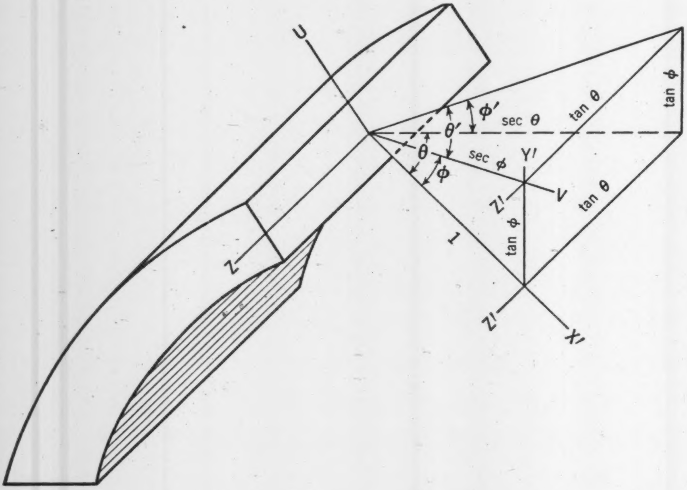

Fig. 5.—Geometry Showing Relationship Between θ and θ'

Analysis shows that the fascia section extends for a distance of three or four times the depth of the section—and, for safety, the maximum depth should be used. Thus, for a haunch depth of 5 ft (which is very common and which increases with the skew) the two fascias account for 40 ft of the width. Also, the crown steel extends for only a short distance; hence, the saving in steel would be slight. There would be no saving in concrete as it usually would not be practical to vary the depth of the section.

Furthermore, sound engineering judgment would indicate that the maximum section of the fascia should be used for the entire width. Although the writer believes that the distribution formulas (Eqs. 18 to 21) are based on logical assumptions, no exact quantitative theory or tests are available. Such tests as have been made are qualitative in nature and rather crude and of little value, even though they do tend to indicate that, at the fascia, the stresses are parallel to the boundary.[7] From the nature of the problem, the stress distribution at the fascia is more certain; and, since the difference becomes appreciable only for the larger skews, it seems to the writer that caution would indicate the use of the larger fascia stresses. In any case, until a theory or accurate test data become available, refinements do not seem to be justified.

The so-called skew stresses (total) are of such a nature, that, even if it is considered necessary to compute them, sufficiently accurate results for design could be obtained from charts that could be easily prepared.

Analysis shows that horizontal shears of considerable magnitude exist in a skew frame; but, since they are more in the nature of column shears, they are of no serious consequence. It can be shown, in the case of a column that, on a plane at 45° with the axis, shears exist which are equal to one half of the compression stress, or as high as 400 lb per sq in. or 500 lb per sq in. The largest shear found was at the crown of a frame with a 50° skew; it varied from about 200 lb per sq in. in the compression area to 100 lb per sq in. (in the opposite direction) in the tension area. At the haunch, the shears were of the same magnitude as that permitted in a beam fully reinforced for tension. In both cases, these high shears were at the center of width, *b*.

It is recommended, therefore, that the larger the skew, the greater the relative thickness of concrete should be; that is, as the skew increases, less compression reinforcement should be used, thus resulting in thicker crown sections. Skews of much more than 50° become impractical for purely geometric reasons; and, for this upper limit, it seems wise to make the crown sufficiently deep so that it can be designed without any compression steel. Although the single-span frame with a 50° skew, previously mentioned, was designed with rather heavy compression reinforcement and showed no signs of distress after almost 20 years of service, the practice is not recommended—unless a complete design is made.

Splayed Bridges and Bridges with Curved Faces

The history of the skew frame or arch bridge is usually prefaced by the statement that this type of structure was made necessary by the development of the high-speed automobile. The trend of fitting the bridge to the highway rather than the reverse practice, which was so common even about 1925, has now become more marked. Since entrances and exits to and from an express highway are logically to be found at grade separations, bridges with nonparallel faces are becoming quite common. The practice of fitting a straight bridge to a curved highway is gradually being abandoned. The result has been that bridges of the shapes (in plan view) shown in Figs. 6(*b*), 6(*c*), and 6(*d*) are being designed and built fairly frequently.

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[7] Discussion by B. L. Weiner of "An Analysis of Multiple-Skew Arches on Elastic Piers," by J. Charles Rathbun, *Transactions*, ASCE, Vol. 98, 1933, p. 46.

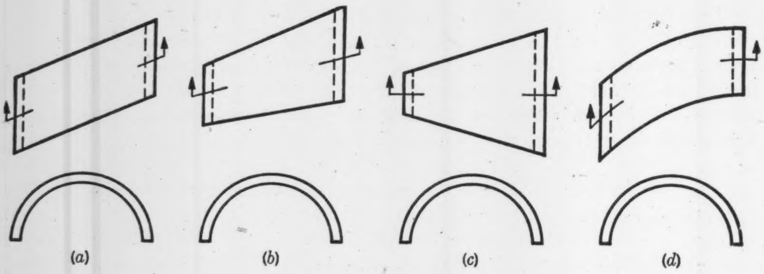

(a) (b) (c) (d)

FIG. 6

There is nothing implicit in the unabridged skew-arch theory which prevents its application to any of the structures with parallel abutments shown in Fig. 6. A modification of the definition of the "skew angle" (and the consequent trigonometric changes) is all that is necessary. With the simplified design already described, the representative "sample" frame for analyzing an equivalent right-angle structure of an identical square span is the same in

TABLE 1.—INFLUENCE VALUES FOR VERTICAL LOADS[a]

Section (1)	x_A (2)	y (3)	Δs (4)	t (5)	t^3 (6)	$y\frac{\Delta s}{I_c}$[b] (7)	$y^2\frac{\Delta s}{I_c}$ (8)	V_e[c] (9)	$\frac{\Delta x}{4.67}$ (10)	$V_e\frac{\Delta x}{4.67}$ (11)	$P_e\frac{\Delta x}{9.33}$ (12)	$\frac{M_e}{4.67}$[d] (13)	H (14)	$P_e x_A$ (15)
								0				992	0.437	
13	31.8	25.0	4.67	3.00	27.0	51.8	1,291		1	51.8	25.9			1,649
								51.8	↗			965	0.426	
12	27.2	24.9	4.67	3.10	29.8	46.8	1,168		1	98.6	23.4			1,270
								98.6	↗			890	0.393	
11	22.5	24.8	4.67	3.31	36.3	38.2	947		1	136.8	19.1			860
								136.8	↗			773	0.341	
10	17.8	24.5	4.67	3.68	49.8	27.6	675		1	164.4	13.8			492
								164.4	↗			622	0.275	
9	13.2	24.2	4.67	4.19	73.6	18.5	447		1	182.9	9.3			243
								182.9	↗			448	0.199	
8	8.5	23.9	4.67	4.82	112.0	12.0	286		1	194.9	6.0			102
								194.9	↗			259	0.115	
7	3.8	23.4	4.67	5.60	175.6	7.5	176		1	202.4	3.8			29
								202.4	↗			61	0.027	
6					∞				0.60	121.4				
								202.4	↗					
5	−1.1	17.9	4.54	5.30	148.9	6.5	117		−0.07	−14.6	−0.2			−7
								208.9	↗					
4	−0.8	13.3	4.54	4.65	100.5	7.3	97		−0.07	−15.1	−0.2			−6
								216.2	↗					
3	−0.5	8.8	4.54	4.00	64.0	7.5	66		−0.07	−15.7	−0.3			−4
								223.7	↗					
2	−0.2	4.3	4.54	3.35	37.6	6.2	26		−0.07	−16.1	−0.2			−1
								229.9	↗					
1	0	1.0	2.00	7.00	343	0.1	0		0	0	0			0
								230.0	↗					
Σ						230.0	$\begin{Bmatrix} 5{,}296 \\ 10{,}592 \end{Bmatrix}$			1,091.7	100.4			4,627

[a] See Fig. 7. [b] $y\frac{\Delta s}{I_c} = P_e$ is the "elastic load." [c] V_e is the "shear" resulting from the "elastic loads." [d] M_e is the moment resulting from the "elastic load."

TABLE 2.—DEAD-LOAD MOMENTS IN BARREL OF FRAME[a]
(IN KIPS AND FEET)

Section (1)	x_A (2)	x_B (3)	y (4)	P(b) (5)	HF(c) (6)	H(d) (7)	$P\left(\frac{x_B}{L}\right)$(e) (8)	$P\left(\frac{x_A}{L}\right)$(b) (9)	V (10)	Δx (11)	$V\,\Delta x$ (12)	$P\frac{\Delta x}{2}$ (13)	M_s(f) (14)	$H y$ (15)	M (16)
			23.0						−30.61				0	339	−339
6′R	67.6	0.7		1.89	0.015	0.028	0.02	1.87		1.50	− 43.1	1.4			
			23.2						−28.72	↗			44	342	−298
7R	64.5	3.8		5.56	0.070	0.389	0.31	5.25		4.67	−107.9	13.0			
			23.7						−23.16	↗			165	349	−184
8R	59.8	8.5		4.91	0.156	0.765	0.61	4.30		4.67	−85.3	11.5			
			24.1						−18.25	↗			262	355	− 93
9R	55.1	13.2		4.40	0.239	1.054	0.85	3.55		4.67	− 64.6	10.3			
			24.4						−13.85	↗			337	360	− 23
10R	50.5	17.8		3.97	0.310	1.230	1.03	2.94		4.67	− 46.1	9.3			
			24.7						− 9.88	↗			392	363	+ 29
11R	45.8	22.5		3.64	0.370	1.348	1.20	2.44		4.67	− 29.0	8.5			
			24.9						− 6.24	↗			430	366	+ 64
12R	41.1	27.2		3.40	0.411	1.400	1.35	2.05		4.67	− 13.2	7.9			
			25.0						− 2.84	↗			451	368	+ 83
13R	36.5	31.8		3.28	0.432	1.416	1.53	1.75		4.67	+ 2.1	7.7			
			25.0						+0.44	↗			457	369	+ 88
13L	31.8	36.5		3.22	0.432	1.390	1.72	1.50		4.67	+ 17.1	7.5			
									+ 3.66	↗			447	368	+ 79
12L	27.2	41.1		3.25	0.411	1.340	1.95	1.30		4.67	+ 32.2	7.6			
									+ 6.91	↗			422	366	+ 56
11L	22.5	45.8		3.39	0.370	1.253	2.27	1.12		4.67	+ 48.1	7.9			
									+10.30	↗			382	363	+ 19
10L	17.8	50.5		3.61	0.310	1.119	2.67	0.94		4.67	+ 65.1	8.4			
									+13.91	↗			325	360	− 35
9L	13.2	55.1		3.94	0.239	0.942	3.18	0.76		4.67	+ 83.5	9.2			
									+17.85	↗			251	355	−104
8L	8.5	59.8		4.38	0.156	0.685	3.84	0.54		4.67	+ 104.0	10.2			
									+22.23	↗			157	349	−192
7L	3.8	64.5		4.93	0.070	0.345	4.65	0.28		4.67	+126.8	11.5			
									+27.16	↗			42	342	−300
6′L	0.7	67.6		1.66	0.015	0.025	1.64	0.02		1.50	+ 43.2	1.2			
									+28.82	↗			0	339	−339
Σ				59.43		14.729	28.82	30.61[a]							

[a] See Fig. 7. [b] P is the weight of a section of dead load. [c] HF is the H-influence value or factor scaled from the influence line in Fig. 7. [d] Col. 7 = Col. 5 × Col. 6. [e] Cols. 8 and 9 are equal to vertical reactions R_A and R_B, respectively. [f] M_s is the "simple beam" moment.

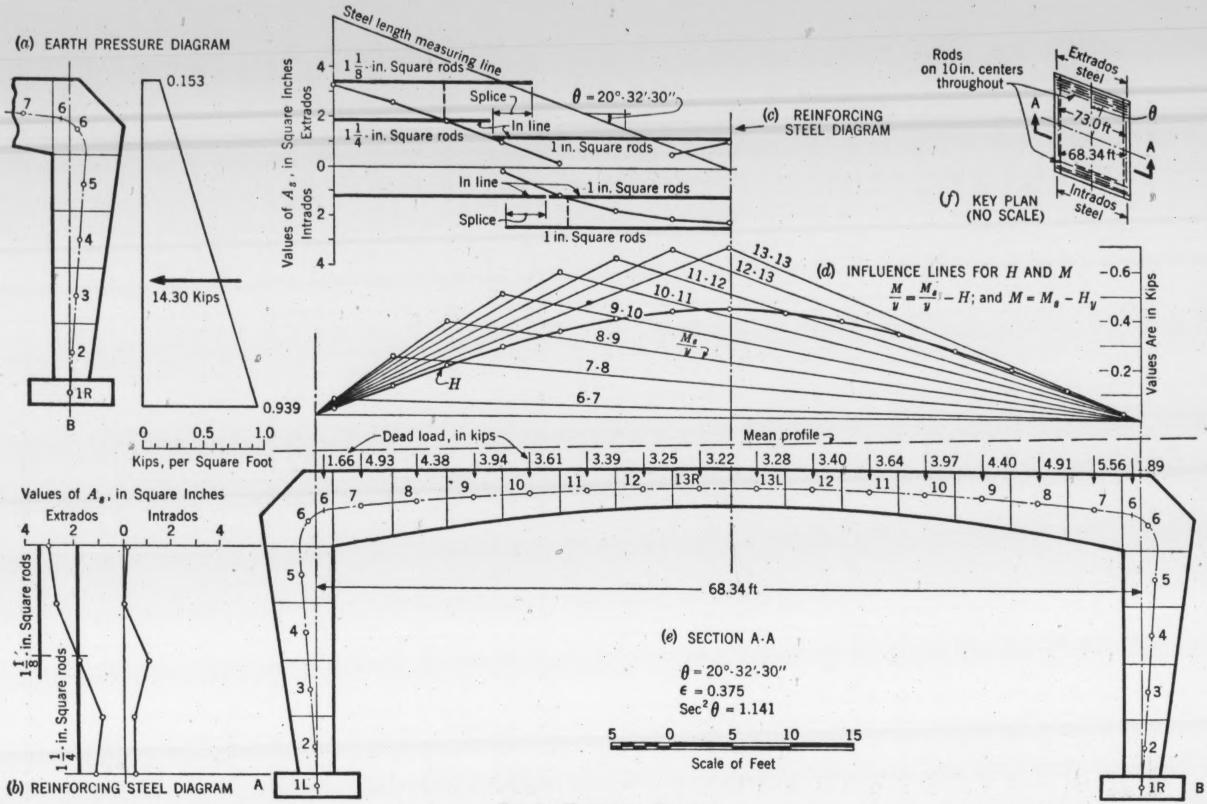
cross section for all four structures shown in Fig. 6. One modification is necessary, however.

It was stated previously that *b*, the skew width (measured parallel to the abutments), is canceled from both sides of the equations. It was also pointed

TABLE 3.—DEAD-LOAD MOMENTS AND NORMAL THRUSTS IN "LEGS"[a]
(IN KIPS AND FEET)

Section No.	x	y	(a) LEFT "LEG"				(b) RIGHT "LEG"		
			$R_A\,x_A$(b)	$H\,y$	M	N	$R_B\,x_B$	M	N
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(4)	(6)	(7)
5–6	−1.25	20.2	−36	298	−334	34	−38	−336	36
4–5	−0.95	15.6	−27	230	−257	38	−29	−259	39
3–4	−0.65	11.1	−19	164	−183	41	−20	−184	43
2–3	−0.35	6.5	−10	96	−106	43	−11	−107	44
1–2	0	2.0	0	30	− 30	45	0	− 30	46

[a] See Fig. 7. [b] $R_A\,x_A = M_s$, the "simple beam" moment.



(a) EARTH PRESSURE DIAGRAM
0.153
14.30 Kips
0.939
0 0.5 1.0
Kips, per Square Foot
B
1R
7 6 6 5 4 3 2
(b) REINFORCING STEEL DIAGRAM
Values of A_s, in Square Inches
Extrados
Intrados
4 2 0 2 4
$1\frac{1}{8}$-in. Square rods
$1\frac{1}{4}$-in. Square rods
Steel length measuring line
$1\frac{1}{8}$-in. Square rods
$1\frac{1}{4}$-in. Square rods
1 in. Square rods
1 in. Square rods
1 in. Square rods
Splice
In line
In line
Splice
$\theta = 20°\text{-}32'\text{-}30''$
(c) REINFORCING STEEL DIAGRAM
Values of A_s, in Square Inches
Extrados
Intrados
4 2 0 2 4
Rods on 10 in. centers throughout
Extrados steel
Intrados steel
73.0 ft
68.34 ft
θ
A
A
(f) KEY PLAN (NO SCALE)
(d) INFLUENCE LINES FOR H AND M
$\frac{M}{y} = \frac{M_s}{y} - H$; and $M = M_s - H_y$
13·13
12·13
11·12
10·11
9·10
8·9
7·8
6·7
$\frac{M_s}{y}$
H
-0.6
-0.4
-0.2
Values Are in Kips
Dead load, in kips
Mean profile
1.66 4.93 4.38 3.94 3.61 3.39 3.25 3.22 3.28 3.40 3.64 3.97 4.40 4.91 5.56 1.89
6 7 8 9 10 11 12 13R 13L 12 11 10 9 8 7 6
6 5 4 3 2 1L
6 5 4 3 2 1R
A
B
68.34 ft
(e) SECTION A-A
$\theta = 20°\text{-}32'\text{-}30''$
$\epsilon = 0.375$
$\text{Sec}^2\,\theta = 1.141$
5 0 5 10 15
Scale of Feet

FIG. 7.—NUMERICAL EXAMPLE

TABLE 4.—Temperature Change Moments[a] (in Kips and Feet)

Section No. (1)	y (2)	$H\,y$[b] (+30°) (3)	$H\,y$[b] (−40°) (4)
13–13	25.0	−18	24
12–13	25.0	−18	24
11–12	24.9	−18	24
10–11	24.7	−18	24
9–10	24.4	−18	23
8– 9	24.1	−18	23
7– 8	23.7	−17	23
6– 7	23.2	−17	22
5– 6	20.2	−15	19
4– 5	15.6	−11	15
3– 4	11.1	−8	11
2– 3	6.5	−5	6
1– 2	2.0	−2	2
6′	23.0	−16	22

[a] See Fig. 7. [b] $H\,y$ is the final moment (since the vertical reactions are zero).

TABLE 5.—"Simple Beam" Earth-Pressure Moments in "Legs"[a] (in Kips and Feet)

Section No. (1)	x_B (2)	y (3)	$R\,x_B$ (4)	$Q\,y$ (5)	$-0.939\frac{y^2}{2}$ (6)	$0.005\,y^3$ (7)	M_s[b] (8)
5R	−1.1	17.9	2.3	256.0	−150.2	28.6	136.7
4R	−0.8	13.3	1.7	190.0	−83.0	11.8	120.5
3R	−0.5	8.8	1.0	126.0	−36.3	3.4	84.1
2R	−0.2	4.3	0.4	61.5	−8.7	0.4	53.6
1R	0.0	1.0	0.0	14.3	−0.5	0.0	13.8
5–6R	−1.25	20.2	2.6	289.0	−191.0	41.2	141.8
4–5R	−0.95	15.6	2.0	223.0	−114.0	18.9	129.9
3–4R	−0.65	11.1	1.4	159.0	−57.7	6.9	109.6
2–3R	−0.35	6.5	0.7	93.0	−19.8	1.4	75.3
1–2R	0.0	2.0	0.0	28.6	−1.9	0.0	26.7

[a] See Fig. 7. [b] M_s is the "simple beam" moment.

TABLE 6.—Earth-Pressure Horizontal Reactions[a] (in Kips and Feet)

Section No. (1)	x_A (2)	M_s[b] (3)	$y\frac{\Delta s}{I}$ (4)	$M_s\,y\frac{\Delta s}{I}$ (5)
1R		13.8[c]	0.1	....
2R		53.6	6.2	332
3R		84.1	7.5	630
4R		120.5	7.3	880
5R		136.7[c]	6.5	889
6R		....	....	....
7R	64.5	134.2	7.5	1,006
8R	59.8	124.3	12.0	1,490
9R	55.1	114.9	18.5	2,120
10R	50.5	105.0	27.6	2,900
11R	45.8	95.5	38.2	3,650
12R	41.1	85.5	46.8	4,000
13R	36.5	76.0	51.8	3,930
13L	31.8	66.1	51.8	3,420
12L	27.2	56.5	46.8	2,640
11L	22.5	46.8	38.2	1,790
10L	17.8	37.0	27.6	1,025
9L	13.2	27.4	18.5	507
8L	8.5	17.7	12.0	212
7L	3.8	7.9	7.5	59
6L	....	....	....	....
5L	−1.1	−2.3	6.5	−15
4L	−0.8	−1.7	7.3	−12
3L	−0.5	−1.0	7.5	−8
2L	−0.2	−0.4	6.2	−2
1L	0.0	0.0	0.1	0
Σ				31,444

[a] See Fig. 7. [b] M_s is the "simple beam moment." [c] See Table 5.

TABLE 7.—Earth-Pressure Moments[a] (in Kips and Feet)

Section No. (1)	x_A (2)	y (3)	M_s[b] (4)	$H_A\,y$ (5)	M (6)
2– 1R		2.0	26.7[c]	5.9	20.8
3– 2R		6.5	75.3	19.3	56.0
4– 3R		11.1	109.6	33.0	76.6
5– 4R		15.6	129.9	46.3	83.6
6– 5R		20.2	141.8[c]	60.0	81.8
7– 6R	66.8	23.2	139.0	69.0	70.0
8– 7R	62.2	23.7	128.7	70.5	58.2
9– 8R	57.5	24.1	119.8	71.5	48.3
10– 9R	52.8	24.4	110.0	72.5	37.5
11–10R	48.2	24.7	100.4	73.5	26.9
12–11R	43.5	24.9	90.5	74.0	16.5
13–12R	38.8	25.0	80.8	74.2	6.6
13R–13L	34.2	25.0	71.1	74.2	−3.1
12–13L	29.5		61.4	74.2	−12.8
11–12L	24.8		51.5	74.2	−22.7
10–11L	20.2		42.0	73.5	−31.5
9–10L	15.5		32.2	72.5	−40.3
8– 9L	10.8		22.4	71.5	−49.1
7– 8L	6.2		12.9	70.5	−57.6
6– 7L	1.5		3.1	69.0	−65.9
5– 6L			−2.6	60.0	−57.4
4– 5L			−2.0	46.3	−44.3
3– 4L			−1.4	33.0	−31.6
2– 3L			−0.7	19.3	−18.6
1– 2L			0.0	5.9	−5.9
6′R	68.3	23.0	142.1	68.3	+73.8
6′L	0	23.0	0	68.3	−68.3

[a] See Fig. 7. [b] M_s is the "simple beam moment." [c] See Table 5.

out that this was merely a numerical convenience and not a necessity. For Figs. 6(*b*), 6(*c*), and 6(*d*), b cannot be canceled out. However, the equations can still be divided by the width of some section (or by some arbitrary number, for that matter) to reduce the size of the numbers handled. If the former procedure is followed, the value of I used will, in effect, be divided by a magnitude b_0 and the resulting total stresses instead of being divided by b will now be divided by b_0. However, to obtain the total stress at any section, the stress computed will have to be multiplied by b_0/b and the result can then be treated as in the analysis of a skew frame.

TABLE 8.—Summary of Basic Stresses;

Line	Loading	(a) Barrel of the Frame (ϕ = 0°; θ' = θ; and sec² θ'											
		6–7L		7–8L		8–9L		9–10L, R		10–11L, R		11–12R	
		M	*N*	*M*	*N*	*M*	*N*	*M*	*N*	*M*	*N*	*M*	*N*
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
1	Dead load (DL)........	−300	15	−192	15	−104	15	− 35L − 23R	15 15	+ 29R + 19L	15 15	+ 64	15
	Earth Pressure (EP)—												
2	Near side............	+ 70	3	+ 58	3	+ 48	3	+ 38	3	+ 27	3	+ 17	3
3	Far side.............	− 66	3	− 58	3	− 49	3	− 40	3	− 32	3	− 23	3
	Live Load (LL)—												
4	+*M*................							+ 19	1	+ 22	1	+ 23	2
5	−*M*................	− 45	2	− 33	2	− 23	2	− 16	2	− 10	1		
	Temperature Change ($\Delta\tau$)												
6	+30°................	− 17	1	− 17	1	− 18	1	− 18	1	− 18	1	− 18	1
7	−40°................	+ 22	−1	+ 23	−1	+ 23	−1	+ 23	−1	+ 24	−1	+ 24	−1
8	$\overline{\text{DL}}+2\overline{\text{EP}}$ (optional).....	−300	15	−192	15	−105	21	− 37	21	+ 29	15	+ 64	15
9[a]	Line 8 + $\overline{\text{LL}}$............	−345	17	−225	17	−128	23	− 53	23	+ 51	16	+ 87	17
10[a]	$\overline{\text{DL}}+\overline{\text{LL}}+1\,\overline{\text{EP}}+\Delta\tau$.....	−428	21	−300	21	−194	21	−109L + 57R	21 18	−102R − 41 L	18 20	+128	19

[a] Lines 9 and 10 are cases

For the design of the sections of the bridges shown in Figs. 6(*b*) and 6(*c*), the two faces are designed separately, each for its own particular skew angle. The change in the reinforcing steel, from the direction parallel to one face to the direction parallel to the other, can be made either by "fanning" the rods near the middle of the structure or by intersecting the two sets of steel at some convenient section, according to the practice of the individual designer.

In the case of the curved bridge shown in Fig. 6(*d*), the skew angle varies at every point; but this is no great hardship. As the radii of the curves will be fairly large (for practical cases), the variation will not be very rapid and only a few skew angles need to be used in the design of the sections. Even if the maximum angle is used, no great loss in economy will result. If the two faces of the bridge are concentric arcs, the reinforcing steel can be curved and placed parallel to the faces of the structure. If the faces are not parallel, modifications similar to those suggested for splayed bridges can be used by the designer. The practice of curving the steel is not new. Bridges with curved soffits require steel bent in a vertical plane; curving in a horizontal plane should present no difficulty.

DESIGN OF A SINGLE-SPAN, SKEW, RIGID FRAME; NUMERICAL EXAMPLE

The data for the design of an actual rigid frame are given in Fig. 7. Hinged ends are assumed, and a correction is made, subsequently (in accordance with current practice), for the fixity of the footings. Therefore, the right-frame, variable section design is a typical example of standard office procedure, and Tables 1 to 9 need little further explanation. "Elastic loads" are used to obtain the constants for computing influence lines. The temperature stresses are modified as explained in the text. Table 9 demonstrates the final step, the design of the sections, which (except for the correction required by the modified skew angle θ') is little different from a right-frame design.

SEE FIG. 7 (IN KIPS AND FEET)

= 1.141)				(b) "LEGS" ($\phi = 90°$; $\theta' = 0°$; AND $\sec^2 \theta' = 1$)													
12–13R		13–13		6′R		5–6R		4–5R		3–4R		2–3R		1–2R		Line
M (15)	*N* (16)	*M* (17)	*N* (18)	*M* (19)	*N* (20)	*M* (21)	*N* (22)	*M* (23)	*N* (24)	*M* (25)	*N* (26)	*M* (27)	*N* (28)	*M* (29)	*N* (30)	
+ 83	15	+ 88	15	−399	33	−336	36	−259	39	−184	43	−107	44	−30	46	1
+ 7	3	− 3	3	+ 74	−2	+ 82	−2	− 84	−2	+ 77	−2	+ 56	−2	+21	−2	2
− 13	3	− 3	3	− 68	2	− 57	2	− 44	2	− 32	2	− 19	2	− 6	2	3
+ 23	2	+ 23	2													4
				− 48	6	− 43	6	− 34	6	− 24	6	− 14	6	− 4	6	5
− 18	1	− 18	1	− 16	0	− 15	0	− 11	0	− 8	0	− 5	0	− 2	0	6
+ 24	−1	+ 24	−1	+ 22	0	+ 19	0	+ 15	0	+ 11	0	+ 6	0	+ 2	0	7
+ 83	15	+ 88	15	−399	33	−336	36	−259	39	−184	43	−107	44	−30	46	8
+106	17	+111	17	−387	39	−379	42	−293	35	−208	49	−121	50	−34	52	9[a]
+137	19	+135	16	−471	41	−451	44	−348	47	−248	51	−145	52	−42	54	10[a]

I and II, respectively.

In Table 1, the data in Cols. 2 and 5 are scaled from the layout in Fig. 7, (x_A and x_B being the abscissas measured from points A and B, respectively, and y being the ordinate measured from line AB), from which the values in Cols. 6, 7, and 8 follow directly by slide-rule computation. The values for $\left(y \frac{\Delta s}{I}\right)$ in Col. 7 can be considered the "elastic loads," P_e, in order to compute the deflections $\Sigma M_s \, y \frac{\Delta s}{I}$ for substitution in the formula:

$$H = \frac{\Sigma M \, y \frac{\Delta s}{I}}{\Sigma y^2 \frac{\Delta s}{I}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

which results in the influence line for vertical loads. In Col. 9, V_e is the vertical "elastic shear" resulting from the "elastic loads," P_e. Since the structure is symmetrical, the sum of the loads for half the bridge is the reaction—that is, 230.0 from Col. 7. In the elastic load method, the "elastic moment" at any point is the deflection at that point. Cols. 9 to 13, inclusive, show the computation of these moments by shear and moment increments. The horizontal reactions, H, in Col. 14 are obtained by dividing the values of M_e from Col. 13 by $\Sigma y^2 \dfrac{\Delta s}{I} = 10{,}592$ from Col. 8.

TABLE 9.—DESIGN

Description	Case I
Concrete stress, f_c, in pounds per square inch	800
Steel stress, f_s, in pounds per square inch	20,000
$K_1 = \left(K = \dfrac{M_s}{b\,d^2}\right)$ for a balanced section, in pounds per square inch	116
$a = \dfrac{f_s\,j}{12}$, which for a balanced section, is	1.49

Section No. (1)	Case (2)	$M \sec^2 \theta'$ (K') (3)	$N \sec^2 \theta'$ (K) (4)	t (in.) (5)	d (in.) (6)	r (ft)[a] (7)	$N' r$ (kip-feet) (8)	M_s (kip-feet)[b] (9)	K (lb per sq in.) (10)
13–13	I	127.0	17.1	36	33.5	1.29	22.1	149	133
13–13	II	154.2	19.4	36	33.5	1.29	25.0	179	160
12–13	I	121.0	19.4	37	34.5	1.33	25.8	147	124
12–13	II	156.5	21.7	37	34.5	1.33	28.9	185	156
11–12	II	141.8	21.7	38	35.5	1.37	29.8	172	137
10–11	II	116.6	20.6	42	39.5	1.54	31.8	1[illegible]	96
10–11	II	−46.7	22.8	42	39.5	1.54	35.1	82	53
9–10	II	65.0	20.6	47	44.5	1.75	36.0	101	51
9–10	II	−124.5	24.0	47	44.5	1.75	42.0	167	84
8– 9	II	−221.5	24.0	54	51.5	2.04	49.0	270	102
7– 8	II	−342	24.0	62	59.5	2.37	56.9	399	113
6– 7	I	−396	19.3	72	69.5	2.79	53.8	450	93
6– 7	II	−487	24.0	72	69.5	2.79	66.9	554	115
5– 6	I	−379	42	67	64.5	2.58	108.7	488	118
5– 6	II	−451	44	67	64.5	2.58	113.6	565	136
4– 5	I	−293	35	59	56.5	2.25	78.8	372	117
3– 4	I	−208	49	51	48.5	1.92	94.0	302	129
2– 3	I	−121	50	43	40.5	1.58	79.0	200	122
1– 2	I	− 34	52	36	33.5	1.29	67.0	101	90
1– 2	I	85	52	36	33.5	1.29	67.0	152	136

[a] $r = \dfrac{t}{2} - 0.21$. [b] $M_s = M' + N' r$. [c] $M_2 = M_s \left(\dfrac{K - K_1}{K}\right)$.

By symmetry, the "elastic moment" at the center line equals $\Sigma(P_e\, x_A)$ which is the value of the deflection $\Sigma \left(M\, y \dfrac{\Delta s}{I}\right)$ at this point, and substituting the summation from Col. 15 into the formula for the horizontal reaction: $H = \frac{4{,}630}{10{,}590} = 0.437$, which is the same as the corresponding value in Col. 14. Thus Col. 15 serves as a check on the numerical work leading up to and including Col. 14.

In Table 2, the horizontal reaction for dead load is obtained by multiplying the weight of a section by the influence line ordinate ($H\ F$) scaled from the diagram in Fig. 7. The computation of the "simple beam moment" and the negative moment $H\ y$ is obvious. The former moment is computed by shear and moment increments. The value of H computed in Table 2 is also used in Table 3.

OF SECTIONS

Case II
1,000 $n = 12$
25,000 $k = 0.324$
145 $j = 0.892$
186 $p = 0.0065$

M_2 (kip-feet)[c] (11)	$C_2 = T_2$ (kips)[d] (12)	KIPS PER SQUARE INCH		SQUARE INCHES			$\frac{N'}{f_s}$ (sq in.) (18)	A_s (sq in.) (19)	Section No.
		f_c (13)	f'_s (14)	A'_s[e] (15)	A_{s1} (16)	A_{s2}[e] (17)			
19.0	7.4	20	6.7	1.10	2.61	0.37	0.85	2.13	13–13
16.8	6.5		8.4		2.61	0.26	0.78	2.09	13–13
9.5	3.6	20	6.8	0.53	2.69	0.18	0.87		12–13
13.1	4.9	25	8.5	0.58	2.69	0.20	0.87	2.02	12–13
		25			2.59		0.87	1.72	11–12
		25			1.97		0.82	1.15	10–11
		25			1.08		0.91	0.17	10–11
		25			1.22		0.82	0.40	9–10
		25			1.97		0.96	1.01	9–10
		25			2.77		0.96	1.81	8– 9
		25			3.56		0.96	2.60	7– 8
		20			4.41		0.77	3.64	6– 7
		25			4.21		0.96	3.25	6– 7
8.3	1.6	20	7.7	0.21	5.03	0.08	2.10	3.01	5– 6
		25			4.70		1.76	2.94	5– 6
3.2	0.7	20	7.6	0.09	4.40	0.04	1.75	2.69	4– 5
30.4	7.9	20	7.4	1.07	3.78	0.40	2.45	1.73	3– 4
9.8	3.1	20	7.1	0.44	3.16	0.16	2.50	0.82	2– 3
		20			1.59		2.60	Nominal	1– 2
22.4	8.7	20	6.7	1.13	2.62	0.44	2.60	0.46	1– 2

[d] $C_2 = T_2 = \frac{M_2}{2\ r}$. [e] A'_s (Col. 15) $= \frac{C_2}{f'_s}$; and A_{s2} (Col. 17) $= \frac{T_2}{f_s}$.

To obtain the horizontal reactions for temperature change, use is made of the relationship:

$$H = H'' \sec^2 \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

in which H is the "basic" horizontal reaction in a skew frame and H'' is the corresponding reaction in a right frame. Since

$$H'' = \frac{c\,t\,L}{\Sigma y^2 \dfrac{\Delta s}{E\,I}} = \frac{E\,c\,t\,L}{\Sigma y^2 \dfrac{\Delta s}{I}} \ldots\ldots\ldots\ldots\ldots\ldots (26)$$

it is easily computed. Using kip and foot units, $E\,c\,t\,L = 432{,}000 \times 0.0000065 \times 100° \times 68.3 = 19{,}200$; $\sec^2 \theta = 1.141$; and $\Sigma\left(y^2 \dfrac{\Delta s}{I_c}\right) = 10{,}590$.

Assuming that I_s, the moment of inertia of the steel, equals 15% of I_c, the moment of inertia of the concrete; and, since $I = I_c + I_s = 1.15\,I_c$, $H = \dfrac{19{,}200 \times 1.15 \times 1.141}{10{,}590} = 2.38$ kips for + 100° F. Therefore, $H = 0.715$ kips for + 30° F; and H = − 0.951 kips for − 40° F. With these values for H, Table 4 is readily computed.

To obtain the loads Q and "simple beam" reactions R_A and R_B for earth pressure acting on one side of the structure (Fig. 7), moments are taken about point B:

$$\begin{aligned} 0.153 \times 13.10 &= 2.00 \text{ kips} \times 17.47 = 34.9 \text{ kip-ft} \\ 0.939 \times 13.10 &= 12.30 \text{ kips} \times 8.73 = 107.2 \text{ kip-ft} \\ Q &= 14.30 \text{ kips} \qquad M_B = 142.1 \text{ kip-ft} \end{aligned}$$

The vertical reaction $R_A = \dfrac{142.1}{68.3} = 2.08$ kips $= -R_B$, from which Table 5 follows directly. This table is a preparatory computation and simplifies the work involved in preparing subsequent tables.

To compute the horizontal reactions for earth pressure acting on one side of the structure, use is again made of Eq. 24. The numerator of this equation is computed in Table 6 using some of the values from Table 5, from which $H_A = \dfrac{31{,}400}{10{,}590} = 2.97$ kips. Since H is now known, the moments due to earth pressure acting on one side of the structure are computed in Table 7. The remaining values computed in Table 5 are brought over to this table.

Table 8 shows a summary of the "basic moments and thrusts," M and N, respectively, for the loading conditions already computed. The live-load moments and thrusts are also given. These are obtained directly from the influence lines shown in Fig. 7 and need no further explanation.

The assumption of hinged footings has been found to be slightly on the side of safety at the crown and immaterial at the springing line. The only sections that need further investigation are at the top of footings, as may be seen from Fig. 8. However, even these sections have sufficient reinforcing steel in the form of dowels to resist the positive moment that may be induced by the fixity of the footing. A close estimate of the steel required for this moment can be made by assuming an average value for the moment at the bottom of the fixed footing to be about −40% of the moment at point 6′ (see Fig. 7): $M_{1-2} = 0.4\,M_{6'} - 1.4\,M_{6'} \times \frac{3}{23} = 0.22\,M_{6'} = 0.22 \times 387$ kip-ft. Therefore $M_{1-2} = 85$ kip-ft, the thrust remaining practically the same as for the hinged condition.

Table 9 shows the "design of the sections" and except for the fact that the "basic moments and thrusts" are multiplied by $\sec^2 \theta'$ (the values of which are given in the table), this computation is otherwise exactly the same as that for a right-angle structure.

Fig. 7 shows a typical main-steel pattern for a rigid frame. The spacing given is measured normal to the steel that is parallel to the fascia. To find the true length of the bars, they are projected onto and scaled along the "steel length measuring line."

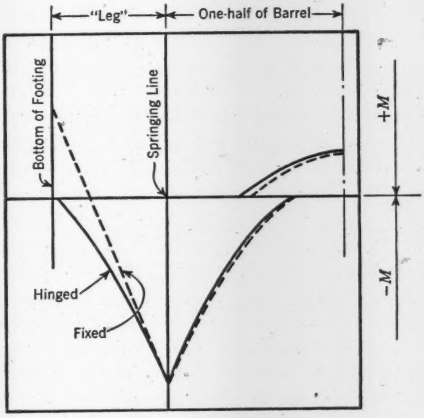


FIG. 8.—RELAITVE MAXIMUM MOMENTS FOR HINGED AND FIXED FOOTINGS

CONCLUSION

A skew frame can be designed with little more labor than that required for a similar right-angle frame. For clarity, the salient features of a design are given in a numerical example (under the heading, "Design of a Single-Span, Skew, Rigid Frame; Numerical Example") based on the following considerations:

1. With more than sufficient accuracy for practical design purposes, the "basic stresses" and the "skew stresses" (total) may be separated and treated as separate elastic systems.
2. The basic stresses, except for temperature and shrinkage, are very closely equal to those in a similar right-angle structure with the same square span and with the same elastic properties. For temperature and shrinkage, the basic stresses obtained from the same right-frame analysis are multiplied by the square of the secant of the skew angle.
3. For the fascia section, only the basic stresses need to be known. The resultant stresses must be parallel to the boundary (free surface or face of bridge) from which it follows that the bending moment and thrust per unit width are multiplied by the square of the secant of the modified skew angle (θ'). The sections are then designed for these increased stresses in the same way as for a right-angle structure.
4. The fascia section is adequate for the entire width of structure from the points of view of both safety and economy.

Regardless of other considerations, the stresses at the fascia must have a resultant that is parallel to the boundary. This requirement follows entirely from considerations of equilibrium and is quite independent of any and all skew-arch theories. The stresses at the fascia being more certain, therefore, sound judgment should dictate the use of the fascia design, which is on the side of safety, for the entire width of structure. Without good experimental data, refinements do not seem to be justified.

Attention is also called to the fact that the total stresses—however computed—must first be reduced to stresses per unit width before the sections can be designed. In 1930, H. M. Westergaard, M. ASCE, described an analogous case of such treatment, including the finding of maximum moments per unit width by an extension of the principles involved in the ellipse of stress.[8]

TRANSVERSE REINFORCEMENT

The question of transverse reinforcement is as much a problem in the design of right-angle frames as it is in the design of skew frames. The latter, as has been shown herein, needs no special treatment; in either case the reinforcement is required for transverse load and stress distribution. As far as dead load is concerned, only nominal reinforcement is required since any unbalance is not likely to be very large even for a large skew and a steep top grade. However, reinforcement for the distribution of unbalanced live load—unbalance in the transverse direction—is necessary.

It is suggested that a fair estimate of the transverse reinforcement can be obtained by considering the horizontal member of the frame as a simply supported slab between points of contraflexure. Using the distance between the points of contraflexure as the span and the average moment of inertia between these two points, the transverse steel necessary to distribute the concentrated live load may be obtained in the same manner as for any ordinary roadway bridge slab. Mr. Westergaard's equations[8] or any of the numerous approximations give reasonable results.

Theoretically, since no live load is applied directly to the "legs," no transverse reinforcement is necessary. However, the same transverse reinforcement used in the barrel is generally also used in the "legs." Since the percentage of steel computed as suggested in the previous paragraph is rather light, there is no loss in using the same steel in the "legs." In fact, the steel required is no greater than the quantity that would normally be used for temperature and shrinkage reinforcement.

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[8] "Computation of Stresses in Bridge Slabs Due to Wheel Loads," by H. M. Westergaard, *Public Roads*, March, 1930, p. 1.

DISCUSSION

Maurice Barron,[9] M. ASCE.—In several respects, this paper corroborates and confirms the methods, procedures, and conclusions of a previous paper and discussion on this subject by the writer. It is interesting to compare the conclusions and arguments presented by the author with similar items in the writer's paper:

1a (Weiner)	1b (Barron)[10]
"1. With more than sufficient accuracy for practical design purposes, the 'basic stresses' and the 'skew stresses' (total) may be separated and treated as separate elastic systems."	"It has been demonstrated that for all practical design purposes: "1. With few exceptions, the rectangular elastic system is independent of the torsional elastic system; "2. The rectangular redundants and the rectangular stresses for all applied loads are independent * * * of the skew angle; * * *"
2a (Weiner)	2b (Barron)[10]
"2. The basic stresses, except for temperature and shrinkage, are very closely equal to those in a similar right-angle structure with the same square span and with the same elastic properties. For temperature and shrinkage, the basic stresses obtained from the same right-frame analysis are multiplied by the square of the secant of the skew angle."	"5. The rectangular redundants due to temperature change vary as the $\sec^2$—that is, as $(1 + \epsilon^2)$; * * * "6. The torsional stresses may be written directly from the rectangular stresses by multiplying an appropriate rectangular stress by an appropriate geometric or elastic constant, thereby completely eliminating the torsional elastic analysis; * * *"
3a (Weiner)	3b (Barron)[10]
"* * * It will be shown that, in most cases, the 'skew stresses' need not be computed at all for design purposes. Designs that are adequate for both safety and economy can be made by applying a rather simple correction to a right-frame design. "The right-frame analysis for the total stresses can be made by any of the methods in common use and this phase of the problem, except for one detail, therefore, needs no further discussion * * *"	"12. For practical office procedure no torsional analysis is required, and the longitudinal steel may be determined from the rectangular design using the 'turning process'; * * * "10. Any method that will determine the rectangular stresses satisfactorily may be used as the basis for determining the torsional stresses (such other methods include moment distribution and slope deflection); * * *"

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[9] Cons. Engr., Farkas & Barron, New York, N. Y.

[10] "Reinforced Concrete Skewed Rigid-Frame and Arch Bridges," by Maurice Barron, *Proceedings-Separate No. 18*, ASCE, April, 1950.

4a (Weiner)

"It has been found in the general case, however, that, where the neutral surfaces of the several members are neither straight nor vertical and horizontal, there is little error in considering the two sets of stresses independently and in analyzing them as two separate elastic systems. * * * The approximation holds good for any number of spans and for any shape of frame or arch that would be encountered in practical design."

4b (Barron)[11]

"* * * The evidence shows that the two elastic systems of single-span skewed structures are practically independent for vertical and horizontal loads and equally independent of the shape of the structure. A double-span structure may be considered as a single span with an elastic bracket at the center, to the ends of which are applied forces equal to the redundant reactions. This logic may be extended to include three or more spans."

There are several other significant points of agreement in the two papers. It should be noted that the author states his proposed method has been checked by comparing the theoretical results with previous designs, and the presentation of substantiating data would be helpful. For example, the opening statement of the paper is "No new theories are presented in this paper." As a matter of fact the paper makes use of several supplementary sources of new information and ideas, several of which are not properly cited.

There are three contradictions in this paper which should be reconciled so that an inexperienced reader will not be confused:

5a.—In the third paragraph of the "Introduction"—

> "For 'small' skews, the expedient of designing on the skew span has often been used. A consideration of the arguments in this paper will show that this expedient is entirely incorrect; it leads to results that either are on the side of danger or are very wasteful."

whereas (item 5b), in the fourth paragraph—

> "The thickness of the members and the amount of reinforcing steel do not appear excessive or unusual compared to that required for a right-angle structure of span equal approximately to the skew span. Such skew structures have been built and have been in service from 6 years to 20 years without showing signs of distress."

6a.—In the first paragraph of "Characteristics of Skew Frames"—

> "The practical design of most structures—as distinct from the theoretical analysis for research purposes—may be divided into two steps:
>
> 1. The computation of the total moments, thrusts, and shears; and
> 2. The design of the sections to resist the total stresses computed in step 1."

whereas (item 6b) in the second paragraph—

> "In the case of wide structures such as slabs, total stresses have little significance and are practically useless for design purposes."

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[11] Discussion by Maurice Barron of "Simplified Analysis of Skewed Reinforced Concrete Frames and Arches," by Richard M. Hodges, *Transactions*, ASCE, Vol. 109, 1944, p. 966.

7a.—In the last paragraph of "Characteristics of Skew Frames"—

> "In theory, then, the skew frame is treated as a whole in three dimensions, and dividing by *b* is merely an arithmetical convenience."

whereas (item 7b), in the paragraph immediately preceding—

> "It is found, however, that the skew width *b*, measured parallel to the abutments, can be factored out of all the terms of the equations but one, the so-called 'factor of torsion.' "

A comparison of the contents of other papers and discussions with the contents of this paper reveals a repetition of contradictions that have caused confusion since the early 1930's. The author can perform a very worthy service at this time by reconciling the conflicts. The following four instances are indicative:

8a.—In the paragraphs following Eq. 23, Mr. Weiner states:

> "* * * the fascia section extends for a distance of three or four times the depth of the section * * * the maximum section of the fascia should be used for the entire width."

In 1931, he advocated[12] a fascia extension of only one and one-half times the depth of the sections (item 8b):

> "The fascia section extends for a distance (parallel to the abutments) of about one and one-half times the depth of the section; and where cross-steel is required, it should extend for at least a bond length of the cross-steel."

What are the reasons for the wider fascia?

9a.—Under the heading "Stresses in the Interior Parts of a Section," Mr. Weiner states that:

> "* * * the design of the fascia, only, is sufficient. * * * designs, for single-span and multiple-span frames and full centered 'arches' (frames with curved soffits) and for skew angles up to 50°, also showed that adequate designs can be made without recourse to complete skew-arch analysis."

yet (item 9b), in his discussion of the paper by Richard M. Hodges[13] he implies quite the opposite.

10a.—In the last paragraph of "Stresses in the Interior Parts of a Section," Mr. Weiner recommends that:

> "* * * the larger the skew, the greater the relative thickness of concrete should be; that is, as the skew increases, less compression reinforcement should be used, thus resulting in thicker crown sections. Skews of much more than 50° become impractical for purely geometric reasons; and, for this upper limit, it seems wise to make the crown sufficiently deep so that it can be designed without any compression steel. Although the single-span frame with a 50° skew, previously mentioned, was designed with

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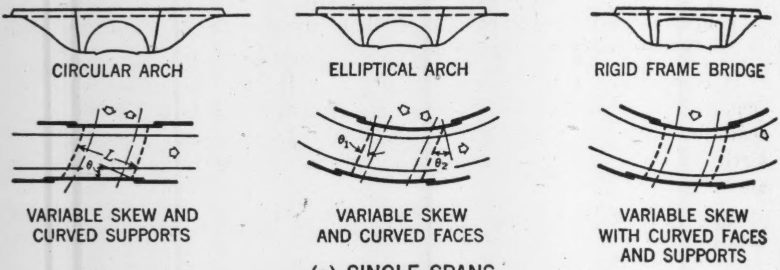
[12] "Design of a Reinforced Concrete Skew Arch," by Bernard L. Weiner, *Transactions*, ASCE, Vol. 96, 1932, p. 1274.

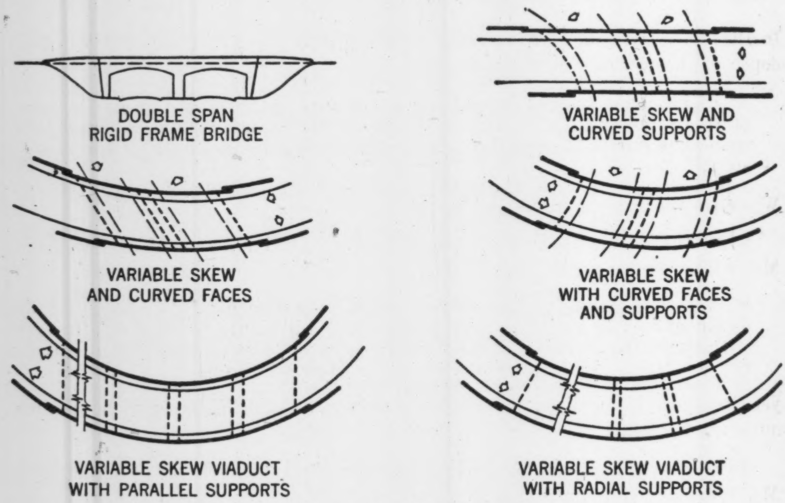
[13] Discussion by Bernard L. Weiner of "Simplified Analysis of Skewed Reinforced Concrete Frames and Arches," by Richard M. Hodges, *Transactions*, ASCE, Vol. 109, 1944, pp. 948 and 953.

rather heavy compression reinforcement and showed no signs of distress after almost 20 years of service, the practice is not recommended—unless a complete design is made."

This does not "square" (item 10b) with the logic advanced by Phillips H. Lovering, Assoc. M. ASCE, in his discussion[14] of the Hodges paper nor with


CIRCULAR ARCH
ELLIPTICAL ARCH
RIGID FRAME BRIDGE
L
θ3
θ1
θ2
VARIABLE SKEW AND CURVED SUPPORTS
VARIABLE SKEW AND CURVED FACES
VARIABLE SKEW WITH CURVED FACES AND SUPPORTS

(a) SINGLE SPANS


DOUBLE SPAN RIGID FRAME BRIDGE
VARIABLE SKEW AND CURVED SUPPORTS
VARIABLE SKEW AND CURVED FACES
VARIABLE SKEW WITH CURVED FACES AND SUPPORTS
VARIABLE SKEW VIADUCT WITH PARALLEL SUPPORTS
VARIABLE SKEW VIADUCT WITH RADIAL SUPPORTS

(b) MULTIPLE SPANS

Fig. 9

Mr. Hodges' reply emphasizing the fact that temperature effects for structures with large skew are the controlling factors and that "it is best to thicken the knee."

11a.—Finally, under the heading "Characteristics of Skew Frames," Mr. Weiner declares:

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[14] Discussion by Phillips H. Lovering of "Simplified Analysis of Skewed Reinforced Concrete Frames and Arches," by Richard M. Hodges, *Transactions*, ASCE, Vol. 109, 1944, p. 966.

> "The analysis and the design of a rectangular frame or arch, for example, are really based on the assumption that the frame or arch can be divided into narrow strips, each one of which may be designed as a narrow beam—because of symmetry along the width of the structure."

which is quite inconsistent with his comparable statement[15] (under the heading "General Remarks") in discussing the Hodges paper (item 11b):

> "* * * In spite of the misleading statements that have been made time and time again, the skew arch must be analyzed as a complete unit and the division by *b* is merely a minor arithmetic convenience which has no structural significance."

Concerning item 11, a little different viewpoint will clear up the misunderstanding concerning an analysis or design using 1-ft strips. Neither the right nor skewed structure is really considered, by most designers, to be cut into strips. The structure is considered in its entirety. Since the width of the structure is a linear function in every stress equation and in every load equation, using values per foot of width avoids the use of large numbers, permits the use of published design aids, and helps the designer in many other ways.

Until the writer demonstrated[10] that the rectangular elastic system is almost independent of the torsional elastic system and was able to determine the effects of each term quantitatively in the "exact" analysis, the effects of variable skew could not be determined, and these structures could not be analyzed, designed, or constructed. In his treatment of these commercially patented structures, Mr. Weiner does not show how the variable skew affects the foundation restraints or how total stresses are determined. Boundary design for the maximum skew can be shown to be inadequate. Fig. 9 shows several types of variable skewed curved-in-plan structures. Other combinations of variable skew, variable width, and variable span are shown elsewhere.[16]

There are few designs available for either single-span or double-span heavily skewed structures based on exact analysis. One of these rare specimens is a double-span structure designed under the author's supervision, which involved the simultaneous solution of eighteen elastic equations (three sets of six equations each). The author's double-span analysis has been used as a "guinea pig" on which to check the conclusions stated in the writer's paper.[10] In discussing these comparisons with the author, the writer requested an application of the same checks to his conclusions. The use of elastic weights[16] as advocated in this paper is a departure from the author's former preference[17,7,18] for the method of partial summation. The author repeatedly stresses his preference for "ellipse of stress" theory. For sections with large ratios of b/t, Mr. Rathbun has pointed out that the ellipse degenerates into a straight line or into an extremely flat ellipse. Mr. Hodges and others have also shown that this refinement is not warranted.

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[15] Discussion by Bernard L. Weiner of "Simplified Analysis of Skewed Reinforced Concrete Frames and Arches," by Richard M. Hodges, *Transactions*, ASCE, Vol. 109, 1944, p. 953.

[16] "The Rigid-Frame Bridge," by Arthur G. Hayden and Maurice Barron, John Wiley & Sons, Inc., New York, N. Y., 1950.

[17] "Design of a Reinforced Concrete Skew Arch," by Bernard L. Weiner, *Transactions*, ASCE, Vol. 96, 1932, p. 1212.

[18] Discussion by Bernard L. Weiner of "Simplified Analysis of Skewed Reinforced Concrete Frames and Arches," by Richard M. Hodges, *Transactions*, ASCE, Vol. 109, 1944, p. 946.

The determination of the effect of re-entrance (negative slope of legs) in rigid-frame bridges is also a refinement that is not warranted. It is common knowledge that the assumed median line does not truly represent the structure as a line diagram. Some tables of the paper take the re-entrant effect into account, and other tables neglect this effect. This is an unnecessary complication and will be confusing to all but the experienced designers.

In the computations, different sections for analysis and design are used. By using the same section for both phases of the work several steps can be eliminated. This will result in a simpler and more practical office design. For example, Col. 6 of Table 2 need not be scaled influence ordinates but the (H F)-values could be copied directly from Table 1, providing Table 1 was also simplified by omitting Col. 12 and finding the elastic moment at the point of application of the load rather than between loads.

The division of axes can be improved by having a point at the center line of the frame, a critical design section.

The shear increment method for finding moments due to elastic loads or to applied loads has not been clearly demonstrated. In Table 1 values in Cols. 10, 11, and 12 should be placed halfway between points because they are shears and moment increments between points. The statement, that in the "* * * elastic load method, the 'elastic moment' at any point is the deflection at that point," is not complete. What is the deflection, in what direction, and due to what load? A more complete explanation and better illustration of the elastic weight method is shown elsewhere.[16]

The computations are much simpler, with less chance of error, if fixed loads such as dead loads, and earth pressure are considered as a single system and not as superimposed individual loads. The simple-span moments may be determined by shear increment method and the moments due to the total redundant may be added to find the final value.

No proofs or data are presented to justify a generalization that the approximate method proposed "* * * holds good for any number of spans or for any shape of frame or arch that would be encountered in practical design."

In Table 3, the definition of M_s as the "simple beam" moment is incorrectly applied. The re-entrant structure used is far from a simple beam. The same principle applies to Table 5 and Fig. 6. The peaks of the influence lines for simple spans are not at the same position as shown in the analysis.

An error in design procedure is the result of applying the transformation which has been derived for external forces to the unit stresses. In Table 9, it should be noted that K_1 for a balanced section is applied both to the reinforcing and to the concrete. The statement that $C_2 = T_2 = \frac{M_2}{2\,r}$ is incorrect. This equation is correct for T_2 but incorrect for C_2. Since moments and thrusts have been turned as shown in Cols. 3 and 4, it is incorrect to determine compressive stress in the concrete from these turned moments. In other words, vectorally adding a unit compressive stress and a unit shear stress in concrete is incorrect. Using the author's method, if the reinforcement is placed perpendicular to the abutments, a different value for C_2 would result.

Eq. 25 is given without proof and is based upon a demonstration presented elsewhere. This formula expresses an approximation applicable to no other structure but an idealized frame with vertical legs and horizontal vault. The author presents no way of evaluating the error or approximation for arches or for high-vaulted rigid frames. It is improper to generalize Eq. 25 as is done in this paper. By including the effect of the reinforcement in the moment of inertia in the example, another unnecessary refinement has been added: I and E always appear as a product ($E\,I$) and the uncertainties in evaluating E do not justify great refinement in I.

The writer cannot agree with the author's estimate of the moment at the bottom of the footing of a rigid-frame bridge being equal to 40% of the moment at the point 6. The vertical reaction of the example is approximately 30 kips; 40% of the moment at point 6, divided by 30 kips, gives an eccentricity of more than 5 ft to the foundation reaction. Unless the footing is anchored to rock so as to develop tension within its base,[19] this is an impossibility. The writer's investigation of several structures founded on soil results in an eccentricity of less than 2 in.

By extending the mathematical equations into the design phase, it has been shown[10] that the required steel areas are functions of the stresses and that the skew correction can be applied to the final answer rather than to the stresses as represented by Eqs. 18, 19, 20, and 21. When numerical values are substituted prematurely, the mathematical and structural significance is lost. Concerning Eq. 18, it should be observed that the first and last terms are insignificant when compared with the middle term and that for all practical design purposes

$$t_s = \epsilon' \frac{t_v}{b} \qquad (27)$$

Also, in Eq. 20*b*, $m_v = \epsilon' m_s$ and, therefore,[10] Eq. 20*b* reduces to Eq. 20*a*.

The statement that "* * * horizontal shears * * * are more in the nature of column shears, they are of no serious consequence" is questionable. Shearing stresses on a design section result from the cross shear, t_s, the radial shear, t_u, and the torsion moment, m_v. For a design section of a skewed bridge, the shearing stresses are the result of external forces and are not the shearing stresses on a plane oblique with the axial column load. For structures with large skew, these shear stresses have exceeded 1,000 lb per sq in., twice the maximum value stated by the author. The writer contends[10] that these stresses are frequently the controlling stresses regardless of how much reinforcement is provided.

Mr. Westergaard's equations, recommended in the section on transverse reinforcement, apply only to live loads on flat plates of uniform thickness. They do not consider earth pressure or temperature change, arch action, or the variable I. The justification for using these equations in determining transverse reinforcement is very obscure.

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[19] "The Rigid-Frame Bridge," by Arthur G. Hayden and Maurice Barron, John Wiley & Sons, Inc., New York, N. Y., 1950, p. 42.

The conclusions stated in this paper are generally correct, but it is improper and unnecessary to generalize the problem of skewed structures based on a few numerical comparisons of "exact" and approximate designs or upon equations derived for an idealized rigid frame. If a better method were not available, such comparisons and generalizations would be proper. However, it has been shown[10] how a complete separation of the rectangular and torsional systems may be affected and how each skew effect may be evaluated as a secondary quantity. The complexity associated with the analysis and design of skew arches and rigid-frame bridges is a self-imposed complexity.

Arthur G. Hayden,[20] M. ASCE.—It appears that structural engineers are at the end of the long trail leading to the clarification and simplification of the analysis of skewed rigid-frame bridges. The writer wrestled with this problem unsuccessfully in the 1920's, and it remained for Richard M. Hodges[4] to take the first long step in this direction. However, neither he nor the writer nor anyone else succeeded in separating the skew effects so that they could be evaluated separately, thus avoiding lengthy tabulations and the solution of complicated systems of simultaneous equations. It remained for Maurice Barron to do this, and he succeeded; but his conclusions[21] were at first controversial and met with opposition for some time. His missionary work converted one sceptic after another before his conclusions were finally accepted, and it is gratifying that Mr. Weiner, in his restatement of the problem, corroborates Mr. Barron in every respect.

As a result of these labors in the field of structural analysis the ordinary designer of bridges for grade separations has a power that he did not formerly possess.

Bernard L. Weiner,[22] M. ASCE.—In scientific research the "work" is the important thing; who did what, is of minor importance. However, since both Messrs. Barron and Hayden—the former in great detail—attempt to show how various writers have reached more or less the same conclusions, it would be well to consider the chronology of events briefly.

The present paper is an extension of a previous paper by the writer published in 1932.[23] Part III of this earlier paper shows mathematically that the two elastic systems can be separated in one type of structure—where the median lines of the members of a single-span frame are straight and perpendicular to each other. It also shows that for the more general type of single-span frame in which the members are neither straight nor perpendicular to each other, separation of the two elastic systems results in very little error. This analysis forecasts the possibility of extending the approximation to full-centered arches, to high-vaulted rigid frames, and to multiple spans. The writer has since analyzed such skew structures and found the same approximations to be valid.

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[20] St. Michaels, Md.

[21] "The Rigid-Frame Bridge," by Arthur G. Hayden and Maurice Barron, John Wiley & Sons, Inc., New York, N. Y., 1950.

[22] Cons. Engr., New York, N. Y.

[23] "Design of a Reinforced Concrete Skew Arch," by Bernard L. Weiner, *Transactions*, ASCE, Vol. 96, 1932, p. 1212.

It would be utterly impossible to publish all such analyses, even if it were not altogether too time-consuming for a practicing engineer to attempt to "dig out" all available numerical data. Such a task should properly be undertaken by one of the numerous college research and experiment stations.

It can merely be stated that, where an extended analysis was made, it corroborated the writer's forecast in the 1932 paper that the two elastic systems were separable. The authors who followed confirmed the findings of the writer rather than the other way around.

As a matter of fact, the separation of the two systems is really of little importance in the arguments presented in this paper, although it is a valuable tool for some purposes. The foregoing answers all of Mr. Hayden's discussion and the first part of Mr. Barron's.

Mr. Barron states that there are three contradictions in the paper, which he identifies as paragraphs 5, 6, and 7:

5. Although the writer states that the expedient of designing on the "skew span" is both incorrect and unnecessary (item 5a) it should be obvious that this expedient does give a rough approximation of the results to be derived from a more logical analysis. Hence the statement referred to as item 5b is not a contradiction. A checker often must resort to the expedient of "roughing out" an approximate result, and such an approximation makes an excellent independent check.

6. Here (item 6a) the question is raised as to why the "total stresses" are computed in an arch or frame while it is stated at the same time (item 6b) that in a "wide structure" such total stresses have very little meaning. This is an interesting point and applies to right structures as well as to skew structures. The answer to this question also anticipates one of Mr. Barron's later points. In the design of arches and frames it is really implicitly assumed that when the structure is loaded with as many trucks (concentrated loads) as it can physically accommodate, the resulting stresses will be greater than the local stresses produced by a very few local loads. Although this is also a study that could very well be undertaken by a research and experiment station, the question is academic to a certain extent since the live-load stresses are only a comparatively small fraction of the dead-load stresses. The latter, of course, are more or less uniformly distributed across the width of the structure. In loading the structure with all the trucks it can carry, line loads are really assumed. In fact, the "equivalent loads" used in modern bridge specifications consist of uniform loads and "line concentrations." Under this type of loading or the line loads of trucks, the total stresses are of value, or "have meaning."

7. The point made here (item 7a) is that the factor b can be factored out of all expressions except the "factor of torsion" (item 7b), and yet it is stated that it is factored out as an arithmetical convenience. The reason for this, as explained in the paper, is that in the "factor of torsion," b appears in the form $\left(\frac{1}{b\,t^3} + \frac{1}{b^3\,t}\right)$; and as the second term is small compared to the first, it can be neglected. The quantity b can thus be factored out, but this does

not mean that the structure can be analyzed in two dimensions. The question still remains whether a stress divided by *b* represents a stress per unit width. It does so only if the stress in question is uniformly distributed across the width, which is not the case for all stresses.

Mr. Barron states that the writer's 1932 paper advocates that the fascia section be extended for a width of 1.5 times the depth of the section (item 8b) and now advocates a value of 3 instead. However, it is also stated[24] that when b/t is greater than 3, the hyperbolic functions in the St. Venant theory of torsion reach their limiting value. Hence, beyond this distance the torsion stresses are constant to the opposite fascia. At 1.5 times the depth, these functions are substantially constant but not quite. Since in actual pilot designs, it made practically no difference, the writer has been using the larger outside value of three.

The writer did not do what Mr. Barron seems to imply in this part of his discussion (item 9). This brings up the main point of difference between the writer and authors of the various other papers on the subject of skew structures.

No matter what form of equations is used—either "exact" or "approximate" —to obtain the total stresses, it can be assumed for the present purpose that the same results will be obtained with sufficient accuracy for design purposes. The design of the sections—that is, the computation of the stresses in the concrete and the determination of the amount and location of reinforcing steel —brings up the question of stress distribution. It is on this question that there is disagreement. In his discussion of the paper by Mr. Hodges, the writer raised this question of stress distribution and stated that a thorough knowledge of the distribution of the total stresses was necessary. The writer did not say, as Mr. Barron seems to imply, that a complete and "exact" analysis for the total stresses was necessary for the design of a skew frame.

However, since making this statement, the writer has come to the conclusion that what is needed is a thorough study of the entire subject by a group of engineers who will have sufficient time to do the job completely, to set up design standards based on all the material written to date, and to make additional studies if necessary.

It is the writer's further judgment that, more than anything else, tests are needed to determine the stress distribution across a section. To make such a study for a full-centered arch or high-vaulted rigid frame, based on the mathematical theory of elasticity, seems beyond the scope of present-day knowledge. Some mathematical guides are available, however. The University of Illinois, Engineering Experiment Station, has published studies on skew slabs from which clues might be obtained. In the final analysis, the total stresses that are needed depend on the unit stresses that are found to exist in skew structures and they also depend on how these unit stresses can be obtained from the total stresses. All that can be hoped for in any case is close approximations.

If the writer's recommendations (item 10a) do not "square" with Mr. Lovering's logic as stated by Mr. Barron (item 10b), it is possible that this logic may be at fault. Although it is theoretically true that thickening the section near the knee will decrease the moments near the middle part of the frame, this process is so slow as to be of little practical value. At least, this has been the writer's experience. The writer fails to follow the logic in the argument that it is the temperature stresses, not the skew angle, that are the controlling factor. Since the temperature stresses increase rapidly with the skew, the writer fails to see the point. However, the problem of revising the structure so as to reduce excessive temperature stresses is a matter to be decided by the designer in each individual case, and no general conclusions seem to be possible.

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[24] "Design of a Reinforced Concrete Skew Arch," by Bernard L. Weiner, *Transactions*, ASCE, Vol. 96, 1932, p. 1259.

The question (item 11) of whether the structure can be analyzed in two dimensions or whether it must be analyzed in three dimensions has already been discussed. As already stated herein, theoretically b cannot be factored out, but actually or rather, practically, it can be factored since $1/b^3\,t$ can be neglected. Dividing by b does not give the stresses per unit width if they are not uniformly distributed.

The writer can not agree with Mr. Barron that the effects of the skew could not be determined before his demonstration became available. Also, Mr. Barron's reference to commercially patented structures is unfortunate. The writer does not care to enter into a discussion of this subject.

Mr. Barron is in error in stating that there are few single-span or double-span designs available, based on exact analysis. His reference to eighteen equations must refer to a three-span bridge, not a two-span bridge. There are six equations per span.

Mr. Barron is evidently referring to three pilot designs made under the writer's supervision—two two-span rigid frames and one three-span bridge. The former required twelve equations each, and the latter eighteen, as already stated.

The writer fails to understand the reference to b/t in Mr. Barron's discussion of the ellipse of stress. The ellipse of stress, referred to by the writer, is the standard transformation formulas for finding the stresses on any plane if the stresses on an elementary parallelopiped of dimensions dx, dy, and dz are known. These formulas are so called because they can be put in the form of the ellipse equation. The ellipse of stress also leads to the determination of the principal planes. Since this ellipse of stress refers to stresses at a point, b/t does not enter into it.

The discussion by Mr. Barron of the writer's method of analyzing a single-span right rigid frame is irrelevant. It is specifically stated that a designer may use any method he chooses for making the basic design. It might be emphasized, however, that there are reasons for the setup used, and the writer has found from experience that it has marked value. Placing the loads between sections, for instance, eliminates an error in the summations at the section where the influence load is placed. Oddly enough, it was Mr. Hayden, co-author with Mr. Barron of the book on rigid frames who taught the writer some of the "tricks" which displease Mr. Barron. The objections to the sloping legs, etc., as being unnecessary refinements, are not valid. It is sometimes easier to be "exact" than to be approximate, which is the case here. It requires more effort to approximate on the basis of an idealized structure than it does to analyze the actual structure.

It was not the writer's intention to treat elastic loads. Neither Mr. Barron nor the writer need bother describing "elastic loads." There are good descriptions of this method in many standard texts on statically indeterminate stress analysis.

As to design of the sections, since the transformation proposed by the writer results in finding a principal plane, the writer fails to see the logic in the statement that shears are being added to compression stresses. In any case, once the stresses are resolved on a plane and the steel placed perpendicular to that plane the design is correct. The only question is whether the original transformation was correctly made. If the steel were placed perpendicular to the abutments, the problem would be different and much more difficult as shown by the writer in his 1932 paper. Fortunately the section normal to the fascia (at the fascia) is a principal plane—as was shown on general principles independent of any and all skew arch theories—and, therefore, the design differs in no way from ordinary designs of reinforced concrete for bending and direct stress.

Mr. Barron's statement that it is impossible to develop the fixing moment in a rigid-frame footing is correct. As Mr. Hayden has shown, it makes very little difference in most of the structures whether the footings for the rigid frame are considered fixed or hinged. Hence, if the frame is designed so that it is safe for both conditions, it follows that it is safe for the true condition of partial fixity. Since there is no loss in economy in this method of design, there is no need to find the actual degree of fixity which is equivalent to stating that there is no need to find the actual eccentricity of the thrust. Since it is easier, the analysis is usually made on the basis of the hinged condition, and there is no need for more than a rough approximation of the fixing moment.

Mr. Westergaard's equations apply exactly as Mr. Barron states they do. Since the writer finds that transverse steel is not needed for the design of the dead loads and line-load live load stresses, the only transverse steel needed is for distributing actual wheel loads. Therefore, the problem is the same in a right arch as in a skew arch. The writer has simply suggested a crude method for roughing out the amount of transverse steel needed. If there is a better method even for a right arch, the writer would like to know about it.

Finally, if the writer's contention is correct and the fascia design used for the entire width is both safe and economical, only a right arch-analysis is necessary.

It is to be hoped that some college research and experiment station will undertake this problem, make some tests on stress distribution, review all that has been written on the subject, and finally set up a set of recommendations so that skew arch design will be put on a rational basis that can be accepted by the profession.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2475

INFLUENCE OF MEMBERSHIP TRENDS ON SOCIETY RESPONSIBILITIES

ADDRESS AT THE SUMMER CONVENTION, LOUISVILLE, KENTUCKY, JUNE 13, 1951

By Gail A. Hathaway,[1] President, ASCE

The history and current status of the Civil Engineering profession has been reported by Presidents of the Society in annual addresses for many years. These messages have been published in the annual *Transactions* and are recommended for rereading by all members and prospective members who are interested in a better understanding and appreciation of the American Society of Civil Engineers, its objectives, and attainments. The deficiencies to be overcome are also highlighted in these addresses to the end that future efforts may be focused upon their elimination.

In this address it is desired to review briefly some of the facts already presented by my predecessors in the office of President, and to discuss certain problems that have become of acute importance within recent years and months.

The profession of engineering has been described as "the art of applying the laws of the natural sciences to the utilization of the materials and forces of nature for the benefit of mankind and the art of organizing the human effort required in connection therewith."[2] The profession has become one of the largest and most rapidly growing in the Nation. The professional engineer provides technical leadership in industry. He investigates, reports upon, designs, and supervises construction on a wide range of projects, products, machines, and processes that affect the daily life of every individual in our social structure. Many are engaged in research and the teaching profession, and a large group perform services principally for the purpose of selling equipment and materials. There is an increasing number being utilized in business and management. Let us hope that this trend continues at an accelerated rate. Of considerable importance is the increasing number of engineers engaged in public service either with City, County, State, or the Federal Government as well as quasi-public organizations. Engineering plays a most prominent and essential role in the military defense of our country.

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[1] Special Asst. to Chf. of Engrs., U. S. Army, Washington, D. C.

[2] "Career in Engineering," by Lowell O. Steward, Iowa State College Press, Ames, Iowa, 1947.

The increasing importance and utilization of engineers in our economic life is demonstrated by the variation in the ratio of workers per engineer during the period 1890 to 1940, which has dropped from about 290 in 1890 to 78 in 1940.[3] This ratio has been decreasing more slowly in recent years but furnishes ample evidence of the increasing need for engineers.

In 1890, the five major fields of engineering: namely, civil, mechanical, electrical, chemical, and mining and metallurgical, included only 27,000 engineers. This number has increased to almost 400,000 in 1951. Civil Engineering, the oldest of the engineering professions, had a membership of 47,000 in 1910, 56,500 in 1920, 88,500 in 1930, 89,000 in 1940, and over 100,000 at the present time. Large numbers left the Civil Engineering branch of the profession during the depression years, 1930 to 1940. However, over the years the profession has experienced a remarkable growth and has occupied an increasingly important place in the economic structure of the Nation.

It is of interest to compare the growth of the Society with that of the Civil Engineering profession. In 1910 approximately 11% of all civil engineers were members of the Society. During the 20-yr period 1920 to 1940 the percentage varied between 16% to 18%. In 1945 membership in the Society increased rapidly until at present one out of every three civil engineers is a member.

The most important factor in the Society's growth within recent years is the significant increase in the number of Junior members. On April 9, 1951, almost 44% of the 32,000 members were Juniors. Because of the importance of the Junior membership, it is desired to review certain Society actions that had a major bearing on the progressive increases.

Charles Warren Hunt, in referring to the first meeting of the Society on November 5, 1852, at the office of the Croton Aqueduct Department,[4] Rotunda Park, New York, N. Y., states as follows:

> "The Constitution was taken up by sections, discussed, amended and adopted, and this Constitution remained in operation without change until 1868. It provided that the permanent place for the transaction of the business of the Society should be in the city of New York, for one grade of membership only, to which were eligible 'Civil, Geological, Mining and Mechanical Engineers, Architects and other persons who, by profession, are interested in the advancement of science.'
>
> "* * * In 1872–73 the entrance fees and dues were raised to the amounts in force (January 1, 1897) and the Fellowship fee was increased to $250. This was again reduced to $150, November 4th, 1874. The question of the establishment of Chapters and of the admission of students was much discussed, but no action taken. The letter ballot, to enable non-residents to vote in all elections, was introduced November 5, 1873, and many slight modifications made in the Constitution."

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[3] "Employment Outlook for Engineers," Bulletin No. 968, U. S. Dept. of Labor, Bureau of Labor Statistics, Occupational Outlook Series, 1949.

[4] "Historical Sketch of the American Society of Civil Engineers 1852-1897," by Charles Warren Hunt, ASCE, New York, N. Y., 1897.

One of the slight modifications made in the Constitution on November 5, 1873, was a provision for the grade of Junior. A Junior was entitled to all privileges of the Society, except the right to vote and to hold office. Membership records show 7 Juniors in 1874. There were 36 Juniors in 1875 and the number progressively increased with minor variations until a peak of 840 was reached on February 10, 1915 (see Table 1). The number then gradually declined to 429 on February 10, 1921, representing about 4.3% of the total Society membership. This decline in Junior membership demonstrated the need for more effective contacts for interesting young engineers in Society

TABLE 1.—ANNUAL SUMMARY OF MEMBERSHIP IN THE AMERICAN SOCIETY OF CIVIL ENGINEERS

Date[a] (1)	Honorary Member (2)	Corresponding Member (3)	Member (4)	Associate Member (5)	Associate or Affiliate[b] (6)	Junior and Junior Member[c] (7)	Fellow[d] (8)	Subscriber (9)	Total (10)
1853[a]	6	1	48	....	....	....	....	....	55
4	6	1	47	....	....	....	....	....	54
....	....	....	....	....	....	....	....	....	....
1869	....	....	....	....	....	....	....	....	160
1870	....	....	191	....	1[b]	....	51	....	243
1871	....	....	215	....	5	....	59	....	279
1872	4	....	278	....	7	....	48	....	337
1873	6	....	329	....	17	....	62	....	414
1874	7	....	355	....	15	7	61	....	445
1875	7	....	371	....	18	36	60	....	492
1876	6	3	417	....	16	51	59	....	552
1877	6	3	436	....	16	56	57	....	574
1878	6	3	454	....	17	57	56	....	593
1879	6	3	464	....	18	54	56	....	601
1880	12	3	473	....	20	48	55	....	611
1881	11	3	513	....	24	54	52	....	657
1882	10	3	565	....	30	58	53	37	756
1883	9	3	589	....	31	67	57	39	795
1884	10	3	657	....	34	77	57	40	878
1885	8	3	701	....	33	89	54	40	928
1886	9	3	760	....	44	106	57	40	1,019
1887	8	3	838	....	45	120	56	40	1,110
1888	10	3	944	....	48	144	54	40	1,243
1889	8	3	1,010	....	55	169	51	39	1,335
1890	7	3	1,080	....	61	215	51	38	1,445
1891	5	3	1,108	49	63	221	49	38	1,536
1892	8	3	1,134	109	69	238	48	35	1,644
1893	9	3	1,172	146	68	253	47	35	1,733
1894	9	3	1,189	186	73	266	47	35	1,808
1895	8	3	1,240	237	74	278	47	33	1,920
1896	8	3	1,252	282	82	318	41	32	2,018
Feb. 10, 1897[a]	9	3	1,271	293	81	294	41	33	2,025
1898	9	3	1,283	326	81	285	41	....	2,028
1899	8	3	1,336	382	88	257	38	....	2,112
Feb. 10, 1900	9	3	1,376	439	93	263	38	....	2,221
Feb. 11, 1901	9	3	1,421	535	102	245	35	....	2,350
Feb. 10, 1902	10	2	1,512	629	112	255	34	....	2,554
1903	9	2	1,570	728	121	262	33	....	2,725
1904	9	2	1,670	820	127	298	28	....	2,954
Feb. 10, 1905	10	2	1,819	918	123	350	27	....	3,249
1906	12	2	1,957	1,034	130	409	25	....	3,569
1907	11	2	2,084	1,220	143	479	24	....	3,963
1908	8	2	2,227	1,460	156	576	23	....	4,452
1909	7	2	2,380	1,634	162	642	22	....	4,849

TABLE 1.—*Continued*

Date[a] (1)	Honorary Member (2)	Corresponding Member (3)	Member (4)	Associate Member (5)	Associate or Affiliate[b] (6)	Junior and Junior Member[d] (7)	Fellow* (8)	Subscriber (9)	Total (10)
Feb. 10, 1910	8	2	2,582	1,845	166	690	21	....	5,314
1911	9	2	2,784	2,112	172	758	21	....	5,858
1912	8	2	2,930	2,424	176	804	20	....	6,364
1913	6	2	3,076	2,714	180	809	18	....	6,805
1914	5	1	3,287	3,008	171	815	17	....	7,304
Feb. 10, 1915	5	1	3,408	3,270	168	840	15	....	7,707
1916	5	1	3,435	3,483	165	793	14	....	7,896
1917	4	1	3,526	3,796	170	715	13	....	8,225
1918	3	1	3,666	4,126	165	643	10	....	8,614
1919	3	1	3,794	4,419	164	577	10	....	8,968
Feb. 10, 1920	3	1	3,990	4,697	166	506	10	....	9,373
1921	3	....	4,243	5,025	174	429	10	....	9,884
1922	6	....	4,479	5,164	164[b]	452	10	....	10,275
Mar. 1, 1923	11	....	4,703	5,250	167	496	9	....	10,636
1924	13	....	4,843	5,327	159	616	9	....	10,967
Mar. 1, 1925	13	....	4,960	5,365	158	771	8	....	11,275
1926	15	....	5,075	5,266	151	861	8	....	11,376
1927	14	....	5,242	5,518	147	1,181	8	....	12,110
1928	15	....	5,429	5,695	139	1,618	7	....	12,903
1929	16	....	5,571	5,899	138	2,043	7	....	13,674
Mar. 1, 1930	18	....	5,673	6,047	144	2,329	7	....	14,218
1931	16	....	5,826	6,207	133	2,624	7	....	14,813
1932	18	....	5,857	6,343	123	2,885	5	....	15,231
1933	19	....	5,809	6,312	114	2,955	5	....	15,214
1934	18	....	5,758	6,256	108	3,090	4	....	15,234
Mar. 1, 1935	18	....	5,683	6,097	99	3,046	3	....	14,946
Mar. 2, 1936	21	....	5,718	6,173	95	3,188	2	....	15,197
Mar. 1, 1937	24	....	5,656	6,049	87	3,427	1	....	15,244
1938	25	....	5,642	6,162	78	3,664	1	....	15,572
1939	27	....	5,639	6,331	74	3,936	1	....	16,008
Feb. 1, 1940	31	....	5,584	6,336	70	4,144	1	....	16,166
Feb. 3, 1941	35	....	5,670	6,534	69	4,535	1	....	16,844
Feb. 2, 1942	37	....	5,746	6,828	68	4,858	1	....	17,538
1943	36	....	5,876	7,119	71	5,436	1	....	18,539
Feb. 1, 1944	36	....	6,023	7,392	74	6,074	1	....	19,600
Feb. 1, 1945	36	....	6,189	7,743	76	6,458	1	....	20,503
Feb. 1, 1946[a]	38	....	6,349	8,109	78	6,591	1	....	21,166
1947	39	....	6,506	8,364	77	6,433[c]	1	....	21,420
1948	42	....	6,887	8,919	75	6,866	1	....	22,790
Feb. 10, 1949	44	....	7,251	9,384	72	8,289	1	....	25,041
Feb. 9, 1950	39	....	7,514	9,667	74	10,810[c]	1[d]	....	28,105
Feb. 9, 1951	39	....	7,789	10,041	68	13,777	1	....	31,715

[a] Data for the period 1853 to 1896 are from "Historical Sketch of the American Society of Civil Engineers, 1852–1897," by Charles Warren Hunt, American Society of Civil Engineers, Publication, New York, N. Y., 1897, Page 79. Data for the periods 1897 to 1945 and 1946 to 1951 were compiled from annual yearbooks of the Society and from "Civil Engineering," respectively. [b] The grade of associate was changed to affiliate on October 5, 1921. [c] Juniors officially acquired the right to vote on October 7, 1947, and on April 19, 1950, the grade of Junior was changed to Junior Member. [d] The grade of Fellow was discontinued on April 19, 1950.

membership and had an important influence upon the establishment of Student Chapters.

The authorization in 1920 of a Student Chapter at Stanford University, in Stanford, Calif., by the Board of Direction, followed by the establishment of eight more during the same year was, without doubt, one of the most forward-looking steps ever taken by the Society. The purpose of this action, as stated in the Minutes of the Board, was "To afford the beginnings of professional association." In 1921 thirty additional Student Chapters were organized, and the Board amended the By-Laws and adopted regulations for their recognition and government. The number has grown until on March 1, 1951, there were 131 Student Chapters having a membership of approximately 11,000 students.

The rapid increase in Junior members in proportion to the total membership, ranging from 4.3% of Juniors in 1921 to 43.5% in February, 1951, was in large part due to the formation of Student Chapters. Principal credit for this growth can be given to the "Faculty Advisers," "Contact Members" appointed by the Local Sections, the Student Chapter Committee, and the assistance provided by Local Sections. Credit must also be attributed to the professional standing and prestige of the Society in attracting and holding the interest of the young engineer after his graduation. The number of Junior members dropped from the Society's rolls during recent years is extremely small. The number of Junior resignations would greatly increase if the new graduate, after becoming a member, found later that the benefits of Society membership did not fulfill his expectations.

The adoption of a Constitutional amendment in 1947 giving Junior members the right to vote on Society matters reflected a growing appreciation of the importance of our younger members.

As the proportion of Junior membership to the total has progressively increased to the present ratio of approximately 50%, the average age of the composite membership has become correspondingly lower and the interests and needs of the majority have been changed appreciably by this trend. Most of these Junior members are salaried employees, and the majority are destined to remain in the employee classification throughout their careers. The report of the Engineers Joint Council on the 1946 survey of the engineering profession, entitled "The Engineering Profession in Transition," contains data showing that the classification of workers in the Civil Engineering profession is slightly more than 86% employee, including 37.5% in private engineering, and 48.7% in public engineering. The report classifies about 8% of the Civil Engineering profession as employers and 5% as independent consultants, with the remaining 1% being classified as retired, nonengineering, student, and unemployed.

Because the economic welfare of the majority of engineers is directly dependent upon salaries paid for engineering services, it is logical to expect that their professional Societies should play a prominent part in assuring that adequate scales of salaries are established and adhered to. Inasmuch as a large percentage of engineers engaged in private practice are themselves employers of engineers, it is essential that the fees received by them be adequate to support proper salary payments to their engineer-employees and still leave a reasonable profit for services rendered.

To some individuals the subject of improving the economic status of engineers may connote a selfish interest on the part of the profession. However, a broader viewpoint demonstrates that a proper interest in assuring economic returns to engineers in equitable relation to services performed is essential to a maintenance of the high level of engineering achievement upon which our national economy and way of life so heavily depends. Although many talented individuals would undoubtedly continue to enter and follow the engineering profession in spite of inadequate compensation, records demonstrate that the increasingly unfavorable economic status of engineers in relation to other professions and trades is having a strong influence in reducing the supply of engineers below a safe level for national requirements. Certainly the American Society of Civil Engineers has an important role to play in assuring proper recognition of professional engineers, and in exerting responsible influence to assure that the engineer is fairly compensated, while at the same time protecting the profession and the nation against adverse effects that might be expected if engineers should feel impelled to seek economic relief through trade unions. For these reasons it is desired to discuss briefly certain problems pertaining to the economic status of engineers, and their effects on the engineering profession.

The situation with respect to Non-Federal Government employees in engineering appears to be the most critical. For example, let us consider the economic compensation of approximately 20,000 professional engineers employed by the 48 state highway organizations in the United States, in relation to the importance of services performed. During the fiscal year, ending June 30, 1951, it is estimated that these engineers will have designed and supervised the construction of new projects and maintained existing projects involving expenditures in excess of $2,500,000,000. If county and city projects are included, the value of the work handled would exceed $3,500,000,000. The economic return to these highly trained professional engineers that perform such a splendid service for the Nation is surprisingly low. Despite the fact that highway work is of such vast importance, only a few States pay their principal highway engineers as much as $10,000 a year. It is estimated that the median salary for professional engineers engaged in highway work in 1950 was $4,200 a year. Because of these low salaries, the highway commissions of the various states are having difficulty in recruiting young engineers and, in fact, are losing talented young engineers to private industry which is offering from 30% to 40% more in salaries. Similar difficulties with respect to recruiting of young engineers exist in the Federal service.

The difficulty in recruiting young engineers in various segments of the profession is further complicated by the critical shortage of engineers. Following World War II there was a sharp increase in the total enrollment of engineering colleges in the United States which lead to the erroneous belief that engineers were in over supply. Notwithstanding the large graduating classes during the past few years, which produced a peak of almost 53,000 engineering graduates in 1950, there still remains a critical shortage of engineers for national requirements. Although the actual military need for engineering graduates has not been determined, the highly technical nature of modern warfare will require many engineering graduates and experienced engineers in the Armed Services.

In the fall of 1950 the decline in engineering enrollment was much more severe than was experienced in total enrollment in all colleges and universities. The enrollment in engineering schools dropped 18% from the fall of 1949 as compared to a decrease of 9% in over-all college enrollment of male students.[5]

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[5] "Engineering Enrollments and Degrees 1950," by R. C. Story and H. H. Armsby, *Circular No. 287*, Federal Security Agency, Office of Education, March 1, 1951.

The enrollment of engineering freshmen in the fall of 1950 dropped below that of 1940, a pre-World War II year.

Because of the existing shortage and the absolute necessity of efficiently utilizing engineers during the present emergency, Engineers Joint Council, at the request of the National Security Resources Board, established an Engineering Manpower Commission in September, 1950. The objective of the Commission is to develop programs to promote the most effective utilization of the engineering profession in the national health, safety, and interest. The Engineering Manpower Commission of Engineers Joint Council has made recommendations with respect to the importance of engineering to the national economy, the critical shortage of engineers and engineering students, the Selective Service, Universal Military Service, and extended R.O.T.C. proposals. The Commission has issued a pamphlet on "The Critical Shortage of Engineers," distributed a leaflet setting forth the salient facts of the engineering shortage to more than 23,000 high school principals throughout the country and has prepared a pamphlet entitled "Utilizing Engineering Manpower, Deferment Procedures."

In order to fulfill the requirements of industry and organizations utilizing professional engineers it is estimated that a minimum of 30,000 engineering graduates will be needed annually over the next 10 years. Henry H. Armsby, Associate Chief for Engineering Education, U. S. Office of Education, estimates[6] that the young engineering graduates who can reasonably be expected to be available for employment in the engineering field, after requirements of the military services are met, may drop to as low as 12,000 in 1952. During the period from 1954 to 1957 the number will be about 17,000 per year. His estimate is based on current enrollments in engineering colleges, upon pre-Korea trends, and upon the probable effects of the recently announced Selective Service policies with respect to the deferment of college students.

In the preceding paragraphs I have endeavored to stress the importance of the engineering profession in our economic structure and have summarized data showing its pronounced growth during the past 40 years. The Society has enjoyed a continuous growth, but over the years there have been some important changes in the membership, particularly with respect to age and types of employment.

Professional engineers can approach the future with the knowledge that requirements for their services are expanding; however, many do not receive adequate compensation for the services rendered. With about 44% of the membership of our Society now classified as Junior members, most of them serving as employees, it is apparent that the Society must give more consideration to the economic status of engineers. In that connection, I should like to quote a section of the address of the late Henry Earl Riggs, Past-President, ASCE, presented before the annual convention in Salt Lake City, Utah, on July 20, 1938. Professor Riggs was speaking of the problems facing the profession, particularly with reference to unionization of the younger engineer, matters of legislation and welfare confronting engineers, and the necessity for engineers of the United States to emphasize that engineering is a profession. The quotation follows:

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[6] *Circular No. 296*, U. S. Office of Education, Washington, D. C., June, 1951.

> "And if it be true that these problems face the profession today, then it is equally true that the profession must speak through its societies. The societies can no longer devote themselves exclusively to matters of technical theory, construction methods and materials, or of design and research. They must give more thought to registration and the enforcement of registration laws; to the merit system in civil service; to proper classification of employees, and not only to the fixing of adequate salaries for each class, but to insisting that those salaries be paid. They will have to give thought to such matters as political discrimination against engineers, unjust discharge, and many other matters that involve the relationships of the engineer employee of City, State or Federal Government."

The foregoing comments apply with equal force today. Although the Society has continuously maintained high standards of professional ethics and membership qualifications, has contributed greatly to the technical advancement of the engineering profession, and has been effective in many efforts to protect and advance the general welfare of its members, aggressive leadership in the formulation and implementation of policies for improvement of the economic status of engineers has not been assumed. I urge the adoption of such a program, knowing full well that it would be conducted in a dignified, professional manner, commensurate with the highest traditions of the Society. Because of the importance of the professional engineer in our social and economic structure, I am confident that proper economic return for his services can be obtained if the problem is attacked vigorously by the engineering profession as a whole.

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OF THE

AMERICAN SOCIETY OF CIVIL ENGINEERS

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